Symmetries in string theory

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Chapter 3

BRST quantization of strings

In the previous chapter, we described the classical string and the formalism of conformal field theory. We now turn to the quantization of the string, and at many places conformal field theory techniques will be helpful. The constraints encountered in the preceding chapter must somehow be implemented at the quantum level. We will use the BRST formalism to do this. Section 3.1 describes some aspects of the quantization of gauge theories with emphasis on the BRST formalism. In section 3.2 we discuss the BRST quantization of the bosonic string. We also mention the possibility of non-critical strings. In the last section of this chapter, the BRST quantization of W-strings is considered. The latter section contains a summary of the papers [22, 21, 20, 39].

3.1 BRST quantization of gauge theories

Classically, the presence of gauge symmetry means that general solutions of the equations of motion involve arbitrary functions of the coordinates. This indicates that not all fields in the classical action describe independent physical degrees of freedom. In a standard canonical quantization of a gauge theory, one would thus also quantize unphysical degrees of freedom. The presence of such unphysical degrees of freedom is usually reflected in the existence of negative norm states in the Hilbert space. Therefore, it is important to identify the true physical degrees of freedom in a gauge theory. One way to get rid of the unphysical degrees of freedom is to use the gauge transformations to fix a gauge at the classical level. This means that one imposes certain conditions on the fields. These conditions should be accessible: in any gauge equivalence class of field configurations there must be a representative which satisfies the conditions. Also, one would like to fix the gauge completely, such that these representatives are unique. Any non-trivial gauge transformation then leads out of the gauge. Next, one should identify a complete set of independent physical degrees of freedom. Only these physical degrees of freedom need to be quantized. However, there are some problems with this seemingly straightforward approach. First of all, it is not always easy to find good gauge conditions which eliminate all gauge symmetry. Usually, it is also a difficult task to obtain a complete set of independent physical degrees of freedom. Moreover, after imposing gauge conditions, some global symmetries (e.g. Lorentz invariance) of the theory may no longer be manifest.

It is often possible to choose gauge conditions which are manifestly symmetric under the global symmetries of the theory. However, such covariant gauge conditions usually do not fix the gauge completely. Therefore, quantization may still be problematic, in that for example negative norm states may be present in the Hilbert space. Additional constraints must be imposed to eliminate them.

A procedure which does not suffer from the problems mentioned above is BRST quantization. This way of dealing with the gauge symmetry is named after Becchi, Rouet, Stora and Tyutin [15], and is based on a global symmetry of an effective action that is obtained by adding a gauge-fixing term plus compensating ghost terms to the original gauge invariant action. The gauge invariance of the original action is replaced by this global symmetry generated by the BRST charge.

The BRST charge associated to the BRST invariance of the effective action depends on the particular gauge-fixing that is used. In the Hamiltonian formulation, however, the BRST charge does not depend on any kind of gauge-fixing. Already at the classical level the gauge symmetry may be encoded in terms of the BRST charge by enlarging the phase space to include ghost variables. One of the main advantages of the BRST approach is then that covariance is preserved because no gauge conditions need to be imposed. Instead, the unphysical degrees of freedom are eliminated by the procedure of taking the cohomology of the BRST charge. This assumes that the BRST charge is nilpotent, which reflects the closure of the gauge symmetry algebra. All degrees of freedom, including the ghosts, are to be quantized and the cohomology of the quantum BRST charge yields the physical spectrum. Below, we summarize some important ideas of the BRST approach. In particular, we discuss the construction of the classical BRST charge, which will be useful later when we consider BRST charges for gauge theories based on W-algebras. Extensive discussions of the quantization of gauge theories in the Hamiltonian formalism can be found in [110, 111].

Before turning to the BRST formalism, let us first look at a simple example of a gauge theory, Maxwell's theory, to illustrate part of the preceding discussion. Details may be found in most textbooks on quantum field theory. The action is given by

$$S = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} \,, \tag{3.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This action is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \,, \tag{3.2}$$

where Λ is an arbitrary function of the coordinates. The infinite set of Noether currents associated to the gauge symmetry (3.2) is given by

$$j^\mu_\Lambda = F^{\mu\nu} \partial_\nu \Lambda \,. \tag{3.3}$$

The momenta conjugate to A_μ are

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}\,, \tag{3.4}$$

and this yields the primary constraint $T_1 \equiv \Pi^0 = 0$. Using this, we find that the canonical Hamiltonian is

$$\begin{aligned} H &= H_0 - \int d^3x A_0 \partial_i \Pi^i\,, \\ H_0 &= \int d^3x \left(\tfrac{1}{2}\Pi^i\Pi_i + \tfrac{1}{4}F_{ij}F^{ij}\right)\,. \end{aligned} \tag{3.5}$$

There is another constraint which follows from the consistency condition $\dot{\Pi}^0 = 0$. Constraints which arise upon using the equations of motion are called secondary. In this case we have

$$T_2 \equiv -\frac{\delta H}{\delta A_0} = \partial_i \Pi^i = 0\,, \tag{3.6}$$

which is in fact Gauss' law $\vec{\nabla}\cdot\vec{E} = 0$. There are no further constraints. The Poisson brackets among the constraints are trivial, $\{T_a, T_b\} = 0$. Constraints that form a closed algebra under Poisson brackets are called first-class. They are the generators of gauge transformations. One might at first wonder why there are two generators (with two independent gauge parameters), but this is because (3.2) involves the time derivative of Λ which, in the Hamiltonian formulation, is treated independently from Λ and its spatial derivatives.

It is possible to fix the gauge completely by introducing two gauge conditions. For example, the conditions $A_0 = 0$ and $\partial_i A^i = 0$ eliminate all gauge invariance[1]. This selects out the physical degrees of freedom, namely the transverse polarizations of the photon. These degrees of freedom can then be quantized. Unfortunately, though, Lorentz invariance is no longer manifest since a Lorentz transformation leads out of the gauge. In principle, one should then express the Lorentz generators in terms of the physical degrees of freedom and check whether the algebra is still intact quantum-mechanically.

Another possibility is to impose a covariant gauge such as $\partial_\mu A^\mu = 0$, known as the Lorentz gauge. In the Lagrangian formalism this can be done by changing the action to

$$S = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)\,. \tag{3.7}$$

The extra term eliminates the constraints. However, in order that we are still dealing with the same theory, we have to impose $\partial_\mu A^\mu = 0$. In the quantum theory it then turns out that in the Hilbert space one has to impose

$$\partial^\mu A_\mu^{(+)}|\psi\rangle = 0\,, \tag{3.8}$$

[1]To ensure that constraints plus gauge conditions are effectively zero inside brackets, the original Poisson bracket should be replaced by a Dirac bracket.

as a condition for physical states, where the + denotes the positive energy part. This ensures that expectation values $< \psi|\partial^\mu A_\mu|\psi >$ vanish. This is the Gupta-Bleuler quantization procedure. It eliminates all unphysical negative norm states, while Lorentz invariance is manifest. Due to the fact that the condition $\partial^\mu A_\mu = 0$ does not fix the gauge completely, there may still be states containing the scalar and longitudinal photons in a certain combination which satisfies the constraint (3.8), but these have zero norm and decouple.

3.1.1 The BRST formalism

We start with a classical description. As is well-known, and as we have seen in the example above, gauge invariance leads to constrained Hamiltonian dynamics. We assume that the constraints, denoted by T_a[2], are first-class, with $\{T_a, T_b\} = f^c_{ab} T_c$, and generate all gauge transformations. We also assume that the constraints are independent.

In the BRST approach, the gauge symmetry is replaced by a global fermionic symmetry in an extended phase space[3] consisting of the original phase space together with the ghosts c^a and their momenta b_a, with $\{b_a, c^b\} = -\delta^b_a$. The ghosts are anticommuting for bosonic constraints and commuting for fermionic constraints. The generator Q of the symmetry is called the BRST charge. The essential property of Q is its nilpotency, $\{Q, Q\} = 0$. This enables one to define a differential (BRST) complex, consisting of extended phase space functions, with a ghost number[4] grading, in which Q acts as the differential. It is further required that Q is Grassmann odd, has ghost number one, is real (Hermitian), and has the form $Q = c^a T_a + ...$, where the dots denote terms nonlinear in the ghost variables c^a. The BRST charge Q is defined such that its ghost number zero cohomology, $H^0(Q)$, coincides with the set of (equivalence classes of) gauge invariant functions on the constraint surface $T_a = 0$. In more physical terms, the physical gauge invariant functions are those that satisfy $\{Q, f\} = 0$, and two functions f_1 and f_2 are equivalent if $f_1 = f_2 + \{Q, g\}$ for some g. Effectively, what happens is that in going to the cohomology the ghost variables 'kill' the gauge degrees of freedom. All of this is carried out without loss of covariance since no gauge-fixing is applied.

