Dualities of strings and branes

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Chapter 6

World Volume Actions

In this chapter we will study the world volume theory, and more in particular the world volume actions of the extended objects we encountered in the previous chapters. These effective actions describe the dynamics of the objects and their energy-momentum tensor occurs as a source term in the equations of motion of the solutions. Since there exist all kinds of duality relations between the different solutions, the same relations should connect the various world volume actions to each other.
The aim of this chapter is to see how some of these dualities are realized. In Section 6.1, we derive via dimensional reduction the form of the world volume actions of the fundamental string, the $D2$-brane, the solitonic five-brane and the $D4$-brane from the world volume actions of the $M2$ and $M5$-brane. In Section 6.2 we investigate the T-duality map between the world volume actions of the ten-dimensional gravitational wave and the fundamental string and in Section 6.3 we use the T-duality between the solitonic five-brane and the Kaluza-Klein monopole to construct the world volume action of the monopole.

Part of the work presented here can also be found in [94].

6.1 Type IIA Branes from $D = 11$

In section 3.2 we have seen that there exists a direct relation between $D = 11$ supergravity (2.37) and Type IIA theory (2.33): the latter can be obtained from the former via a dimensional reduction over a circle S^1. The different extended objects that appear as solutions of the Type IIA theory can be interpreted as direct and double dimensionally reduced objects from eleven-dimensional supergravity [156], as shown in Figure 3.1.

This implies of course also that the world volume actions of the Type IIA extended objects, presented in Section 2.3, should be related to the world volume actions of the $D = 11$ supergravity solutions. In this section, we will show that the world volume actions for the fundamental string (2.46) and for the $D2$-brane (2.60) can be obtained from the $M2$-action and the $S5$ and $D4$-action from the action of the $M5$.

We will not discuss in this section the world volume actions of the gravitational wave or the Kaluza-Klein monopole and the ten-dimensional objects they reduce to. The ten-dimensional gravitational wave and monopole will be discussed in Section 6.2 and Section 6.3 respectively, where they are constructed making use of the T-duality relations with the fundamental string and the solitonic five-brane.

6.1.1 The Membrane Action

Let us consider the bosonic part of the $M2$-brane action of eleven-dimensional supergravity, given by [30]:

$$
\begin{aligned}
S_{M2} = -\tfrac{1}{2}\int d^3\sigma\ &\sqrt{|\det(\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu})|} \\
&+ \tfrac{1}{6}\int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\hat C_{\hat\mu\hat\nu\hat\rho}\ .
\end{aligned}
\tag{6.1}
$$

The $\hat X^{\hat\mu}$ ($\hat\mu = 0,1,...,10$) are the target space embedding coordinates and the $\sigma^{\hat\imath}$ ($\hat\imath = 0,1,2$) the world volume coordinates on the brane. The $D = 11$ supergravity background fields induce a metric and a three-form gauge field on the world volume:

$$
\begin{aligned}
\hat g_{\hat\imath\hat\jmath} &= \partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu}\ , \\
\hat C_{\hat\imath\hat\jmath\hat k} &= \partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\hat C_{\hat\mu\hat\nu\hat\rho}\ .
\end{aligned}
\tag{6.2}
$$

The world sheet action for the fundamental string is obtained via dimensional reduction of (6.1) over a world volume direction [59]. Therefore we identify one of the embedding coordinates with a world volume direction

$$
\sigma^{\hat\imath} = (\sigma^i, \sigma), \qquad \hat X^{\hat\mu} = (X^\mu, \sigma), \tag{6.3}
$$

where the world volume indices now run over $i = 0,1$ and the target space indices $\mu = 0,...9$. Using the reduction rules (3.72) between ten and eleven dimensions, the induced metric and gauge field can be expressed in terms of the ten-dimensional fields as:

$$
\begin{aligned}
\hat g_{\sigma\sigma} &= (-e^{4\phi/3})\ , \\
\hat g_{i\sigma} &= \partial_i X^\mu(-e^{4\phi/3}A^{(1)}_\mu)\ , \\
\hat g_{ij} &= \partial_i X^\mu\partial_j X^\nu(e^{-2\phi/3}g_{\mu\nu} - e^{4\phi/3}A^{(1)}_\mu A^{(1)}_\nu)\ , \\
\varepsilon^{\hat\imath\hat\jmath\hat k}\ \partial_{\hat\imath}\hat X^{\hat\mu}&\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\ \hat C_{\hat\mu\hat\nu\hat\rho} = 3\ \varepsilon^{ij}\ \partial_i X^\mu\partial_j X^\nu\ B_{\mu\nu}\ .
\end{aligned}
\tag{6.4}
$$

Making use of the formula (3.24), it is easy to see that the square root of the determinant in (6.1) reduces as

$$
\sqrt{|\det\hat g_{\hat\imath\hat\jmath}|} = \sqrt{|\det g_{ij}|}, \tag{6.5}
$$

so that the reduced action is the world sheet action for a fundamental string (2.27):

$$S_{F1} = -\tfrac{1}{2}\int d^2\sigma\ \sqrt{|\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})|} + \tfrac{1}{2}\int d^2\sigma\ \varepsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}\ . \qquad (6.6)$$

