Symmetries in string theory

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Chapter 2

Strings and conformal field theory

This chapter starts with a description of the classical bosonic string. Special attention is paid to the gauge symmetries of the classical action and to the constraints that result after gauge-fixing. References for section 2.1 and for a general introduction to string theory are [104, 145, 108]. The connection with conformal field theory, which we review in section 2.2, is emphasized. The possibility of extended conformal symmetries is also discussed. More information on conformal field theory may be found in [96, 57, 61]. In section 2.3 we briefly discuss several known string models.

2.1 Classical string action

Let us start by considering a string moving in D-dimensional Minkowski space-time labeled by coordinates X^μ. Space-time indices $\mu, \nu, ...$ are raised and lowered with the Minkowski metric $\eta = \mathrm{diag}[-1, 1, 1, ...1]$. The string sweeps out a two-dimensional surface in space-time, the world-sheet, which is parametrized by $\sigma^a = (\tau, \sigma)$ where $\sigma^0 = \tau$ is the time-evolution parameter and $\sigma^1 = \sigma$ runs along the length of the string.

The dynamics of the string may be deduced from an action proportional to the area of the world-sheet,

$$S_{NG} = -T \int dA = -T \int d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})}\,, \tag{2.1}$$

where $X^\mu(\tau, \sigma)$ describes the embedding of the world-sheet into space-time. The expression in the determinant is the metric on the world-sheet induced by the space-time metric. This action, called the Nambu-Goto action, is the direct analogue of the action of a relativistic point particle, which is proportional to the length of its world-line. The constant T is called the string tension, and it has dimension one over length squared in units in which the action is dimensionless.

The action (2.1) is not always a convenient starting point for string theory due to the presence of the square root. Another action, which is classically equivalent to (2.1) and does not contain the square root, is

$$S = -\frac{T}{2}\int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \,, \tag{2.2}$$

where $h = \det h_{ab}$. This is called the Polyakov action[1]. The world-sheet metric h_{ab} is treated here as an independent variable. Its equation of motion is $T_{ab} = 0$ where we defined the world-sheet energy-momentum tensor

$$T_{ab} \equiv -\frac{1}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} = \tfrac{1}{2}\partial_a X^\mu \partial_b X_\mu - \tfrac{1}{4} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu \,. \tag{2.3}$$

Taking the determinant of $T_{ab} = 0$, one finds $\sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu = 2\sqrt{-\det \partial_a X^\mu \partial_b X_\mu}$, and substitution in the Polyakov action gives back the Nambu-Goto action.

The Polyakov action is the usual starting point for first quantization of the string where one considers a sum over two-dimensional surfaces (string trajectories) in space-time. Note that from the world-sheet point of view, however, it defines a two-dimensional field theory of scalar fields X^μ coupled to two-dimensional gravity.

Let us discuss the symmetries of the Polyakov action. First of all, it is invariant under reparametrizations of the world-sheet, as it should be. For any diffeomorphism $\sigma^a \to \sigma'^a(\sigma)$ [2] the action is invariant under

$$h'_{ab}(\sigma') = \frac{\partial \sigma^c}{\partial \sigma'^a}\frac{\partial \sigma^d}{\partial \sigma'^b} h_{cd}(\sigma) \,, \qquad X'^\mu(\sigma') = X^\mu(\sigma) \,. \tag{2.4}$$

There is an additional local invariance due to the fact that in two dimensions the combination $\sqrt{-h} h^{ab}$ is invariant under local rescalings

$$h'_{ab}(\sigma) = \Lambda(\sigma) h_{ab}(\sigma) \,. \tag{2.5}$$

This clearly is a symmetry of the Polyakov action and is called Weyl invariance. So there are three gauge invariances in total, two diffeomorphisms and one Weyl rescaling. Later we will also discuss the possibility of string models based on even more local symmetries.

The action has global symmetries as well. Because space-time is Minkowskian, the action is invariant under the D-dimensional Poincaré transformations

$$h'_{ab}(\sigma) = h_{ab}(\sigma) \,, \qquad X'^\mu(\sigma) = \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu \,, \tag{2.6}$$

where a^μ is a constant vector and Λ is a constant $O(D-1,1)$ matrix. For a string moving in a more general background, e.g. in some curved instead of Minkowskian space-time, the global symmetry is generally smaller.

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[1] This form of the world-sheet action was introduced in [51]

[2] Here and in the following, the argument σ denotes possible dependence on both $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, unless otherwise stated.

Reparametrization invariance implies the existence of conserved currents $j_b^f = f^a(\sigma)T_{ab}$ for any function $f^a(\sigma)$. Note that these currents vanish on-shell. It is a consequence of Weyl invariance that the trace of the energy-momentum tensor is zero, even without using the equations of motion, since for Weyl rescalings $\delta S = \int d^2\sigma \frac{\delta S}{\delta h_{ab}(\sigma)}\lambda(\sigma)h_{ab}(\sigma) = -T\int d^2\sigma\sqrt{-h}\lambda(\sigma)T^a{}_a(\sigma)$ with λ an arbitrary (infinitesimal) function, so $T^a{}_a$=0 because $\delta S = 0$. This can be verified directly in (2.3). Invariance under global Poincaré transformations leads to the space-time energy-momentum and angular momentum currents

$$
\begin{aligned}
P_\mu^a &= -T\sqrt{-h}h^{ab}\partial_b X_\mu \,, \\
J_{\mu\nu}^a &= X_\mu P_\nu^a - X_\nu P_\mu^a \,.
\end{aligned}
\tag{2.7}
$$

The conservation law $\partial_a P_\mu^a = 0$ is the equation of motion for X^μ, from which $\partial_a J_{\mu\nu}^a = 0$ also follows.

We may now use the local invariances to choose a convenient gauge in which the subsequent analysis simplifies. Since there are three gauge invariances, the two-dimensional metric can be gauged away completely, at least locally. Thus we may take $h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This is known as the conformal gauge. In this gauge the action takes the form

$$
S = -\frac{T}{2}\int d^2\sigma \eta^{ab}\partial_a X^\mu \partial_b X_\mu \,. \tag{2.8}
$$

However, this gauge-fixing does not eliminate the complete gauge symmetry; there are residual symmetries. If a reparametrization $\sigma \to \sigma'(\sigma)$ is such that

$$
\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\eta_{cd} = \Lambda(\sigma)\eta_{ab} \,, \tag{2.9}
$$

we can undo this transformation by a corresponding Weyl rescaling. The conformal gauge is then preserved. The residual symmetry transformations satisfying (2.9) constitute the two-dimensional conformal group. The action (2.8) is invariant under this group[3]. Conformal invariance has proven to be a very important principle in string theory and the 'language' of conformal field theory, described in the next section, is useful for many calculations in string theory, see e.g. the review papers [90, 174].

The equation of motion for X^μ derived from (2.8) is simply the free wave equation

$$
(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = 0 \,, \tag{2.10}
$$

whose general solution is

$$
X^\mu(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \,. \tag{2.11}
$$

Here both X_R and X_L are arbitrary functions of their arguments; they are called right-moving and left-moving string coordinates, respectively. It is useful to introduce the

[3]Equation (2.8) represents an action for free scalar fields and therefore has even more symmetries besides conformal (and Poincaré) invariance.

so-called light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, in which the components of the metric become $\eta_{++} = \eta_{--} = 0\,,\ \eta_{+-} = \eta_{-+} = -\frac{1}{2}$, and $\partial_\pm \equiv \frac{\partial}{\partial\sigma^\pm} = \frac{1}{2}(\partial_0 \pm \partial_1)$. The equation of motion then reads

$$\partial_+\partial_- X^\mu = 0\,. \tag{2.12}$$

We should also take care of the boundary terms that arise in the variation of the action. In the variational principle, the initial and final configurations of the string are fixed, so the fields are not varied at the spacelike boundaries $\tau = \tau_i$ and $\tau = \tau_f$. For a closed string, which has no ends, there are no other boundary terms, but we should impose periodicity $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$. The open string world-sheet has additional boundaries: the trajectories of the endpoints of the string. The corresponding boundary terms in the variation of the action are required to vanish which gives for open strings the boundary condition $\frac{\partial}{\partial\sigma} X^\mu = 0$ at both ends.