Given any gauge symmetry, one can construct a classical BRST charge Q. This can be done using a procedure which we now sketch. The expansion of Q in ghost variables is

$$Q = \sum_{p=0}^{r} c^{b_{p+1}} c^{b_p} ... c^{b_1} U^{(p)\, a_1 \ldots a_p}_{b_1 \ldots b_{p+1}} b_{a_p} b_{a_{p-1}} ... b_{a_1} \,, \tag{3.9}$$

which is the most general ghost number one expression in the extended phase space. Here $U^{(p)}$ are called the structure functions of order p (they are independent of the ghost variables), and r is called the rank of the set of constraints $\{T_a\}$. The zeroth

[2]For simplicity we suppress coordinate dependence, and in the case of a field theory we assume that the index a includes the spatial coordinates.

[3]This is sometimes called the BFV extended phase space named after Batalin, Fradkin and Vilkovisky.

[4]The (total) ghost number assignments are $G(c^a) = 1\,,\ G(b_a) = -1$ and zero for the other fields. This can be stated equivalently in terms of Poisson brackets with the ghost number charge $b_a c^a$.

order structure functions are required to be the generators T_a so that the first term in (3.9) represents a general gauge generator with parameters replaced by ghosts. The other structure functions can be found by requiring Q to be nilpotent under the Poisson bracket. Each term in $\{Q,Q\}$ contains $n+1$ ghosts c and $n-1$ ghost-momenta b. All these terms have to vanish separately. For $n=1$ this amounts to

$$c^a c^b \left(\{T_a, T_b\} + 2U_{ab}^{(1)\,c} T_c \right) = 0\,. \tag{3.10}$$

The solution, whose existence is guaranteed by the closure of the constraint algebra, is $U_{ab}^{(1)c} = -\frac{1}{2} f_{ab}^c$. For $n=2$, nilpotency implies

$$\begin{aligned} c^c c^b c^a \left(D_{abc}^{(1)\,d} + 4U_{abc}^{(2)\,de} T_e \right) b_d = 0\,, \\ D_{abc}^{(1)\,d} = -f_{[ab}^e f_{c]e}^d + \{f_{[ab}^d, T_{c]}\}\,. \end{aligned} \tag{3.11}$$

For a Lie algebra, where f_{ab}^c are constants, the second term in $D^{(1)}$ of course vanishes, and the first term vanishes due to the Jacobi identities. So in that case there is no second order structure function, and a nilpotent BRST operator is given by

$$Q = c^a T_a + \tfrac{1}{2} f_{ab}^c c^a c^b b_c\,. \tag{3.12}$$

However, in general (and in particular in the case of W-algebras), the structure constants can be field-dependent. The Jacobi identity, which reads $D_{abc}^{(1)\,d} T_d = 0$, then implies that $D_{abc}^{(1)\,d} = X_{abc}^{de} T_e$ for some fully antisymmetric tensor X, and (3.11) is solved by $U^{(2)} = -\frac{1}{4} X$. This procedure continues (if necessary) for the higher order structure functions, and Jacobi identities always guarantee the existence of appropriate structure functions that constitute a nilpotent BRST charge.

Quantization changes the Poisson bracket constraint algebra into a commutator algebra

$$\{T_a, T_b\} = f_{ab}^c T_c \;\rightarrow\; [\hat{T}_a, \hat{T}_b] = i\hbar \hat{f}_{ab}^c \hat{T}_c + \hbar^2 \hat{C}_{ab}\,, \tag{3.13}$$

where the structure constants may have changed by order $\hbar$ quantum corrections, and there may be terms $\hat{C}_{ab}$ (e.g. central charges) that break the naive closure of the quantum algebra. But even if $\hat{C}_{ab} \neq 0$, BRST quantization may still be consistent. The reason is that the ghost variables may not only cancel gauge degrees of freedom, but possibly also anomalies. This happens, for example, with the central charge anomaly in the Virasoro algebra of the 26-dimensional bosonic string, as will be shown in the next section. It is another advantage of the BRST approach. The quantum BRST operator $\hat{Q}$ should involve the quantum structure constants, and is required to be nilpotent in order to be able to identify the physical states with the cohomology classes of $\hat{Q}$. However, due to problems like normal-ordering, it is usually not an easy task to obtain the BRST operator; it is not even guaranteed to exist. Fortunately though, the classical BRST charge often guides us to the corresponding quantum operator.

Before moving on to the application to string theory, we first continue our example of the free electromagnetic field to illustrate the BRST formalism. For the constraints

T_1 and T_2 we introduce corresponding anticommuting ghost fields (c_1, b_1) and (c_2, b_2), respectively, and the classical BRST charge is then simply given by

$$Q = \int d^3x \left(c_1 \Pi^0 + c_2 \partial_i \Pi^i \right) . \tag{3.14}$$

There are no higher order ghost terms since the constraint algebra is abelian. The classical cohomology can then be shown to consist only of the transverse degrees of freedom, see e.g. [111]. A BRST invariant Hamiltonian is given by H_0 in (3.5). Changing the gauge corresponds to adding an appropriate BRST exact expression $\{Q, \Psi\}$ to H_0, where Ψ is called a gauge-fixing fermion. This produces another BRST invariant Hamiltonian.

In the temporal gauge $A_0 = 0$, which eliminates the constraint $\Pi^0 = 0$, the remaining degrees of freedom are (Π^i, A_i)[5]. The residual symmetry is $A_i \to A_i + \partial_i \Lambda(x^j)$, and associated currents are $F^{\mu i}\partial_i \Lambda$. The charges are then given by

$$q_\Lambda = \int d^3x\, F^{0i} \partial_i \Lambda = \int d^3x\, \partial_i \Pi^i \Lambda \, . \tag{3.15}$$

They generate the residual symmetry transformations and may be compared with the L_n and $\bar{L}_n$ Virasoro charges that generate residual transformations in the conformal gauge of a two-dimensional generally covariant plus Weyl invariant theory. Now one only needs (c_2, b_2) to eliminate the longitudinal degree of freedom. For this minimal sector, the BRST charge is given by $Q = \int d^3x\, c_2 \partial_i \Pi^i$. In the subsequent discussion of the BRST formulation of the bosonic string in the conformal gauge, the reader may notice some similarities with the formulation of electromagnetism in the temporal gauge.

3.2 Critical and non-critical bosonic strings

We next study the quantization of the bosonic string using the BRST formalism [92, 126, 119]. Let us first note that we are dealing with first-quantized string theory. That is, in a path integral formulation one sums over all possible paths of a single string between two fixed string configurations. However, from the world-sheet point of view, the quantization of the bosonic string may be regarded as second quantization of the two-dimensional field theory of scalar fields $X^\mu(\sigma, \tau)$ and two-dimensional metric $h_{ab}(\sigma, \tau)$ (in the Polyakov formulation).

We start with a classical BRST analysis. The Polyakov action of the classical bosonic string defines a two-dimensional gauge theory as we described in section 2.1. Let us repeat here the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \, . \tag{3.16}$$

The gauge invariances are two-dimensional general covariance and Weyl symmetry. We know then that there are constraints among the phase space variables. To begin with,

[5] The Dirac bracket is simply the Poisson bracket with (Π^0, A_0) ignored.

there is a primary constraint

$$p_{ab} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}^{ab}} = 0\,. \tag{3.17}$$

The secondary constraint that results from the condition that (3.17) is conserved in time, is the Virasoro constraint,

$$\dot{p}_{ab} = -\frac{\partial \mathcal{H}}{\partial h^{ab}} = \frac{\partial \mathcal{L}}{\partial h^{ab}} = -\frac{1}{4\pi}\sqrt{-h}T_{ab} = 0\,, \tag{3.18}$$

where the energy-momentum tensor is given in (2.3). To get rid of the primary constraint, we impose the gauge condition $h_{ab} = \eta_{ab}$[6], as we did in section 2.1.

Note that the imposition of the conformal gauge condition $h_{ab} = \eta_{ab}$ is similar to the temporal gauge-fixing in electromagnetism. They both eliminate the primary constraint, associated with the Lagrange multiplier fields h_{ab} and A_0 in the respective theories. Whereas in electromagnetism this breaks manifest Lorentz invariance, in the string case we do not lose space-time Lorentz invariance.

Now we are left with the constraint $T_{ab} = 0$ and, as argued in chapter two, in the present conformal gauge $h_{ab} = \eta_{ab}$ the energy-momentum tensor is composed of one holomorphic component $T(z)$ and one anti-holomorphic component $\bar{T}(\bar{z})$ on the complex plane. Alternatively, if one starts from the Nambu-Goto action (2.1), the Virasoro constraints (which are the string-equivalents of the mass-shell constraint $p^2+m^2 = 0$ of the relativistic particle) are primary and there are no other constraints, because there is no independent metric variable. At the Lagrangian level, we already saw in section 2.1 how the Virasoro constraint comes in after substituting the conformal gauge condition into the action: the equation of motion $T_{ab} = 0$ is then lost and has to be imposed as a separate constraint.

