Dualities of strings and branes

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Chapter 5

Solutions

In this chapter we will study the extended object solutions of the ten- and eleven-dimensional low energy effective action, presented in Section 2.3, and more particularly, solutions that consist of more then one object: the so-called intersections or superpositions of various extended objects.

The interest in these intersections lays in the fact that, after dimensional reduction, they give rise to various new single-brane solutions in lower dimensions. As we will see later on in this chapter, these lower-dimensional single-brane solutions differ in the way they are coupled to the dilaton, which is determined by the number of branes in the original ten- or eleven-dimensional intersection. So in order to have an overview of the lower-dimensional p-branes, it is necessary to have a classification of the p-brane intersections that reduce to these. A special interest has risen recently in those intersections that reduce to black holes, because the number of micro-states of a black hole (which is a measure for its entropy) is determined by the number of intersections that reduce to this black hole [154].

In Section 5.1 we study the conditions two objects should satisfy in order to form a stable configuration, we classify the different intersection classes and compute the amount of supersymmetry of the intersections. In Section 5.2 we use the conditions for stable two-object intersections to construct multiple intersections, consisting of more then two objects per configuration. Again we will give a classification of the intersection classes for different numbers of objects involved, and determine the maximum number of objects in a configuration. In Section 5.3 we construct new, lower-dimensional solutions from the obtained ten and eleven-dimensional intersections.

The results presented in this chapter are a summary of [15, 20, 21].

5.1 Pair Intersections of Extended Objects

In this section we will study the conditions for two fundamental objects to combine into a two-object intersection. We start with the intersection of two D-branes in Subsection 5.1.1, and generalize the results to any two fundamental objects in Subsection 5.1.2.

5.1.1 D-brane Pair Intersections

The elementary Dirichlet p–brane solutions in ten dimensions are characterized by a single function H that depends on the $(9-p)$ transverse coordinates and is harmonic with respect to these variables. In the string-frame metric, the solution with p $(0 \leq p < 9)$ is given by (2.58):

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(R-R)}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m : p+1, ..., 9). \end{cases} \tag{5.1}$$

For even (odd) p this metric corresponds to a solution of IIA (IIB) supergravity.

We have seen in Section 3.1 that T-duality relates the various D-branes to each other. If one assumes an isometry direction x, the only non-trivial T-duality rule involving the metric is given by (3.46)[1]:

$$\tilde{g}_{xx} = 1/g_{xx}\,. \tag{5.2}$$

Clearly, under this duality transformation the metric of a Dirichlet p-brane becomes that of a $(p+1)$-brane if the duality is performed over one of the transverse directions of the p-brane. In other words, one of the transverse directions of the p-brane has become a world volume direction of the $(p+1)$-brane. It is of course also possible to perform T-duality in an orthogonal direction and change a world volume coordinate into a transverse one. However, in this case one has be careful, since then we have to suppose that the harmonic function after dualization depends on the direction in which we have dualized and it is not guaranteed that this is the case.

It is convenient to represent every coordinate that corresponds to a world volume direction by $\times$ and every direction transverse to the brane by $-$. We thus obtain the following representation of the metric of a Dp-brane solution:

$$ds^2 = \underbrace{\times | \;\; \times \;\; ... \;\; \times}_{p+1} \overbrace{- \;\; - \;\; ... \;\; -}^{9-p}. \tag{5.3}$$

Note that the first $\times$ on the left hand side represents the time direction, which is necessarily a world volume direction. It is easy to see that acting with T-duality on this metric, a $-$ changes into a $\times$ or vice versa. This representation will turn out to be very useful in the study of intersection solutions.

[1]The T-duality rules for the R-R fields can be found in [18].

We will study a special type of intersections: the so-called *orthogonally intersecting threshold BPS bound states*. These are intersections where each participating brane corresponds to an independent harmonic function H_i in the solution. Furthermore, the branes intersect each other orthogonally and the forces between the different branes vanish [161], so that there is no potential energy. The total energy of the intersection is the sum of the energy of each brane separately. The precise form of such a solution is given by the *harmonic function rule* [160], which prescribes how products of powers of the harmonic functions H_i of the intersecting branes must occur in the composite solution. In particular, it implies that if one removes one of the N branes of the configuration (i.e., one of the H_i is set equal to one), a solution with $(N-1)$ intersecting branes is obtained. Solutions satisfying these intersection conditions have been studied extensively in the literature [124, 160, 106, 15, 70, 102, 125, 43, 161, 4, 123, 20, 6, 5, 21]. We will not consider non-threshold bound states, branes at angles, rotating branes or transversely boosted branes. For this we refer to [134, 48, 121, 34, 44, 159].

Let us now study in detail the pair intersections and derive the conditions necessary to form a stable solution of the equation of motion. An Ansatz describing the (string frame) metric of a $D(p+r)$-brane intersecting a $D(p+s)$-brane over p coordinates, is given by [160]:

$$
\begin{aligned}
ds^2 = (H_1 H_2)^{-1/2}\, ds^2_{p+1} - \Big(\frac{H_1}{H_2}\Big)^{1/2}\, dx_s^2 \\
-\Big(\frac{H_2}{H_1}\Big)^{1/2}\, dx_r^2 - (H_1 H_2)^{1/2}\, dx_m^2\ .
\end{aligned}
\tag{5.4}
$$

The harmonic function H_1 describes the $(p+r)$-brane, while H_2 describes the $(p+s)$-brane. It is easy to see that this Ansatz satisfies the harmonic function rule: the metric of a single D-brane is recovered upon setting the other harmonic function equal to one. We will denote this solution of a $D(p+r)$-brane and a $D(p+s)$-brane intersecting over p coordinates as

$$
\left(p\,|\,D(p+r),\ D(p+s)\right)
\tag{5.5}
$$

We see that the coordinates naturally split into three parts: (1) the overall world volume coordinates $x_i, (i = 0, ..., p)$, which are common to the two branes, (2) the overall transverse coordinates x_m, with $m = 1, ..., 9-p-r-s$, which are orthogonal to both branes and (3) the other coordinates x_a with $a : 1, ..., n = r+s$ which are called relative transverse coordinates and are transverse to one brane but parallel to the other one. Using the notation of (5.3), we can write an intersection of the type (5.4) as:

$$
ds^2 = \left\{ \begin{array}{c|ccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - \\ \times & \times & - & - & - & - & - & - & - & - \end{array} \right. \quad \begin{array}{l} : H_1 \\ : H_2\ . \end{array}
\tag{5.6}
$$
$$
\underbrace{\qquad}_{x_i} \qquad \underbrace{\qquad\qquad}_{x_a} \qquad \underbrace{\qquad\qquad}_{x_m}
$$

Every column represents a direction x_μ, which can be either common world volume (x_i), relative transverse (x_a) or overall transverse (x_m).