Now we can write down the general solution of (2.12) which respects the boundary conditions for the string coordinates. For a closed string the Fourier expansion is

$$\begin{aligned} X_R^\mu(\sigma^-) &= \tfrac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\alpha_n^\mu e^{-in\sigma^-}\,, \\ X_L^\mu(\sigma^+) &= \tfrac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\bar{\alpha}_n^\mu e^{-in\sigma^+}\,. \end{aligned} \tag{2.13}$$

The normalizations of the various modes have been chosen for later convenience. Besides the zero modes x^μ and p^μ there are also positive and negative frequency modes which describe the oscillations of the string. This is what makes strings very different from point particles, of course. The α and $\bar{\alpha}$ oscillators are referred to as right and left-moving modes, or right and left-movers, respectively. Reality of X^μ implies the relations

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu \quad \text{and} \quad (\bar{\alpha}_n^\mu)^* = \bar{\alpha}_{-n}^\mu\,. \tag{2.14}$$

For the open string, the general solution is

$$X^\mu(\sigma, \tau) = x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma\,. \tag{2.15}$$

Here we used the convention that σ runs from 0 to π along the open string[4]. Notice that there is only one set of oscillators α, because the boundary conditions relate right and left-moving waves on the string.

This is not yet the end of the story, since we still have to impose $T_{ab} = 0$ on the solutions (2.13) and (2.15). We missed this equation of motion because the world-sheet metric h_{ab} was replaced by the fixed metric η_{ab} in our gauge-fixed action (2.8). This is something to be aware of in gauge theories: upon fixing a gauge and substituting it in the action, one may lose equations of motion which then have to be implemented separately as

[4]We use the conventions of [145]. It is often convenient to choose a system of units in which $T = \frac{1}{\pi}$ for the open string and $T = \frac{1}{4\pi}$ for the closed string. We shall do this later on.

constraints. In the conformal gauge the components of the energy-momentum tensor in the $\sigma^{\pm}$ coordinate system, $T_{\pm\pm} = \frac{\partial\sigma^a}{\partial\sigma^{\pm}}\frac{\partial\sigma^b}{\partial\sigma^{\pm}}T_{ab}$, are

$$\begin{aligned} T_{--} &= \tfrac{1}{4}(T_{00} - 2T_{01} + T_{11}) = \tfrac{1}{2}\partial_- X^\mu \partial_- X_\mu \,, \\ T_{++} &= \tfrac{1}{4}(T_{00} + 2T_{01} + T_{11}) = \tfrac{1}{2}\partial_+ X^\mu \partial_+ X_\mu \,, \\ T_{+-} &= T_{-+} = \tfrac{1}{4}(T_{00} - T_{11}) = 0\,. \end{aligned} \tag{2.16}$$

Note that $T_{+-} = T_{-+} = 0$ expresses the Weyl invariance of the original theory. Conservation of energy-momentum $\partial^a T_{ab} = 0$ now translates into $\partial_- T_{++} = 0$ and $\partial_+ T_{--} = 0$. This indeed follows immediately from (2.16) and the equations of motion (2.12). Thus the dynamics of the string is described by the solutions (2.13) or (2.15), supplemented by the constraints $T_{++} = 0$ and $T_{--} = 0$.

For later use, let us also briefly discuss the classical string in a Hamiltonian formulation. For the gauge-fixed action (2.8), the momentum conjugate to X^μ is

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu\,. \tag{2.17}$$

We use the conventional notation $\dot{X}^\mu = \partial_0 X^\mu$ and $X'^\mu = \partial_1 X^\mu$. The equal time Poisson brackets are given by

$$\begin{aligned} \{X^\mu(\sigma), X^\nu(\sigma')\} &= \{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\} = 0\,, \\ \{X^\mu(\sigma), \dot{X}^\nu(\sigma')\} &= \frac{1}{T}\eta^{\mu\nu}\delta(\sigma - \sigma')\,. \end{aligned} \tag{2.18}$$

The Hamiltonian for the string in the conformal gauge is

$$H = \int d\sigma(\dot{X}^\mu P_\mu - L) = \frac{T}{2}\int d\sigma(\dot{X}^2 + X'^2)\,. \tag{2.19}$$

Using the fundamental brackets (2.18), we may calculate the Poisson brackets of the energy-momentum tensor. The result is

$$\begin{aligned} \{T_{--}(\sigma), T_{--}(\sigma')\} &= -\frac{1}{2T}\left[2T_{--}(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma T_{--}(\sigma)\delta(\sigma - \sigma')\right]\,, \\ \{T_{++}(\sigma), T_{++}(\sigma')\} &= \frac{1}{2T}\left[2T_{++}(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma T_{++}(\sigma)\delta(\sigma - \sigma')\right]\,, \end{aligned} \tag{2.20}$$

while the other possible brackets vanish. One defines the so-called Virasoro charges as the Fourier modes of the energy-momentum tensor,

$$L_m = 2T\int d\sigma e^{-im\sigma}T_{--}\,, \qquad \bar{L}_m = 2T\int d\sigma e^{im\sigma}T_{++}\,. \tag{2.21}$$

From (2.20) it then follows that the modes satisfy the algebra

$$\begin{aligned} \{L_m, L_n\} &= -i(m-n)L_{m+n}\,, \\ \{\bar{L}_m, \bar{L}_n\} &= -i(m-n)\bar{L}_{m+n}\,, \\ \{L_m, \bar{L}_n\} &= 0\,, \end{aligned} \tag{2.22}$$

under Poisson brackets. This is the classical Virasoro algebra, and L_m and $\bar{L}_m$ are the generators of the infinite-dimensional conformal group in two dimensions. To see this, note that the conformal transformations, which satisfy (2.9), take the infinitesimal form

$$\delta\sigma^+ = \varepsilon f^+(\sigma^+)\,, \qquad \delta\sigma^- = \varepsilon f^-(\sigma^-)\,, \tag{2.23}$$

where $f^\pm$ are arbitrary functions of their argument, and ε is an infinitesimal parameter. The associated conserved currents are given by

$$j_+^f = f^+(\sigma^+)T_{++}\,, \qquad j_-^f = f^-(\sigma^-)T_{--}\,. \tag{2.24}$$

Current conservation is expressed by $\partial_- j_+^f + \partial_+ j_-^f = 0$. The two terms vanish separately, since $\partial_- T_{++} = \partial_+ T_{--} = 0$. Now, if we choose for the functions $f^\pm$ the complete sets $e^{im\sigma^\pm}$ satisfying the periodicity condition of the closed string, we get conserved charges $Q_m = \int d\sigma e^{im\sigma^-} T_{--}$ and $\bar{Q}_m = \int d\sigma e^{im\sigma^+} T_{++}$, closely related to the Virasoro charges L_m and $\bar{L}_m$.