The classical BRST charge can now be obtained following the standard procedure outlined before. Thereto we note that the algebra of first-class constraints is

$$\begin{aligned}
T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}\,,\\
\bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{(\bar{z}-\bar{w})}\,,\\
T(z)\bar{T}(\bar{w}) &= 0\,,
\end{aligned} \tag{3.19}$$

which is the classical Poisson bracket algebra (2.20) expressed in the language of OPEs as explained in equations (2.48) and (2.49). Next, introduce anticommuting ghost variables $(c(z), b(z))$ and $\big(\bar{c}(\bar{z}), \bar{b}(\bar{z})\big)$ with canonical Poisson brackets, in OPE language,

$$b(z)c(w) = \frac{1}{(z-w)}\,, \qquad \bar{b}(\bar{z})\bar{c}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})}\,. \tag{3.20}$$

The BRST charge can now be read off from the constraint algebra, knowing that no higher than first order structure functions appear, since the structure constants of the

[6] This gauge is accessible in a neighborhood of any point on the world-sheet. However, it is in general not accessible globally.

Virasoro algebra are really constant, i.e. not field-dependent. Thus the BRST charge is given by

$$Q = \int d\sigma \left(c(z)T(z) + c(z)\partial c(z)b(z) + \bar{c}(\bar{z})\bar{T}(\bar{z}) + \bar{c}(\bar{z})\bar{\partial}\bar{c}(\bar{z})\bar{b}(\bar{z}) \right) . \tag{3.21}$$

We again restrict ourselves to the holomorphic sector, so from now on we consider the BRST charge

$$Q = \oint \frac{dz}{2\pi i} j(z) , \tag{3.22}$$

where $j(z)$ is the BRST current

$$j(z) = c(z)T(z) + c(z)\partial c(z)b(z) . \tag{3.23}$$

The BRST charge can be shown to be nilpotent as is guaranteed by the method of construction. In the language of OPEs the nilpotency statement translates to the first order pole of the OPE $j(z)j(w)$ being a total derivative. Using (3.19) plus the canonical OPE for the ghosts, one easily finds that the first order pole of $j(z)j(w)$ vanishes.

Note that we can rewrite (3.23) as

$$j(z) = c(z) \left(T(z) + \tfrac{1}{2} T_{gh}(z) \right) , \tag{3.24}$$

where the ghost energy-momentum tensor is defined by $T_{gh} = -2b\partial c - \partial bc$. This implies that the conformal weights of c and b are -1 and 2, respectively. Their mode expansions are

$$c(z) = \sum_{n=-\infty}^{+\infty} c_n z^{-n+1} , \qquad b(z) = \sum_{n=-\infty}^{+\infty} b_n z^{-n-2} . \tag{3.25}$$

Now we are ready to quantize. In subsection 2.2.2 we already saw that because of double contractions among the free scalar fields (the string coordinates), the Virasoro algebra obtains a central charge,

$$< T(z)T(w) >= \frac{D/2}{(z-w)^4} . \tag{3.26}$$

Thus the conformal algebra does not close anymore and this is referred to as the conformal anomaly. As a consequence, under the assumption that the BRST operator is still given by the expression (3.23) now taken to be normal-ordered, it is no longer nilpotent in general,

$$Q^2 = \tfrac{1}{2}\{Q, Q\} = \oint \left(\tfrac{3}{2}\partial^2 c\partial c + (-\tfrac{2}{3} + \tfrac{D}{12})\partial^3 cc \right) . \tag{3.27}$$

However, the integrand becomes a total derivative[7] for $D = 26$ and therefore the BRST operator is nilpotent only if the number of scalar fields is 26, i.e. if the string is moving in 26-dimensional space-time. In fact, the conformal anomaly (3.26) is cancelled by the

[7] This total derivative is absent if one adds the total derivative $\frac{3}{2}\partial^2 c$ to the BRST current j. This in addition makes j a $(1,0)$ primary field [90].

ghosts precisely in $D = 26$, because then the total energy-momentum tensor of matter plus ghosts[8],

$$T_{tot} = T + T_{gh} = -\frac{1}{2}\partial X^{\mu}\partial X_{\mu} - 2b\partial c - \partial bc\,, \tag{3.28}$$

has zero central charge. This follows from the OPE of two ghost energy-momentum tensors

$$\underbrace{T_{gh}(z)T_{gh}}(w) = \frac{-13}{(z-w)^4} + \frac{2T_{gh}(w)}{(z-w)^2} + \frac{\partial T_{gh}(w)}{(z-w)}\,, \tag{3.29}$$

which shows that the ghosts contribute -26 to the central charge cancelling that of the matter energy-momentum tensor formed by 26 string coordinates. For $D = 26$, the quantum theory is independent of the world-sheet metric, as is also indicated by the fact that T_{tot} is BRST trivial, $T_{tot} = \{Q, b\}$.

We have witnessed here one of the virtues of BRST quantization: requiring nilpotency of the BRST operator immediately yields the condition for a consistent quantization, in this case the critical dimension $D = 26$.

The physical operators of the theory correspond to cohomology classes of the BRST operator. This will be discussed in the next chapter. Let us mention one subtlety now. The ghost zero modes c_0 and b_0 commute with the BRST invariant Hamiltonian $L_0 + L_0^{gh}$, and from the relations $c_0^2 = b_0^2 = 0$, $\{c_0, b_0\} = 1$ it then follows that the ghost sector ground state is degenerate, and consists of two states, $|\uparrow>$ and $|\downarrow>$, which satisfy

$$\begin{aligned} c_0|\downarrow> &= |\uparrow>\,, \qquad & c_0|\uparrow> &= 0\,, \\ b_0|\downarrow> &= 0\,, \qquad & b_0|\uparrow> &= |\downarrow>\ . \end{aligned} \tag{3.30}$$

In operator language, the state $|\downarrow>$ corresponds to $c(z)$ (i.e. $c(z=0)|0> = |\downarrow>$ with $|0>$ the $sl(2)$-invariant vacuum) and $|\uparrow>$ corresponds to $\partial c(z)c(z)$. These operators are indeed the ones with lowest possible conformal weight (energy) -1 in the ghost sector. If we take $|\downarrow>$ as our ghost vacuum, physical operators corresponding to string states with momentum p_μ will be of the form

$$\mathcal{O} = cV(X,p) = cP(\partial X)e^{ip\cdot X}\,, \tag{3.31}$$

where P is some polynomial in the derivatives of X^μ, and $V(X,p)$ is called a vertex operator.

There is also a simple interpretation for the operators corresponding to the other ghost vacuum, $\partial ccV(X,p)$. It is related to the anomaly in the ghost number current $J_{gh} = cb$ (which is a primary field, classically),

$$\underbrace{T_{gh}(z)J_{gh}}(w) = \frac{-3}{(z-w)^3} + \frac{J_{gh}(w)}{(z-w)^2} + \frac{\partial J_{gh}(w)}{(z-w)}\,. \tag{3.32}$$

This has the consequence that a correlation function on the sphere, say the two-point function $<\mathcal{O}_2(z)\mathcal{O}_1(w)>$, vanishes unless the total ghost number of the operators in

[8]The ghost energy-momentum tensor can also be derived from the BRST invariant gauge-fixed action involving Faddeev-Popov ghost terms, see e.g. [90].

it equals three. One way to see this is as follows. Inserting the ghost number charge $Q_{gh} = \oint \frac{dx}{2\pi i} J_{gh}(x)$ into the correlator and contracting it around z and w using OPEs gives the total ghost number times the correlator. We can also contract the integral over x to a point on the other side of the sphere, in which case we have to use the conformal transformation $x \rightarrow -\frac{1}{x}$. Then we meet no operator insertions and one would expect to get zero, indicating conservation of ghost number. However, due to the anomalous central term in (3.32), which means that J_{gh} does not transform as a tensor under (global) conformal transformations, the result will instead be three times the correlator. Thus, for a correlation function to be nonzero, the total ghost number of the operators in it must be equal to three. Now for a nonzero two-point function, if $\mathcal{O}_1$ is of the standard form (3.31) with ghost number one, $\mathcal{O}_2$ should belong to the other ghost vacuum with ghost number two.

It is not difficult to show that the condition $[Q, \mathcal{O}] = 0$ with $\mathcal{O}$ of the form (3.31) is equivalent to

$$(L_m - \delta_{m,0})|V>= 0 \quad \text{for } m \geq 0\,, \tag{3.33}$$

where L_m are the modes of the matter energy-momentum tensor and $|V>= V(0)|0>$ with V the vertex operator in (3.31). Here $|0>$ is the $sl(2)$-invariant vacuum, defined in equation (2.57).

The conditions (3.33) are nothing but the positive energy modes of the Virasoro constraint $T(z) = 0$ imposed on the Hilbert space. We see therefore that physical states are highest weight states with weight one with respect to the matter energy-momentum tensor. The vertex operators $V(X,p)$ are therefore primary operators with conformal weight one. The specific form of the L_0 constraint is due to a normal-ordering effect. In fact, L_0 is the only mode which has an ambiguous normal-ordering, which can be seen from (the quantum commutator version of) (2.25) and (2.29). In (3.33) we implicitly defined the normal-ordered operator L_0 to be

$$L_0 = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n\,. \tag{3.34}$$

However, an arbitrary constant $-a$ could be added because of the commutation relations $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$. The correct value $a = 1$ directly follows from BRST quantization as we see in (3.33).

