On the relation between global properties of linear difference and differential equations with polynomial coefficients III

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Abstract

This paper represents the third part of a contribution to the "dictionary" of homogeneous linear differential equations with polynomial coefficients on one hand and corresponding difference equations on the other. In the first part (cf. [5]) we studied the case that the differential equation (D) has at most regular singularities at O and at ∞, and arbitrary singularities in the rest of the complex plane. We constructed fundamental systems of solutions of a corresponding difference equation (Δ), using integral transforms of microsolutions of (D) at its singular points in $\mathbb{C}^*$. In the second part ([3]) we considered differential equations having at most a regular singularity at ∞ and an irregular one at O. We used integral transforms of asymptotically flat solutions of (D) to define a fundamental system of solutions of (Δ), holomorphic in a right half plane, and integral transforms of sections of the sheaf of solutions of (D) modulo solutions with moderate growth as $t \to 0$ in some sector, to define a fundamental system of (Δ), holomorphic in a left half plane. In this, final, part we combine the techniques and results of the preceding papers to deal with the general case.

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Introduction

As before (cf. [3]) we consider linear differential operators $D \in \mathbb{C}[t, \partial]$, where e

$$\partial := t\frac{d}{dt}$$

and linear difference operators $\Delta \in \mathbb{C}[x, \tau^{-1}]$, where τ denotes the shift operator defined by

$$\tau y(x) = y(x+1)$$

We recall that the differential operator

$$D := \sum_{h=0}^{H} \sum_{l=0}^{L} a_{hl} t^h \partial^l \tag{1}$$

is changed to the linear difference operator

$$\Delta := \sum_{h=0}^{H} \sum_{l=0}^{L} a_{hl} (x-h)^l \tau^{-h} \tag{2}$$

by means of the transformation defined by the substitutions

$$t \to \tau^{-1}, \partial \to x$$

From now on, we consider a fixed differential operator D of the type (1) and the associated difference operator Δ defined by (2).

By $\mathbb{C}[[t]]$, $\mathbb{C}\{t\}$ and $\mathcal{O}(\mathbb{C})$ we denote the rings of all formal power series, of all convergent power series and of all power series with infinite radius of convergence, respectively. Throughout this paper we shall identify holomorphic functions in a disk about the origin with their Taylor expansions at O. For any formal power series $\sum_{n=0}^{\infty} y(n)t^n \in \mathbb{C}[[t]]$ we have the equality

$$D\sum_{n=0}^{\infty} y(n)t^n = \sum_{n=0}^{\infty} \sum_{h=0}^{\min(H,n)} \sum_{l=0}^{L} a_{hl}(n-h)^l y(n-h)t^n,$$

which shows that

$$D\sum_{n=0}^{\infty} y(n)t^n \in \mathbb{C}[t]$$

if and only if the sequence $\{y(n)\}_{n=n_0}^{\infty}$ satisfies the corresponding homogeneous linear difference equation (Δ) defined by

$$(\Delta): \qquad \Delta y = 0,$$

for some nonnegative integer n_0. Formal power series of this type occur as asymptotic or Taylor expansions of analytic solutions of inhomogeneous equations of the form $Df = g$, where $g \in \mathbb{C}[t]$. An important class is formed by the power series expansions of Cauchy-Heine transforms of certain solutions of the homogeneous differential equation

$$(D): \qquad D\varphi = 0$$

A Cauchy-Heine transform of a function φ is an integral of the form

$$\int_\gamma \frac{\varphi(\tau)}{2\pi i(\tau - t)} d\tau$$

where γ is a suitable path of integration. By means of partial integration, it can be seen that, if we disregard integrated terms, then

$$D \int_\gamma \frac{\varphi(\tau)}{2\pi i(\tau - t)} d\tau \in \mathbb{C}[t]$$

Similarly, expanding $(\tau - t)^{-1}$ in a Taylor series about $t = 0$, we have

$$D \sum_{n=0}^{\infty} (\int_\gamma \varphi(\tau)\tau^{-n-1} d\tau) t^n \in \mathbb{C}[t]$$

Again by partial integration, disregarding integrated terms, we find that

$$\Delta \int_\gamma \varphi(t) t^{-x-1} dt = 0$$

An integral of this type will be called a Pincherle transform of φ.

The preceding observations show that there is an intimate relationship between solutions of (D), formal power series $\hat{f}$ with the property that $D\hat{f} \in \mathbb{C}[t]$ and solutions of (Δ). In this paper we define three isomorphisms from certain classes of solutions (or microsolutions) of (D) onto classes of formal power series with the above property. We use these isomorphisms in order to construct a fundamental system of solutions of the difference equation (Δ), analytic in a right half plane. In a similar manner, one can construct a fundamental system of solutions of (Δ) which is analytic in a left half plane.

The paper is arranged as follows. In §1 we establish a correspondance between bases of $\mathrm{Ker}(D, \mathbb{C}[[t]]/\mathbb{C}\{t\}) \oplus \mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C})) \oplus \mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and fundamental systems of solutions of (Δ) with a particular type of asymptotic behaviour in a right half plane. In §2 we define three isomorphisms which map linear spaces of analytic solutions or microsolutions of (D) onto the three kernels mentioned above. In §3 these isomorphisms are used to construct a fundamental system of solutions of (Δ), holomorphic in a right half plane. The elements of this fundamental system can be represented by different types of Pincherle transforms of solutions of (D).

1. Newton polygons of D and Δ

The Newton polygon $N(D)$ of a differential operator D of the form (1) is the convex hull in $\mathbb{R}^2$ of the set consisting of the half lines γ_{hl}:

$$\gamma_{hl} = \{(x, y) \in \mathbb{R}^2 : x \leq l, y = h\}$$

where $h \in \{0, \ldots, H\}$ and $l \in \{0, \ldots, L\}$ such that $a_{hl} \neq 0$ (cf. [10]). It has two horizontal edges of infinite length, a finite number of edges with a positive or negative slope and at most one vertical edge (with slope $= \infty$). If $N(D)$ has an edge with slope k, then $l(k)$ and $h(k)$ will denote its projection on the x-axis and the y-axis, respectively. If $N(D)$ has an edge with slope $k > 0$, then the equation (D) has $l(k)$ formal solutions of the type

$$\hat{\phi}(t) = e^{p(t)} t^{\rho} \sum_{l=0}^{s} \hat{g}_l(t) (\log t)^l \tag{3}$$

where $\rho \in \mathbb{C}$, $p \in \mathbb{C}[t^{-1/q}]$ for some $q \in \mathbb{N}$ such that $\deg p = qk$, $s \in \mathbb{N} \cup \{0\}$, $\hat{g}_l \in \mathbb{C}[[t^{1/q}]]$ for each $l \in \{0, \ldots, s\}$, with $\hat{g}_0(0) \neq 0$, and the sets of 'formal invariants' $\{p, \rho, s\}$ are distinct. It is easily seen that a change of variable $t \mapsto s := \frac{1}{t}$ corresponds to a reflection of the Newton polygon with respect to the x-axis. Consequently, if $N(D)$ has an edge with slope $k < 0$, then, at ∞, (D) has $l(k)$ formal solutions of the form (3), with $\hat{g}_l \in \mathbb{C}[[t^{-1/q}]]$ for some $q \in \mathbb{N}$ and each $l \in \{0, \ldots, s\}$, $p \in \mathbb{C}[t^{1/q}]$, and $\deg p = -qk$.