The Poisson brackets (2.18) together with the expansion (2.13) can be shown to yield the algebra of oscillators

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\} = -im\delta_{m+n,0}\eta^{\mu\nu}\,. \tag{2.25}$$

All brackets between barred and non-barred oscillators vanish. The position and momentum of the string have the usual Poisson bracket $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$. It is also conventional to define

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \frac{1}{\sqrt{4\pi T}} p^\mu\,, \tag{2.26}$$

hence (2.25) is also valid for $m = 0$ or $n = 0$. Then

$$\begin{aligned} \partial_- X_R^\mu &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu e^{-in\sigma^-}\,, \\ \partial_+ X_L^\mu &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{+\infty} \bar{\alpha}_n^\mu e^{-in\sigma^+}\,, \end{aligned} \tag{2.27}$$

from which we find that p^μ equals the total conjugate momentum of the string:

$$\int d\sigma P^\mu = T \int d\sigma \dot{X}^\mu(\sigma) = T \int d\sigma (\partial_- X_R^\mu + \partial_+ X_L^\mu) = p^\mu\,. \tag{2.28}$$

Expressed in terms of oscillators, the Virasoro generators are

$$L_m = \tfrac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n\,, \qquad \bar{L}_m = \tfrac{1}{2} \sum_n \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n\,, \tag{2.29}$$

and the Hamiltonian is

$$H = \tfrac{1}{2} \sum_n (\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n) = L_0 + \bar{L}_0\,. \tag{2.30}$$

The notation $A \cdot B$ is used here to mean $A^\mu B_\mu$. The constraints $T_{++} = 0$ and $T_{--} = 0$ translate into $L_m = 0$ and $\bar{L}_m = 0$. Thus we see that $H = 0$ is one of the constraints, and this gives a mass-shell condition

$$M^2 \equiv -p^\mu p_\mu = 2\pi T \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n) \,. \tag{2.31}$$

Using also $L_0 - \bar{L}_0 = 0$, we see that for closed strings the right and left-moving sectors are connected through

$$M^2 = 8\pi T \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 8\pi T \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \,. \tag{2.32}$$

2.2 Conformal field theory and W-algebras

The formalism of conformal field theory has found many uses in physics after the pioneering work of Belavin, Polyakov and Zamolodchikov [16]. For example, in statistical mechanics, the critical behaviour of systems near a second order phase transition point can be described by conformal field theory. Scaling and conformal invariance had been used before to determine some relations between critical exponents, but in [16] it was shown that in two dimensions much more can be done. For some two-dimensional models all critical exponents can be computed exactly using conformal invariance.

Here we are interested primarily in the application of conformal field theory to string theory. We have seen in the previous section that (2.8) is the action of a two-dimensional conformal field theory because the symmetries left over from two-dimensional coordinate invariance and Weyl invariance constitute the conformal group. It turns out that, for consistency reasons, conformal invariance must be maintained upon quantization of the string moving in some background (for which the world-sheet action is not a free field action in general). This means that conformal field theory (CFT) is of great importance for the study of possible string backgrounds. We will return to this point later. Now we wish to review some general results of [16] concerning two-dimensional quantum conformal field theory. Some considerations also apply to the classical theory, though. In the next chapter we discuss the quantization of the string as a conformal field theory with constraints.

The conformal group consists of the transformations that leave the metric invariant modulo a local scale transformation as in equation (2.9). In D-dimensional Minkowski space with $D > 2$, the conformal transformations consist of Poincaré transformations, dilatations (scale transformations) and special conformal transformations, together forming a group that is locally isomorphic to $SO(D,2)$. For more details, see e.g. [96]. It is only in two dimensions that the group of conformal transformations is infinite-dimensional. This is because conformal transformations are those transformations that leave all angles invariant, and in two dimensions these are precisely the complex analytic coordinate transformations

$$z \to f(z) \,, \qquad \bar{z} \to \bar{f}(\bar{z}) \,. \tag{2.33}$$

It is therefore convenient to formulate two-dimensional CFT on the complex plane, and in order to do so in the case of the closed string, we have to map the cylinder (world-sheet) to the complex plane. Complex coordinates on the cylinder are defined by first performing a Wick rotation to Euclidean signature, $\tau \to -i\tau$, and then defining

$$\zeta = i\sigma^- = \tau - i\sigma\,, \qquad \bar\zeta = i\sigma^+ = \tau + i\sigma\,. \tag{2.34}$$

Then we perform the conformal transformation

$$\zeta \to z = e^{\zeta} = e^{\tau - i\sigma}\,, \qquad \bar\zeta \to \bar z = e^{\bar\zeta} = e^{\tau + i\sigma}\,, \tag{2.35}$$

which maps the cylinder to the complex plane with coordinate z. Note that time evolves along the radial coordinate $|z|$; the infinite past corresponds to $z = 0$ and the infinite future to $z = \infty$. Lines of equal time are concentric circles around $z = 0$.

It is sometimes convenient to think of z and $\bar z$ as independent complex variables. So we write both z and $\bar z$ as the arguments of a general field. Holomorphic fields, depending on z only, are written as $A(z)$, and anti-holomorphic fields, depending on $\bar z$ only, as $\bar A(\bar z)$[5]. The analytic substitutions of z and $\bar z$ can then be considered independent, and the conformal group is therefore a direct product of them. At the end one should of course remember that $\bar z$ is the complex conjugate of z.

In a two-dimensional quantum conformal field theory, the complete set of operators decomposes into representations of the conformal group, so-called conformal families $[\phi_m]$. These conformal families consist of a primary operator and an infinite number of secondary operators. Primary operators, denoted here by ϕ_m, are characterized by their special behaviour under conformal transformations. They transform covariantly (as tensors) under (2.33) with $z' = f(z)$ and $\bar z' = \bar f(\bar z)$,

$$\phi'_m(z', \bar z') = \left(\frac{\partial z'}{\partial z}\right)^{-h_m} \left(\frac{\partial \bar z'}{\partial \bar z}\right)^{-\bar h_m} \phi_m(z, \bar z)\,, \tag{2.36}$$

where h_m and $\bar h_m$ are called the conformal weights of ϕ_m. The simplest example of a primary operator is the identity operator with $h = \bar h = 0$. All other operators of a conformal family are called secondary, and among them are all derivatives of any operator in $[\phi_m]$. Their transformations under (2.33) are more complicated.

As we have seen in the previous section, the modes of the energy-momentum tensor are the generators of conformal transformations. In CFT, the energy-momentum tensor has only two independent components, since the trace vanishes as a consequence of scale invariance. In complex coordinates, the two components T_{zz} and $T_{\bar z\bar z}$ (cf. T_{--} and T_{++} in light-cone coordinates) are holomorphic and anti-holomorphic[6] and are denoted simply by $T(z)$ and $\bar T(\bar z)$, respectively.

The variation of a field A under the infinitesimal transformations $z \to z - \varepsilon(z)$ and $\bar z \to \bar z - \bar\varepsilon(\bar z)$ is given by the 'equal-time' commutator

$$\delta_{\varepsilon,\bar\varepsilon} A(w, \bar w) = [Q_{\varepsilon,\bar\varepsilon}, A(w, \bar w)]\,, \tag{2.37}$$

[5]Note that the bar on $\bar A$ has nothing to do with complex conjugation in general.

[6]In the classical theory, this follows from the equations of motion. In the quantum theory it is true inside correlation functions except at points where other operators are inserted.

where the charge $Q_{\varepsilon,\bar{\varepsilon}}$ is given by the spatial integral of the radial (time) component of the corresponding current, which on the complex plane means

$$Q_{\varepsilon,\bar{\varepsilon}} = \frac{1}{2\pi i} \oint \left(dz\, \varepsilon(z) T(z) + d\bar{z}\, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}) \right) . \tag{2.38}$$

The line integral is performed over some circle of fixed radius in the counter-clockwise sense.

Now, as is customary in quantum field theory in Euclidean space, products of operators are only defined if they are time-ordered. In the present case this means that a product $A(z)B(w)$ is only defined for $|z| > |w|$. Thus we define the radial (time) ordering operation R as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |w| > |z| \end{cases} . \tag{2.39}$$

This allows us to evaluate the commutator in (2.37) as follows:

$$\begin{aligned} \delta_{\varepsilon,\bar{\varepsilon}} A(w,\bar{w}) = \frac{1}{2\pi i} \left(\oint_{|z|>|w|} - \oint_{|z|<|w|} \right) \Big[& dz\, \varepsilon(z) R(T(z) A(w,\bar{w})) \\ & + d\bar{z}\, \bar{\varepsilon}(\bar{z}) R(\bar{T}(\bar{z}) A(w,\bar{w})) \Big] \\ = \frac{1}{2\pi i} \oint \left(dz\, \varepsilon(z) R(T(z)A(w,\bar{w})) + d\bar{z}\, \bar{\varepsilon} R(\bar{T}(\bar{z}) A(w,\bar{w})) \right) & , \end{aligned} \tag{2.40}$$

where the two contours in the first line are deformed into a single contour integration of z around w in the last line.