A number of other quantization methods have also been used to quantize the bosonic string. One of these methods is the light-cone gauge quantization. This is an example of a full gauge-fixing prior to quantization. The light-cone gauge is obtained from the conformal gauge by using the residual holomorphic and anti-holomorphic reparametrization invariance to eliminate the two light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$. The remaining physical degrees of freedom X^i, $i = 1, 2, ...D-2$, correspond to transverse excitations. These are to be quantized. Disadvantage of this method is that manifest space-time Lorentz invariance is lost. Therefore one should check Lorentz invariance and the result is that the commutators of space-time angular momentum constitute the Lorentz algebra if and only if $D = 26$ and $a = 1$ [100].

Another quantization method, which goes under the name old covariant quantization, is more like the Gupta-Bleuler quantization of electromagnetism. One adopts the conformal gauge, which is manifestly space-time Lorentz covariant, and imposes the Virasoro constraints on the physical states by requiring the positive energy modes to annihilate physical states:

$$(L_m - a\delta_{m,0})|\psi >= 0 \quad \text{for } m \geq 0\,. \tag{3.35}$$

A careful analysis of the norms in the Hilbert space shows that for $a = 1$ and $D = 26$ there are no negative norm states.

3.2.1 Non-critical strings

In the preceding discussion of BRST quantization of the bosonic string, we selected the conformal gauge $h_{ab} = e^{\phi}\eta_{ab}$ and forgot about the conformal factor e^{ϕ} since it drops out of the classical action due to Weyl symmetry. However, it was shown by Polyakov [159] that, quantum-mechanically, the conformal factor may be ignored only if $D = 26$. In other space-time dimensions it becomes dynamical thus providing a non-trivial model for two-dimensional quantum gravity.

The approach of [159] starts with the Euclidean path integral[9]

$$Z = \int \mathcal{D}h\mathcal{D}X \exp\left[-\frac{1}{8\pi}\int d^2\sigma\sqrt{h}\, h^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu}\right]. \tag{3.36}$$

In order to perform the path integration without overcounting, it is necessary to fix a gauge. Let us first write

$$h_{ab} = e^{\phi(\sigma)}\hat{h}_{ab}\,, \tag{3.37}$$

where $\hat{h}_{ab}$ is a fiducial metric. Since the action is also invariant under diffeomorphisms we have to integrate not over all $\hat{h}_{ab}$ but only over those that are not related by a world-sheet diffeomorphism. In fact, it can be shown that the space of metrics modulo diffeomorphisms and Weyl transformations is a finite-dimensional space (see e.g. [69]), the moduli space of Riemann surfaces. For the sphere (genus 0 Riemann surface), the moduli space consists of a single point only.

Gauge-fixing with respect to the group of diffeomorphisms leads to the introduction of the Faddeev-Popov ghosts. After factoring out the volume of the group of diffeomorphisms we are left with

$$Z = \int d\mu\, \mathcal{D}_{e^{\phi}\hat{h}}X\mathcal{D}_{e^{\phi}\hat{h}}\phi\mathcal{D}_{e^{\phi}\hat{h}}b\mathcal{D}_{e^{\phi}\hat{h}}c \exp\left\{-S_P[X,\hat{h}] - S_{gh}[b,c,\hat{h}]\right\}, \tag{3.38}$$

where $\mathcal{D}_{e^{\phi}\hat{h}}$ indicates that the integration measure is evaluated with respect to the original metric h_{ab}, and $d\mu$ is the measure in the space of fiducial metrics (moduli space). S_P is the Polyakov action and S_{gh} the action for the ghost fields,

$$S_{gh}[b,c,\hat{h}] = \frac{i}{2\pi}\int d^2\sigma\sqrt{\hat{h}}\hat{h}^{ab}b_{bc}\hat{\nabla}_a c^c\,. \tag{3.39}$$

[9] We follow here the discussion as given in [5, 2].

Next, one would like to write the integration measures in (3.38) with respect to the fiducial metric, in order to make the dependence on $\phi(\sigma)$ explicit. For the fields X^μ, it can be shown [2] that

$$\mathcal{D}_{e^\phi \hat{h}} X = e^{(D/48\pi) S_L[\phi,\hat{h}]} \mathcal{D}_{\hat{h}} X \,, \tag{3.40}$$

and for the ghost fields

$$\mathcal{D}_{e^\phi \hat{h}} b \mathcal{D}_{e^\phi \hat{h}} c = e^{(-26/48\pi) S_L[\phi,\hat{h}]} \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \,, \tag{3.41}$$

where $S_L[\phi, \hat{h}]$ is the Liouville action for the conformal factor,

$$S_L[\phi, \hat{h}] = \int d^2\sigma \sqrt{\hat{h}} \left(\frac{1}{2} \hat{h}^{ab} \partial_a \phi \partial_b \phi + R^{(2)} \phi + \lambda e^\phi \right) \,, \tag{3.42}$$

in which λ is the cosmological constant, needed for renormalization purposes [159]. Thus, we get

$$\begin{aligned} Z &= \int d\mu \, \mathcal{D}_{e^\phi \hat{h}} \phi \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \\ &\times \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] - \frac{26 - D}{48\pi} S_L[\phi, \hat{h}] \right\} . \end{aligned} \tag{3.43}$$

For $D \neq 26$, the conformal mode of the metric (the Liouville field) becomes a dynamical field. Hence the realization of the conformal symmetry is very different depending on whether $D = 26$ or $D \neq 26$. Non-critical strings are those with $D \neq 26$, and the Liouville field is necessary in order to satisfy the requirement of conformal invariance. For $D = 26$, corresponding to the critical string, the Liouville field decouples from the action and the integration over ϕ can be absorbed in the normalization of the path integral. The only remainder of the integration over metrics in this case is the integration over moduli $d\mu$,

$$Z = \int d\mu \, \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \, \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] \right\} . \tag{3.44}$$

Under some plausible assumptions the dependence of the measure in (3.43) on the conformal factor can be further extracted, resulting in

$$\begin{aligned} Z &= \int \mathcal{D}_{\hat{h}} \phi \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \\ &\times \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] - \frac{25 - D}{48\pi} S_L^{'}[\phi, \hat{h}] \right\} , \end{aligned} \tag{3.45}$$

where $S_L^{'}[\phi, \hat{h}]$ is given by

$$S_L^{'}[\phi, \hat{h}] = \int d^2\sigma \sqrt{\hat{h}} \left(\tfrac{1}{2} \hat{h}^{ab} \partial_a \phi \partial_b \phi + R^{(2)} \phi + \lambda e^{\gamma\phi} \right) \,, \tag{3.46}$$

with

$$\gamma = \frac{1}{12} \left(25 - D - \sqrt{(25 - D)(1 - D)} \right) . \tag{3.47}$$

This expression was obtained by David and Distler and Kawai [66]. A similar analysis but in the so-called chiral gauge was performed in [129]. The results obtained are reasonable only for $D \leq 1$. For $D > 1$ some of the arguments used in [129, 66] might break down. This $c = 1$ barrier (recall that $D = 1$ corresponds to a matter central charge $c = 1$) is usually viewed as a transition to a strong coupling phase of 2d-gravity for $c > 1$. This phase, which might be related to the existence of tachyonic excitations for the bosonic string in space-time dimensions $D > 2$, is still not well-understood.

If we rescale

$$\phi \to \sqrt{\frac{12}{25-D}}\phi\,, \tag{3.48}$$

the propagator of ϕ becomes the same as that of the X^μ fields. The contribution to the energy-momentum tensor of the rescaled ϕ is

$$T_\phi = -\tfrac{1}{2}\partial\phi\partial\phi + \tfrac{Q}{2}\partial^2\phi, \tag{3.49}$$

where $Q = \sqrt{\frac{25-D}{3}}$. The second term in T_ϕ comes from the term $R^{(2)}\phi$ in S_L' and $\frac{Q}{2}$ is called a background charge. The central charge of T_ϕ can be calculated using OPEs, and one finds $c_\phi = 1 + 3Q^2$. We now observe that the total central charge vanishes,

$$c_{tot} = c + c_\phi + c_{gh} = D + 1 + 3Q^2 - 26 = 0\,, \tag{3.50}$$

consistent with overall conformal invariance. Physical operators can now be obtained using the BRST operator (3.24) with T replaced by $T_X + T_\phi$. Operators of the matter theory thus get 'dressed' by the Liouville field in such a way that their total conformal weight equals one [129, 66].

We can interpret (3.45) in two ways. First, since the Liouville action is an effective action for the world-sheet metric, we may interpret (3.45) as a quantum theory of 2d-gravity described by the Liouville field coupled to D scalar fields. Due to the $c = 1$ barrier, we can only consider $D = 0$ or $D = 1$. However, the discussion leading to (3.45) does not depend on the particular matter CFT, but only on its central charge. Therefore it is also possible to consider $c < 1$ minimal models coupled to 2d-gravity. In the other interpretation ϕ is regarded as one of the string coordinates, hence we have a $(D + 1)$-dimensional non-critical string theory. A special case is $D = 25$, where the Liouville field may be viewed as the 26th coordinate of the critical bosonic string. For more information on non-critical string theory and 2d-gravity, also from the matrix model point of view, the reader may consult [97, 1].