The action for the $D2$-brane can be derived from the action (6.1) via direct reduction over a direction transverse to the brane [137, 157]. To show this we start from the Howe-Tucker formulation [55, 89] of the $M2$ action, which makes use of an auxiliary world volume metric $\hat\gamma_{\hat\imath\hat\jmath}$ and is (classically) equivalent to (6.1) [137]:

$$\begin{aligned} S_{M2} = -\tfrac{1}{2}\int d^3\sigma\ \sqrt{|\hat\gamma|}\ \Big\{\hat\gamma^{\hat\imath\hat\jmath}\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu} - 1\Big\} \\ + \tfrac{1}{6}\int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\partial_{\hat\imath}\hat X^{\hat\mu}\partial_{\hat\jmath}\hat X^{\hat\nu}\partial_{\hat k}\hat X^{\hat\rho}\hat C_{\hat\mu\hat\nu\hat\rho}\ . \qquad (6.7)\end{aligned}$$

Since we are reducing over a direction orthogonal to the brane, the world volume metric on the brane does not change: $\hat\gamma_{\hat\imath\hat\jmath} = \gamma_{\hat\imath\hat\jmath}$. Splitting the embedding coordinates $\hat X^{\hat\mu}$ in the ten-dimensional embedding coordinates X^μ and a world volume scalar S, we find for the reduced action:

$$\begin{aligned} S_1 = -\tfrac{1}{2}\int d^3\sigma\sqrt{|\gamma|}\ \Big\{e^{-\frac{2}{3}\phi}\gamma^{\hat\imath\hat\jmath}g_{\hat\imath\hat\jmath} - e^{\frac{4}{3}\phi}\gamma^{\hat\imath\hat\jmath}F_{\hat\imath}F_{\hat\jmath} - 1\Big\} \\ + \tfrac{1}{6}\int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\Big\{\tfrac{3}{2}C_{\hat\imath\hat\jmath\hat k} + 3B_{\hat\imath\hat\jmath}F_{\hat k} - 3B_{\hat\imath\hat\jmath}A^{(1)}_{\hat k}\Big\}\ , \qquad (6.8)\end{aligned}$$

where $B_{\hat\imath\hat\jmath}$ is the pull-back of $B_{\mu\nu}$ and $F_{\hat\imath}$ is the gauge invariant field strength of the world volume scalar S:

$$F_{\hat\imath} = \partial_{\hat\imath}S + \partial_{\hat\imath}X^\mu A^{(1)}_\mu. \qquad (6.9)$$

In order to relate the action (6.8) to the action of the $D2$-brane, we have to replace $F_{\hat\imath}$ by its world volume Poincaré dual, which is done by considering $F_{\hat\imath}$ as an independent field and imposing its Bianchi identity via the Lagrange multiplier term

$$S_2 = \int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\ V_{\hat\imath}\ \Big\{\partial_{\hat\jmath}F_{\hat k} - \partial_{\hat\jmath}A^{(1)}_{\hat k}\Big\}\ . \qquad (6.10)$$

The equation of motion for $F_{\hat\imath}$

$$F_{\hat\imath} \quad = \quad \tfrac{1}{\sqrt{|\gamma|}}\ e^{-\frac{3}{4}\phi}\gamma_{\hat\imath\hat\jmath}\ \varepsilon^{\hat\jmath\hat k\hat l}\ \big(\partial_{\hat k}V_{\hat l} - \partial_{\hat l}V_{\hat k} - B_{\hat k\hat l}\big) \qquad (6.11)$$

expresses $F_{\hat\imath}$ in terms of its Poincaré dual $\mathcal{F}_{\hat\imath\hat\jmath}$

$$\mathcal{F}_{\hat\imath\hat\jmath} \quad = \quad \partial_{\hat\imath}V_{\hat\jmath} - \partial_{\hat\jmath}V_{\hat\imath} - B_{\hat\imath\hat\jmath}. \qquad (6.12)$$

Substituting (6.9) and (6.11) in the action and redefining the world volume metric $\gamma_{\hat\imath\hat\jmath} \to e^{-2\phi/3}\gamma_{\hat\imath\hat\jmath}$, we find for the dual action in terms of the world volume vector $V_{\hat\imath}$:

$$\begin{aligned} S_{D2} = -\tfrac{1}{2}\int d^3\sigma\ e^{-\phi}\sqrt{|\gamma|}\ \Big\{\gamma^{\hat\imath\hat\jmath}g_{\hat\imath\hat\jmath} + \tfrac{1}{2}\mathcal{F}_{\hat\imath\hat\jmath}\mathcal{F}^{\hat\imath\hat\jmath} - 1\Big\} \\ + \tfrac{1}{4}\int d^3\sigma\ \varepsilon^{\hat\imath\hat\jmath\hat k}\Big\{C_{\hat\imath\hat\jmath\hat k} + 2\mathcal{F}_{\hat\imath\hat\jmath}A_{\hat k}\Big\}\ , \qquad (6.13)\end{aligned}$$

which is the Howe-Tucker form of the world volume action for the $D2$-brane [157]. The world volume vector $V_{\hat{i}}$ is of course the Born-Infeld vector and the factor $e^{-\phi}$ indicates the property of D-branes that their mass is proportional to the inverse of the coupling constant $g = e^{<\phi>}$.