Since everything works the same for holomorphic and anti-holomorphic parts of the theory, we will from now on concentrate on the holomorphic part only. We have learned that the variation of a field A under the infinitesimal transformation $z \to z - \varepsilon(z)$ is given by

$$\delta_\varepsilon A(w,\bar{w}) = \oint \frac{dz}{2\pi i} \varepsilon(z) T(z) A(w,\bar{w}) , \tag{2.41}$$

Here and in the following, we will assume operator products to be radially ordered, and the ordering symbol R will be omitted. Comparing (2.41) with the infinitesimal transformation of a primary field according to (2.36),

$$\delta_\varepsilon \phi_m(z,\bar{z}) = (h_m \partial\varepsilon + \varepsilon\partial) \phi_m(z,\bar{z}) , \tag{2.42}$$

we deduce, using the calculus of residues, that for a primary field ϕ_m

$$T(z)\phi_m(w,\bar{w}) = \frac{h_m \phi_m(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi_m(w,\bar{w})}{z-w} + \text{regular part} . \tag{2.43}$$

We employ the usual notation $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. Also note that the regular part of (2.43) is not determined by the transformation law (2.42). Usually, only the singular part is of interest. It is this part which contains the information of the corresponding equal-time commutator (or Poisson bracket, classically).

Equation (2.43) is our first example of an operator product expansion. The operator product expansion (OPE) is one of the most important tools of conformal field theory. It was originally introduced by Wilson [193] in more general quantum field theories. However, in the case of CFT a lot more can be said about the general form of OPEs. It is assumed that the (infinite) set of all local operators $\{A_i(z,\bar{z})\}$ is complete in the following sense:

$$A_i(z,\bar{z})A_j(0,0) = \sum_k C_{ij}^k(z,\bar{z})A_k(0,0)\,, \tag{2.44}$$

where $C_{ij}^k(z,\bar{z})$ are single-valued functions. This is the operator product algebra of the theory, and it implies that any quantum state can be generated by the action of a linear combination of local operators on the vacuum. One should interpret (2.44) as a prescription valid inside correlation functions, usually valid in a region around $z = 0$ which excludes all insertion points of other operators. From the operator product algebra one can extract all equal-time commutators between fields [193].

The functions $C_{ij}^k(z,\bar{z})$ in (2.44) are severely restricted by conformal invariance. For example, the OPE of two primary fields can be written as

$$\begin{aligned}\phi_m(z,\bar{z})\phi_n(0,0) \quad &= \quad \sum_p \sum_{\{k\}} \sum_{\{\bar{k}\}} C_{mn}^{p;\{k\}\{\bar{k}\}} \\ &\quad \times z^{h_p - h_m - h_n + \sum k_i}\, \bar{z}^{\bar{h}_p - \bar{h}_m - \bar{h}_n + \sum \bar{k}_i}\, \phi_p^{\{k\}\{\bar{k}\}}(0,0)\,,\end{aligned} \tag{2.45}$$

where $\phi_p^{\{k\}\{\bar{k}\}}$ are the secondary fields belonging to the conformal family $[\phi_p]$. They are defined below, in equation (2.61). The sets of positive integers k_i and $\bar{k}_i$ label the secondary fields. The primary fields in this notation are $\phi_p^{\{0\}\{0\}}$. The constants $C_{mn}^{p;\{k\}\{\bar{k}\}}$ can be represented as

$$C_{mn}^{p;\{k\}\{\bar{k}\}} = C_{mn}^p \beta_{mn}^{p;\{k\}} \bar{\beta}_{mn}^{p;\{\bar{k}\}}\,, \tag{2.46}$$

where the factors β and $\bar{\beta}$ can be calculated in terms of the conformal weights of the primary fields involved. These calculations in general get very complicated and an algorithm for their computation has been implemented in a Mathematica [196] package that computes OPEs [185], see also [186]. Thus, only the constants C_{mn}^p are not determined by conformal invariance. Their values are restricted by the requirement of associativity of the operator product algebra. However, the Jacobi identities are too complicated to solve in the general case.

From the conformal properties of the energy-momentum tensor itself (conformal weight two), one can deduce

$$\underbrace{T(z)T}(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,, \tag{2.47}$$

where $\underbrace{A(z)B}(w)$ denotes the contraction of two fields, which is defined to be the singular part of their OPE. In general, the parameter c is nonzero. Note that $T(z)$ is primary only for $c = 0$. The constant c is called the central charge and depends on the particular CFT being studied.

The OPE (2.47) is the quantum version of the classical Poisson bracket algebra (2.20). It encodes the algebra of conformal transformations. Often, a nonzero central charge is a quantum effect, representing an anomaly, i.e. a breakdown of conformal symmetry due to quantization. We will also use OPEs for classical Poisson brackets. This makes calculations in the classical and quantum theory very similar. The Poisson bracket

$$\{A(z), B(w)\} = \sum_{n>0} \{AB\}_n(w) \partial_z^{n-1} \delta(z-w) \tag{2.48}$$

may equivalently be represented by the classical OPE

$$A(z)B(w) = \sum_{n>0} \frac{[AB]_n(w)}{(z-w)^n}\,, \tag{2.49}$$

with the correspondence $\{AB\}_n = \frac{1}{(n-1)!}[AB]_n$.

The modes of T and $\bar{T}$ are defined in the same way as was done in (2.21), apart from a normalization factor. If we define

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\,, \tag{2.50}$$

then L_n here corresponds to the same L_n of (2.21). Comparing (2.50) with (2.38), we see that L_n is the generator of the conformal transformation $z \to z - \varepsilon z^{n+1}$. The inverse of (2.50) expresses T as the Laurent expansion

$$T(z) = \sum_{n=-\infty}^{+\infty} z^{-n-2} L_n\,. \tag{2.51}$$

In general, the mode expansion of an operator $A(z)$ of conformal weight h is defined by

$$A(z) = \sum_{n=-\infty}^{+\infty} z^{-n-h} A_n\,. \tag{2.52}$$

This implies that under a scaling $z \to \lambda z$ the modes scale as $A_n \to \lambda^n A_n$. An OPE can always be represented equivalently by the commutation relations of the modes involved. From (2.47) the following mode algebra is derived:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12} m(m^2-1)\delta_{m+n,0}\,. \tag{2.53}$$

This is the Virasoro algebra. Similar formulas exist for $\bar{L}_m$ and $\bar{T}$, but holomorphic and anti-holomorphic sectors[7] of the theory are independent, since mixed commutators vanish.

The generators $\{L_{-1}, L_0, L_1\}$ span an $sl(2)$ subalgebra of the Virasoro algebra (2.53). The operator L_{-1} simply generates translations; for $z' = z - \varepsilon$ with ε a constant, we

[7] Recall that these correspond to the right and left-moving sectors, respectively, in the string language of the previous section.

have

$$\begin{aligned}\delta_\varepsilon A(w,\bar{w}) &= \oint \frac{dz}{2\pi i}\varepsilon T(z)A(w,\bar{w}) = \oint \frac{dz}{2\pi i}\varepsilon \sum_n (z-w)^{-n-2}L_n(w)A(w,\bar{w}) \\ &= \varepsilon L_{-1}(w)A(w,\bar{w})\,, \end{aligned} \tag{2.54}$$