It can be shown that (cf. [10])

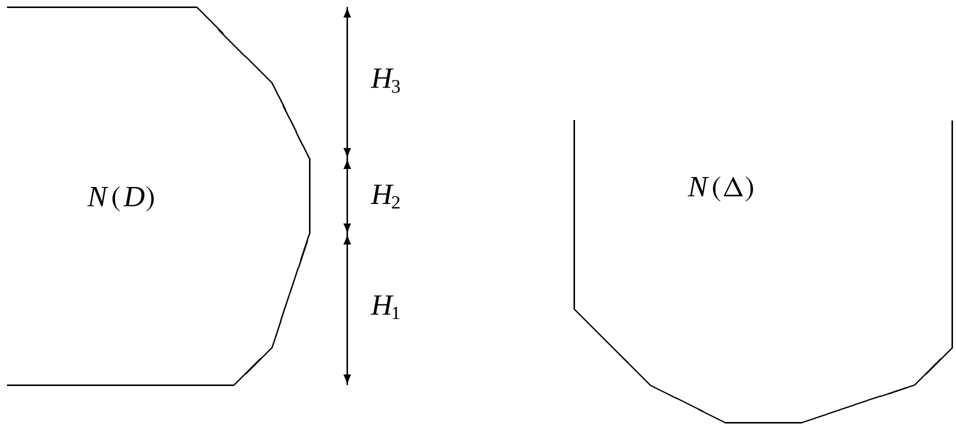

Figure 1.1: The Newton polygons of D and Δ

$$H_1 := \sum_{k>0} h(k) = i_0(D) = \dim \operatorname{Ker}(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\})$$

$$H_2 := h(\infty) = \dim \operatorname{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$$

$$H_3 := \sum_{k<0} h(k) = i_\infty(D) = \dim \operatorname{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$$

Here $i_0(D)$ and $i_\infty(D)$ denote the irregularities of D at O and ∞, respectively. It is easily seen that $H_1 + H_2 + H_3$ equals the order of Δ.

The Newton polygon $N(\Delta)$ of Δ is obtained from $N(D)$ by a reflection in the line $y = -x$ (cf. [1, 2]). The projection on the horizontal axis of an edge of $N(\Delta)$ with slope d equals the number of formal solutions of (Δ) of the form

$$x^{dx+\rho} e^{\tilde{p}(x)} \sum_{l=0}^{s} \hat{h}_l(x)(\log x)^l \tag{4}$$

where $\rho \in \mathbb{C}$, $\tilde{p} \in \mathbb{C}[x^{1/q}]$ for some $q \in \mathbb{N}$, such that $\deg \tilde{p} \leq q$, $s \in \mathbb{N} \cup \{0\}$, $\hat{h}_l \in \mathbb{C}[\![x^{-1/q}]\!]$ for each $l \in \{0, \ldots, s\}$, with $\hat{h}_0(\infty) \neq 0$, and the sets of 'formal invariants' $\{\tilde{p} \pmod{2\pi i x\mathbb{Z}},\ \rho, s\}$ are distinct. Corresponding to each formal solution $\hat{y}$ of this type there is a meromorphic solution y, holomorphic in a right half plane and represented asymptotically by $\hat{y}$ as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (cf. [4]. Similarly, there is a meromorphic solution $\tilde{y}$, holomorphic in a left half plane and represented asymptotically by $\hat{y}$ as $x \to \infty$, in the left half plane.) To a slope $k \neq 0$ of $N(D)$, with vertical projection $h(k)$, corresponds a slope $\frac{1}{k}$ in $N(\Delta)$, with horizontal projection $h(k)$. Thus (Δ) has H_1 solutions y_j^1, $j = 1, \ldots, H_1$, with 'supra-exponential growth' as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, admitting an asymptotic representation of the form (4) with $d > 0$, H_2 solutions y_j^2, $j = 1, \ldots, H_2$, with exponential growth of order 1, admitting an asymptotic representation of the form (4) with $d = 0$, and H_3 solutions with 'supra-exponential decrease', admitting an asymptotic representation of the form (4) with $d < 0$. These solutions form a **fundamental system of solutions of** (Δ). What is more, for some sufficiently large integer $n_0 \geq 0$, the sequences $\{y_j^i(n)\}_{n=n_0}^{\infty}$, $j \in \{1, \ldots, H_i\}$, $i \in \{1, 2, 3\}$, are linearly independent (this can be deduced from the asymptotic behaviour of the corresponding Casorati-determinant, cf. [3]) and $D \sum_{n=n_0}^{\infty} y_j^i(n) t^n \in \mathbb{C}[t]$ for each $j \in \{1, \ldots, H_i\}$ and $i \in \{1, 2, 3\}$. (Similarly, (Δ) has H_1 solutions $\tilde{y}_j^1$, $j = 1, \ldots, H_1$, with supra-exponential decrease, H_2 solutions $\tilde{y}_j^2$, $j = 1, \ldots, H_2$, with exponential growth of order 1, and H_3 solutions with supra-exponential growth in the left half plane, and these solutions form another fundamental system of (Δ).)

Proposition 1.1
1. Ker$(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\})$, Ker$(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and Ker$(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$ each have a basis whose elements can be represented by a series η with the property that $D\eta \in \mathbb{C}[t]$.
2. If $\hat{f} = \sum_{n=0}^{\infty} a_n t^n$ is a formal power series with the property that $D\hat{f} \in \mathbb{C}[t]$, then there exists a meromorphic solution y of (Δ), holomorphic in a right half plane Re $x \geq n_0$, such that $a_n = y(n)$ for all integers $n \geq n_0$.

PROOF. 1 is immediate: the series $\sum_{n=n_0}^{\infty} y_j^i(n) t^n$, $j = 1, \ldots, H_i$ with $i = 1, 2, 3$, where the functions y_j^i denote solutions of (Δ) with the properties mentioned above, are suitable representatives of a basis of Ker$(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\})$,
Ker$(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and Ker$(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$, respectively.

2. If $\hat{f} \in \mathcal{O}(\mathbb{C})$, then $\hat{f} \pmod{\mathbb{C}[t]} \in$ Ker$(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$, hence there are complex numbers c_j^3, $j = 1, \ldots, H_3$, such that $\hat{f} - \sum_{j=1}^{H_3} c_j^3 \sum_{n=n_0}^{\infty} y_j^3(n) t^n \in \mathbb{C}[t]$. Hence

$$a_n = \sum_{j=1}^{H_3} c_j^3 y_j^3(n)$$

for all sufficiently large n. If $\hat{f} \in \mathbb{C}\{t\}$ there exist numbers c_j^2, $j = 1, \ldots, H_2$, such that

$$\hat{g} := \hat{f} - \sum_{j=1}^{H_2} c_j^2 \sum_{n=n_0}^{\infty} y_j^2(n) t^n \in \mathcal{O}(\mathbb{C})$$

As $D\hat{g} \in \mathbb{C}[t]$, there exist $c_j^3,\ j = 1, \ldots, H_3$, such that $\hat{g} - \sum_{j=1}^{H_3} c_j^3 \sum_{n=n_0}^{\infty} y_j^3(n) t^n \in \mathbb{C}[t]$. Consequently,

$$a_n = \sum_{j=1}^{H_2} c_j^2 y_j^2(n) + \sum_{j=1}^{H_3} c_j^3 y_j^3(n)$$

In general, for all sufficiently large n, the coefficients a_n can be written as a (fixed) linear combination of $y_j^i(n)$, $j = 1, \ldots, H_i$, $i = 1, 2, 3$. The second statement of the proposition follows from the fact that each of the y_j^i is a meromorphic function in $\mathbb{C}$, holomorphic in a right half plane. □

In this way, starting from an appropriate fundamental system of solutions of (Δ), one can construct bases of $\mathrm{Ker}(D, \mathbb{C}[[t]]/\mathbb{C}\{t\})$, $\mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and $\mathrm{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$ satisfying the condition mentioned in the first part of proposition 1.1. Conversely, suppose we are given bases $\{\eta_j^i,\ j = 1, \ldots, H_i\}$, $i = 1, 2$ and 3, of $\mathrm{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$, $\mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and $\mathrm{Ker}(D, \mathbb{C}[[t]]/\mathbb{C}\{t\})$, respectively, such that η_j^i can be represented by a series $\sum_{n=0}^{\infty} \eta_{jn}^i t^n$ with the property that $D \sum_{n=0}^{\infty} \eta_{jn}^i t^n \in \mathbb{C}[t]$.

Then, according to proposition 1.1, there exist meromorphic solutions y_j^i of (Δ), $j \in \{1, \ldots, H_i\}$, $i \in \{1, 2, 3\}$, holomorphic in a right half plane, such that, for every sufficiently large integer n, $y_j^i(n) = \eta_{jn}^i$. It is easily seen that the sequences $\{y_j^i(n)\}_{n=n_0}^{\infty}$ are linearly independent and, consequently, $\{y_j^i,\ j = 1, \ldots, H_i, i = 1, 2, 3\}$ is a fundamental system of solutions of (Δ).