6.1.2 The Five-brane Action

The construction of the world volume actions for the $M5$ and the Type IIA $S5$ is a more subtle problem, due to the non-linearity of the kinetic term and the presence of a self-dual world volume two-form $\hat{W}^{+}_{\hat{i}\hat{j}}$ [72, 98]. The equations of motion and a fully covariant action, involving an auxiliary scalar field can be found in [87, 126, 10, 1, 88, 11]. In this section we will restrict ourselves to the non-self-dual action, up to quadratic order in $\hat{W}^{+}_{\hat{i}\hat{j}}$, presented in [23]:

$$\begin{aligned} S_{M5} = \int d^6\sigma \sqrt{|\det \hat{g}_{\hat{i}\hat{j}}|}\, \left(1 + \tfrac{1}{2}\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}}\hat{\mathcal{H}}^{\hat{i}\hat{j}\hat{k}}\right) \\ + \int d^6\sigma\, \varepsilon^{\hat{i}_1\ldots\hat{i}_6}\left(\tfrac{1}{70}\hat{C}_{\hat{i}_1\ldots\hat{i}_6} + \tfrac{3}{4}\partial_{\hat{i}_1}\hat{W}^{+}_{\hat{i}_2\hat{i}_3}\hat{C}_{\hat{i}_4\hat{i}_5\hat{i}_6}\right) . \end{aligned} \quad (6.14)$$

The six-form gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ is the Poincaré dual of $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ in the dual $D = 11$ supergravity [1]. The tensor $\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}}$ is the field strength of the self-dual field $\hat{W}_{\hat{i}\hat{j}}$:

$$\hat{\mathcal{H}}_{\hat{i}\hat{j}\hat{k}} \;=\; \partial_{[\hat{i}}\hat{W}_{\hat{j}\hat{k}]} - \tfrac{1}{2}\hat{C}_{\hat{i}\hat{j}\hat{k}} \; . \quad (6.15)$$

The self-duality condition $\hat{\mathcal{H}} = {}^*\hat{\mathcal{H}}$ does not follow from (6.14), but has to be put in by hand as an extra equation of motion. Note that this procedure is analogous to the one used to write down an action for the Type IIB supergravity theory in Section 2.2 [17].

Double dimensional reduction, using (3.72), gives for the induced metric the same reduction rules as (6.4), only now $\sqrt{\hat{g}_{\hat{i}\hat{j}}} = e^{-\phi}\sqrt{g_{ij}}$. The components of the gauge fields reduce as

$$\hat{W}_{ij} = W_{ij}, \qquad \hat{W}_{i\sigma} = V_i, \qquad \hat{C}_{\mu_1\ldots\mu_5\sigma} = \tfrac{7}{6}\, C_{\mu_1\ldots\mu_5} \quad (6.16)$$

so that in ten dimensions the field strength tensors are of the form

$$\begin{aligned} \mathcal{H}_{ijk} &= 3\,(\partial_{[i}W_{jk]} - \tfrac{1}{2}C_{ijk} - A^{(1)}_{[i}\mathcal{F}_{jk]}), \\ \mathcal{F}_{ij} &= \partial_i V_j - \partial_j V_i - B_{ij}. \end{aligned} \quad (6.17)$$

The action (6.14) reduces then to [23]

$$S = \int d^5\sigma\, \sqrt{|\det g_{ij}|}\, \left\{e^{-\phi} + \tfrac{1}{2}e^{\phi}\mathcal{H}_{ijk}\mathcal{H}^{ijk} - \tfrac{3}{2}e^{-\phi}\mathcal{F}_{ij}\mathcal{F}^{ij}\right\}$$

[1] The construction of a dual formulation of $D = 11$ supergravity in terms of a six-form gauge field is a notorious problem: since the action cannot be written in terms of the field strength tensors only, but contains terms in which the gauge field $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ occurs explicitly, a dual formulations in terms of the dual gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ has not been found yet. One could avoid the problem by considering the dual theory only on-shell, such that $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ can be eliminated via its equations of motion [2], or try to formulate the dual theory making use of an auxiliary field [29]. For our purposes it is sufficient to know that the dual field exists and a dual formulation can be written down.

$$+\int d^5\sigma\ \varepsilon^{i_1\dots i_5}\Big\{\tfrac{1}{10}C_{i_1\dots i_5}+\tfrac{3}{2}(\partial_{i_1}W_{i_2i_3}B_{i_4i_5}-\partial_{i_1}V_{i_2}C_{i_3i_4i_5})\Big\}\ . \tag{6.18}$$

The self-duality condition has reduced to a duality relation $\mathcal{H}=e^{-\phi}\ {}^*\mathcal{F}$ between the world volume one- and two-form, which can be used to consistently eliminate W_{ij} from the equations of motion of (6.18). It can be shown [23] that the then obtained equations follow from the action

$$\begin{aligned} S=&\int d^5\sigma\ e^{-\phi}\ \sqrt{|\det g_{ij}|}\ \Big\{1-\ 3\ \mathcal{F}^2\Big\}\\ &+\int d^5\sigma\ \varepsilon^{(5)}\ \Big\{\tfrac{1}{10}C_{(5)}-3\partial VC_{(3)}+\tfrac{3}{4}BC_{(3)}-\tfrac{3}{2}A^{(1)}\mathcal{F}\mathcal{F}\Big\}\ , \end{aligned} \tag{6.19}$$

which is precisely the action of the $D4$-brane [77] up to quadratic order in the kinetic term.

The action of the solitonic five-brane, obtained via direct reduction of the action of the $M5$, has the same subtleties as the $M5$ action since after dimensional reduction the world volume field $W_{\hat{\imath}\hat{\jmath}}$ still satisfies the self-duality condition $\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}={}^*\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}$. The reduction of (6.14) is straightforward: the induced metric and the world volume field strength reduce as

$$\begin{aligned} \hat{g}_{\hat{\imath}\hat{\jmath}}\ &=\ e^{-\frac{2}{3}\phi}g_{\hat{\imath}\hat{\jmath}}\ -\ e^{\frac{4}{3}\phi}F_{\hat{\imath}}F_{\hat{\jmath}}\ ,\\ \hat{\mathcal{H}}_{\hat{\imath}\hat{\jmath}\hat{k}}\ &=\ 3\ \partial_{[\hat{\imath}}W^+_{\hat{\jmath}\hat{k}]}+\tfrac{1}{2}C_{\hat{\imath}\hat{\jmath}\hat{k}}+3\ B_{[\hat{\imath}\hat{\jmath}}\partial_{\hat{k}]}S=\mathcal{H}_{\hat{\imath}\hat{\jmath}\hat{k}}\ , \end{aligned} \tag{6.20}$$