The labels p, r and s in the configuration (5.4) have to fulfill certain conditions: first of all $p+r+s \leq 9$ for the obvious reason that we only have 9 spatial dimensions to fill.

$n=2$	
(0|0,2)	(0|1,1)
(1|1,3)	(1|2,2)
(2|2,4)	(2|3,3)
(3|3,5)	(3|4,4)
(4|4,6)	(4|5,5)
(5|5,7)	(5|6,6)
(6|6,8)	(6|7,7)
(7|7,9)	(7|8,8)

$n=6$			
(0|0,6)	(0|1,5)	(0|2,4)	(0|3,3)
(1|1,7)	(1|2,6)	(1|3,5)	(1|4,4)
(2|2,8)	(2|3,7)	(2|4,6)	(2|5,5)
(3|3,9)	(3|4,8)	(3|5,7)	(3|6,6)

$n=4$		
(0|0,4)	(0|1,3)	(0|2,2)
(1|1,5)	(1|2,4)	(1|3,3)
(2|2,6)	(2|3,5)	(2|4,4)
(3|3,7)	(3|4,6)	(3|5,5)
(4|4,8)	(4|5,7)	(4|6,6)
(5|5,9)	(5|6,8)	(5|7,7)

$n=8$				
(0|0,8)	(0|1,7)	(0|2,6)	(0|3,5)	(0|4,4)
(1|1,9)	(1|2,8)	(1|3,7)	(1|4,6)	(1|5,5)

Table 5.1: *The tables of the T-duality classes of intersecting configurations of two D-branes in ten dimensions. Via T-duality one can move horizontally and vertically within a table. Only the tables with $n=4$ and $n=8$ correspond to threshold solutions to the equations of motion.*

Furthermore we only want to combine objects which come from the same theory (Type IIA or Type IIB), so r and s have to be both odd or both even. In other words $r+s$ has to be an even number n.

A T-duality transformation on a configuration (5.6) acts in a certain direction, changing in the column corresponding to that direction every $\times$ for a $-$ and vice versa. In general, a $(p\ |p+r,\ p+s)$-configuration can transform under T-duality in two ways: either the T-duality is performed in a relative transverse direction

$$\Big(p \;\Big|\; p+r,\ p+s\Big) \to \Big(p \;\Big|\; p+(r\pm 1),\ p+(s\mp 1)\Big)\,, \tag{5.7}$$

and the duality interchanges a relative transverse direction of one object with a relative transverse direction of the other object. The second possibility is that the T-duality is applied to an overall transverse direction or a common world volume direction

$$\Big(p \;\Big|\; p+r,\ p+s\Big) \to \Big(p\pm 1 \;\Big|\; (p\pm 1)+r,\ (p\pm 1)+s\Big)\,. \tag{5.8}$$

In either case (5.7) or (5.8) the number $r+s=n$ remains constant, so that n can be used to label the four different classes, as given in Table 5.1 [15, 70]. Within each class we can move horizontally or vertically via the T-duality transformations: horizontal movements correspond to a T-duality transformation in a relative transverse direction (5.7), while vertical movements are generated by T-duality transformations of the type (5.8).

For the Ansatz for the dilaton we take the product of the dilaton expressions for each brane separately:

$$e^{-2\phi} = (H_1)^{\frac{p+r-3}{2}} \, (H_2)^{\frac{p+s-3}{2}} \, . \tag{5.9}$$

In this way the harmonic function rule is satisfied in a straightforward way. Furthermore it is guaranteed that (5.9) transforms correctly under the T-duality rule for the dilaton (3.5) to give the right dilaton expression of the T-dual intersection (5.7) or (5.8).

The expression for the R-R gauge fields can easily be obtained by the requirement that, if one of the harmonic functions is set equal to one, the intersecting configuration should reduce to one of the D-brane solution (5.1). The explicit form of the R-R gauge fields is most easily given by using a formulation where the magnetic configurations are described by magnetic (dual) potentials. This leads us to consider the Lagrangian

$$\mathcal{L} = \sqrt{|g|}\left\{ e^{-2\phi}\Big[R - 4(\partial\phi)^2\Big] + \frac{(-)^{p+r+1}}{2(p+r+2)!}F^2_{(p+r+2)} + \frac{(-)^{p+s+1}}{2(p+s+2)!}F^2_{(p+s+2)} \right\} , \tag{5.10}$$

where it is understood that in the field equations one imposes the constraint that $F_{(8-p)}$ is the dual of $F_{(p+2)}$. In particular, $F_{(5)}$ is self-dual. Pseudo-Lagrangians of this form have been discussed in [17]. It is also understood that the two kinetic terms for the gauge fields become identical if $r = s$.

We next distinguish three different cases:

- **Case 1:** Both harmonic functions depend on the overall transverse directions x_m. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rm} = \partial_m H_1^{-1} \, , \qquad F^{(2)}_{0\cdots p1\cdots sm} = \partial_m H_2^{-1} \, . \tag{5.11}$$

- **Case 2:** The function H_1 depends on the overall transverse directions x_m, whereas H_2 depends on its relative transverse directions x_a. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rm} = H_2^{\alpha}\partial_m H_1^{-1} \, , \qquad F^{(2)}_{0\cdots p1\cdots sa} = \partial_a H_2^{-1} \, . \tag{5.12}$$

- **Case 3:** Both harmonic functions depend on their (own) relative transverse directions x_a and x_b. The R-R gauge fields are given by

$$F^{(1)}_{0\cdots p1\cdots rb} = H_2^{\alpha}\partial_b H_1^{-1} \, , \qquad F^{(2)}_{0\cdots p1\cdots sa} = H_1^{\beta}\partial_a H_2^{-1} \, . \tag{5.13}$$

The α in Case 2 and the α, β in Case 3 are arbitrary (real) parameters that cannot be fixed by the Bianchi identities. We will determine them via the equations of motion.

So far, we have only applied T-duality to generate the Ansatz (5.4), (5.9), (5.11–5.13) for intersecting D-brane configurations, without really knowing whether they correspond to solutions to the equations of motion. Our next task is to determine which of these configurations corresponds to a (supersymmetric) solution of the Lagrangian (5.10). Substituting our Ansatz into the vector field and dilaton equation[2], we see that [14]:

- Case 1 can only be a solution for $n = 4$,
- Case 2 for $n = 4$ and $\alpha = 0$
- Case 3 requires that $n = 8$ and $\alpha = \beta = 1$.

Configurations in our Ansatz with $n = 2, 6$ relative transverse directions do not appear as solutions of the equations of motion. Non-threshold bound states with 2 relative transverse directions have been argued to exist [45], but it is not clear whether these solutions are of the form given above.