Here the operators $L_n(w)$ are given by the contour integrals

$$L_n(w) = \oint \frac{dx}{2\pi i}(x-w)^{n+1}T(x)\,, \tag{2.55}$$

these being the Laurent modes of $T(z)$ around the point w. Thus, $L_{-1}(z)A(z,\bar{z}) = \partial A(z,\bar{z})$ for any field $A(z,\bar{z})$, and also $\bar{L}_{-1}(\bar{z})A(z,\bar{z}) = \bar{\partial}A(z,\bar{z})$. In the same way we have for the generators L_0 and $\bar{L}_0$ of dilatations $z' = z - \varepsilon z$ and $\bar{z}' = \bar{z} - \bar{\varepsilon}\bar{z}$, $L_0(z)A(z,\bar{z}) = hA(z,\bar{z})$ and $\bar{L}_0(\bar{z})A(z,\bar{z}) = \bar{h}A(z,\bar{z})$. The sum of the conformal weights $h+\bar{h}$ is called the scaling dimension. The spin of a field is defined by $s = h - \bar{h}$, and characterizes the behaviour under rotations of the complex plane generated by $i(L_0 - \bar{L}_0)$. As in (2.30), the Hamiltonian is given by $H = L_0 + \bar{L}_0$, since this is the generator of translations in the time (radial) direction. Therefore, the scaling dimension of an operator is the energy of the state created by this operator. Finally, the operators L_1 and $\bar{L}_1$ generate special conformal transformations, $z' = z - \varepsilon z^2$ and $\bar{z}' = \bar{z} - \bar{\varepsilon}\bar{z}^2$, infinitesimally. It can be shown that $\{L_{-1}, L_0, L_1; \bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ generate the group $SL(2,\mathbb{C})$ of global conformal transformations

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad a,b,c,d \in \mathbb{C}\,, \quad ad - bc = 1\,. \tag{2.56}$$

These are the only globally defined conformal transformations that map $\mathbb{C}$ one-to-one onto $\mathbb{C}$.

The vacuum state $|0>$ of the theory must satisfy

$$L_n|0>=0 \quad \text{for} \;\; n \geq -1\,. \tag{2.57}$$

Otherwise, the energy-momentum tensor would be singular at $z=0$ ($\tau = -\infty$). The L_n's with $n \geq -1$ generate conformal transformations that are regular at $z = 0$, so (2.57) is a manifestation of conformal invariance of the vacuum state. To obtain a Hilbert space structure, one introduces the vacuum at $z = \infty$ ($\tau = +\infty$), $<0|$. Hermitian conjugation then involves the map $z \to z' = 1/z$. The modes of a primary field $A(z)$ corresponding to a real (Hermitian) field on the Minkowskian cylinder can be shown to satisfy

$$A_n^\dagger = A_{-n}\,. \tag{2.58}$$

Regularity of $T(z)$ at $z = \infty$ implies

$$<0|L_n = 0 \quad \text{for} \;\; n \leq 1\,. \tag{2.59}$$

Note that L_{-1}, L_0 and L_1 annihilate both 'in' and 'out' vacua. We will refer to $|0>$ as the $sl(2)$-invariant vacuum.

2.2.1 Representations of the Virasoro algebra

The Hilbert space of states must form a representation of the symmetry algebra, which in the case of a generic CFT is the Virasoro algebra. Therefore, it is important to know something about the representations of the Virasoro algebra. We mainly discuss here one particular class of representations, namely the conformal families that were already mentioned. They are representations in field space. At the level of states[8], these representations are called Verma modules.

A state created by a primary operator, $|\phi_m> \equiv \lim_{z\to 0}\phi_m(z)|0>$, has the following properties:

$$L_0|\phi_m>= h_m|\phi_m> \quad \text{and} \quad L_n|\phi_m>=0 \quad \text{for} \quad n>0\,. \tag{2.60}$$

The actions of L_{-n}'s with $n>0$ create all secondary states, which are in one-to-one correspondence with the secondary fields of the conformal family $[\phi_m]$. Let us now explain the notation for the secondary fields in (2.45):

$$\phi_m^{\{k\}\{\bar{k}\}} = L_{-k_1}...L_{-k_q}\bar{L}_{-\bar{k}_1}...\bar{L}_{-\bar{k}_r}\phi_m\,, \tag{2.61}$$

where the L_{-k} are the operators defined in equation (2.55), k_i and $\bar{k}_j$ are positive integers and their numbers q and r can be any nonnegative integer. Among the secondary fields, there are so-called quasi-primary fields which are defined by the requirement that $L_1\phi=0$. They transform like primary operators under the global conformal group. The energy-momentum tensor is an example of a quasi-primary operator. Usually, it is enough to consider only primary operators, since conformal Ward identities imply that correlation functions involving secondary fields can always be expressed in terms of correlation functions of primary fields only.

The state $|\phi_m>$ satisfying (2.60) is called a highest weight state of the Virasoro algebra. Together with all descendants (secondaries), it forms a representation of the Virasoro algebra, called a Verma module. Highest weight representations are the physically relevant ones, since they ensure that energy is bounded from below. (For example, using the Virasoro commutation relations, one easily shows that the state corresponding to the operator in (2.61) has $\sum k_i + \sum \bar{k}_j$ units of energy ($L_0+\bar{L}_0$ eigenvalue) relative to the highest weight state.)

An important question is whether or not a particular Verma module constitutes an irreducible representation of the Virasoro algebra. Let us address this question now.

A Verma module can be decomposed into sectors of definite conformal weight, i.e. sectors of states with the same L_0 eigenvalue. Because of the relation $[L_0, L_n]=-nL_n$, the action of L_n on a state increases the conformal weight by $-n$. The level of a state is defined by its conformal weight minus that of the primary from which it stems. At a certain level, there might be states with zero norm, depending on the central charge of the theory and the dimension of the highest weight state. Such zero norm states are called null vectors, and the simplest example is $L_{-1}|0>$. Its norm[9] vanishes, whatever

[8] There is a one-to-one correspondence between states and operators via the relation $|A>=A(0)|0>$.

[9] Norms are calculated using $L_n^\dagger = L_{-n}$. (Actually, (2.58) is generally true for quasi-primary fields.) The Virasoro algebra (2.53) is then used to commute lowering operators to the right to annihilate $|0>$.

the value of the central charge:

$$< 0|L_1 L_{-1}|0 >= 2 < 0|L_0|0 >= 0\,. \tag{2.62}$$

At level 2, a less trivial example is

$$\left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right) |h >\,, \tag{2.63}$$

where $|h >$ is a primary state of conformal weight h. Using the Virasoro algebra, one can show that this state is null in a theory with central charge $c = \frac{2h(5-8h)}{(2h+1)}$. A null vector can be viewed itself as a highest weight state, which means that it defines its own submodule of the Virasoro algebra. A Verma module therefore constitutes an irreducible representation only if it contains no null vectors. It is consistent to divide out a null vector simply by putting it to zero. This eliminates the complete submodule generated by the null vector.

The general formula that tells us when null vectors exist and at what level, is the expression of the Kac determinant [123] (proven in [80]),

$$\det M_N(c,h) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)}\,. \tag{2.64}$$

It is the determinant of the matrix of inner products of all states at level N in the Verma module $M(c,h)$ of a primary field of weight h in a CFT with central charge c. If it is zero, there is a null vector at level N. The product in (2.64) is over all positive integers p, q whose product is less than or equal to N, and α_N is a constant independent of c and h. By $P(N - pq)$ we denote the number of partitions of $N - pq$ into positive integer parts. It represents the multiplicity of the zeroes, since a null vector at level pq gives rise to $P(N - pq)$ null states at the higher level N, these being its descendants. The values $h_{p,q}$ are given by

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}\,, \tag{2.65}$$

where m (in general complex) parametrizes the central charge as

$$c = 1 - \frac{6}{m(m+1)}\,. \tag{2.66}$$

Another important question is whether a Verma module is a unitary representation. Again, we give the result [91] without further arguments. In the region $c \geq 1$ and $h \geq 0$ the Kac determinant never changes sign at any level, so in this region the Verma modules are (or at least can be made) irreducible unitary representations. On the other hand, for $0 < c < 1$ there is only a discrete set of unitary representations, with m in (2.65) and (2.66) taking one of the values $m = 3, 4, 5, \ldots$. If $c < 0$, there will always be negative norm states in a Verma module.