with $F_{\hat{\imath}}$ as in (6.9). The action of the solitonic five-brane, to quadratic order, is of the form [29]:

$$\begin{aligned} S_{S5}=&\int d^6\sigma\ e^{-2\phi}\sqrt{|\det(g_{\hat{\imath}\hat{\jmath}}-e^{2\phi}F_{\hat{\imath}}F_{\hat{\jmath}})|}\ \Big\{1+e^{2\phi}\ \mathcal{H}^2\Big\}\\ &+\int d^6\sigma\ \varepsilon^{(6)}\ \Big\{\tfrac{1}{70}C_{(6)}+\tfrac{1}{10}C_{(5)}\partial S+\tfrac{3}{4}\partial W(C_{(3)}+B\partial S)\Big\}\ . \end{aligned} \tag{6.21}$$

Note that the dilaton factor in front of the kinetic term indicates that the mass of the $S5$ is proportional to the inverse coupling constant squared, the typical behaviour of a solitonic object.

6.2 Wave/String Duality

In Section 3.1 we showed that the gravitational wave ($\mathcal{W}$) and the fundamental string solution ($F1$) were related via a T-duality transformation in the propagation direction of the wave or, the other way round, in the world volume direction of the string. To derive this T-duality we had to use a different procedure then in the original derivation [37], presented in Subsection 3.1.1, since the latter relates only fundamental string backgrounds to other fundamental string backgrounds. In order to relate other than fundamental string solutions to each other, we used the idea of T-duality via dimensional reduction: the gravitational wave and the fundamental string are dual to each other in ten dimensions because they can be mapped onto the same nine-dimensional solution, using two different (T-dual) reduction schemes. The ten-dimensional T-duality rules, obtained by relating the two reduction schemes, map one solution to the other. The dualized coordinate corresponds to the direction over which we have reduced and decompactified. We used the same procedure in Chapter 4 to relate the Type IIA and Type IIB and Heterotic actions in ten and six dimensions.

Concretely, the fundamental string solution (2.44) and the gravitational wave solution (2.64) can be reduced both onto the same nine-dimensional massive 0-brane solution (3.49)

$$m0 = \begin{cases} ds^2 = H^{-1}dt^2 - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{1}{2}} \\ k = H^{-\frac{1}{2}} \\ B_0 = -H^{-1} \\ B_{\mu\nu} = A_\mu = 0 \ , \end{cases} \tag{6.22}$$

if we take for the reduction rules for the $F1$

$$F1: \quad \begin{cases} \hat{g}_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu \ , & \hat{B}_{\mu\nu} = B_{\mu\nu} + A_{[\mu}B_{\nu]} \ , \\ \hat{g}_{x\mu} = -k^2 A_\mu \ , & \hat{B}_{x\mu} = B_\mu \ , \\ \hat{g}_{xx} = -k^2 \ , & \hat{\phi} = \phi + \frac{1}{2}\log k, \end{cases} \tag{6.23}$$

while for the reduction scheme of the $\mathcal{W}$ we use the T-dual version

$$\mathcal{W}: \quad \begin{cases} \hat{g}_{\mu\nu} = g_{\mu\nu} - k^{-2} B_\mu B_\nu \ , & \hat{B}_{\mu\nu} = B_{\mu\nu} + B_{[\mu}A_{\nu]} \ , \\ \hat{g}_{x\mu} = -k^{-2} B_\mu \ , & \hat{B}_{x\mu} = A_\mu \ , \\ \hat{g}_{xx} = -k^{-2} \ , & \hat{\phi} = \phi - \frac{1}{2}\log k. \end{cases} \tag{6.24}$$

In the case of the $F1$ the x-direction is the world volume direction of the string, while for the $\mathcal{W}$ it corresponds to the propagation direction of the wave. It is easy to verify that the combination of the two reduction schemes gives the well-known T-duality rules (3.4).

The T-duality between the wave/string solutions suggests that a same type of duality exists between the world volume actions that describe the dynamics of these solutions. We will use the procedure of T-duality via direct and double dimensional reduction to construct the duality map between the actions.

We start from the Nambu-Goto form for the kinetic term for the fundamental string action together with a Wess-Zumino term (6.6)

$$\mathcal{L}_{F1} \;=\; \sqrt{|\det(\partial_i \hat{X}^{\hat{\mu}} \partial_j \hat{X}^{\hat{\nu}} \hat{g}_{\hat{\mu}\hat{\nu}})|} \;+\; \tfrac{1}{2}\,\varepsilon^{ij}\partial_i \hat{X}^{\hat{\mu}} \partial_j \hat{X}^{\hat{\nu}} \hat{B}_{\hat{\mu}\hat{\nu}} \tag{6.25}$$

The indices $i,\ j$ are the world volume indices $\tau,\ \sigma$. If we assume that the direction in which the string is oriented is compact and the string is wound m times around this direction, we can make a split in the target space coordinates as follows:

$$\hat{X}^{\hat{\mu}}(\tau,\sigma) = \big(X^\mu(\tau),\ m\sigma\big). \tag{6.26}$$

This means that we have identified the world volume direction of the string with the space-time direction in which the string is oriented. Using this split in the coordinates and the reduction rules given in (6.23), we can rewrite the action (6.25) after double dimensional reduction as

$$\begin{aligned}
\mathcal{L}_{F1} &= \left| \begin{matrix} \partial X^\mu \partial X^\nu \, (g_{\mu\nu} - k^2 A_\mu A_\nu) & m\partial X^\mu(-k^2 A_\mu) \\ m\partial X^\mu(-k^2 A_\mu) & m^2(-k^2) \end{matrix} \right|^{\frac{1}{2}} + \; m \, \varepsilon^{\sigma\tau} \partial X^\mu B_\mu \\
&= mk \sqrt{|\det(\partial X^\mu \partial X^\nu g_{\mu\nu})|} \; - \; m \, \partial X^\mu B_\mu, \qquad (6.27)
\end{aligned}$$

which is the action for a massive particle with mass m. With ∂X^μ we mean the partial derivative of X^μ with respect to τ.