It turns out that the Cases 1 and 2 can naturally be combined into a more general configuration where H_1 only depends on the overall transverse directions, as before, but where H_2 is given by the sum of two harmonics $H_2^{(a)}, H_2^{(b)}$, which depend on the overall and relative transverse directions, respectively, i.e.

$$H_2(x_m, x_b) = H_2^{(a)}(x_m) + H_2^{(b)}(x_b)\,. \tag{5.14}$$

We will now investigate the supersymmetry of these solutions. For a single D-brane the supersymmetry condition is $\delta\lambda = \delta\psi_\mu = 0$, where λ is the dilatino and ψ_μ the gravitino in the IIA/IIB supergravity multiplet. Their variations (in the string frame) are given by (compare to (2.49)):

$$\begin{aligned} \delta\psi_\mu &= \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon + \frac{(-)^p}{8(p+2)!}\ e^\phi\ F_{\mu_1\ldots\mu_{p+2}}\ \gamma^{\mu_1\cdots\mu_{p+2}}\gamma_\mu\ \epsilon'_{(p)} = 0\,, \\ \delta\lambda &= \gamma^\mu(\partial_\mu\phi)\epsilon + \frac{3-p}{4(p+2)!}\ e^\phi F_{\mu_1\cdots\mu_{p+2}}\ \gamma^{\mu_1\cdots\mu_{p+2}}\ \epsilon'_{(p)} = 0\,, \end{aligned} \tag{5.15}$$

where $\epsilon'_{(p)} = \epsilon$ for $p = 0, 4, 8$; $\epsilon'_{(p)} = \gamma_{11}\epsilon$ for $p = 2, 6$; $\epsilon'_{(p)} = i\epsilon$ for $p = 7$ and $\epsilon'_{(p)} = i\epsilon^\star$ for $p = 1, 5$. Substituting the single D-brane solution into the above equation leads to the condition

$$\epsilon + \gamma_{01\cdots p}\ \epsilon'_{(p)} = 0\,, \tag{5.16}$$

which defines a projection operator on ϵ that breaks half of the supersymmetry.

Now consider the intersection of a $(p+r)$-brane with a $(p+s)$-brane. Then the two supersymmetry conditions corresponding to the $(p+r)$-brane and $(p+s)$-brane are given by

$$\begin{aligned} \epsilon + \gamma_{01\cdots p+r}\epsilon'_{(p+r)} &= 0\,, \\ \epsilon + \gamma_{01\cdots p+s}\epsilon'_{(p+s)} &= 0\,, \end{aligned} \tag{5.17}$$

[2]The case that only intersecting 3-branes are involved is special since for this case the dilaton equation is trivially satisfied. By applying T–duality one can relate this case to the other cases and show that the same restrictions as given below apply.

respectively. Each one breaks half of the supersymmetry. Combining the two supersymmetry conditions we get

$$\epsilon'_{(p+r)} = (-)^{\frac{1}{2}r(r+1)}\gamma_{r+s}\epsilon'_{(p+s)}\,. \tag{5.18}$$

We now distinguish four cases in which the two spinors in the above equation are given by $(\epsilon, \epsilon), (\epsilon, \gamma_{11}\epsilon), (i\epsilon, i\epsilon)$ or $(i\epsilon, i\epsilon^\star)$, respectively. All four cases lead to the consistency condition that $\gamma_{r+s}^2 = 1$, or

$$n = 4 \quad \text{or} \quad 8\,. \tag{5.19}$$

This reproduces the result of [129], where it is stated that the only supersymmetric (1/4 of the supersymmetry is unbroken) pair intersections are the ones with $r + s = 0 \; mod \; 4$.

We next extend this analysis and consider the Killing spinor equation that follows from $\delta\lambda = 0$ for the case that we substitute the complete intersecting configuration and not only the separate D-brane configurations. In the string–frame we obtain the following equation from $\delta\lambda = 0$:

$$\begin{aligned} \gamma^\mu(\partial_\mu\phi)\,\epsilon \quad &+ \quad \tfrac{1}{4}(3-p-r)\; e^\phi F^{(1)}_{0\cdots p+r\mu}\gamma^{0\cdots p+r\mu}\;\epsilon'_{(p+r)} \\ &+ \quad \tfrac{1}{4}(3-p-s)\; e^\phi F^{(2)}_{0\cdots p+s\mu}\gamma^{0\cdots p+s\mu}\;\epsilon'_{(p+s)} = 0\,. \end{aligned} \tag{5.20}$$

Substituting the explicit form of the general intersecting configuration (5.4), (5.9), (5.11-5.13) into the above Killing spinor equation leads, for case 1 to $n = 4$, for Case 2 to $n = 4, \alpha = 0$ and for Case 3 to $n = 8, \alpha = \beta = 1$ [14]. This nicely agrees with our earlier finding that only these configurations can be solutions to the equations of motion.

Summarizing, we come to the following conclusions: there exist three types of D-brane pair intersections in ten dimensions satisfying the Ansatz (5.4), (5.9), (5.11-5.13), each conserving one quarter of the original supersymmetry. The three types of intersections differ in the dependence of the harmonic function on the coordinates and in the number n of relative transverse directions in the intersection, which labels the T-duality classes of intersections:

1. both harmonic functions depend on the overall transverse coordinates x_m. The only allowed intersections are the ones that have $n = 4$ relative transverse directions. The gauge fields are of the form (5.11).

2. one of the harmonics depends on the overall transverse coordinates, while the other depends on its relative transverse directions. Also here the only allowed intersections are the ones in the $n = 4$ class. The gauge fields are of the form (5.12) with $\alpha = 0$.

3. both harmonic functions depend on their relative transverse directions. Now the intersections must have $n = 8$ relative transverse coordinates and the gauge fields are of the form (5.13) with $\alpha = \beta = 1$.