For correlation functions involving primary fields with null descendants[10], one can derive linear differential equations. This provides further restrictions on such correlation functions. In particular, it is possible to derive selection rules for OPEs, known as fusion rules, which determine the possible primary fields that can appear on the RHS of OPEs involving degenerate fields. The fusion rules for the degenerate primaries $\phi_{p,q}$ of conformal weight $h_{p,q}$ may be expressed as

$$\phi_{p_1,q_1} \times \phi_{p_2,q_2} = \sum_{p_3=|p_1-p_2|+1}^{p_1+p_2-1} \sum_{q_3=|q_1-q_2|+1}^{q_1+q_2-1} \phi_{p_3,q_3}\,, \tag{2.67}$$

where p_3 (q_3) runs over even integers, provided p_1+p_2 (q_1+q_2) is odd and vice versa. Equation (2.67) shows that the degenerate conformal families form a closed operator algebra. For special values of the central charge in the region $0 < c < 1$, namely the series (2.66) with $m = 3, 4, 5, ...$, the bounded 'grid' of conformal families $[\phi_{p,q}]$ with $p = 1, 2, ...m-1$ and $q = 1, 2, ...m$, forms a closed operator product algebra. This 'truncation from above' of the fusion rules is a consequence of the existence of additional null vectors at these values of c. The families $[\phi_{p,q}]$ are called completely degenerate in this case and contain two independent null vectors, due to the symmetry $(p, q) \to (m-p, m-q+1)$ of the highest weights (2.65). Since there are only a finite number of conformal families involved in such operator algebras, the models composed of these conformal families are called minimal. In general, minimal models exist for central charges

$$c = 1 - \frac{6(r-s)^2}{rs}\,, \tag{2.68}$$

with r and $s > r$ relatively prime positive integers[11]. However, only for $s = r+1$ these minimal models are unitary.

The case $m = 2$ in (2.66), corresponding to $c = 0$, yields a trivial minimal model. It contains only the identity operator. All its descendants are null, because $L_{-1}\mathbf{1} = 0$ and $L_{-2}\mathbf{1} = T(z)$ is null as well for $c = 0$. A number of minimal models have been shown to describe statistical models undergoing a second order phase transition. As an example, the first non-trivial member of the minimal model series ($m = 3$) has central charge $c = \frac{1}{2}$ and describes the critical point of the Ising model. In chapter 4 we will see that minimal models also arise in the study of W-string spectra.

2.2.2 The example of free scalar fields

We shall now illustrate part of the formalism introduced above by a simple application: the closed bosonic string of the previous section. The action in conformal gauge (2.8) is simply the action of D free massless bosons. From now on we take for the string tension $T = \frac{1}{4\pi}$ and write everything in $(z, \bar{z})$ coordinates. The action then becomes

$$S = \frac{1}{4\pi}\int d^2z \partial X^\mu \bar{\partial} X_\mu\,, \tag{2.69}$$

[10]Such primary fields are called degenerate.
[11]Note that this corresponds to (2.66) with rational $m = r/(s-r)$.

and the propagator of the field X^μ is calculated from

$$\frac{1}{2\pi}\partial_z\partial_{\bar z} < X^\mu(z,\bar z)X^\nu(w,\bar w) >= -\eta^{\mu\nu}\delta^2(z-w,\bar z-\bar w)\,. \tag{2.70}$$

The solution is

$$< X^\mu(z,\bar z)X^\nu(w,\bar w) >= -\eta^{\mu\nu}\log|z-w|^2\,. \tag{2.71}$$

The field $X^\mu(z,\bar z)$ is not a well-defined field due to the logarithmic infrared divergence. However, the equations of motion still imply $X^\mu(z,\bar z) = X^\mu(z) + \bar X^\mu(\bar z)$ everywhere except at coinciding points in correlation functions, and ∂X^μ and $\bar\partial\bar X^\mu$ are well-defined conformal fields with propagators[12]

$$\underbracket{\partial X^\mu(z)\partial X^\nu}(w) = \frac{-\eta^{\mu\nu}}{(z-w)^2}\,, \qquad \underbracket{\bar\partial\bar X^\mu(\bar z)\bar\partial\bar X^\nu}(\bar w) = \frac{-\eta^{\mu\nu}}{(\bar z-\bar w)^2}\,. \tag{2.72}$$

The holomorphic component of the energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial X^\mu(z)\partial X_\mu(z) : \,. \tag{2.73}$$

Normal-ordering, denoted by : ... :, is necessary because of the singularity in the OPE (2.72). It is defined here as

$$: \partial X^\mu(w)\partial X^\nu(w) :\equiv \lim_{z\to w}\left(\partial X^\mu(z)\partial X^\nu(w) + \frac{\eta^{\mu\nu}}{(z-w)^2}\right)\,. \tag{2.74}$$

This is equivalent to taking the constant $O((z-w)^0)$ part of the OPE. Henceforth, we will use this same normal-ordering prescription for composite operators, which in the general case is expressed by [9]

$$: A(z)B(z) :\equiv \oint \frac{dw}{2\pi i}\frac{A(z)B(w)}{z-w}\,. \tag{2.75}$$

We shall usually omit the normal-ordering symbols. The normal-ordering prescription is not associative, so when a product of more than two operators at the same point is considered, the convention is to start normal-ordering from the right,

$$: A(z)B(z)C(z) :\equiv : A(z) \; : B(z)C(z) :: \,. \tag{2.76}$$

In order to calculate OPEs between composite operators, we also need the following Wick rule [9]

$$\underbracket{A(z) \; : B}C : (w) = \oint \frac{dx}{2\pi i}\frac{1}{x-w}\left(\underbracket{A(z)B}(x)C(w) + B(x)\underbracket{A(z)C}(w)\right)\,. \tag{2.77}$$

Using the Wick rule, we calculate the OPE of $\partial X^\mu(z)$ with the energy-momentum tensor and see that it is a primary of weight $(h,\bar h) = (1,0)$:

$$\underbracket{T(z)\partial X^\mu}(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w}\,. \tag{2.78}$$

[12] The propagator or two-point function is the term in the OPE which contains the unit operator.

Now the OPE of T with itself can be calculated using the Wick rule once again, and the result is

$$\underbrace{T(z)T}(w) = \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \tag{2.79}$$

Thus, the central charge is D and each scalar field contributes 1 to it. The fourth order pole comes from double contractions in

$$T(z)T(w) = \frac{1}{4}\ :\partial X^\mu \partial X_\mu : (z)\ :\partial X^\nu \partial X_\nu : (w)\,. \tag{2.80}$$

There are two double contractions that can be made, resulting in $\frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}/(z-w)^4$, which is the first term in (2.79).

We can construct other primaries from the field $X^\mu(z,\bar{z})$. The operators

$$V_p(z,\bar{z}) =: e^{ip\cdot X(z,\bar{z})} : \tag{2.81}$$

are primary with conformal dimension $(\frac{p^2}{2}, \frac{p^2}{2})$. Classically, however, a field $e^{ip\cdot X}$ would have no conformal dimension, because X^μ is dimensionless. The conformal dimension of (2.81) is a quantum effect, and is due to double contractions in the OPE $T(z)V_p(w,\bar{w})$. The operators (2.81) are called vertex operators. In string theory their insertion on the world-sheet represents an incoming string in its ground state with momentum p_μ.

2.2.3 W-algebras

The Virasoro algebra plays an important role in string theory because it is the underlying world-sheet symmetry algebra of the bosonic string. It is interesting now, to think about the possibility of extending the world-sheet symmetry. We have seen that the Virasoro algebra is the algebra of modes of the (quasi-)primary spin-two generator $T(z)$. Extended conformal algebras are obtained by the inclusion of additional generators. Then the Virasoro algebra encoded in the OPE $T(z)T(w)$ constitutes a subalgebra. In fact, extensions by generators of spins less than two have been studied for quite a long time. Examples include the $N=1$ superconformal algebra which has an additional spin-$\frac{3}{2}$ fermionic current and the $N=2$ superconformal algebra which has two fermionic spin-$\frac{3}{2}$ currents and a spin-1 current besides the energy-momentum tensor. We will normally use the name W-algebra for an extended conformal algebra with at least one current of spin greater than two. Some other and more precise definitions of W-algebras exist in the literature, but for our purpose the description just given is sufficient. For a review of W-symmetry in conformal field theory, see [49]. There is also a recent volume of reprints on W-symmetry [50].