The world volume action of a gravitational wave is given by

$$\mathcal{L}_{\mathcal{W}} = \tfrac{1}{2}\sqrt{|\gamma|} \; \gamma^{-1} \; \partial \hat{X}^{\hat{\mu}} \; \partial \hat{X}^{\hat{\nu}} \; \hat{g}_{\hat{\mu}\hat{\nu}}. \qquad (6.28)$$

Direct dimensional reduction via the reduction scheme (6.24) gives an action of the form

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|\gamma|} \; \gamma^{-1} \Big[\partial X^\mu \partial X^\nu g_{\mu\nu} \; - \; k^{-2}(\partial S + B_\mu \partial X^\mu)^2 \Big], \qquad (6.29)$$

where S the world sheet scalar coming from the compact dimension: $S = \hat{X}^x$. It can be eliminated via its equations of motion

$$\partial \, \Big[\sqrt{|\gamma|}\gamma^{-1} \; k^{-2} \; (\partial S + B_\mu \partial X^\mu) \Big] = 0, \qquad (6.30)$$

or equivalently

$$\partial S = \sqrt{|\gamma|} \; \mu \; k^2 - B_\mu \partial X^\mu, \qquad (6.31)$$

where μ is a constant that corresponds to the momentum in the compactified direction. Before we can substitute this expression directly in (6.28) we have to verify whether this is consistent with the other equations of motion. It turns out that the substitution can be done if we add to the Lagrangian (6.28) a total derivative term [8, 31], giving

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|\gamma|}\gamma^{-1}\partial X^\mu \partial X^\nu g_{\mu\nu} \; + \; \tfrac{1}{2}\sqrt{|\gamma|} \; \mu^2 k^2 \; - \; \mu \; B_\mu \partial X^\mu. \qquad (6.32)$$

Again we can go to the Nambu-Goto formulation eliminating the world line metric γ via its equation of motion

$$\gamma^{-1} = \frac{\mu^2 k^2}{\partial X^\mu \partial X^\nu g_{\mu\nu}}. \qquad (6.33)$$

This yields the nine-dimensional action for a massive particle with mass μ

$$\mathcal{L} = \mu \; k \; \sqrt{|\det(\partial X^\mu \partial X^\nu g_{\mu\nu})|} \; - \; \mu \; \partial X^\mu B_\mu, \qquad (6.34)$$

which is exactly the same Lagrangian as was obtained via T-dual double dimensional reduction from the fundamental string, provided that we identify the constants m and μ. Physically this means that the T-duality interchanges the winding number of the string with the momentum of the wave in the dualized direction.

Note that the world volume action (6.28) of the gravitational wave coincides with the world volume action for a massless particle. As a matter of fact the massless particle is the source of the gravitational wave solution: a massless particle, moving at the speed of light, drags along a gravitational wave as it moves through space. This can be seen best if we rewrite the gravitational wave solution (2.64) in light cone coordinates

$$u = \tfrac{1}{\sqrt{2}}(t+z), \qquad v = \tfrac{1}{\sqrt{2}}(t-z). \tag{6.35}$$

The solution (2.64) then takes the form

$$ds^2 = 2dudv + 2(1-H)du^2 - dx_m^2. \tag{6.36}$$

Consider now the action of a massless particle coupled to gravity

$$S = -\tfrac{1}{\kappa}\int d^{10}x\sqrt{|g|}\ R \ \ - \tfrac{T}{2}\int d\sigma\sqrt{|\gamma|}\gamma^{-1}\partial X^\mu\partial X^\nu g_{\mu\nu}, \tag{6.37}$$

and the equation of motion of $g_{\mu\nu}$

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\tfrac{\kappa T}{2\sqrt{|g|}}\int d\sigma\sqrt{|\gamma|}\gamma^{-1}\partial X_\mu\partial X_\nu\ \delta(X^\mu - x^\mu). \tag{6.38}$$

The gravitational wave solution (6.36) satisfies this equation of motion, if we choose the following parametrisation for the embedding coordinates:

$$U = 0, \qquad V = \tau, \qquad X^m = 0, \tag{6.39}$$

which are the embedding coordinates of a massless particle moving at the speed of light. The equation of motion then reduces to

$$\partial_m\partial^m H = -\tfrac{\kappa T}{2}\ \delta(u)\ \delta(x^m) \tag{6.40}$$

which has as a solution

$$H(u,x^m) = -\frac{\kappa T}{2}\ \frac{\delta(u)}{\sqrt{|x^m x_m|}^6}\,. \tag{6.41}$$

6.3 The Five-brane/Monopole Duality

Let us now make use of the known T-duality between the solitonic five-brane ($S5$) and the Kaluza-Klein monopole ($\mathcal{KK}_{10}$) in ten dimensions in order to construct a world volume action for the monopole. We will present the bosonic part of the world volume action for the Heterotic ($N=1$ supersymmetric) monopole. For the kinetic part of the eleven-dimensional and ten-dimensional Type IIA monopole action we refer to [28].