REMARK. Any solution y of (Δ) which is holomorphic in a right half plane, is a meromorphic function in $\mathbb{C}$. This follows immediately from the relation

$$\sum_{l=0}^{L} a_{Hl}(x-H)^l y(x-H) = -\sum_{h=0}^{H-1} \sum_{l=0}^{L} a_{hl}(x-h)^l y(x-h)$$

provided the polynomial $a_H := \sum_{l=0}^{L} a_{Hl}(x-H)^l$ does not vanish identically. The poles of y are determined by the zeroes of this polynomial.

Example 1. $D = \sum_{h=0}^{H} a_h t^h$ is a differential operator of order 0. Its Newton polygon has one vertical slope of length $H_2 \leq H$. Obviously, $H_1 = H_3 = 0$. The corresponding difference operator Δ has constant coefficients: $\Delta = \sum_{h=0}^{H} a_h \tau^{-h}$, and $N(\Delta)$ has a horizontal slope of length H_2. Let us assume that $H_2 = H$ and that $\sum_{h=0}^{H} a_h t^h$ has H distinct zeroes $\lambda_j \neq 0$, $j = 1, \ldots, H$. Thus D is of the form

$$D = c(t-\lambda_1)\ldots(t-\lambda_H)$$

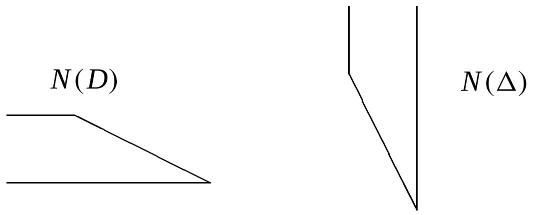

Figure 1.2: The Newton polygons of D and Δ in example 1.3

where $c \in \mathbb{C}^*$. The difference equation (Δ) has a fundamental system of solutions $\{\lambda_j^{-x};\ j = 1, \ldots, H\}$ and $\mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ is spanned by the equivalence classes modulo $\mathcal{O}(\mathbb{C})$ of the series $\eta_j,\ j = 1, \ldots, H$, defined by

$$\eta_j(t) = \sum_{n=0}^{\infty} \lambda_j^{-n} t^n$$

$\triangleleft$

Example 2. $D = \partial^2 + t - 1$. $N(D)$ has one non–horizontal edge with slope $= -\frac{1}{2}$, and the length of its projection on the y-axis equals 1. Thus $H_1 = H_2 = 0$ and $H_3 = 1$. D has a regular singularity at O and an irregular singularity at ∞. The equation (D) has two formal solutions $\hat{\phi}_1$ and $\hat{\phi}_2$ of the form

$$\hat{\phi}_1(t) = \exp(2it^{1/2})t^{-1/4}\hat{g}(t)$$

$$\hat{\phi}_2(t) = \hat{\phi}_1(te^{2\pi i}) = -i\exp(-2it^{1/2})t^{-1/4}\hat{g}(te^{2\pi i}),$$

where $\hat{g} \in \mathbb{C}[\![t]\!]$, $\hat{g}(\infty) = 1$. $\mathrm{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$ is the linear space generated by the equivalence class modulo $\mathbb{C}[t]$ of the series

$$\sum_{n=1}^{\infty} \frac{(-t)^n}{(n-1)!(n+1)!}$$

The corresponding difference operator $\Delta = \tau^{-1} + x^2 - 1$ is of order 1. Its Newton polygon has an edge with slope -2 and the equation (Δ) has the solution

$$y(x) = \frac{e^{\pi i x}}{\Gamma(x)\Gamma(x+2)}$$

admitting the asymptotic representation $\hat{y}(x) = e^{(\pi i+2)x}x^{-2x-1}\hat{h}(x)$, where $\hat{h} \in \mathbb{C}[[x^{-1}]]$, as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. $\triangleleft$

2. Three basic isomorphisms

In this section we define three isomorphisms which, roughly speaking, map sets of analytic solutions or microsolutions of (D) onto sets of (formal) power series solutions of corresponding inhomogeneous equations, more precisely, onto the linear spaces $\operatorname{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$, $\operatorname{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ and $\operatorname{Ker}(D, \mathbb{C}[[t]]/\mathbb{C}\{t\})$. These mappings may serve to define particular bases of the three kernels, having the property mentioned in proposition 1.1. With slight adaptations, these isomorphisms will be used, in §3, to construct fundamental systems of solutions of (Δ).

Definition 2.1 *Let $I \subset \mathbb{R}$ be an open interval. By $S(I)$ we denote the sector of the Riemann surface of the logarithm, to be denoted by $\mathbb{C}_\infty$, defined by*

$$S(I) = \{t \in \mathbb{C}_\infty : \arg t \in I\}$$

Let $R > 0$. By $S(I, R)$ we denote the set

$$S(I, R) = \{t \in S(I) : |t| < R\}$$

and by $S_\infty(I, R)$ the set

$$S_\infty(I, R) = \{t \in S(I) : |t| > R\}$$

We shall call $S(I)$ a 'large' and both $S(I, R)$ and $S_\infty(I, R)$ a 'small' sector (of $\mathbb{C}_\infty$). Similarly, if $U := I \pmod{2\pi\mathbb{Z}}$, we define sectors $S(U)$, $S(U, R)$ and $S_\infty(U, R)$ of $\mathbb{C}$, analogously to $S(I)$, $S(I, R)$ and $S_\infty(I, R)$, respectively.

Definition 2.2 *By $\mathcal{A}$ ($\mathcal{A}_\infty$) we denote the sheaf on $\mathbb{R}$ of (germs of) holomorphic functions in a small sector with vertex at O (∞) and by $\mathcal{A}_\theta$ ($(\mathcal{A}_\infty)_\theta$) its stalk at $\theta \in \mathbb{R}$.*

Let $k > 0$. By $\mathcal{A}^{\leq -k}$ ($\mathcal{A}_\infty^{\leq -k}$), $\mathcal{A}^{<-k}$ ($\mathcal{A}_\infty^{<-k}$), $\mathcal{A}^{\leq k}$ ($\mathcal{A}_\infty^{\leq k}$) and $\mathcal{A}^{<k}$ ($\mathcal{A}_\infty^{<k}$) we denote the sheaves on $\mathbb{R}$ of holomorphic functions with at least exponential decrease of order k, with 'supra-exponential decrease of order k', with at most exponential growth of order k and with 'subexponential growth of order k', respectively, in some sector with vertex at O (∞). More precisely, if $I \subset \mathbb{R}$ is an open interval, $\mathcal{A}^{\leq -k}(I)$, $\mathcal{A}^{<-k}(I)$, $\mathcal{A}^{\leq k}(I)$ and $\mathcal{A}^{<k}(I)$ are the sets of all functions f with the property that, for any

closed interval $I' \subset I$, there exists a positive number R such that f is holomorphic on the sector $S(I', R)$ of the Riemann surface of $\log t$, and

$$\sup_{t \in S(I',R)} |f(t)| e^{c|t|^{-k}} < \infty$$

for some $c > 0$, for all $c > 0$, for some $c < 0$ and for all $c < 0$, respectively.

By $\mathcal{A}^{\leq 0}$ ($\mathcal{A}_\infty^{\leq 0}$) we denote the sheaf on $\mathbb{R}$ of holomorphic functions with moderate (i.e. at most polynomial) growth in some sector with vertex at O (∞). By $\mathcal{A}^{<0}$ ($\mathcal{A}_\infty^{<0}$) we denote the sheaf on $\mathbb{R}$ of holomorphic functions that are asymptotically equal to zero as $t \to 0$ ($t \to \infty$) in some sector with vertex at O (∞).

Let $k \geq 0$. By $\mathcal{A}_D^{\leq -k}$, $\mathcal{A}_D^{<-k}$, $\mathcal{A}_D^{\leq k}$, $\mathcal{A}_D^{<k}$, $\mathcal{A}_{\infty,D}^{\leq -k}$, etc., we denote the subsheaves of $\mathcal{A}^{\leq -k}$, $\mathcal{A}^{<-k}$, $\mathcal{A}^{\leq k}$ and $\mathcal{A}^{<k}$, $\mathcal{A}_\infty^{\leq -k}$, etc., consisting of solutions of (D).