Zamolodchikov was the first to construct W-algebras in [198]. He argued that if among the primary fields in the theory there is a field Q_s with conformal dimensions $(h,\bar{h}) = (s,0)$ (and therefore spin s), where s is some integer or half-integer number, then there is an additional infinite symmetry in such a theory. A field with $\bar{h}=0$ is often called a chiral field; it is necessarily holomorphic, $\bar{\partial}Q_s = 0$. Therefore, an infinite number of conserved currents take the form

$$j_s^f(z) = f(z)Q_s(z)\,, \qquad \bar{\partial} j_s^f(z) = 0\,, \tag{2.82}$$

where $f(z)$ is an arbitrary analytic function. These currents generate the additional symmetry. The set of all holomorphic fields generates what is called the chiral algebra $\mathcal{A}$ of the theory. Similarly, the anti-chiral algebra $\bar{\mathcal{A}}$ is generated by the set of anti-holomorphic fields. The full symmetry algebra is the direct product

$$\mathcal{G} = \mathcal{A} \otimes \bar{\mathcal{A}} . \tag{2.83}$$

The Hilbert space of the theory should therefore decompose into representations of $\mathcal{G}$,

$$\mathcal{H} = \bigoplus_{m,\bar{m}} [\phi_m] \otimes [\bar{\phi}_{\bar{m}}] . \tag{2.84}$$

Of special interest are the so-called rational conformal field theories (RCFTs) in which the sum in (2.84) is over a finite number of representations of $\mathcal{G}$.

We have seen that minimal models are RCFTs with the Virasoro algebra as chiral algebra. Minimal models with W-symmetry also exist. Although they usually contain an infinite number of conformal families (Virasoro representations), they decompose into a finite number of representations of the W-algebra. As usual, we shall restrict ourselves to the chiral part $\mathcal{A}$ of the symmetry algebra.

Most W-algebras are nonlinear. This means that normal-ordered products of generators appear in the commutator (OPE) of two generators. To see this, consider a W-algebra whose maximal spin generator is Q_s with spin s. The OPE $Q_s(z)Q_s(w)$ generally produces (in the second order pole) operators with spin $2s-2$, which is greater than s for $s > 2$. Such operators consist of normal-ordered products of some currents. This argument cannot be applied to W-algebras with an infinite number of currents with increasing spins. Indeed, linear W-algebras with an infinite number of higher spin currents are known [11, 163].

A characteristic example of a W-algebra is the W_3 algebra [198]. It is the algebra generated by the energy-momentum tensor $T(z)$ together with an additional spin-three current $W(z)$. In terms of OPEs, the algebra is given by

$$\begin{aligned}
\underbrace{T(z)T(w)} &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} , \\
\underbrace{T(z)W(w)} &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} , \\
\underbrace{W(z)W(w)} &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
&\quad + \frac{3}{10}\frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15}\frac{\partial^3 T(w)}{z-w} \\
&\quad + \frac{16}{22+5c}\left(\frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{z-w}\right) ,
\end{aligned} \tag{2.85}$$

where $\Lambda =: TT : -\frac{3}{10}\partial^2 T$. The only nonlinear term is the normal-ordered product of two energy-momentum tensors. The first equation in (2.85) gives the Virasoro algebra, while the second equation expresses the fact that W is a primary field of spin three. Also for more general W-algebras, it is customary to work in a basis in which all generators (except $T(z)$) are primary with respect to $T(z)$. The last equation of (2.85) tells us that the OPE of two spin-three generators gives rise to the conformal family of the unit operator. Up to a normalization, the coefficients in this equation are all determined by conformal invariance (cf. β-coefficients in (2.46)). The W_3 algebra exists (satisfies the Jacobi identities) for any value of the central charge c.

The W_3 algebra has been generalized to a series of W-algebras involving higher spin generators by Fateev and Lukyanov [78]. These are called W_N algebras and consist of generators of spins $2, 3, 4...N$, the spin-two generator being the energy-momentum tensor. In [78] the generators of a W_N algebra are constructed out of free scalar fields. This has the advantage that if a closed algebra is obtained (closure of W_N algebras was proven in [144]), it automatically satisfies the Jacobi identities. In the following chapters we investigate the possibility of string models based on W_N symmetries.

Although we will not use W-algebras other than the W_N series, we wish to spend a few words on some other known W-algebras, and especially their construction. There are essentially three ways by which W-algebras are constructed, see [49] for a detailed review and [50] for reprints and an extensive list of references.

The first approach is to try to write down an extended algebra by proposing a number of extra generators with given spins and closing the algebra. Usually, the extra generators are assumed to be primary and therefore the form of the OPEs is already fixed by conformal invariance as in (2.45). The difficult step in this approach is to guarantee that the algebra is associative, thereby restricting the OPE coefficients C^p_{mn}. The W_3 algebra (2.85) was in fact obtained in this way in [198]. Other W-algebras with two and three generators, among them the W_4 algebra, were constructed according to this approach in the papers [35] and [127].

A second, much more systematic approach of constructing W-algebras is the so-called Drinfeld-Sokolov reduction. This associates a W-algebra to any embedding of $sl(2)$ into a simple Lie algebra $\mathbf{g}$, see [10, 38] and references therein. This results in a large class of W-algebras. The W_N algebras are obtained if one takes the so-called principal embedding of $sl(2)$ into $sl(N)$.

A third approach is to start from a known model of a CFT and to see if there are extended symmetries in that model. Additional currents are then formed from the fields in the model. For example, in a free field theory, currents may be constructed out of the free fields. The closure of the algebra should be checked and the Jacobi identities are automatically satisfied. Another example is given by the Casimir algebras of reference [9]. In this paper it was shown that W-algebras can be obtained from an affine Lie algebra $\hat{\mathbf{g}}$ by the construction of the Casimir invariants in terms of the affine currents. The W_N algebras are the Casimir algebras of $\widehat{sl(N)}$.

Let us end this section with some remarks on representations of W-algebras. Highest weight representations of W-algebras are defined in the same way as those for the Virasoro algebra. The highest weight state is now characterized by its weights under all generators of the W-algebra. For the W_N algebra with generators $W^{(k)},\ k = 2, 3, ...N$, of spin k, one defines a highest weight state $|w> \equiv |w^{(2)}, w^{(3)}, ...w^{(N)}>$ by the requirement

$$\begin{aligned} W_0^{(k)}|w\rangle &= w^{(k)}|w\rangle\,, \\ W_n^{(k)}|w\rangle &= 0 \quad \text{for} \quad n>0\,. \end{aligned} \tag{2.86}$$

The modes of the currents are defined by the usual expansion $W^{(k)}(z) = \sum_{-\infty}^{\infty} z^{-n-k} W_n^{(k)}$. The W_N representations are then given by the set of states obtained by acting with linear combinations of strings of 'raising operators' $W_{-n}^{(k)}$ $(n>0)$ on the highest weight state.

As mentioned before, minimal models for W_N algebras also exist. Their central charges and operator contents were given in [79] for W_3 and in [78] for W_N in general. The central charges of unitary W_N minimal models are given by

$$c = (N-1)\left(1 - \frac{N(N+1)}{m(m+1)}\right)\,, \tag{2.87}$$

with $m \in \{N, N+1, N+2, ...\}$. These minimal models describe the critical behaviour of certain $\mathbb{Z}_N$ symmetric statistical systems [79, 78]. An interesting feature is that unlike Virasoro minimal models, W_N minimal models also exist for $c \geq 1$. Thus, an infinite number of Virasoro primary fields in $c \geq 1$ models may sometimes be rearranged into a finite number of W-algebra primary fields.