In the next subsection we will try to generalize these results to M-brane intersections in eleven dimensions and ten-dimensional intersections that also involve other objects than D-branes.

	common wv.	relative trv.	overall trv.
$(0\|M2,M2)$	–	$(0\|F1,D2)$	$(0\|D2,D2)$
$(1\|M2,M5)$	$(0\|F1,D4)$	$(1\|F1,S5)$ $(1\|D2,D4)$	$(1\|D2,S5)$
$(3\|M5,M5)$	$(2\|D4,D4)$	$(3\|D4,S5)$	$(3\|S5,S5)$
$(1\|M2,\mathcal{W})$	$(0\|F1,D0)$	$(1\|F1,W)$	$(1\|D2,W)$
$(1\|M5,\mathcal{W})$	$(0\|D4,D0)$	$(1\|D4,W)$	$(1\|S5,W)$
$(2\|M2,\mathcal{KK})$	$(1\|F1,KK)$	$(2\|D2,KK)$	$(2\|D2,D6)$
$(5\|M5,\mathcal{KK})$	$(4\|D4,KK)$	$(5\|S5,KK)$	$(5\|S5,D6)$
$(0\|M2,\mathcal{KK})$	–	$(0\|F1,D6)$ $(0\|D2,KK)$	$(0\|D2,D6)^*$
$(3\|M5,\mathcal{KK})$	$(2\|D4,KK)$	$(3\|D4,D6)$ $(3\|S5,KK)$	$(3\|S5,D6)^*$
$(1\|\mathcal{W},\mathcal{KK})$	$(0\|D0,KK)$	$(1\|W,KK)$	$(1\|W,D6)$
$(4\|\mathcal{KK},\mathcal{KK})^a$	$(3\|KK,KK)^a$	$(4\|D6,KK)^*$	$(4\|D6,D6)$
$(4\|\mathcal{KK},\mathcal{KK})^b$	$(3\|KK,KK)^b$	$(4\|D6,KK)$	$(4\|D6,D6)^*$

Table 5.2: *Pair intersections in $D = 11$ and their reductions to $D = 10$ with dependence on overall transverse coordinates: the first column represents the pair intersections in $D = 11$, reductions to non-trivial solutions in $D = 10$, obtained by compactification in different directions (common world volume, relative transverse and overall transverse) with respect to the branes, are indicated in the remaining columns. The $D = 10$ solutions marked with * are not of the usual harmonic form.*

5.1.2 General Pair Intersections

The results of subsection 5.1.1 can be easily be uplifted to eleven dimensions, since the relations between the ten-dimensional Type IIA D-brane solutions and the solutions of $D = 11$ supergravity are known [156] (see Figure 3.1). On the other hand, dimensional reduction of the intersections in $D = 11$ yields new ten-dimensional intersections that do not only contain D-branes, but also fundamental strings, solitonic five-branes, waves and monopoles.

In Tables 5.2 and Table 5.3 we summarize the results on the pair intersections [21]. The two independent harmonic functions of the pairs in Table 5.2 depend on the overall transverse coordinates[3]. For the pairs in Table 5.3 both harmonic functions must depend on the relative transverse coordinates.

In the first three rows of Table 5.2 we list the intersections of $M2$- and $M5$-branes

[3]Here we will not consider the case where one of the harmonic functions in the intersections depends on the relative transverse directions. Their intersections are the same as for the case where the two harmonic functions depend on the overall transverse coordinates.

[125, 160] and their reductions to ten dimensions:

$$(0|M2,M2) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \end{array} \right. \tag{5.21}$$

$$(1|M2,M5) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & \times & - & \times & \times & \times & \times & - & - & - & - \end{array} \right. \tag{5.22}$$

$$(3|M5,M5) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & \times & \times & \times & - & - & \times & \times & - & - & - \end{array} \right. \tag{5.23}$$

As an example, we will discuss the $(1|M2,M5)$ configuration and its different compactifications to ten dimensions. Reduction over x_1 gives $(0|F1,D4)$ in ten dimensions. For the relative transverse directions the possibilities are: either reduction over x_2, giving $(1|F1,S5)$, or reduction over one of the directions $x_3,\ldots,x_6$, giving $(1|D2,D4)$. Finally, one can impose an isometry in one of the overall transverse directions by restricting the dependence of the harmonic functions to three coordinates. Reduction over such a direction gives $(1|D2,S5)$.

The next two rows represent the addition of a wave (2.64) to the $D=11$ M-branes. The z-direction of the wave must be placed in the world volume of the M-brane. The dependence of the harmonic functions is only on the directions transverse to the M-brane, so that the wave does not propagate. The metric for these two $D=11$ pairs can be represented by[4]:

$$(1|M2,\mathcal{W}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & z & - & - & - & - & - & - & - & - & - \end{array} \right. \tag{5.24}$$

$$(1|M5,\mathcal{W}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & z & - & - & - & - & - & - & - & - & - \end{array} \right. \tag{5.25}$$

The next four rows in Table 5.2 denote the pairs involving one M-brane and one Kaluza-Klein monopole (2.65). The metric for these four cases takes on the form

$$(2|M2,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.26}$$

$$(5|M5,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & \times & \times & \times & \times & \times & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.27}$$

$$(0|M2,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & - & - & \times & - & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.28}$$

$$(3|M5,\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & \times & - & - & \times & \times & \times & \times & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.29}$$

As we see, there are two possibilities. The z-direction of the Kaluza-Klein monopole can be placed either in a direction transverse to ($(2|M2,\mathcal{KK})$ and $(5|M5,\mathcal{KK})$) or in the world volume of the M-brane ($(0|M2,\mathcal{KK})$ and $(3|M5,\mathcal{KK})$). The solutions (5.26) and (5.27) have also been given in [160, 43]. For these, the reduction to $D=10$ is straightforward. Note that the reduction over an overall transverse direction can be

[4]Note that we extend the notation $(p|p+r;p+s)$ to include waves and monopoles with the understanding that the world volume directions of the "$\mathcal{W}$-brane" are given by t,z (see (2.64)), and the transverse directions of the "$\mathcal{KK}$-brane" are given by the isometry direction z and the coordinates in which the Kaluza-Klein vector is oriented. These directions (called x_m in (2.65)) will be denoted by A_m.

either over a direction indicated by z, or, by imposing an additional isometry, in the direction of a component of the vector field.

In the solutions (5.28) and (5.29) the harmonic functions depend only on the two overall transverse coordinates, so that the Kaluza-Klein monopole has one additional isometry direction (indicated by A_1). In both of these solutions the reduction over the relative transverse A_1 and z directions yields, after a coordinate transformation, the same result[5].

The last three rows of Table 5.2 correspond to intersections of Kaluza-Klein monopoles and waves. The possibilities are shown in (5.30-5.32)[6]. Note that there are two ways to intersect two Kaluza-Klein monopoles, both with a five-dimensional common world volume. In solution (5.31) the two harmonic functions depend on a single coordinate (x^1), in (5.32) on two coordinates (x^1, x^2).

$$(1|\mathcal{W},\mathcal{KK}) = \left\{ \begin{array}{c|cccccccccc} \times & - & - & - & - & z_1 & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z_2 & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.30}$$

$$(4|\mathcal{KK},\mathcal{KK})^a = \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \\ \times & B_1 & \times & \times & z & B_5 & B_6 & \times & \times & \times & \times \end{array} \right. \tag{5.31}$$

$$(4|\mathcal{KK},\mathcal{KK})^b = \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & B_1 & B_2 & \times & \times & B_5 & z_2 & \times & \times & \times & \times \end{array} \right. \tag{5.32}$$

The solution (5.31) solves the equations of motion, since it is the known ten-dimensional solution $(4|D6, D6)$ lifted up to $D = 11$. The configuration (5.32) must be a solution because, after reduction over a common world volume direction, it can be related to a known solution involving two solitonic five-branes via the following T-duality chain in $D = 10$:

$$(3|S5, S5) \rightarrow (3|S5, KK) \rightarrow (3|KK, KK)^b \,. \tag{5.33}$$

Similarly, the intersection of a wave and a Kaluza-Klein monopole can be obtained from ten dimensions by first constructing an intersection in $D = 10$ of a $D1$-brane with the solitonic five-brane and performing a T-duality in the direction of the string:

$$(0|D1, S5) \rightarrow (0|D0, KK) \,, \tag{5.34}$$

and by lifting this to eleven dimensions.