3.3 BRST quantization of W-strings

We now come to the study of W-string theories, i.e. string theories based on W-algebras instead of the Virasoro algebra. W-algebras were briefly discussed in subsection 2.2.3. A first question we could ask ourselves is what the analogue of the Polyakov action is. The Polyakov action describes the Weyl invariant coupling of 2d-gravity to matter, and the Virasoro algebra is the symmetry algebra that remains after fixing the conformal gauge. Generalizing the Virasoro algebra to a W-algebra, we know that the W-algebra is the symmetry algebra in a conformal type gauge, resulting from a covariant action describing the W-Weyl invariant coupling of W-gravity to matter. Although W-algebras themselves are known explicitly, it is not a simple matter to obtain such corresponding covariant actions that generalize the Polyakov action. Anyway, a consistent quantization of matter coupled to gravity leads to a string theory, critical or non-critical depending on whether or not the world-sheet gravity field(s) decouple. The next task would then be to study the physical properties of these string theories, such as the spectrum of physical states.

Gauging (the reverse of fixing a gauge) the Virasoro algebra leads to the Polyakov action. Thus one should try to gauge a W-algebra in order to obtain the W-gravity coupling. Let us quickly see how this might be done by considering a simple example of gauging a classical version of the W_3 algebra. For a review, see [115]. See also [36].

Consider a free field theory of scalar fields with action

$$S_0 = \frac{1}{4\pi}\int d^2z \partial\phi_i \bar{\partial}\phi^i \, . \tag{3.51}$$

From the equations of motion $\bar{\partial}\partial\phi_i = 0$, it follows that the chiral part of the symmetry algebra is generated by $\{\partial\phi_i\}$ (and the anti-chiral part by $\{\bar{\partial}\phi_i\}$). In particular, the energy-momentum tensor is given by

$$T = -\tfrac{1}{2}\partial\phi_i\partial\phi^i \, . \tag{3.52}$$

As discussed in the previous chapter, it generates holomorphic coordinate transformations (sometimes referred to as semi-local or semi-rigid symmetries). Gauging this chiral copy of the Virasoro algebra is accomplished by the minimal linear coupling to a spin-two gauge field which enters the action as a Lagrange multiplier imposing the Virasoro constraint. Gauging of both chiral and anti-chiral copies of the Virasoro algebra ultimately yields the Polyakov action.

We could, however, just as well gauge a larger part of the chiral algebra. Consider another conserved current, say, the spin-three current

$$W = \tfrac{1}{3}d_{ijk}\partial\phi^i\partial\phi^j\partial\phi^k \, . \tag{3.53}$$

In order to gauge the corresponding symmetries, the algebra generated by the currents (T, W) must close. This restricts the coefficients d_{ijk} to solutions of

$$d^k_{(ij}d_{lm)k} = \kappa\delta_{(ij}\delta_{lm)} \, . \tag{3.54}$$

The classical OPE of W with itself then closes on the square of T,

$$W(z)W(w) = \frac{-4\kappa\Lambda(w)}{(z-w)^2} + \frac{-2\kappa\partial\Lambda(w)}{z-w} \, , \tag{3.55}$$

with $\Lambda = T^2$ and κ a constant. This, together with the OPEs involving T, is a classical version of Zamolodchikov's W_3 algebra (2.85), and we will refer to it as the w_3 algebra.

The infinitesimal transformation parameters associated to $(T(z), W(z))$ are arbitrary holomorphic functions $(\varepsilon(z), \lambda(z))$ and the transformations they generate are

$$\delta\phi^i(z,\bar{z}) = \oint \frac{dw}{2\pi i} \left(\varepsilon(w)T(w) + \lambda(w)W(w)\right) \phi^i(z,\bar{z}) = \varepsilon(z)\partial\phi^i - \lambda(z) d^i{}_{jk}\partial\phi^j\partial\phi^k \, . \tag{3.56}$$

To gauge this chiral algebra we have to promote this symmetry to one where (ε, λ) can be arbitrary functions (not necessarily holomorphic). This can be done by employing the Noether procedure in which the gauge fields are minimally coupled to the currents. Denoting the spin-two and spin-three gauge fields by h and B, the coupling is given by

$$S = S_0 + \frac{1}{2\pi} \int d^2z \, (hT + BW) \, . \tag{3.57}$$

With appropriate transformation rules for the gauge fields, this action is invariant under chiral w_3 gauge transformations.

It has been shown that for gauging any chiral algebra, a linear coupling to the gauge fields suffices [114]. The action (3.57) is the action of scalar matter coupled to chiral w_3 gravity, but it may also be viewed as the action of covariant w_3 gravity coupled to matter in the so-called chiral gauge. To obtain a covariant non-chiral coupling of w_3-gravity, one should also gauge the anti-holomorphic components[10] $(\bar{T}, \bar{W})$ which satisfy the same w_3 algebra. For such non-chiral gaugings it is not enough to have linear couplings to the gauge fields, and in fact one should add higher and higher order terms to the action to make it gauge invariant. In the case of ordinary gravity, we know that this process can be circumvented, since the coupling to gravity is well-known and takes the form as in the Polyakov action. But for the higher-spin gauge field an *explicit* closed form of the coupling is not known. This is related to the fact that W-geometry is not really well-understood. However, closed forms for non-chiral W-gravity coupled to matter actions are known [173]. These involve auxiliary fields which, after elimination using the equations of motion, again yield an infinite power series in the gauge fields.

Here we do not really need covariant actions of W-gravity coupled to W-matter, since our purpose is at first to obtain the spectrum of physical states of a W-string. This can be obtained by constructing the BRST operator corresponding to the W-algebra (which should be viewed as the algebra of first-class constraints that remain after going to the conformal type gauge). If a nilpotent BRST operator can be constructed, the physical spectrum can in principle be computed through the cohomology.

3.3.1 BRST analysis of W-symmetries

Let us start with the construction of the BRST operator for the W_3 string. Its classical constraint algebra w_3 (in general we write w_N for the centerless classical versions of W_N), takes the form

$$T(z)T(w) \;\; = \;\; \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \, ,$$

[10]It is a consequence of generalized Weyl (W-Weyl) symmetry that the W generators are traceless symmetric tensors which in two dimensions implies that they have only two independent components.

$$
\begin{aligned}
T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}\,, \qquad (3.58)\\
W(z)W(w) &= \frac{\lambda T^2(w)}{(z-w)^2} + \frac{\lambda T\partial T(w)}{z-w}\,,
\end{aligned}
$$

where λ is a constant which depends on the normalization of W. This symmetry is realized by a set of free scalar fields, as in (3.52) and (3.53). We introduce two ghost pairs (c_2, b_2) and (c_3, b_3) corresponding to the spin-two and spin-three symmetries, respectively. They satisfy the standard Poisson bracket

$$
c_i(z)b_j(w) = \frac{\delta_{ij}}{z-w}\,. \qquad (3.59)
$$

It is not difficult to write down the classical BRST current for the algebra in (3.58). Following the procedure described in the previous section, we see that in spite of the fact that the algebra is nonlinear, no second or higher order structure functions are needed. The result is

$$
\begin{aligned}
j(z) &= c_2(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}) + c_3 W + \tfrac{\lambda}{2}c_3\partial c_3 b_2 T\,,\\
T_{c_k,b_k} &= -k b_k \partial c_k + (1-k)\partial b_k c_k\,, \qquad (3.60)
\end{aligned}
$$

with T_{c_k,b_k} the spin-k ghosts' contribution to the total energy-momentum tensor. The BRST charge is given by $\oint \frac{dz}{2\pi i} j(z)$. Note the appearance of T in the cubic ghost term of the BRST current. Here it plays the role of structure constant (first order structure function). It originates from the nonlinearity in the OPE of W with itself: one T is interpreted as generator, the other as structure constant.

A quantum extension of (3.60) for $\lambda = 2b^2$, with $b^2 = \frac{16}{22+5c}$ as in (2.85), is obtained by normal-ordering and adding the following quantum corrections:

$$
j(z) \;\to\; j(z) + \tfrac{25\lambda}{192}c_3(9\partial c_3 \partial^2 b_2 + 15\partial^2 c_3 \partial b_2 + 10\partial^3 c_3 b_2)\,. \qquad (3.61)
$$

The expression for the quantum BRST operator was first found by Thierry-Mieg [187]. The construction of BRST operators for more general quadratically nonlinear Lie algebras has been described in [172]. For the quantum W_3 BRST operator to be nilpotent the central charge must be fixed to $c = 100$. This is the W_3 analogue of the $c = 26$ requirement for the ordinary (Virasoro) bosonic string. The reason for this is that the ghost fields contribute -100 to the central charge (-26 from (c_2, b_2) and -74 from (c_3, b_3)) which can be checked by considering the OPEs of the ghost energy-momentum tensors. This central charge must be cancelled by that of the matter W_3 algebra so that the conformal anomaly disappears.