By S^1 we denote the circle of directions in $\mathbb{C}$: $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. With a slight abuse of notation we will also use the symbol $\mathcal{A}$ to denote the sheaf on S^1 of germs at O of holomorphic functions in a sector of the complex plane.

*A collection $\{U_\nu\}_{\nu=1}^N$ of connected open subsets of S^1 with the property that $\cup_{\nu=1}^N U_\nu = S^1$ will be called a **good covering of** S^1 if $U_\nu \cap U_\mu \neq \emptyset$ if and only if $|\mu - \nu| \leq 1$ and the sets U_ν are ordered cyclically.*

The main result of this section is the following theorem.

Theorem 2.1
(i) Ker$(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\}) \simeq H^1(S^1, \mathcal{A}_D^{<0})$ (Malgrange, cf. [6])
(ii) Ker$(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C})) \simeq \oplus_{\lambda\in\Lambda}Ker(D, \mathcal{C}(\lambda))$, where Λ denotes the set of singular points of D in $\mathbb{C}^*$.
(iii) Ker$(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t]) \simeq \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$

In the following three subsections we successively define three isomorphisms corresponding to parts (iii), (ii) and (i) of theorem 2.1.

2.1 The isomorphism Ker$(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t]) \simeq \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$

We begin by defining an isomorphism from $\mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0}(S^1)$ to $\mathcal{O}(\mathbb{C})/\mathbb{C}[t]$. Every $\Phi \in \mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0}(S^1)$ can be represented by a 0-cochain of $\mathcal{A}_\infty$ with respect to a good covering $\{U_\nu\}_{\nu=1}^N$ of S^1, i.e. a collection of functions $\{\phi_\nu \in \mathcal{A}_\infty(U_\nu)\}_{\nu=1}^N$, with the property that $\phi_\nu - \phi_{\nu+1} \in \mathcal{A}_\infty^{\leq 0}(U_\nu \cap U_{\nu+1})$ for all $\nu \in \{1, \ldots, N\}$, if we set

$U_{N+1} = U_1$. Without loss of generality we may assume that ϕ_ν is holomorphic in the sector $S_\nu := S_\infty(U_\nu, R)$ with $R > 0$, for all $\nu \leq N$. For $\nu \in \{1, \ldots, N\}$, let γ_ν denote the half line in $\mathbb{C}$ from $Re^{i\theta_\nu}$ to ∞ with direction $\theta_\nu \in U_\nu \cap U_{\nu+1}$ and let C_ν denote the arc of the circle $|t| = R$ from $\theta_{\nu-1}$ to θ_ν, where $\theta_0 := \theta_N$. For sufficiently large integers M the function $\alpha_M^{(3)}(\Phi)$ defined by

$$\alpha_M^{(3)}(\Phi)(t) = \sum_{\nu=1}^{N} t^M \left\{ \int_{\gamma_\nu} \frac{(\phi_\nu - \phi_{\nu+1})(\tau)}{2\pi i(\tau - t)\tau^M} d\tau + \int_{C_\nu} \frac{\phi_\nu(\tau)}{2\pi i(\tau - t)\tau^M} d\tau \right\}$$

is holomorphic in the disk $|t| < R$. It is easily seen that it is independent of the choice of the 0-cochain $\{\phi_\nu\}_{\nu=1}^N$. With the aid of Cauchy's theorem one verifies that it does not depend on R. Thus $\alpha_M^{(3)}(\Phi) \in \mathcal{O}(\mathbb{C})$. By $\alpha^{(3)}(\Phi)$ we denote its equivalence class modulo $\mathbb{C}[t]$, which is independent of M.

Proposition 2.1 *The mapping $\alpha^{(3)} : \mathcal{A}_\infty / \mathcal{A}_\infty^{\leq 0}(S^1) \to \mathcal{O}(\mathbb{C})/\mathbb{C}[t]$ defined above is a bijection with inverse $\mu^{(3)}$ defined by : $\mu^{(3)}(F) = f \pmod{\mathcal{A}_\infty^{\leq 0}}$, where f is any representative of $F \in \mathcal{O}(\mathbb{C})/\mathbb{C}[t]$.*

PROOF. First, we show that $\mu^{(3)} \circ \alpha^{(3)} = I$, i.e. $\alpha_M^{(3)}(\Phi) \pmod{\mathcal{A}_\infty^{\leq 0}} = \Phi$ for every $\Phi \in \mathcal{A}_\infty / \mathcal{A}_\infty^{\leq 0}(S^1)$ and sufficiently large M. Let Φ be represented by a 0-cochain $\{\phi_\nu\}_{\nu=1}^N$ of $\mathcal{A}_\infty$ with the properties mentioned above. Let $t \in \mathbb{C}$, $|t| < R$ and let $R' > 0$ such that $R' < |t|$. Let $\nu \in \{1, \ldots, N\}$ such that $t \in S(U_\nu)$. From Cauchy's theorem we deduce that

$$\alpha_M^{(3)}(\Phi)(t) - \phi_\nu(t) = \sum_{\nu=1}^{N} t^M \left\{ \int_{\gamma'_\nu} \frac{(\phi_\nu - \phi_{\nu+1})(\tau)}{2\pi i(\tau - t)\tau^M} d\tau + \int_{C'_\nu} \frac{\phi_\nu(\tau)}{2\pi i(\tau - t)\tau^M} d\tau \right\}$$

where γ'_ν and C'_ν are defined in the same way as γ_ν and C_ν above, but with R replaced by R'. It is easily seen that the function on the right-hand side belongs to $\mathcal{A}_\infty^{\leq 0}(U_\nu)$.

Now suppose that $f \in \mathcal{O}(\mathbb{C})$ and let $F = f \pmod{\mathbb{C}[t]}$. Then $\mu^{(3)}(F) = f \pmod{\mathcal{A}_\infty^{\leq 0}}$. $\mu^{(3)}(F)$ is represented by the 0-cochain $\{\phi_\nu\}_{\nu=1}^N$ where $\phi_\nu = f$ for $\nu = 1, \ldots, N$. Thus $\alpha_M^{(3)} \circ \mu^{(3)}(F)$ is an ordinary Cauchy-integral:

$$\alpha_M^{(3)} \circ \mu^{(3)}(F) = t^M \int_C \frac{f(\tau)}{2\pi i(\tau - t)\tau^M} d\tau$$

where C denotes a circle about O of radius $R > |t|$, described in the positive sense. Hence $\alpha^{(3)} \circ \mu^{(3)}(F) = F$. □

If $\Phi \in \mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0}(S^1)$ can be represented by a 0-cochain of $\mathcal{A}_{\infty,D}$, then it is easily seen that $D\alpha_M^{(3)}(\Phi) \in \mathbb{C}[t]$ for all sufficiently large integers M. Thus $\alpha^{(3)}$ induces a mapping from $\mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$ to $\mathrm{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$ which will again be denoted by $\alpha^{(3)}$.

Proposition 2.2 *The mapping*

$$\alpha^{(3)} : \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1) \to Ker(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$$

is a bijection.