In Table 5.3 we consider intersections in which the two harmonic functions depend on the relative coordinates. There is one pair involving only $M5$ [70], and five pairs involving Kaluza-Klein monopoles. Some of these configurations and their generalization to non-orthogonal intersections were discussed in [69].

Below we present the metric of these pairs in the usual, short-hand way. The pairs involving Kaluza-Klein monopoles are each related to known solutions through $D = 10$, so that we can be sure that they solve the equations of motion. For example, $(2|\mathcal{KK},\mathcal{KK})$

[5] For a more detailed discussion of the possible dependences of the harmonic function of the monopole on one or two coordinates only, we refer to [21]. In general one can say that upon reduction of the monopole solution $\mathcal{KK}_D$ over any transverse direction z or x_m one always finds a magnetic $(D-5)$-brane.

[6] Solution (5.30) was presented in [160].

	common wv.	relative trv.	overall trv.
$(1\|M5,M5)$	$(0\|D4,D4)$	$(1\|D4,S5)$	$(1\|S5,S5)$
$(0\|M2,\mathcal{KK})$	–	$(0\|D2,KK)$ $(0\|F1,D6)^*$	$(0\|D2,D6)$
$(1\|M5,\mathcal{KK})$	$(0\|D4,KK)$	$(1\|S5,KK)$ $(1\|D4,D6)$	–
$(3\|M5,\mathcal{KK})$	$(2\|D4,KK)$	$(3\|S5,KK)$ $(3\|D4,D6)^*$	$(3\|S5,D6)$
$(2\|\mathcal{KK},\mathcal{KK})$	$(1\|KK,KK)$	$(2\|D6,KK)$	–
$(4\|\mathcal{KK},\mathcal{KK})$	$(4\|KK,KK)$	$(4\|D6,KK)$ $(4\|D6,KK)^*$	$(4\|D6,D6)^*$

Table 5.3: *Pair intersections in $D = 11$ and their reductions to $D = 10$ with dependence on relative transverse coordinates. The reductions indicated by a * are not expressed in a standard way in terms of harmonic functions.*

can be reduced to $(1|KK,KK)$ in ten dimensions and applying T-duality twice, in the directions z_1 and z_2, we find

$$(1|KK,KK) \rightarrow (1|S5,KK) \rightarrow (1|S5,S5)\,, \tag{5.35}$$

and this can be lifted up to $(1|M5,M5)$, which is a known solution. The intersections of Table 5.3 are of the form:

$$(1|M5,M5) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & \times & - & - & - & - & \times & \times & \times & \times & - \end{array} \right. \tag{5.36}$$

$$(0|M2,\mathcal{KK}) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.37}$$

$$(1|M5,\mathcal{KK}) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & \times & \times & \times & - & - & - & - & - \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.38}$$

$$(3|M5,\mathcal{KK}) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & \times & \times & \times \\ \times & A_1 & A_2 & A_3 & z & \times & \times & \times & \times & \times & \times \end{array} \right. \tag{5.39}$$

$$(2|\mathcal{KK},\mathcal{KK}) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & z_2 & B_6 & B_7 & B_8 & \times & \times \end{array} \right. \tag{5.40}$$

$$(4|\mathcal{KK},\mathcal{KK}) \;=\; \left\{ \begin{array}{c|cccccccccc} \times & A_1 & A_2 & A_3 & z_1 & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & B_3 & B_4 & B_5 & z_2 & \times & \times & \times & \times \end{array} \right. \tag{5.41}$$

Let us finally make a remark about the ten-dimensional intersections that are not characterized by the usual harmonic functions. They come from regular eleven-dimensional intersections, but fall out of the usual Ansatz by the way we have reduced to $D = 10$. They are indicated in Table 5.2 and Table 5.3 by a *.

As an example, consider the reduction of (5.31). The harmonic functions depend on x_1, the gauge field components can be chosen to be all zero except A_2 and B_5, which then depend on x_3 and x_6, respectively. It is not difficult to see that this satisfies the condition (2.66) on the off-diagonal components of the metric.

Reduction over z gives $(4|D6, D6)$, but also reduction over x_2 is possible. This gives a $D = 10$ configuration which has the properties of $(4|D6, KK)$, but the fields do not have the standard harmonic form. It is given by [21]:

$$\begin{aligned} ds^2 &= \varphi^{-1/2}(dt^2 - dx^2_{(7-10)} - H_2 dx^2_{(5-6)}) \\ &\quad - H_2^{-1}\varphi^{1/2}\ \big((dz + B_5 dx_5)^2 + (H_1^2 H_2 + A_2^2)(dx_3^2 + H_2 dx_1^2)\big)\ , \\ e^{2\phi} &= \varphi^{-3/2}\,, \\ C_z &= \frac{\varphi A_2}{H_1 H_2}\,, \qquad C_5 = \frac{\varphi A_2 B_5}{H_1 H_2}\,, \end{aligned} \tag{5.42}$$

where

$$\varphi = H_1 H_2 / (A_2^2 + H_1^2 H_2)\,. \tag{5.43}$$

The nonzero components of the R-R-vector field in $D = 10$ are denoted by C_μ. Note that φ is indeed not harmonic in x_1, x_3. If $H_2 = 1$ and $B_5 = 0$, φ does become harmonic, and we obtain a standard $D6$ solution, after the coordinate transformation

$$d(u + iv) = (H + iA_2)\, d(x_1 + ix_3). \tag{5.44}$$

Conversely, for $H_1 = 1$, $A_2 = 0$ a standard Kaluza-Klein monopole is obtained in $D = 10$.

5.2 Multiple Intersections of Extended Objects

In this section we will generalize the results we found in the previous sections to intersections consisting of more then two extended objects. The conditions for pair intersections will form the basis to construct the multiple intersections. We will follow the same strategy as before, namely first we will construct the multiple D-brane and multiple M-brane intersections, and then see where we can add other objects.