As in the classical case, the various terms in (3.61) can be associated to terms in the algebra, which has now become the quantum W_3 algebra. The field realizations of T and W have to be normal-ordered, and counterterms that cancel anomalies have to be introduced. These counterterms are such that the renormalized currents satisfy the W_3 algebra [160, 114].

The classical currents (T, W) of equations (3.52) and (3.53) no longer generate a closed algebra quantum-mechanically, since double and triple contractions give rise to terms not proportional[11] to T or W. Such terms can be constant (central charges), or field-dependent. The central terms are sometimes called universal anomalies. The field-dependent terms are called matter-dependent anomalies, and they are characteristic of nonlinearly realized algebras such as is the case here, see (3.56). They do not occur for the ordinary bosonic string, as only a central charge anomaly arises in that case. These anomalies have been calculated for chiral w_3 gravity where they correspond to the non-invariance of the effective action (usually called the induced action), obtained by integrating out the matter fields, under chiral w_3 gauge transformations. For a review, see [115][12]. One can add local counterterms to the action (3.57) of chiral w_3 gravity of the form

$$S_c = \frac{1}{2\pi}\int d^2z\left(a_i\partial^2\phi^i h + (e_{ij}\partial\phi^i\partial^2\phi^j + f_i\partial^3\phi^i)B\right) , \qquad (3.62)$$

for some constants (a_i, e_{ij}, f_i). This means that the currents are modified to

$$\begin{aligned} T &= -\tfrac{1}{2}\partial\phi_i\partial\phi^i + a_i\partial^2\phi^i , \\ W &= \tfrac{1}{3}d_{ijk}\partial\phi^i\partial\phi^j\partial\phi^k + e_{ij}\partial\phi^i\partial^2\phi^j + f_i\partial^3\phi^i . \end{aligned} \qquad (3.63)$$

These appear to be the most general terms that can be added without introducing new fields or dimensionful couplings. It has been shown [164] that all anomalies (universal and matter-dependent) cancel if the coefficients (a_i, e_{ij}, f_i) are such that (3.63) generate the quantum W_3 algebra with central charge $c = 100$, and if ghost variables (c_2, b_2) and (c_3, b_3) are taken into account appropriately. These conditions are equivalent to the existence of a nilpotent BRST operator, as we have seen.

As in the case of the ordinary Virasoro string, one might also consider the non-critical string where the matter does not yield a $c = 100$ realization of the W_3 algebra. Then, not all anomalies are cancelled, and the W-gravity gauge fields become dynamical quantum-mechanically. Much work has been done on the computation of the effective action for quantum W-gravities. In particular, the relation between affine Lie algebras and W-algebras has been exploited to learn more about these effective actions. See for example [36] and references therein. For W_N algebras, much evidence has been provided for the suggestion that the effective W_N gravity actions in the conformal gauge are $sl(N)$ Toda actions, i.e. straightforward generalizations of the Liouville action. These are realized in terms of $N-1$ scalar fields which can be interpreted as coming from generalized conformal factors that become propagating fields quantum-mechanically due to anomalous W-Weyl symmetry.

Both matter and gravity sectors (in the non-critical case) separately should realize the W-symmetry. However, it is not obvious how the complete matter-coupled-to-gravity system realizes the W-symmetry. Due to the nonlinearity of the W-algebra it is impossible to obtain new realizations by just adding two commuting realizations of

[11] Terms proportional to the generators that are not present in the classical algebra do not affect the closure of the algebra but represent a quantum deformation of the classical algebra. Such 'anomalies' are cancelled simply by modifying the transformation laws of the gauge fields.

[12] More recent discussions of chiral W_3 gravity and its anomalies have been given in [148, 190]. In [148], instead of the conformal gauge a derivative gauge condition is employed which leads to a simplification of the Hamiltonian BRST formulation. In [190], the anomalies of chiral W_3 gravity are computed as an illustration of the Lagrangian Batalin-Vilkovisky quantization method. See also [67].

the currents. However, for the classical w_3 algebra we can add two independent copies of the algebra to form a new algebra in the following way. Define

$$T = T_M + T_L \quad \text{and} \quad W = W_M + iW_L\,, \tag{3.64}$$

for two commuting copies (T_M, W_M) and (T_L, W_L) of the w_3 algebra. The Poisson bracket of W with itself then reads

$$W(z)W(w) = \frac{(T_M - T_L)T}{(z-w)^2} + \frac{\frac{1}{2}\partial((T_M - T_L)T)}{(z-w)}\,, \tag{3.65}$$

with structure constant $T_M - T_L$. This algebra is called modified w_3 [26]. The corresponding BRST current is simply (3.60) with T replaced by $T_M - T_L$ in the last term. A corresponding quantum BRST operator was found in [30, 26]. Its current is

$$\begin{aligned} j &= c_2(T_M + T_L + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}) + c_3(\frac{W_M}{b_M} \pm i\frac{W_L}{b_L}) \\ &+ c_3\partial c_3 b_2(T_M - T_L) + \nu c_3\partial^2 c_3 \partial b_2 + \tfrac{5}{3}\nu c_3 \partial^3 c_3 b_2\,, \end{aligned} \tag{3.66}$$

where $(b_{M,L})^2 = \frac{16}{22+5c_{M,L}}$ and $\nu = \frac{c_M - 50}{32}$. It is nilpotent if and only if $c_M + c_L = 100$. We should stress that there is no closed quantum algebra that can be extracted from (3.66). This does not seem to be a problem, since it is the nilpotency of the BRST operator that allows for a consistent quantization.

The construction following (3.64) can be generalized to w_n algebras as follows. Suppose we have two commuting copies of the w_n algebra, generated by spin-k ($k = 2, 3, ...N$) currents v_M^k and v_L^k. Then if we define new spin-k currents

$$v^k = v_M^k + i^{k-2} v_L^k\,, \tag{3.67}$$

we obtain a new closed algebra. This is however not a w_N algebra. Instead, the algebra involves structure constants that are functions of the separate currents v_M and v_L as in (3.65). The reason that this construction gives a closed algebra can be traced back to the fact that w_N algebras only have second-order poles in their classical OPEs (plus first order poles that involve an additional derivative). Since quantum W_N algebras have higher-order poles as well, this construction will not work in that case, as observed for the modified BRST operator for W_3 [30]. We expect that nilpotent BRST operators based on these modified w_N algebras can nevertheless be constructed also for $N > 3$, although explicit expressions will be complicated. For the non-critical W_3 string we assume that (3.66) is the correct BRST operator. Evidence for this has also been given in [37].

Only a few BRST operators for nonlinear W-algebras are known explicitly. Besides the critical and non-critical BRST operators for the W_3 algebra, of the W_N series only the critical W_4 BRST operator is known explicitly [113, 200, 20]. Already for $N = 4$ does the complicated nonlinear structure of the algebra make it very elaborate to compute explicit expressions such as the BRST operator. Moreover, if these expressions have been obtained, they are usually too involved to use in further calculations.

It is expected that BRST operators for arbitrary W_N algebras exist and that they are unique up to (quantum) canonical transformations[13]. Canonical transformations may be used to simplify the form of the BRST operator and the subsequent cohomology analysis. In the case of an explicit realization of the currents, for example in terms of free bosons, a larger number of different canonical transformations can be performed, since the fields in the realization may also be used to build generating functions. In the next subsection we describe realizations of W_N algebras and after that we will see that canonical transformations involving scalar fields of the realization can indeed be used to simplify the BRST analysis of W-algebras [21, 20, 39].

Finally, we mention that also some BRST operators for other than W_N nonlinear algebras are known. These are the BRST operators for a number of $W_{2,N}$ algebras, which are nonlinear algebras generated by an energy-momentum tensor and a spin-N current. A BRST operator for the $W_{2,4}$ algebra has been given in [200], in a realization-independent basis. For $N = 4, 5, 6$ and 7, BRST operators in a realization-dependent basis have been constructed in [139, 142]. A BRST operator for a non-critical $W_{2,4}$ model has been given in [141], see also [153]. Some progress in constructing BRST charges for higher-spin strings ($W_{2,N}$ and W_N strings) has also been reported in [87].

3.3.2 Realizations of W-algebras

Since we are interested in the construction of string theories based on W-algebras, the realizations that are most relevant to us are free bosonic realizations. Fortunately, the W_N algebras, nonlinear algebras generated by currents of spins $2, 3, ...N$, were originally constructed in terms of free bosons [78]. Let us recall this construction. For more details we refer to [78, 144, 65, 136, 32].