PROOF. The surjectivity can be deduced from the surjectivity of the mapping

$$D : \mathcal{A}_\infty^{\leq 0} \to \mathcal{A}_\infty^{\leq 0}$$

(cf. [8, 11]) as follows. Let $f \in \mathcal{O}(\mathbb{C})$ and suppose that $Df = g \in \mathbb{C}[t]$. As $g \in \mathcal{A}_\infty^{\leq 0}(S^1)$, there exist a good covering $\{U_\nu\}_{\nu=1}^N$ of S^1 and functions $f_\nu \in \mathcal{A}_\infty^{\leq 0}(U_\nu)$ with the property that $Df_\nu = g$, $\nu = 1, \ldots, N$. Hence it follows that, for each $\nu \in \{1, \ldots, N\}$, there is a function $y_\nu \in \mathcal{A}_{\infty,D}(U_\nu)$, such that $f = y_\nu + f_\nu$. Obviously,

$$y_\nu - y_{\nu+1} = f_{\nu+1} - f_\nu \in \mathcal{A}_{\infty,D}^{\leq 0}(U_\nu \cap U_{\nu+1})$$

Thus the 0-cochain $\{y_\nu\}_{\nu=1}^N$ defines an element $Y \in \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$. With the aid of proposition 2.1 we find that $\alpha_M^{(3)}(Y) \pmod{\mathcal{A}_\infty^{\leq 0}} = Y \pmod{\mathcal{A}_\infty^{\leq 0}} = f \pmod{\mathcal{A}_\infty^{\leq 0}}$ and this implies that $\alpha_M^{(3)}(Y) - f \in \mathbb{C}[t]$ for all sufficiently large integers M and thus $\alpha^{(3)}(Y) = f \pmod{\mathbb{C}[t]}$. □

2.2 The isomorphism Ker$(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C})) \simeq \oplus_{\lambda\in\Lambda}Ker(D, \mathcal{C}(\lambda))$

We recall the definition of a microfunction at λ used in [5, 7].

Definition 2.3 *Let $\lambda \in \mathbb{C}$ or $\lambda \in \mathbb{C}_\infty$, and let $r > 0$. We use the following notations. $\mathcal{O}(\lambda, r)$ is the ring of holomorphic functions in the disk $D(\lambda, r)$ with center λ and radius r. $\tilde{\mathcal{O}}(\lambda, r)$ is the ring of holomorphic functions on the universal covering $\tilde{D}(\lambda, r)$ of $D(\lambda, r) - \{\lambda\}$ with respect to some fixed base point. By T_λ we denote the action of the monodromy on $\tilde{\mathcal{O}}(\lambda, r)$ (i.e. the mapping obtained by analytic continuation of the elements of $\tilde{\mathcal{O}}(\lambda, r)$ on a positive loop about λ). We put*

$$\mathcal{C}(\lambda, r) = \tilde{\mathcal{O}}(\lambda, r)/\mathcal{O}(\lambda, r)$$

and denote by can_λ the projection from $\tilde{\mathcal{O}}(\lambda, r)$ onto $\mathcal{C}(\lambda, r)$. Furthermore,

$$var_\lambda \; : \; \mathcal{C}(\lambda, r) \to \tilde{\mathcal{O}}(\lambda, r)$$

is the unique mapping with the property that

$$var_\lambda \circ can_\lambda = T_\lambda - I.$$

The set $\mathcal{C}(\lambda)$ *of microfunctions at* λ *is defined as the inductive limit*

$$\mathcal{C}(\lambda) = \lim_{r\to 0} \; \mathcal{C}(\lambda, r).$$

Furthermore, we write

$$\mathcal{O}(\lambda) = \lim_{r\to 0} \; \mathcal{O}(\lambda, r)$$

and

$$\tilde{\mathcal{O}}(\lambda) = \lim_{r\to 0} \; \tilde{\mathcal{O}}(\lambda, r).$$

Let $\lambda \in \Lambda$ and $\Phi \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$. Let $\tilde{\varphi} \in \tilde{\mathcal{O}}(\lambda, r)$ such that $\mathrm{can}_\lambda \tilde{\varphi} = \Phi$. Then $D\tilde{\varphi} \in \mathcal{O}(\lambda)$, hence $D\mathrm{var}_\lambda \Phi = (T_\lambda - I) D\tilde{\varphi} = 0$. This shows that $\mathrm{var}_\lambda \Phi \in \mathrm{Ker}(D, \tilde{\mathcal{O}}(\lambda))$. Consequently, $\mathrm{var}_\lambda \Phi$ can be continued analytically along any path that avoids the singular points of (D). In the remaining part of this subsection λ is considered to be a point of $\mathbb{C}_\infty$, i.e. we choose some determination of $\arg \lambda \in \mathbb{R}$. Let $0 < \epsilon < r$ and let $C_{\lambda,\epsilon}$ be a small positive loop about λ, starting at $\lambda_1 := \lambda(1 + \epsilon)$. Let λ_2 be a point with the property that $\arg \lambda_2 = \arg \lambda$ and $|\lambda_2| > \max_{\lambda' \in \Lambda} |\lambda'|$. Let γ denote a path in $\mathbb{C}_\infty$ from λ_1 to λ_2 consisting of segments of the half line $\arg t = \arg \lambda$ and positive semi-circles about the singular points of (D) on (λ_1, λ_2). $\mathrm{var}_\lambda(\Phi)$ can be continued analytically through γ to a function $\phi \in \mathcal{A}_{\infty,D}(\mathbb{R})$. In [3] we have proved the following result.

Proposition 2.3 *Let* $\phi \in \mathcal{A}_{\infty,D}(\mathbb{R})$. *There is a finite number of directions* $\theta_i \in \mathbb{R}$ *and functions* $\phi_i \in \mathcal{A}_{\infty,D}(\mathbb{R})$, $i \in 1, \ldots, N$, *such that* $\phi_{i\theta_i} \in (\mathcal{A}^{\leq 0}_{\infty,D})_{\theta_i}$ *and* $\phi = \sum_{i=1}^N \phi_i$.

Let γ_i denote the path from λ_2 to ∞ consisting of the arc from λ_2 to $|\lambda_2|e^{i\theta_i}$ and the half line from $|\lambda_2|e^{i\theta_i}$ to ∞ with direction θ_i. For all sufficiently large integers M the expression

$$-\frac{t^M}{2\pi i}\left(\int_{C_{\lambda,\epsilon}} \frac{\tilde{\varphi}(\tau)}{(\tau - t)\tau^M} d\tau + \int_{\gamma} \frac{\phi(\tau)}{(\tau - t)\tau^M} d\tau + \sum_{i=1}^{N} \int_{\gamma_i} \frac{\phi_i(\tau)}{(\tau - t)\tau^M} d\tau\right)$$

defines a function $CH_M(\Phi)$, holomorphic in the disk $|t| < |\lambda|$, independent of the choice of the representative $\tilde{\varphi}$ and of ϵ and λ_2. It is easily verified that $DCH_M(\Phi) \in \mathbb{C}[t]$. Moreover, its equivalence class modulo $\mathcal{O}(\mathbb{C})$ is independent of M. This equivalence class will be denoted by $CH(\Phi)$. It does not depend on the choice of the solutions ϕ_i and the directions θ_i: suppose that $\phi = \sum_{i=1}^{N'} \phi_i'$, where $\phi_i' \in \mathcal{A}_{\infty,D}(\mathbb{R})$, $i \in 1, \dots, N'$, and $\phi_{i\theta_i'}' \in (\mathcal{A}_{\infty,D}^{\leq 0})_{\theta_i'}$ and let γ_i' be defined similarly to γ_i. From the fact that $\sum_{i=1}^{N} \phi_i = \sum_{i=1}^{N'} \phi_i'$ it can be deduced that

$$\sum_{i=1}^{N} \int_{\gamma_i} \frac{\phi_i(\tau)}{(\tau - t)\tau^M} d\tau \; - \sum_{i=1}^{N'} \int_{\gamma_i'} \frac{\phi_i'(\tau)}{(\tau - t)\tau^M} d\tau \in \mathcal{O}(\mathbb{C})$$

Now let $\Phi = \oplus_{\lambda \in \Lambda} \Phi_\lambda$, where $\Phi_\lambda \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$. We define

$$\alpha^{(2)}(\Phi) = \sum_{\lambda \in \Lambda} CH(\Phi_\lambda)$$

Proposition 2.4 *The mapping* $\alpha^{(2)} : \oplus_{\lambda \in \Lambda} Ker(D, \mathcal{C}(\lambda)) \to Ker(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ *is an isomorphism.*