Our strategy will be similar to the one in the previous chapter: because of the T-duality we know that a correct action for the monopole is one that, upon a T-dual compactification, reduces to the nine-dimensional form of the $S5$. The reduction of the monopole must be performed over the isometry direction z, since the monopole solution (2.65) transforms into the five-brane solution (2.54) after dualization in this direction.

The reduction of the $N=1$ five-brane world volume action is straightforward. Starting from the action (2.55)

$$
\begin{aligned}
S_{(S5)} &= -\tfrac{T}{2}\int d^6\sigma\ e^{-2\hat\phi}\sqrt{|\det(\partial_i\hat X^{\hat\mu}\partial_j\hat X^{\hat\nu}\hat g_{\hat\mu\hat\nu})|} \\
&\quad +\tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\dots i_6}\ \partial_{i_1}\hat X^{\hat\mu_1}\dots\ \partial_{i_6}\hat X^{\hat\mu_6}\hat C_{\hat\mu_1\dots\hat\mu_6}\,, \qquad (6.42)
\end{aligned}
$$

and using the reduction rules

$$
\begin{array}{ll}
\hat X^\mu = X^\mu\,, & \hat X^x = S \\
\hat g_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu\,, & \hat C_{x\mu_1\dots\mu_5} = D_{\mu_1\dots\mu_5},\,, \\
\hat g_{x\mu} = -k^2 A_\mu\,, & \hat C_{\mu_1\dots\mu_6} = C_{\mu_1\dots\mu_6} + 6A_{[\mu_1}D_{\mu_2\dots\mu_6]}\,, \\
\hat g_{xx} = -k^2\,, & \hat\phi = \phi + \tfrac12\log k,
\end{array} \qquad (6.43)
$$

we find for the reduced five-brane action

$$
\begin{aligned}
S &= -\tfrac{T}{2}\int d^6\sigma\ e^{-2\phi}k^{-1}\sqrt{|\det(g_{ij}-k^2F_iF_j)|} \\
&\quad +\tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\dots i_6}\ \Big[C_{i_1\dots i_6} - 6\ D_{i_1..i_5}F_{i_6}\Big], \qquad (6.44)
\end{aligned}
$$

where

$$
\begin{aligned}
g_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \\
F_i &= \partial_i S + A_\mu \partial_i X^\mu. \qquad (6.45)
\end{aligned}
$$

Our task is now to find an action for the monopole that, upon the reduction (6.43) gives an action that is T-dual to (6.44), or equivalently, gives the same action upon T-dual reduction.

However, due to the presence of the isometry direction z in the monopole solution, a subtlety occurs in the counting of the degrees of freedom: it turns out that this z-direction can not be interpreted as a world volume direction [91]. We are therefore dealing with a five-brane, (i.e. with a six-dimensional world volume), that has an extra isometry in its transverse space. Its degrees of freedom are then, just as in the case of the solitonic five-brane, given by the scalar multiplet of an $N=1$ supersymmetric field theory in six dimensions, consisting of four scalars. Naively, one could think that these four scalars again, as for the five-brane, correspond to the four transversal coordinates $(\hat X^i, Z)$, the collective coordinates for the position of the monopole. However, since Z is an isometry direction, it does not correspond to a degree of freedom (being an isometry, the position of the monopole in the Z-direction is not determined) and therefore it should not be taken in account the counting.

So on the one hand, we have to find a way to get rid of this extra degree of freedom Z in a proper way (note that this cannot be done by a simple extra gauge fixing of a world volume diffeomorphisms, since this would turn the monopole into a six-brane), yet on the other hand we have to introduce a new scalar in order to obtain the $N=1, D=6$ scalar multiplet. We will do this via a *gauged sigma model*, where we will gauge the isometry direction, eliminating the Z degree of freedom and introducing a scalar S to get the counting right [28].

Our proposal for the kinetic term of the monopole action is

$$S_{\mathcal{KK}} = \tfrac{T}{2} \int d^6\sigma \; \hat{k}^2 \; e^{-2\hat{\phi}} \; \sqrt{\left|\det\left(\partial_i \hat{X}^{\hat{\mu}} \partial_j \hat{X}^{\hat{\nu}} \; \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} - \hat{k}^{-2} \hat{\mathcal{F}}_i \hat{\mathcal{F}}_j\right)\right|}. \tag{6.46}$$

The vector $\hat{k}^{\hat{\mu}}$ is a Killing vector associated with the isometry direction z, and

$$\hat{k}^2 = -\hat{k}^{\hat{\mu}} \hat{k}^{\hat{\nu}} \hat{g}_{\hat{\mu}\hat{\nu}}. \tag{6.47}$$

In coordinates adapted to the isometry direction, $\hat{k}^{\hat{\mu}}$ will be of the form $\hat{k}^{\hat{\mu}} = \delta^{\hat{\mu} z}$. Furthermore we introduced a "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$ and a scalar $\hat{S}$ via

$$\begin{aligned} \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} &= \hat{g}_{\hat{\mu}\hat{\nu}} \; + \; \hat{k}^{-2} \; \hat{k}_{\hat{\mu}} \; \hat{k}_{\nu}, \\ \hat{\mathcal{F}}_i &= \partial_i \hat{S} \; + \; \partial_i \hat{X}^{\hat{\mu}} \; \hat{k}^{\hat{\nu}} \; \hat{B}_{\hat{\mu}\hat{\nu}}. \end{aligned} \tag{6.48}$$