The currents of the W_N algebra can be obtained from the following differential operator of order N,

$$(\sqrt{2}\alpha_0)^N D_N = \prod_{m=1}^{N} \left(\sqrt{2}\alpha_0\partial - \vec{h}_m \cdot \partial\vec{\phi}(z)\right) , \tag{3.68}$$

where $\vec{\phi}$ is an $(N-1)$-dimensional vector of scalar fields, and $\vec{h}_m\,,\ m = 1, 2, ...N$ are $(N-1)$-dimensional vectors satisfying

$$\vec{h}_m \cdot \vec{h}_n = \delta_{mn} - \frac{1}{N}\,, \qquad \sum_{m=1}^{N} \vec{h}_m = 0\,. \tag{3.69}$$

Expanding this differential operator in powers of ∂, we can write

$$D_N = \sum_{k=0}^{N} (\sqrt{2}\alpha_0)^{-k} U_k(z) \partial^{N-k}\,, \tag{3.70}$$

[13] At the classical level, it is known that the BRST charge associated to a constraint surface is unique up to canonical transformations, see e.g. [111].

for currents $U_k(z)$ with spin k. As always, normal-ordering is understood. Comparing (3.70) with (3.68) we find $U_0 = 1$, $U_1 = 0$ and, after some rearrangements,

$$U_2(z) = -\tfrac{1}{2}\partial\vec{\phi}(z)\cdot\partial\vec{\phi}(z) - \sqrt{2}\alpha_0\vec{\rho}\cdot\partial^2\vec{\phi}(z)\,, \tag{3.71}$$

where $\vec{\rho} = \sum_{i=1}^{N-1}\vec{\lambda}_i$, the sum of the $sl(N)$ fundamental weights $\vec{\lambda}_i$, is the $sl(N)$ Weyl vector. The central charge of $T \equiv U_2$ is given by

$$c = N - 1 + 24(\alpha_0)^2\rho^2 = (N-1)(1 + 2N(N+1)(\alpha_0)^2)\,. \tag{3.72}$$

Equation (3.70) yields a realization of the W_N algebra for arbitrary central charge parametrized by α_0. In (3.71) we recognize an energy-momentum tensor in terms of $N-1$ scalars. It has been shown [144] that the currents $\{U_k(z)\}\,,\ k = 2, 3, ...N$ generate a closed operator product algebra which is quadratically nonlinear. The transformation from the algebra of free scalar fields to the algebra generated by $\{U_k(z)\}$ is the quantum Miura transformation. For that reason we call the resulting realizations of W_N algebras Miura realizations. We first must note, however, that the currents $\{U_k(z)\}$ are not primary with respect to $T(z) = U_2(z)$, whereas usually the W_N algebras are assumed to be generated by T plus a set of primary currents of spins $3, 4, ...N$. To obtain a primary spin-k current starting from $U_k(z)$, one has to add appropriate terms involving derivatives and composites of lower-spin currents to it. The virtue of a primary basis is that OPEs for primary fields take the relatively simple form given in equation (2.45). A less attractive consequence is that the algebra in the primary basis contains higher than quadratic nonlinearities.

For $N > 3$, the explicit realizations and OPEs become rather awkward. The OPEs of the W_4 algebra have been given explicitly in [35] and [127]. The complete W_5 algebra has been given in [112, 199].

The N weights $\vec{h}_m^{(N)}$ satisfying (3.69) may be defined recursively as [65, 136]

$$\vec{h}_m^{(N)} = \left(\vec{h}_m^{(N-1)}, \frac{1}{\sqrt{N(N-1)}}\right)\,, \tag{3.73}$$

for $1 \leq m \leq N-1$, starting from $N = 2$ with $\vec{h}_1^{(2)} = \left(\frac{1}{\sqrt{2}}\right)$. In other words, the first $N-2$ components of the first $N-1$ vectors are precisely the $\vec{h}_m^{(N-1)}$ vectors of W_{N-1}. The N^{th} vector $\vec{h}_N^{(N)}$ is fixed by the second condition in (3.69). This enables one to re-express the W_N currents in terms of W_{N-1} currents and the scalar field ϕ_{N-1} which is the last component of $\vec{\phi}$ in the W_N realization. This is a special property of the Miura realizations of W_N algebras and turns out to have some restrictive implications on the spectrum of physical states of W_N strings based on these realizations. Explicit expressions for W_N currents in terms of W_{N-1} currents plus a scalar field have been given in [136].

It is known that two classical limits of the Miura realization of W_N algebras can be considered, one being a truncation of the other. First, a classical limit which in general still involves central charges is obtained by letting $\hbar \to 0$[14] and $\alpha_0 \to \pm\infty$ ($c \to \infty$) such that the product $\hbar c = x$ remains finite. This results in a closed Poisson bracket algebra. A truncation to a classical limit without central charges (or background charges) is obtained by now letting $x \to 0$. In this classical limit, which gives the w_N algebras mentioned before, only terms of highest order in the scalar fields (and thus with lowest number of derivatives) survive. In [20], we considered this second classical limit of the Miura transformation. We exploited there the relation between Miura realizations of W_N and W_{N-1} to redefine the W_N currents in such a way that a nested subalgebra structure arises[15]. This structure is induced by the fact that the spin-k currents w_N^k of the redefined w_N algebra only depend on the scalar fields $\{\phi_{N-1}, \phi_{N-2}, ...\phi_{k-1}\}$. For example, the redefined spin-N current is proportional to $(\partial\phi_{N-1})^N$ and therefore automatically defines a subalgebra with field-dependent structure constant,

$$w_N^N(z) w_N^N(w) = c(N) \left(\frac{1}{(z-w)^2} + \frac{\frac{1}{2}\partial}{(z-w)} \right) (\partial\phi_{N-1})^{N-2} w_N^N \,, \tag{3.74}$$

where $c(N)$ is some constant that depends only on N. In the redefined basis, closed expressions have been obtained for all currents of the w_N algebra [20]. It is now clear that the highest-spin current acts only within the Fock space generated by $\partial\phi_{N-1}$. For w_3 this means that the Miura transformation provides reducible representations only (if we also recall that the energy-momentum tensor is diagonal in the scalar fields).

Let us now consider free boson realizations of the quantum W_3 algebra in somewhat more detail. Classically, the condition for a realization (3.53) of the w_3 algebra (3.58) is the quadratic relation for the d_{ijk} tensor given in (3.54). In a different context, an interesting relationship amongst solutions of (3.54) and Jordan algebras was shown to exist in [106]. The classification of such algebras then leads to two classes of classical w_3 realizations: the 'generic' realizations existing for any value n of scalar fields, and four 'magical' solutions with $n = 5, 8, 14$ and 26. It was shown by Romans [168] that all generic solutions can be extended by adding quantum corrections as in (3.63) to realizations of Zamolodchikov's W_3 algebra. In particular, starting from a certain ansatz for the generators, the following n-scalar realization was found:

$$\begin{aligned} T &= -\frac{1}{2}AA - \sqrt{3}\alpha_0 \partial A + T_X \,, \qquad (3.75) \\ W &= \frac{1}{3}AAA + \sqrt{3}\alpha_0 A\partial A + \alpha_0^2 \partial^2 A + 2AT_X + \sqrt{3}\alpha_0 \partial T_X \,, \end{aligned}$$

where A is the derivative of a scalar field. The other $n-1$ scalars are represented by T_X which commutes with A and satisfies a Virasoro algebra with central charge $c_X = \frac{1}{4}c + \frac{1}{2}$. Note that for $c = -2$, T_X is null and can be set to zero, and this is in fact the only central charge for which a one-scalar realization of W_3 exists [43]. The background charge parameter α_0 is related to the central charge c via $c = 2(1 + 24\alpha_0^2)$. The resulting realization coincides for $n = 2$ with the Miura realization derived from (3.70) above, where A should be identified with $\partial\phi_2$. These two-scalar realizations were first obtained

[14] In the string sigma model context, the role of Planck's constant is played by $\alpha' = \frac{1}{4\pi T}$. A scalar field has the dimension of $\sqrt{\hbar}$.

[15] A similar structure is also found in certain linearizing algebras for W_N [130].

in [79]. In fact, all realizations (3.75) are essentially the Miura realization for W_3 which, as we mentioned before, can be written in terms of an explicit scalar field and an energy-momentum tensor that can be realized by the fields of any conformal field theory with central charge c_X. The four 'magical' w_3 realizations cannot be extended to realizations of the quantum W_3 algebra [168, 149, 82, 190, 83]. This is unfortunate, especially since in contrast to the generic (Miura) w_3 realizations the magical realizations are irreducible. The critical and non-critical W_3 BRST operators need realizations of the quantum W_3 algebra and therefore can't be based on the magical realizations of w_3. A more general ansatz for a nilpotent quantum extension of the classical BRST charge (3.60) has been considered in [83] and it seems that the (reducible) Romans realizations are forced upon us for the construction of a W_3 string in terms of scalar fields. In the next chapter we will see that as a consequence, the W_3 string is rather similar to an ordinary bosonic string or two[16]. Nevertheless, some interesting structures are present in the spectrum of W-strings based on Romans realizations.

In [22], we investigated the possibility of additional realizations of the W_3 algebra if a null spin-four field is allowed to occur in the OPE of W with itself. Four two-scalar realizations with nonvanishing but null spin-four fields were obtained for fixed values of the central charge; two $c = -2$ realizations and two $c = \frac{4}{5}$ realizations. One of the $c = -2$ realizations is up to a null energy-momentum tensor precisely the $c = -2$ one-scalar realization of W_3 [43]. Together with one of the $c = \frac{4}{5}$ realizations, it has the property that it can be written in terms of an energy-momentum tensor plus a single scalar field. These realizations are therefore generalizable to multi-scalar realizations. The $c = \frac{4}{5}$ realization with this property is the only two-scalar W_3 realization that has one real and one imaginary background charge or equivalently, in a real basis, has one timelike and one spacelike scalar. This realization appears in the physical state analysis of $W_{2,4}$ [138] and W_4 [40] strings.