PROOF. The inverse mapping is defined as follows. Let $\hat{f}$ be a convergent power series with the property that $D\hat{f} \in \mathcal{O}(\mathbb{C})$. Then $\hat{f}$ defines a function f, holomorphic at O, that can be continued analytically along any path avoiding the singular points of (D). Let f_λ denote the element of $\tilde{\mathcal{O}}(\lambda)$ obtained by analytic continuation of f on a path from O to λ consisting of segments of the half line $\arg t = \arg \lambda$ and negative semi-circles about the singular points of (D) on (O, λ), and let $\Phi_\lambda := \mathrm{can}_\lambda f_\lambda$. For sufficiently large integers M, $CH_M(\Phi_\lambda)$ can be represented by an expression of the form

$$-t^M \left(\int_{\Gamma_\lambda} \frac{f_\lambda(\tau)}{2\pi i(\tau - t)\tau^M} d\tau + \sum_{i=1}^{N} \int_{\gamma_i} \frac{\phi_i(\tau)}{2\pi i(\tau - t)\tau^M} d\tau \right)$$

where ϕ_i and γ_i are defined as above, and Γ_λ denotes a positive loop about λ, consisting of $-\gamma$, $C_{\lambda,\epsilon}$ and γ (cf. [5, Remark 2.4]). By deformation of the paths of integration it is easily verified that, for any positive number R, the difference $f - \sum_{\lambda \in \Lambda} CH_M(\Phi_\lambda)$ can be continued analytically to the disk $|t| < R$. Hence $\hat{f} \pmod{\mathcal{O}(\mathbb{C})} = \sum_{\lambda \in \Lambda} CH(\Phi_\lambda) = \alpha^{(2)}(\Phi)$ and thus the mapping $\mu^{(2)}$ defined by $\mu^{(2)}(\hat{f} \pmod{\mathcal{O}(\mathbb{C})}) = \oplus_{\lambda \in \Lambda} \Phi_\lambda$ is a right inverse of $\alpha^{(2)}$.

Conversely, let $\Phi = \oplus_{\lambda \in \Lambda} \Phi_\lambda$, where $\Phi_\lambda \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$ and let $f = \sum_{\lambda \in \Lambda} CH_M(\Phi_\lambda)$, where M is some sufficiently large integer. For any $\lambda' \in \Lambda$, the function $CH_M(\Phi_{\lambda'})$ is analytic in the sector $\arg\lambda' \le \arg(t-\lambda') < \arg\lambda' + 2\pi$. Hence it follows that the branch $CH_M(\Phi_{\lambda'})_\lambda$ defined above is regular at λ for all $\lambda' \neq \lambda$. Moreover, one easily verifies that $CH_M(\Phi_\lambda) - \tilde{\varphi}_\lambda$ is regular at λ for any representative $\tilde{\varphi}_\lambda$ of Φ_λ. Hence it follows that $f_\lambda - \tilde{\varphi}_\lambda$ is regular at λ. Consequently, $\mathrm{can}_\lambda f_\lambda = \Phi_\lambda$ for each $\lambda \in \Lambda$, hence $\mu^{(2)}(\hat{f} \pmod{\mathcal{O}(\mathbb{C})}) = \oplus_{\lambda\in\Lambda}\Phi_\lambda$ and thus $\mu^{(2)}$ is a left inverse of $\alpha^{(2)}$ as well. □

2.3 The isomorphism $\mathrm{Ker}(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\}) \simeq H^1(S^1, \mathcal{A}_D^{<0})$

Let $\Phi \in H^1(S^1, \mathcal{A}^{<0})$ be represented by a 1-cocycle $\{\phi_\nu \in \mathcal{A}^{<0}(U_\nu \cap U_{\nu+1})\}_{\nu=1}^N$, where $\{U_\nu\}_{\nu=1}^N$ is a good covering of S^1 and $U_{N+1} = U_1$. For each $\nu \in \{1, \dots, N\}$ let $\theta_\nu \in \mathbb{R}$ such that $\theta_\nu \pmod{2\pi\mathbb{Z}} \in U_\nu \cap U_{\nu+1}$ and $\theta_\nu - \arg\lambda \notin 2\pi\mathbb{Z}$ for any $\lambda \in \Lambda$, and let $r < \min_{\lambda\in\Lambda}|\lambda|$. The function $CH^r(\phi_\nu)$ defined by

$$CH^r(\phi_\nu)(t) = \int_0^{re^{i\theta_\nu}} \frac{\phi_\nu(\tau)}{2\pi i(\tau - t)} d\tau$$

is analytic in $\{t \in \mathbb{C} : \arg t \neq \theta_\nu\}$ and admits an asymptotic expansion $\widehat{CH}^r(\phi_\nu) \in \mathbb{C}[\![t]\!]$ as $t \to 0$ in this sector (cf. [9]). It is well-known that the mapping $\alpha^{(1)} : H^1(S^1, \mathcal{A}^{<0}) \to \mathbb{C}[\![t]\!]/\mathbb{C}\{t\}$ defined by

$$\alpha^{(1)}(\Phi) = \sum_{\nu=1}^N \widehat{CH}^r(\phi_\nu) \pmod{\mathbb{C}\{t\}}$$

is an isomorphism (cf. [6, 9]) and so is the induced mapping from $H^1(S^1, \mathcal{A}_D^{<0})$ to $\mathrm{Ker}(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\})$, which will again be denoted by $\alpha^{(1)}$. We will show that $\alpha^{(1)}(\Phi)$ has a representative with the property that its image under the differential operator D is a polynomial.

Let $R > \max_{\lambda\in\Lambda}|\lambda|$. For each $\nu \in \{1, \dots, N\}$ let $\gamma_\nu \subset \mathbb{C}_\infty$ denote the segment from O to $Re^{i\theta_\nu}$. Obviously, ϕ_ν can be continued analytically through γ_ν to an element of $\mathcal{A}_{\infty,D}(\mathbb{R})$ which will again be denoted by ϕ_ν. According to proposition 2.3, there is a finite number of directions $\theta_{\nu i} \in \mathbb{R}$ and functions $\phi_{\nu i} \in \mathcal{A}_{\infty,D}(\mathbb{R})$ with moderate growth as $t \to \infty$, $\arg t = \theta_{\nu i}$, $i = 1, \dots, N_\nu$, such that $\phi_\nu = \sum_{i=1}^{N_\nu} \phi_{\nu i}$. Let $\gamma_{\nu i}$ denote the path from $Re^{i\theta_\nu}$ to ∞ consisting of the arc from $Re^{i\theta_\nu}$ to $Re^{i\theta_{\nu i}}$ and the half line from $Re^{i\theta_{\nu i}}$ to ∞ with direction $\theta_{\nu i}$. For all sufficiently large integers M the following expression:

$$t^M \left(\int_{\gamma_\nu} \frac{\phi_\nu(\tau)}{2\pi i(\tau - t)\tau^M} d\tau + \sum_{i=1}^{N_\nu} \int_{\gamma_{\nu i}} \frac{\phi_{\nu i}(\tau)}{2\pi i(\tau - t)\tau^M} d\tau \right)$$

defines a function $CH_M(\phi_\nu)$, independent of R, holomorphic in $\{t \in \mathbb{C} : \arg t \neq \theta_\nu\}$, and admitting an asymptotic expansion $\widehat{CH}_M(\phi_\nu) \in t^M\mathbb{C}[\![t]\!]$ as $t \to 0$ in this sector. Moreover, $CH_M(\phi_\nu) - CH^r(\phi_\nu)$ is analytic in the disk $|t| < r$ and thus $\widehat{CH}_M(\phi_\nu) - \widehat{CH}^r(\phi_\nu) \in \mathbb{C}\{t\}$. By means of partial integration it is easily verified that $D\widehat{CH}_M(\phi_\nu) = DCH_M(\phi_\nu) \in \mathbb{C}[t]$. Let

$$\alpha_M^{(1)}(\Phi) = \sum_{\nu=1}^{N} \widehat{CH}_M(\phi_\nu)$$

Obviously, $D\alpha_M^{(1)}(\Phi) \in \mathbb{C}[t]$. Moreover, $\alpha_M^{(1)}(\Phi) \pmod{\mathbb{C}\{t\}} = \alpha^{(1)}(\Phi)$.