2.3 Some different types of string theories

In the first section of this chapter we described the classical action of the ordinary bosonic string. It is the simplest among the string theories in that it is the direct one-dimensional analogue of the relativistic particle. In the next chapter we will demonstrate the well-known result that a consistent quantization requires this string to move in a space-time of dimension $D = 26$. In order to make contact with four-dimensional physics, 22 of the 26 dimensions would somehow have to be 'compactified' such that they are invisible at low energies. Also, the bosonic string turns out to have a tachyon in its physical spectrum (see chapter 4), and no fermions. This means that the bosonic string can never yield a realistic theory of known elementary particles and their interactions. Therefore, other types of strings have been constructed. The superstring has much better physical properties; there are no tachyons and it has fermions as well as bosons in its physical spectrum. The supersymmetry of the superstring also improves the convergence of scattering amplitudes. It still needs to live in an 'unphysical' space-time dimension $D = 10$, though. However, this is not an insurmountable problem, in principle. Six of the ten dimensions may be compactified (in accordance with the equations of the theory, of course) to an internal space of incredibly small size that can't be observed by any of our instruments. Different compactification procedures give rise to different possible gauge interactions and matter content of the four-dimensional low-energy theory; some of them even contain the standard model gauge group and particles.

The classical action of the superstring is given by

$$S = -\frac{1}{8\pi} \int d^2\sigma \{\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\}\,, \tag{2.88}$$

where ψ^μ is a Majorana spinor and ρ^a are two-dimensional Dirac matrices. From the space-time point of view, ψ^μ is a vector. This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= \bar{\varepsilon}\psi^\mu\,, \\ \delta\psi^\mu &= -i\rho^a \partial_a X^\mu \varepsilon\,, \end{aligned} \tag{2.89}$$

where ε is an infinitesimal constant anti-commuting Majorana spinor. The action (2.88) should be compared to the bosonic string action in the conformal gauge (2.8). Here it is a gauge-fixed version of a locally supersymmetric action, i.e. the action of two-dimensional supergravity coupled to the 'matter' fields X and ψ. As a consequence of the gauge-fixing, the equations of motion of (2.88) have to be supplemented by two constraints:

$$J_a = 0 \quad \text{and} \quad T_{ab} = 0\,. \tag{2.90}$$

The current J_a is the conserved Noether current associated to the supersymmetry (2.89), and T_{ab} is the energy-momentum tensor. Together, J and T generate the $N = 1$ superconformal algebra. The superstring is the best known example of a string based on extended conformal symmetry.

It is known that superstring theory not only possesses world-sheet supersymmetry but that it also gives rise to a space-time supersymmetric physical spectrum (and amplitudes). This is normally an $N = 2$ supersymmetry, i.e. there are two space-time supersymmetry generators. However, space-time supersymmetry is not at all manifest in the approach based on the action (2.88). A world-sheet action in which space-time supersymmetry is manifest from the beginning was introduced by Green and Schwarz, see [104] for additional information.

We now list the well-known consistent superstring theories. They all need to live in ten-dimensional space-time.

- Type I: a theory of open[13] strings. The open string boundary conditions break one half of the supersymmetries, so the remaining space-time supersymmetry is $N = 1$. Gauge charges can be attached to the ends of an open string. The only acceptable gauge group turns out to be $SO(32)$.
- Type IIA: a theory of closed strings only, and unbroken $N = 2$ space-time supersymmetry. The two supersymmetries are of opposite chirality. There is no freedom to introduce Yang-Mills fields. They can only appear after compactification.
- Type IIB: same as type IIA, except that the two supersymmetries have the same chirality.

[13]Closed strings can be formed in the interacting theory by the joining of the end-points of an open string.

- Heterotic string: closed strings only. Here the fact that left and right-moving sectors of the theory are independent is utilized. Supersymmetry is introduced only in one of the sectors. The other sector is purely bosonic and gives rise to Yang-Mills fields. Consistency demands either $SO(32)$ or $E_8 \times E_8$ gauge symmetry. The heterotic string has $N = 1$ space-time supersymmetry.

Strings based on extended world-sheet supersymmetry ($N = 2$ and $N = 4$) have also been considered. Their physical interpretation, if any, does not seem to be very clear yet. In the previous section we discussed, among other things, different extensions of conformal symmetry, namely W-symmetry. It is expected to be possible to construct string theories based on W-algebras [33]. The full world-sheet action should then have a local W-symmetry (W-gravity). In the gauge-fixed version of this action, the vanishing of the currents of the W-algebra must be imposed as constraints. In the next two chapters we will describe some aspects of W-strings.

So far, we discussed the possibility of string theories based on different world-sheet gauge algebras. Another property that might distinguish different string theories has also been mentioned: the world-sheet topologies allowed, i.e. open or closed strings[14]. We have not yet discussed the possibility to choose different realizations of the world-sheet gauge algebras corresponding to different (extended) conformal field theories. For the Virasoro algebra we described an explicit realization in terms of the string coordinates X^μ in equation (2.73), corresponding to the action (2.69). We can also study string theory in more general backgrounds or, in other words, take a different conformal field theory as a starting point. However, it is believed that taking different CFTs corresponds to having different vacua of the same string theory.

As an example, let us consider the bosonic string again, and let it propagate in a more general background described by the nonlinear sigma model

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\{g_{\mu\nu}(X)h^{ab}\partial_a X^\mu \partial_b X^\nu + B_{\mu\nu}(X)\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\}\,, \qquad (2.91)$$

where $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ are the background metric, antisymmetric tensor and dilaton field. $R^{(2)}$ is the world-sheet curvature scalar and ε_{ab} is the antisymmetric symbol. The string tension has been expressed in terms of the constant $\alpha' = \frac{1}{2\pi T}$. This action is reparametrization invariant and the first two terms are also invariant under Weyl rescalings of the metric. It has been shown that for a consistent quantization the theory should be Weyl invariant at the quantum level, and this is only true if the background fields satisfy certain equations[15] [54]. In fact, any conformal field theory[16] may serve as a string background. However, the bulk of CFTs do not seem to have a space-time interpretation. Nevertheless, it is clear that a classification of CFTs is important for string theory: it amounts to a classification of all possible string vacua. The problem then arises which CFT should be regarded as the true vacuum and why.

[14] See [155] for a nice exposition.

[15] These background field equations are given, to lowest order, in section 5.1.

[16] Apart from some restrictions such as the critical value of the central charge and modular invariance.

This is an unsolved problem. However, we will see in chapter 5 that not all seemingly different string backgrounds are inequivalent. There are certain transformations called T-dualities that map one CFT to another one describing the same string dynamics.

Not only is it known that certain string backgrounds are equivalent, there are also indications that even string theories based on different world-sheet algebras can be equivalent. It was first shown by Berkovits and Vafa [29] that the $N = 1$ superstring in a special background is equivalent to the bosonic string. Later, similar relations were found between strings based on different N-extended superconformal algebras or even different W-algebras. However, the meaning of this is still not very clear, since all these relations are only valid in very special backgrounds. In section 4.4 we consider some examples.

Perhaps more promising are some of the recently conjectured strong/weak coupling dualities in string theory. They relate the weakly coupled phase of one string (or a more general extended object) to the strongly coupled phase of another. This will be further discussed in section 5.3.

Finally, as another class of strings we should mention the non-critical strings. In the case of the bosonic string it is impossible to respect both diffeomorphism and Weyl invariance in the quantum theory, unless we are in the critical dimension. However, one might choose to give up Weyl invariance, in which case the two-dimensional metric becomes a dynamical field quantum mechanically[17] [159]. In this way, it is possible to obtain strings moving in dimensions below the critical one. Such strings are called non-critical. It would for example be interesting to look at non-critical strings in four space-time dimensions. However, for the bosonic string there are serious problems for $D > 2$, related to the presence of tachyons in the spectrum. As these problems seem to be shifted to higher dimensions in the case of W-strings, this is one of the motivations for studying (non-critical) W-strings. More on this in section 3.2.1.

[17] Therefore, non-critical strings might also teach us something about two-dimensional quantum gravity.