The action (6.46) is a gauged sigma model, because of the symmetries

$$\delta \hat{X}^{\hat{\mu}} = \Lambda \; \hat{k}^{\hat{\mu}}, \qquad \delta \hat{S} = -\hat{k}^{\hat{\mu}} \Sigma_{\hat{\mu}}, \qquad \delta \hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}} \hat{\Sigma}_{\hat{\nu}]}, \tag{6.49}$$

under which the two terms in (6.46) are separately invariant. This symmetry occurs because of the presence of the Killing vector $\hat{k}^{\hat{\mu}}$, which effectively projects the z-direction and the corresponding field $Z(\sigma)$ out of the action. This can be seen in the contraction of the $\hat{k}^{\hat{\mu}}$ with the "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$:

$$\begin{aligned} \hat{k}^{\hat{\mu}} \; \hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}} &= \hat{k}^{\hat{\mu}} \hat{g}_{\hat{\mu}\hat{\nu}} + \hat{k}^{-2} \hat{k}^{\hat{\mu}} \hat{k}_{\hat{\mu}} \hat{k}_{\hat{\nu}} \\ &= \hat{k}^{\hat{\mu}} \hat{g}_{\hat{\mu}\hat{\nu}} - \hat{k}^{-2} \hat{k}^2 \; \hat{k}^{\hat{\rho}} \hat{g}_{\hat{\rho}\hat{\nu}} \; = 0. \end{aligned} \tag{6.50}$$

The "metric" $\hat{\mathcal{G}}_{\hat{\mu}\hat{\nu}}$ is therefore effectively a nine-dimensional metric, written in a ten-dimensional covariant form. Reducing the action (6.46) over the isometry direction, using the reduction rules (6.23), we find for the reduced monopole action

$$S = -\tfrac{T}{2} \int d^6\sigma \; e^{-2\phi} k \; \sqrt{\left|\det \; g_{ij} - k^{-2} \tilde{\mathcal{F}}_i \tilde{\mathcal{F}}_j)\right|}, \tag{6.51}$$

where

$$\tilde{\mathcal{F}}_i = \partial_i S + B_\mu \partial_i X^\mu. \tag{6.52}$$

This is indeed precisely the action (6.44) of the reduced five-brane up to a T-duality transformation (3.31)

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = k^{-1}. \tag{6.53}$$

In order to construct the Wess-Zumino term of the monopole action, we first have to get a closer look at the Wess-Zumino term of the reduced five-brane action (6.44) and to see what is the origin of each field. As we know, the ten-dimensional six-form gauge field $\hat{C}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ is the Poincaré dual of the axion $\hat{B}_{\hat{\mu}\hat{\nu}}$ and reduces in nine dimensions to a six-form and a five-form field, which are the Poincaré duals of the nine-dimensional winding vector B_μ and axion $B_{\mu\nu}$ respectively. Since under T-duality the Kaluza-Klein vector A_μ gets interchanged with the winding vector B_μ, we expect that under the same transformations also the Poincaré dual of the winding vector, the six-form $C_{\mu_1\ldots\mu_6}$, will get interchanged with some six-form $A_{\mu_1\ldots\mu_6}$, being the Poincaré dual of the Kaluza-Klein vector. The axion does not transform under T-duality, so it is logical to suppose that also its dual, the five-form $D_{\mu_1\ldots\mu_5}$ will be invariant.

Taking this in account, we make the following Ansatz for the ten-dimensional Wess-Zumino term:

$$S = \frac{T}{6!}\int d^6\sigma\varepsilon^{i_1\ldots i_6}\hat{k}^{\hat{\mu}_1}\ \partial_{i_1}\hat{X}^{\hat{\mu}_2}\ldots\partial_{i_6}\hat{X}^{\hat{\mu}_7}\Big[\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7} - 6\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}(\partial_{\hat{\mu}_7}\hat{S} - \hat{k}^{\hat{\nu}}\hat{B}_{\hat{\mu}_7\hat{\nu}})\Big]\,, \tag{6.54}$$

where

$$\begin{aligned}
\partial_{[\hat{\mu}_1}\hat{A}_{\hat{\mu}_2\ldots\hat{\mu}_8]} &= \tfrac{1}{2!7!\sqrt{|\hat{g}|}}\hat{k}^2 e^{-2\hat{\phi}}\varepsilon_{\hat{\mu}_1\ldots\hat{\mu}_{10}}[\hat{k}^2\hat{F}^{\hat{\mu}_9\hat{\mu}_{10}}(\hat{A}) + 2\hat{k}^{\hat{\nu}}\hat{B}_{\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\rho}\hat{\mu}_9\hat{\mu}_{10}}],\\
\partial_{[\hat{\mu}_1}\hat{D}_{\hat{\mu}_2\ldots\hat{\mu}_7]} &= \tfrac{1}{8!\sqrt{|\hat{g}|}}e^{-2\hat{\phi}}\varepsilon_{\hat{\mu}_1\ldots\hat{\mu}_{10}}\hat{H}^{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}},\\
\hat{A}_{\hat{\mu}} &= \hat{k}^{-2}\hat{k}^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}}\,.
\end{aligned} \tag{6.55}$$

The vector $\hat{A}_{\hat{\mu}}$ is the uplifting of the Kaluza-Klein vector A_μ, which can be written in a ten-dimensional form via the Killing vector $\hat{k}^{\hat{\mu}}$. The seven-form $\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7}$ and the six-form $\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}$ are the Poincaré dual of $\hat{A}_{\hat{\mu}}$ and $\hat{B}_{\hat{\mu}\hat{\nu}}$ and the uplifting of $A_{\mu_1\ldots\mu_6}$ and $D_{\mu_1\ldots\mu_5}$ respectively . It is not difficult to show, using the definitions (6.55), that the latter are the only non-zero components of $\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7}$ and $\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6}$.