3. A fundamental system of solutions of (Δ)

In this section we define a fundamental system $\{y_j^i : j \in \{1, \dots, H_i\}, i \in \{1, 2, 3\}\}$ of meromorphic solutions of (Δ), using different types of Pincherle transforms of solutions of (D). The (formal) power series $\{\sum_{n=n_0}^{\infty} y_j^i(n)t^n : j \in \{1, \dots, H_i\}\}$ are representatives of a basis of $\mathrm{Ker}(D, \mathbb{C}[\![t]\!]/\mathbb{C}\{t\})$ if $i = 1$, of $\mathrm{Ker}(D, \mathbb{C}\{t\}/\mathcal{O}(\mathbb{C}))$ if $i = 2$, and of $\mathrm{Ker}(D, \mathcal{O}(\mathbb{C})/\mathbb{C}[t])$ if $i = 3$.

Let $\{\Phi_j^1,\ j = 1, \dots, H_1\}$ be a basis of $H^1(S^1, \mathcal{A}_D^{<0})$. For each $j \in \{1, \dots, H_1\}$, let Φ_j^1 be represented by a 1-cocycle $\{\phi_{j\nu} \in \mathcal{A}_D^{<0}(U_\nu \cap U_{\nu+1})\}_{\nu=1}^N$, where $\{U_\nu\}_{\nu=1}^N$ is a good covering of S^1 and $U_{N+1} = U_1$. Using the notations of §2.3, with an additional subscript j to distiguish between different basis-elements whenever necessary, we define a function y_j^1 as follows

$$y_j^1(x) = \sum_{\nu=1}^{N} \left(\int_{\gamma_\nu} \phi_{j\nu}(\tau)\tau^{-x-1}d\tau + \sum_{i=1}^{N_{j\nu}} \int_{\gamma_{j\nu i}} \phi_{j\nu i}(\tau)\tau^{-x-1}d\tau \right) \tag{5}$$

y_j^1 is a meromorphic function in $\mathbb{C}$, holomorphic in the right half plane $\mathrm{Re}\, x \geq M$ for a sufficiently large integer M. By means of partial integration it can be verified that y_j^1 is a solution of (Δ). For each $j \in \{1, \dots, H_1\}$, $\alpha^{(1)}(\Phi_j^1)$ can be represented by the formal power series $\frac{1}{2\pi i} \sum_{n=M}^{\infty} y_j^1(n)t^n$.

REMARK. In general, the function y_j^1 depends on the choice of the 1-cocycle representing Φ_j^1 and on the choice of the numbers θ_ν, $\nu = 1, \dots, N$, which determine the branches of τ^{-x-1} in the integrals on the right-hand side of (5). A different choice of determination of θ_ν for some ν will result in the corresponding terms in the right-hand side of (5) being multiplied with some power of $e^{2\pi i x}$, but this has no effect on the values of the coefficients $y_j^1(n)$ in the power series representing $\alpha^{(1)}(\Phi_j^1)$.

By choosing an appropriate basis of $H^1(S^1, \mathcal{A}_D^{<0})$ and suitable 1-cocycles we can achieve that, for each $j \in \{1, \ldots, H_1\}$, y_j^1 admits an asymptotic representation of the form (4) with $d > 0$, as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This can be seen by a slight adaptation of the argument used in [3]. (Note that the asymptotic behaviour of y_j^1 is determined by the asymptotic behaviour of

$$\sum_{\nu=1}^{N} \int_0^{re^{i\theta_\nu}} \phi_{j\nu}(\tau)\tau^{-x-1}d\tau,$$

the contributions of the remaining integrals in (5) being of exponential order ≤ 1.)

Now, let $\lambda \in \Lambda$ and let $\Phi \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$. The Pincherle transform $\mathcal{P}(\Phi)$ of Φ is defined by

$$\mathcal{P}(\Phi)(x) = \int_{C_{\lambda,\epsilon}} \tilde{\phi}(\tau)\tau^{-x-1}d\tau + \int_{\gamma} \phi(\tau)\tau^{-x-1}d\tau + \sum_{i=1}^{N} \int_{\gamma_i} \phi_i(\tau)\tau^{-x-1}d\tau$$

where we have used the notations of §2.2. $\mathcal{P}(\Phi)$ is a meromorphic function, holomorphic in a right half plane. Moreover, $\lambda^x\mathcal{P}(\Phi)(x)$ has subexponential growth as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (cf. [5]. The asymptotic behaviour is essentially determined by the first two integrals, since the contribution from the terms $\int_{\gamma_i} \phi_i(\tau)\tau^{-x-1}d\tau$ can be made to decrease more rapidly than any exponential function of order 1 by choosing a sufficiently large number R). For any $\Phi \in \oplus_{\lambda\in\Lambda}\mathrm{Ker}(D, \mathcal{C}(\lambda))$ of the form $\Phi = \oplus_{\lambda\in\Lambda}\Phi_\lambda$, where $\Phi_\lambda \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$, we define the Pincherle transform $\mathcal{P}(\Phi)$ by

$$\mathcal{P}(\Phi) = \sum_{\lambda\in\Lambda} \mathcal{P}(\Phi_\lambda)$$

It is easily seen that $\alpha^{(2)}(\Phi)$ can be represented by the convergent power series $-\frac{1}{2\pi i}\sum_{n=M}^{\infty} \mathcal{P}(\Phi)(n)t^n$. Now let $\{\Phi_j^2, j = 1, \ldots, H_2\}$ be a basis of $\oplus_{\lambda\in\Lambda}\mathrm{Ker}(D, \mathcal{C}(\lambda))$ and let

$$y_j^2 := -\mathcal{P}(\Phi_j^2), \;\; j = 1, \ldots, H_2$$

Note that, for a microfunction $\Phi \in \mathrm{Ker}(D, \mathcal{C}(\lambda))$, $\mathcal{P}(\Phi)$ depends on the determination of $\arg\lambda$. If this argument is increased by 2π, then the Pincherle transform $\mathcal{P}(\Phi)$ is multiplied by a factor $e^{-2\pi i x}$.

The third case is slightly more complicated. For any $\Phi \in \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$, $\alpha^{(3)}(\Phi)$ is represented by a power series $\frac{1}{2\pi i}\sum_{n=M}^{\infty} \eta_n t^n$, where

$$\eta_n = \sum_{\nu=1}^{N} \left\{ \int_{\gamma_\nu} (\phi_\nu - \phi_{\nu+1})(\tau)\tau^{-n-1} d\tau + \int_{C_\nu} \phi_\nu(\tau)\tau^{-n-1} \right\} \tag{6}$$

Here, the paths of integration γ_ν and C_ν are half lines in $\mathbb{C}$ and arcs of a circle in $\mathbb{C}$, respectively (cf. §2.1). In this case however, due to the multiformity of τ^{-x-1}, the expression obtained by replacing the integer n in (6) with a complex variable x does not, for arbitrary determinations of $\arg\tau$ on γ_ν, define a solution of (Δ). This difficulty can be overcome by considering, instead of global sections of the sheaf $\mathcal{A}_{\infty,D}/\mathcal{A}^{\leq 0}_{\infty,D}$ on S^1, sections with **compact support** of the corresponding sheaf on $\mathbb{R}$, i.e. elements of $H^0_c(\mathbb{R}, \mathcal{A}_{\infty,D}/\mathcal{A}^{\leq 0}_{\infty,D})$.