For simplicity we will limit ourselves from now on to intersections of the first type, namely intersections that have $n = 4$ relative transverse dimensions and where the harmonic functions depend on the overall transverse directions. For multiple intersections of the other types we refer to [20, 69].

5.2.1 Multiple D-brane Intersections

The construction of multiple D-brane intersections is completely determined by the harmonic function rule and the conditions for pair intersections of D-branes. Since our Ansatz describes threshold BPS bound states, which do not exert forces on each other, one can always remove all but two D-branes in the multiple intersection to infinity, without cost of energy. The remaining pair intersection should of course satisfy the conditions found in the previous section.

One can therefore follow an iterative procedure by adding to a given configuration an extra brane, such that is has $n = 4$ relative transverse directions with all other branes.

In the language of (5.6), this means that we add an extra row with $\times$'s and $-$'s, such that the new row has four different entries of $\times$ and $-$ with every other row.

To streamline the construction, it is useful to characterize an intersection by the contents of the columns (components of the metric) corresponding to the relative transverse coordinates. These columns will be the building blocks of the intersections. For an N-brane intersection, a certain column will consist of k $\times$'s and $(N-k)$ $-$'s. Since T-duality replaces in a column all $\times$'s for $-$, it is not difficult to construct a T-duality invariant quantity: we define n_k as the number of columns with k $\times$'s or k $-$, where $k \leq [N/2]$. The square brackets indicate the integer part of $N/2$.

In an N-brane intersection ($N \geq 2$) there are $\frac{1}{2}N(N-1)$ pairs of intersecting branes. The total number of differences between $\times$ and $-$ in the N-brane intersection is therefore four times the number of pairs, or $2N(N-1)$. On the other hand, a column with k $\times$'s contributes $k(N-k)$ differences. Then we must have

$$\sum_{k=1}^{[N/2]} k(N-k)n_k = 2N(N-1)\,, \tag{5.45}$$

with $\sum_k n_k < 9$. Given N, this is an equation for the n_k.

Let us give a few examples. For $N=2$ there is only one type of building block, namely $k=1$. Equation (5.45) for this case reduces to the equation $n_1 = 4$, which is the condition for a stable threshold BPS bound state found in the previous section. For $N=3$ there is again only one type of building block ($k=1$) and we find $n_1 = 6$. For $N=4$, there are two types of building blocks, with $k=1$ and with $k=2$. Thus (5.45) reduces to $3n_1 + 4n_2 = 24$ which has 3 solutions namely $(n_1, n_2) = (8,0), (4,3)$ and $(0,6)$. For $N=5$ there are again two types of building blocks with $k=1,2$ and we find $4n_1 + 6n_2 = 40$ leading to 2 solutions given by $(n_1, n_2) = (4,4)$ and $(1,6)$.

Clearly, (5.45) is only a necessary condition for the existence of a solution. Given a set of n_k allowed by (5.45), it is not clear that one can actually realize such a solution in terms of the available building blocks and consistent with condition for pair intersections. This is because (5.45) is just an expression for the total number of differences in $\times$ and $-$, but does not contain information about how the configurations should be realized in terms of the no-force condition. Indeed it turns out that there exist solutions of (5.45) that do not correspond to intersecting configurations. However, (5.45) remains useful as a tool in the classification of multiple D-brane intersections.

Note that the numbers $(n_1, n_2, ...n_{[N/2]})$ form a good label for the classification: by construction the n_k's are invariant under T-duality and the set $(n_1, n_2, ...n_{[N/2]})$ labels a unique D-brane configuration,[7] up to T-duality and interchanges of rows and columns. The latter are in fact nothing else then a relabeling of the space-time coordinates and the harmonic functions.

The construction of multiple intersections of D-branes is now straightforward: start adding branes in all possible ways to a known intersection, such that the harmonic

[7] Although this has not been proved rigorously, the uniqueness can be seen in a case by case analysis.

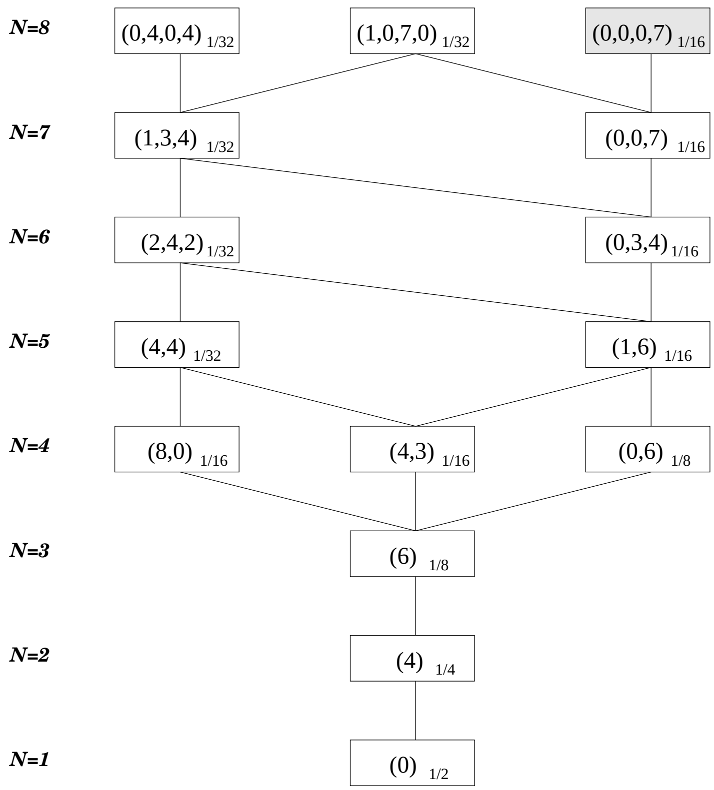

Figure 5.1: *D-brane intersections with $n = 4$ in 10 dimensions: the numbers $(n_1, n_2, \ldots)$ label the number of times a building block with $(1, 2, \ldots)$ world volume directions is used. The subscript in the Figure indicates the amount of supersymmetry preserved in each solution. The number N indicates the number of independent harmonics. The lines between solutions indicate how one configuration follows from another by adding (or truncating) a harmonic function. The configuration (0,0,0,7) cannot be extended to 11 dimensions in terms of $M2$- and $M5$-branes only.*

function rule and equation (5.45) are satisfied. The label $(n_1, n_2, ...n_{[N/2]})$ will tell to which T-duality class the new intersection belongs.