The Wess-Zumino term (6.54) transforms as a total derivative under the gauge transformations

$$\begin{aligned}
\delta\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_7} &= \partial_{[\hat{\mu}_1}\Lambda_{\hat{\mu}_2\ldots\hat{\mu}_7]} + \partial_{[\hat{\mu}_1}\,,\,_{\hat{\mu}_2\ldots\hat{\mu}_6}\hat{k}^{\hat{\nu}}\hat{B}_{|\hat{\nu}|\hat{\mu}_7]}\,,\\
\delta\hat{D}_{\hat{\mu}_1\ldots\hat{\mu}_6} &= \partial_{[\hat{\mu}_1}\,,\,_{\hat{\mu}_2\ldots\hat{\mu}_6]},\\
\delta\hat{S} &= -\hat{k}^{\hat{\mu}}\Sigma_{\hat{\mu}},\\
\delta\hat{B}_{\hat{\mu}\hat{\nu}} &= \partial_{[\hat{\mu}}\Sigma_{\hat{\nu}]}.
\end{aligned} \tag{6.56}$$

Again the contractions with the Killing vector $\hat{k}^{\hat{\mu}}$ take care of the fact that the z-direction is projected out of the action (6.54). Reduction over the isometry direction z gives

$$S = \tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1\ldots i_6}\ \partial_{i_1}X^{\mu_1}\ldots\partial_{i_6}X^{\mu_6}\Big[A_{\mu_1\ldots\mu_6} - 6\ D_{\mu_1\ldots\mu_5}(\partial_{\mu_6}S + B_{\mu_6})\Big], \tag{6.57}$$

which can be mapped onto the Wess-Zumino term of the $S5$-brane, via the T-duality transformation

$$\tilde{B}_\mu = A_\mu, \qquad \tilde{A}_{\mu_1\ldots\mu_6} = C_{\mu_1\ldots\mu_6}. \tag{6.58}$$

Besides the fact that the world volume action (6.46)-(6.54) is T-dual to the $S5$ action, we will give another evidence in favour of this action: the constructed action also serves as a source term for the monopole solution (2.65)[2]. For simplicity we only look at the purely gravitational part. This leads to the following action[3]

$$S = -\tfrac{1}{\kappa}\int d^{10}x\sqrt{|g|}\ R - \tfrac{T}{2}\int d^6\sigma\sqrt{|\gamma|}\gamma^{ij}\partial_i X^\mu \partial_j X^\nu k^2 \mathcal{G}_{\mu\nu} \tag{6.59}$$

Varying this action with respect to $g^{\mu\nu}$ (and taking care of all the metric factors hidden in $\mathcal{G}_{\mu\nu}$ and k^2), we find for the equation of motion:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{\kappa T}{2\sqrt{|g|}}\int d^6\sigma\sqrt{|\gamma|}\gamma^{ij}\partial_i X^\rho \partial_j X^\lambda \delta(X^\mu - x^\mu)\ \times \tag{6.60}$$

$$\times\Big[-k_\mu k_\nu g_{\rho\lambda} - k^2 g_{\mu\rho}g_{\nu\lambda} + k_\nu k_\lambda g_{\mu\rho} + k_\rho k_\nu g_{\mu\lambda}\Big] \tag{6.61}$$

For the monopole solution (2.65) and the parametrisation

$$X^i = \sigma^i, \qquad Z = X^m = 0 \tag{6.62}$$

the equation (6.61) reduces to

$$\partial^2 H = \frac{\kappa T}{2}\delta(x^m). \tag{6.63}$$

The solution to this equation is given by

$$H(x^m) = \frac{\kappa T}{2}\frac{1}{\sqrt{|x^m x_m|}}. \tag{6.64}$$

We therefore can conclude that the (gravitational part) of the world volume action for the monopole is indeed a source for the ten-dimensional monopole solution.

Gauged sigma models have been used lately [110, 122, 29] to give an eleven-dimensional interpretation to the world volume actions of p-brane solutions in the background of massive Type IIA supergravity [133]. The relation between massive Type IIA theory and eleven-dimensional supergravity is a notorious problem, but it turns out that a massive version of $D = 11$ supergravity can be formulated if one assumes an isometry direction, characterized by a Killing vector $\hat{k}^{\hat{\mu}}$. The world volume action of the p-brane in massive Type IIA theory can be described in terms of the massive $D = 11$ supergravity background fields, if one gauges the isometry direction, using gauge transformations that involve a mass parameter m. Reduction over the direction associated to the isometry gives rise to the world volume actions for the p-branes in a massive ten-dimensional background.

An interesting question now is what does the world volume action for a massive eleven-dimensional Kaluza-Klein monopole look like, if the massless monopole is already described by a gauged sigma model? It turns out that there are two possibilities [25]: one

[2]Strictly speaking source terms are only needed for singular objects. For non-singular objects, such as the Kaluza-Klein monopole, a source term can be introduced if in a certain coordinate frame (non-physical) coordinate singularities appear. This is the case considered here.

[3]At this point we omit the hatted notation for ten-dimensional fields.

can extend the gauging (6.49) of the action (6.46)-(6.54) to the massive gauge transformations of [29] and reduce over the isometry direction in order to obtain the massive $D6$-brane action, or one can assume an extra isometry direction with a new Killing vector $\hat{h}^{\hat{\mu}}$. The massive Type IIA monopole action is then obtained by a massive gauging of and reduction over the new isometry direction. The massless limit can be taken consistently by setting the mass parameter $m = 0$.

Knowing the world volume action of the (massive) Type IIA monopole, it would be interesting to perform a (massive) T-duality transformation and see if one can obtain a world volume action for the Type IIB solitonic five-brane.