The other $c = -2$ and $c = \frac{4}{5}$ realizations are only known as two-scalar realizations and also appear in [12] as specific truncations of a nonlinear W_∞ algebra. They can also be derived from the second realization mentioned in a footnote of [79]. The explicit form of all these solutions can be found in [22]. A family of modulo spin-four W_3 realizations for generic central charge was also found in [22]. These are extensions of the Romans solutions by a null energy-momentum tensor. We note that the modulo spin-four realizations do not satisfy in their classical limit the closure condition (3.54), and therefore, their classical limits do not seem to correspond to the w_3 algebra.

Recently, another method of finding realizations for W-algebras has been proposed in [130]. There it was shown that W-algebras can be embedded into linear conformal algebras. For example, for the W_3 algebra, the linearizing algebra W_3^{lin} (in a non-primary basis) consists of currents $(\hat{J}, \hat{T}, \hat{W})$ of spins $1, 2$ and 3, respectively. The W_3 algebra is then obtained from W_3^{lin} by an invertible redefinition of the currents, of the form $T = \hat{T}$ and $W = \hat{W} + a_1 \hat{J}\hat{T} + a_2 (\hat{J})^3 + a_3 \hat{J}\partial\hat{J} + a_4 \partial\hat{T} + a_5 \partial^2 \hat{J}$ for some coefficients a_i. This implies that given a realization of the linear algebra W_3^{lin}, one obtains a realization

[16]The analysis of [83] does not completely rule out the possibility of W_3 strings based on irreducible scalar field realizations. Also, one may try to include other than free scalar fields in the construction of an 'irreducible W_3 string'.

of the W_3 algebra through this redefinition. One of the advantages of a linear algebra is that two independent realizations give rise to another realization simply by adding the currents of both realizations. Some W_3 realizations have been obtained [130] starting from realizations of the linearizing algebra. However, apart from the Miura realizations these seem to involve vertex operators (see also [169]). A number of modulo null fields realizations of W_3 has also been obtained using the linearizing method [17]. Starting from realizations of bigger algebras, at certain central charges it may happen that all currents except the spin-two and spin-three currents become null (see also [117]). This, then, gives rise to a modulo null fields realization of the W_3 algebra. In this way all previously known modulo spin-four null field realizations and some more can be obtained [17].

We have seen that multi-scalar W_3 realizations can be constructed for arbitrary values of the central charge. In particular, $c = 100$ Romans realizations exist for an arbitrary number of scalar fields. This is important for the construction of a W_3 string. However, it is not difficult to see that any of the known realizations of W_3 for $c = 100$ necessarily involves nonzero background charges. For the Virasoro algebra one simply takes 26 free scalar fields without any background charge to get the critical central charge, but for W_3 taking 100 free scalar fields without background charges does not give a realization. This is somewhat disappointing from a physical point of view, since background charges break Lorentz invariance. Also, the distinction between critical and non-critical W-strings is not as clear as it is for the ordinary bosonic string.

3.3.3 Simplifications and canonical transformations

In order to calculate BRST operators for W-algebras other than W_3 one should find some simplification method. Otherwise, the construction becomes soon too complicated because of the increasing number of nonlinearities. It turns out that the BRST analysis can be simplified by performing certain canonical transformations which in the case of explicit realizations may also involve the fields of this realization.

In [135], it was found that a particular transformation of the fields that enter the description of a W_3 string leads to a great simplification in the analysis of the physical states. Consequently, it was found in [21] that this transformation could be interpreted as a redefinition of the constraints at the classical level. Then in [20] it was shown that a similar redefinition could be used to simplify the BRST analysis of W_N algebras with $N > 3$. These redefinitions lead to a nested subalgebra structure as mentioned before.

To illustrate the idea, we consider the W_3 algebra, as usual. The classical Miura realization for the classical w_3 algebra is given by

$$\begin{aligned} T &= -\tfrac{1}{2}\partial\phi\partial\phi + T_X \,, \\ W &= \frac{1}{6}\partial\phi\partial\phi\partial\phi + \partial\phi T_X \,, \end{aligned} \tag{3.76}$$

in terms of one explicit scalar field ϕ and an arbitrary classical stress tensor T_X (with zero central charge). Now define new generators

$$\tilde{T} = T\,, \qquad \tilde{W} = W - \partial\phi T = \frac{2}{3}(\partial\phi)^3\,. \tag{3.77}$$

Now the main idea of the redefinition is already clear: the redefined spin-three generator only depends on a single scalar field and therefore generates a subalgebra by itself. Indeed, the $W(z)W(w)$ OPE has changed to (cf. equation (3.74))

$$\tilde{W}(z)\tilde{W}(w) = \frac{-6\partial\phi\tilde{W}}{(z-w)^2} + \frac{-3\partial(\partial\phi\tilde{W})}{z-w}\,. \tag{3.78}$$

The corresponding BRST charge can be read off directly from the algebra, with the structure 'constants' appearing in the cubic ghost terms. No higher-order ghost terms are needed. This BRST charge, and also its quantum extension [135], has the nice property that it can be written as a sum of two nilpotent charges Q_1 and Q_2 given, classically, by

$$Q_1 = \oint dz c_2(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2})\,, \qquad Q_2 = \oint dz c_3(W - 3\partial\phi\partial c_3 b_3)\,,$$
$$\{Q_1, Q_1\} = \{Q_2, Q_2\} = \{Q_1, Q_2\} = 0\,, \tag{3.79}$$

where we have dropped the tildes on T and W. In somewhat more physical words, (3.77) shows that whereas gravity (via the metric) couples to all fields, the spin-three gauge field does not.

Alternatively, we can describe a redefinition of the constraint algebra by a canonical transformation in the extended phase space. The generating function of the canonical transformation in the case of the redefinition in equation (3.77) turns out to be $G = -\partial\phi c_3 b_2$. Its action on an extended phase space function F is, in OPE language,

$$F(w) \;\rightarrow\; F(w) + \oint \frac{dz}{2\pi i} G(z)F(w) + \frac{1}{2!}\oint \frac{dz}{2\pi i} G(z) \oint \frac{dx}{2\pi i} G(x)F(w) + \,\ldots\,. \tag{3.80}$$

The BRST current in (3.60) transforms into the one given in (3.79) under this canonical transformation.

Similar redefinitions can be carried out for generic w_N algebras. Starting from the Miura realization in terms of $N-1$ scalar fields, we can perform a redefinition of the generators such that the highest spin current only involves one scalar field, the next highest spin current involves this scalar field plus one other scalar field, etc. Only the energy-momentum tensor then depends on all $N-1$ scalar fields. This necessarily induces a nested subalgebra structure where the k highest spin currents form a subalgebra for $k = 1, 2, \ldots N-1$. Of course, the energy-momentum tensor, which is not affected by the redefinition, always generates a subalgebra by itself. This also results in a nested structure of the BRST charge Q. After the redefinition we can write $Q = \sum_{i=2}^{N} Q_i$, and define nilpotent BRST charges $Q_N^k \equiv \sum_{i=k}^{N} Q_i$ associated to the various subalgebras in the nested basis,

$$\{Q_N^k, Q_N^k\} = 0 \;\text{ for }\; k = 2, 3, \ldots N\,, \qquad \{Q_2, Q_2\} = 0\,, \tag{3.81}$$

where Q_i denotes the 'spin-i contribution' to the BRST charge [20]. This generalizes equation (3.79). The nested basis makes the calculation of the BRST charge simpler since it can now be done in steps, starting with the highest-spin part Q_N and going downwards. The same redefinitions can be performed in one of the sectors of the modified w_N algebras. For modified w_3 this results in the same nested structure as in (3.79). To obtain the quantum BRST operator one can parametrize all possible quantum corrections to the classical expression and then demand nilpotency. For the explicitly known cases (critical and non-critical W_3 and critical W_4), the nested basis survives quantization.

In [20], the BRST operator of the W_4 algebra has been found by making use of the nested basis. This BRST operator will be used in the next chapter to study the physical spectrum of the W_4 string. In references [113, 200] the BRST operator of the W_4 algebra was found in the original basis, where only generators appear as structure constants, but these expressions are very lengthy which makes further calculations practically impossible. One difference is that second and third-order structure functions are nonzero in the original basis, while in the nested basis there are at most terms cubic in the ghost variables (first-order structure functions). However, it should be admitted that also in the nested basis higher-order ghost terms are expected for W_N with $N > 4$.

With the BRST operator in the nested basis, it is possible to study the physical states of the W_4 string [40]. As noticed originally [135] for W_3, the physical state analysis simplifies dramatically in the new basis. Another advantage of the nested basis is that it elucidates an apparent relation of W_N strings with minimal models [65]. This will be discussed at length in sections 4.2 and 4.3.