We can repeat this analysis till $N = 8$, for which we find three different (T-inequivalent) configurations. At this point the procedure stops. Although (5.45) has solutions for $N = 9$, it turns out to be impossible to add a ninth brane such that it has $n = 4$ relative transverse directions with the eight other branes. An overview of the different intersection classes and their relations is given in Figure 5.1, the three $N = 8$ configurations are given by (all other configurations with $N < 8$ can be obtained via truncation of harmonic function in the above configurations) [20]:

$$
(0,4,0,4): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & - & \times & \times & - & \times & - & - & \times & - \\
\times & \times & - & \times & - & - & \times & - & \times & - \\
\times & - & \times & \times & - & - & \times & \times & - & -
\end{array} \right. \tag{5.46}
$$

$$
(1,0,7,0): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & - & \times & \times & - & \times & - & - & \times & - \\
\times & - & \times & - & \times & \times & - & \times & - & - \\
\times & - & \times & \times & - & - & \times & \times & - & -
\end{array} \right. \tag{5.47}
$$

$$
(0,0,0,7): \quad \left\{ \begin{array}{c|cc|cc|cc|cc|c}
\times & - & - & - & - & - & - & - & - & - \\
\times & \times & \times & \times & \times & - & - & - & - & - \\
\times & \times & \times & - & - & \times & \times & - & - & - \\
\times & - & - & \times & \times & \times & \times & - & - & - \\
\times & \times & - & \times & - & \times & - & \times & - & - \\
\times & \times & - & - & \times & - & \times & \times & - & - \\
\times & - & \times & \times & - & - & \times & \times & - & - \\
\times & - & \times & - & \times & \times & - & \times & - & -
\end{array} \right. \tag{5.48}
$$

At this stage one should still check whether the above configurations satisfy the Einstein equation and the dilaton equations of motion. This can be done for the three $N = 8$ configurations, using the computer. This implies that the intersections with $N \geq 5$ are also solutions. For lower N the number of overall transverse coordinates increases, so that the harmonic functions can depend on more coordinates. One can check that the equations of motion indeed allow this.

Let us now consider the supersymmetry of the solutions. Just as for the pair intersections, the solution is supersymmetric if $\delta\lambda = \delta\psi_\mu = 0$ (5.15). Each brane contributes a projection operator (5.16) on ϵ, and each time we add a new projection operator, half of the remaining supersymmetry gets broken. However, sometimes it is possible to add a D-brane in such a way that its projection operator is not independent, but given by a product of previous operators [76, 106, 70]. In that case no additional supersymmetry generator is broken. In Figure 5.1 we see this happen for example in the $N = 4$ intersection. For $N = 3$ we have one 0-brane and two 4-branes which preserve 1/8th of the supersymmetry because of the three independent projection conditions

$$
\begin{aligned}
(1 + \gamma_0)\epsilon &= 0\,, \\
(1 + \gamma_{01234})\epsilon &= 0\,, \\
(1 + \gamma_{01256})\epsilon &= 0\,.
\end{aligned} \tag{5.49}
$$

From Figure 5.1 we see that there are three different ways to add a fourth brane. Two of them break an extra half of the remaining supersymmetry (configurations (8,0) and (4,3)), since in these cases the new brane introduces an independent projection operator. The third way (corresponding to configuration (0,6)) is by adding a 4-brane oriented in such a way that its projection operator

$$
(1 + \gamma_{03456})\epsilon = 0 \tag{5.50}
$$

is exactly the product of the previous three operators (5.49). In this way no extra conditions on the Killing spinor arise and no more supersymmetry gets broken.

The construction of projection operators for supersymmetry is another way of building up Figure 5.1. Apparently supersymmetry and the equations of motion go hand in hand: supersymmetry protects the stability of a configuration and vice versa, all stable solutions are supersymmetric. For a more systematic approach on how supersymmetry can be used to obtain intersections, we refer to [54]. The amount of unbroken supersymmetry of each configuration can be found in Figure 5.1.

By using T-duality one can express all intersections in terms of $D2$- and $D4$-branes, except the $N = 8$ $(0,0,0,7)$ solution. Writing an intersection in terms of $D2$- and $D4$-branes has the advantage that an uplifting to eleven dimensions is straightforward in terms of $M2$- and $M5$-branes. As we will see in the next subsection, the uplifting of the $N = 8$ $(0,0,0,7)$ solution is a little more involved, since it requires the presence of a eleven-dimensional gravitational wave. This solution is indicated by a grey box in Figure 5.1.

5.2.2 Multiple Intersections in Eleven Dimensions

Intersections consisting of $D2$- and $D4$-branes can be rewritten straightforwardly in eleven dimensions in terms of $M2$- and $M5$-branes. However, as we have seen in the previous subsection, not all intersection classes can be written in as a purely $D2$-$D4$ intersection. From the point of view of the relation between $D = 11$ supergravity and Type IIA theory, we would like to have to have an eleven dimensional interpretation for these solutions as well.

In general, if there is really a one to one map between eleven-dimensional supergravity and Type IIA theory, then we expect all intersections that involve $D0$- and $D6$-branes to be directly related to an eleven-dimensional solution, and not indirectly via a T-duality transformation to a $D2$-$D4$ intersection.

In this subsection we will give a classification of the eleven-dimensional intersections that reduce to intersections of D-branes in ten dimensions with the harmonic functions depending on the overall transverse directions.. We first give a classification of $M2$-$M5$ intersections and relate them to the $D2$-$D4$ intersection of the previous subsection. Then we will see how we can add wave and monopole solutions, in order to give a $D = 11$ interpretation for the other D-brane intersections.

The M-brane pair intersections that satisfy the eleven-dimensional equivalent of the $n = 4$ condition, are the ones we found in (5.21-5.23): $(0|M2, M2), (1|M2, M5)$ and $(3|M5, M5)$. Next, we add further $M2$-branes and/or $M5$-branes, always satisfying this intersection condition for each pair. Like in $D = 10$, we find that this procedure stops at $N = 8$. We will not present the details of our constructive procedure but instead present the results below. In this way we recover the $M2$-$M5$ intersections which are the direct uplifting of the D-brane intersections found in the previous subsection, but also some extra one, which cannot be reduced to pure D-brane intersections in ten dimensions. One can go from M-branes in $D = 11$ to D-branes in $D = 10$ only if there is one specific direction, such that all $M2$-branes are reduced to $D2$-branes, and all $M5$-branes to $D4$-branes. This will not be true in general: some configurations (which have $N \geq 4$) in $D = 11$ will only reduce to $D = 10$ intersections that involve NS-NS branes. Although these intersections do not have direct relevance for our original motivation (construct the eleven-dimensional version of the D-brane intersections), we will list them here anyway for the sake of completeness. In this way we can give a complete classification of intersecting M-branes with overall transverse dependence of the harmonic functions.