Definition 3.1 *Let $\Phi \in H^0_c(\mathbb{R}, \mathcal{A}_\infty/\mathcal{A}^{\leq 0}_\infty)$ be represented by a collection of functions $\{\phi_\nu\}_{\nu=-\infty}^{\infty}$, where ϕ_ν is holomorphic on a sector S_ν of the form: $S_\infty((\alpha_\nu, \beta_\nu), R)$ with $R > 0$ and $\alpha_\nu < \beta_{\nu-1} < \alpha_{\nu+1} < \beta_\nu$ for all $\nu \in \mathbb{Z}$. Suppose that $\phi_\nu \equiv 0$ for all $\nu < 1$ and all $\nu > N \in \mathbb{N}$. For all $\nu \in \mathbb{Z}$, let $\theta_\nu \in (\alpha_{\nu+1}, \beta_\nu)$, let γ_ν denote the half line from $Re^{i\theta_\nu}$ to ∞ with direction θ_ν and let C_ν denote the arc $\{t \in \mathbb{C}_\infty : |t| = R,\ \theta_{\nu-1} < \arg t < \theta_\nu\}$, directed in the positive sense. For all sufficiently large integers M we define*

$$\tilde{\alpha}^{(3)}_M(\Phi)(t) = t^M \sum_{\nu=0}^{N} \left\{ \int_{\gamma_\nu} \frac{(\phi_\nu - \phi_{\nu+1})(\tau)}{2\pi i(\tau - t)\tau^M} d\tau + \int_{C_\nu} \frac{\phi_\nu(\tau)}{2\pi i(\tau - t)\tau^M} d\tau \right\}$$

and

$$\mathcal{P}(\Phi)(x) = \sum_{\nu=0}^{N} \left\{ \int_{\gamma_\nu} (\phi_\nu - \phi_{\nu+1})(\tau)\tau^{-x-1} d\tau + \int_{C_\nu} \phi_\nu(\tau)\tau^{-x-1} d\tau \right\}$$

Proposition 3.1 *Let $\Phi \in H^0_c(\mathbb{R}, \mathcal{A}_\infty/\mathcal{A}^{\leq 0}_\infty)$. $\mathcal{P}(\Phi)$ is analytic in a right half plane $Re\, x \geq M$ for some integer M and decreases supra-exponentially as $x \to \infty$ in this half plane. Moreover, $\tilde{\alpha}^{(3)}_M(\Phi) \in \mathcal{O}(\mathbb{C})$ and*

$$\tilde{\alpha}^{(3)}_M(\Phi)(t) = \frac{1}{2\pi i} \sum_{n=M}^{\infty} \mathcal{P}(\Phi)(n) t^n$$

The proof of this proposition is straightforward and is left to the reader. For any $\theta \in \mathbb{R}$, let $\dot{\theta} := \theta \pmod{2\pi\mathbb{Z}}$ denote its projection on S^1. If $I \subset \mathbb{R}$ is an open interval of length less than 2π, any $\phi \in \mathcal{A}_\infty(I)$ defines, in an obvious manner, a function $\dot{\phi} \in \mathcal{A}_\infty(\dot{I})$. Now, let $\Phi \in H^0_c(\mathbb{R}, \mathcal{A}_\infty/\mathcal{A}^{\leq 0}_\infty)$ be represented by a 0-cochain

$\{\phi_\nu\}_{\nu=-\infty}^{\infty}$ of $\mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0}$ defined on a covering $\{I_\nu = (\alpha_\nu, \beta_\nu)\}_{\nu=-\infty}^{\infty}$ of $\mathbb{R}$ as in definition 3.1. Suppose that $\phi_\nu \equiv 0$ for all $\nu < 1$ and all $\nu > N \in \mathbb{N}$. By refining the covering, if necessary, we can achieve that the set $\{\dot{I}_\nu\}_{\nu=1}^{N}$ consists of a number of identical copies of a good covering $\{U_\mu\}_{\mu=1}^{N'}$ of S^1. For each $\mu \in \{1, \ldots, N'\}$ we define

$$\tilde{\phi}_\mu := \sum_{\nu:\dot{I}_\nu = U_\mu} \dot{\phi}_\nu$$

The collection of functions $\{\tilde{\phi}_\mu\}_{\mu=1}^{N'}$ represents an element $\dot{\Phi} \in \mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0}(S^1)$, independent of the choice of cochain representing Φ. Moreover one readily verifies that, for each $\Phi \in H_c^0(\mathbb{R}, \mathcal{A}_\infty/\mathcal{A}_\infty^{\leq 0})$,

$$\alpha^{(3)}(\dot{\Phi}) = \tilde{\alpha}_M^{(3)}(\Phi) \pmod{\mathbb{C}[t]}$$

By a slight adaptation of the argument used in [3, §4, with t replaced by $\frac{1}{t}$] we can prove the existence of H_3 sections $\Phi_j \in H_c^0(\mathbb{R}, \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0})$, with the property that the function y_j^3 defined by

$$y_j^3 := \mathcal{P}(\Phi_j)$$

is a solution of (Δ) admitting an asymptotic representation of the form (4) with $d < 0$ as $x \to \infty$, $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Moreover, the equivalence classes modulo $\mathbb{C}[t]$ of the sequences $\{y_j^3(n)\}_{n=M}^{\infty}$, $j = 1, \ldots, H_3$, are linearly independent. Hence we deduce the following result.

Proposition 3.2 *There exist sections $\Phi_j \in H_c^0(\mathbb{R}, \mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0})$, $j = 1, \ldots, H_3$, such that $\{\dot{\Phi}_j : j = 1, \ldots, H_3\}$ is a basis of $\mathcal{A}_{\infty,D}/\mathcal{A}_{\infty,D}^{\leq 0}(S^1)$.*

The above discussion shows that, for some sufficiently large integer M, the sequences $\{y_j^i(n)\}_{n=M}^{\infty}$, $j = 1, \ldots, H_i$, $i = 1, 2, 3$, are linearly independent. This leads to the following conclusion.

Theorem 3.1 *The functions y_j^i, $j = 1, \ldots, H_i$, $i = 1, 2, 3$, defined above, constitute a fundamental system of meromorphic solutions of (Δ), analytic in a right half plane.*

Concluding remarks

The above argument remains essentially unchanged if we replace the ring $\mathbb{C}[[t]]$ of formal power series and its various subrings ($\mathbb{C}\{t\}$, $\mathcal{O}(\mathbb{C})$, etc.) with the ring

$\mathbb{C}[\![t]\!][t^{-1}]$ of formal Laurent series and the corresponding subrings. This has the advantage that the results of the paper, modified accordingly, can be extended to the set of differential operators with rational coefficients. In particular, they can be applied to the differential operator $\tilde{D}$ obtained from D by the change of variable $t \mapsto s := \frac{1}{t}$ and the associated difference operator $\tilde{\Delta}$, which is related to Δ through the change of variable $x \mapsto \xi := -x$. Thus we see that, analogously to proposition 1.1, $\mathrm{Ker}(D, \mathbb{C}[\![t^{-1}]\!][t]/\mathbb{C}\{t^{-1}\}[t])$, $\mathrm{Ker}(D, \mathbb{C}\{t^{-1}\}[t]/\mathcal{O}(\hat{\mathbb{C}}^*)[t])$ and $\mathrm{Ker}(D, \mathcal{O}(\hat{\mathbb{C}}^*)[t]/\mathbb{C}[t, t^{-1}])$ each have a basis whose elements can be represented by a series η with the property that $D\eta \in \mathbb{C}[t, t^{-1}]$. Here, $\hat{\mathbb{C}}^* := \mathbb{C}^* \cup \infty$.
Moreover, if $\hat{f} = \sum_{n=-\infty}^{0} a_n t^n$ is a formal power series in t^{-1}, with the property that $D\hat{f} \in \mathbb{C}[t, t^{-1}]$, then there exists a meromorphic solution y of (Δ), holomorphic in a left half plane $\mathrm{Re}\, x \le n_0$, such that $a_n = y(n)$ for all integers $n \le n_0$. Furthermore, we have the following three isomorphisms, analogous to those of theorem 2.1:
(i) $\mathrm{Ker}(D, \mathbb{C}[\![t^{-1}]\!][t]/\mathbb{C}\{t^{-1}\}[t]) \simeq H^1(S^1, \mathcal{A}^{<0}_{\infty,D})$
(ii) $\mathrm{Ker}(D, \mathbb{C}\{t^{-1}\}[t]/\mathcal{O}(\hat{\mathbb{C}}^*)[t]) \simeq \oplus_{\lambda\in\Lambda}\mathrm{Ker}(D, \mathcal{C}(\lambda))$
(iii) $\mathrm{Ker}(D, \mathcal{O}(\hat{\mathbb{C}}^*)[t]/\mathbb{C}[t, t^{-1}]) \simeq \mathcal{A}_D/\mathcal{A}_D^{\le 0}(S^1)$. With the aid of suitable bases of $H^1(S^1, \mathcal{A}^{<0}_{\infty,D})$, $\oplus_{\lambda\in\Lambda}\mathrm{Ker}(D, \mathcal{C}(\lambda))$ and $\mathcal{A}_D/\mathcal{A}_D^{\le 0}(S^1)$, we can construct a fundamental system of solutions of (Δ), **holomorphic in a left half plane**.

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