To characterize the configurations, we use again the contents of the columns in the representation of the metric. For an N-intersection each column can have $1, \dots, N$ $\times$'s, indicating world volume directions. The numbers of columns with k world volume directions label the solutions, in the notation $\{n_1, \dots, n_N\}$ (using curly brackets). It is convenient to classify, in a first stage, the eleven-dimensional intersections up to T-duality. T-duality works as follows in $D = 11$ [26]: two $D = 11$ solutions are called T-dual if, upon reduction to $D = 10$ dimensions they lead to T-dual D-brane configurations. These T-dual $D = 11$ solutions can be represented by the labels $(n_1, \dots, n_{[N/2]})$ (using round brackets) which were used in the previous section to label T-dual D-brane configurations. Of course, this notation can only be used for $D = 11$ intersections that can be reduced to D-branes only. For the other classes we will stick to the curly bracket notation.

The results we find in $D = 11$ can be represented in three different ways [20]. First of all, in Figure 5.2 we present the solutions up to T-duality in $D = 11$. For those M-brane intersections that reduce to one of the D-brane intersections given in Figure 5.1, we use the same notation $(n_1, \cdots, n_{[N/2]})$ as in the previous Section. The gray rectangles indicate the solutions which necessarily contain NS-NS-branes in $D = 10$, and for those the $D = 11$ notation $\{n_1, \cdots, n_N\}$ is used. Secondly, in Table 5.4 more details are given about the contents of Figure 5.2 by showing all $D = 11$ solutions that correspond to the same $D = 10$ D-brane intersection. Finally, we give the $N = 8$ intersections explicitly:

$$\{0,4,0,5,0,0,0,0\}_{1/32} : \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \\ \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & - & - & - & - & - & - & \times & \times & - & - \\ \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & - & \times & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \end{array} \right. \tag{5.51}$$

$$\{1,0,6,1,1,0,0,0\}_{1/32} : \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \\ \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & - & - & - & - & - & - & \times & \times & - & - \\ \times & - & \times & - & \times & \times & - & - & \times & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \end{array} \right. \tag{5.52}$$

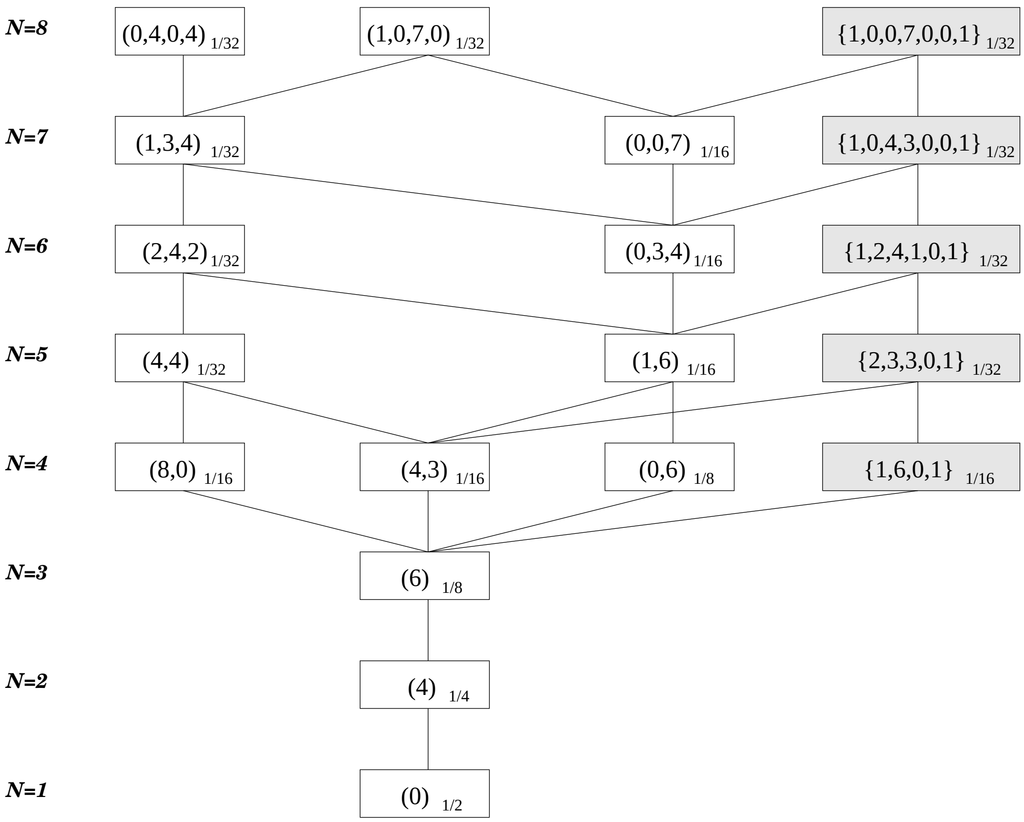

Figure 5.2: *M-brane intersections with $n = 4, 5$ in 11 dimensions: the numbers $(n_1, \cdots, n_{[N/2]})$ are the same labels used in $D = 10$, and indicate to which D-brane intersection the $D = 11$ solution reduces. The configurations in gray rectangles only reduce to $D = 10$ intersections involving NS-NS branes. For these configurations we use the eleven-dimensional notation $\{n_1, \cdots, n_N\}$ explained in the text. The subscripts indicate the amount of residual supersymmetry.*

$$\{1,0,0,7,0,0,0,1\}_{1/32} : \quad \left\{ \begin{array}{c|cc|cc|cc|cc|cc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & \times & - & \times & \times & \times & \times & - & - & - & - \\ \times & \times & - & \times & \times & - & - & \times & \times & - & - \\ \times & \times & - & - & - & \times & \times & \times & \times & - & - \\ \times & \times & - & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & - & \times & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & - & \times & \times & - \\ \times & \times & - & \times & - & - & \times & - & \times & \times & - \end{array} \right. \qquad (5.53)$$

Again the explicit form of all other intersections with $N < 8$ can be obtained via truncation of these configurations. It can be checked that these intersections indeed solve the equations of motion.

As in $D = 10$, the complete structure of the $D = 11$ intersections can be recovered

N=8	**(0,4,0,4)**	**(1,0,7,0)**	**{1,0,0,7,0,0,0,1}**	
	$[2^4,5^4]\{0,4,0,5,0,0,0,0\}$	$[2^4,5^4]\{1,0,6,1,1,0,0,0\}$	$[2^1,5^7]\{1,0,0,7,0,0,0,1\}$	
N=7	**(1,3,4)**	**(0,0,7)**	**{1,0,4,3,0,0,1}**	
	$[5^7]\{1,0,4,0,3,0,1\}$ $[5^7]\ \{0,3,0,4,0,1,1\}$ $[2^3,5^4]\{1,2,4,1,1,0,0\}$ $[2^3,5^4]\{1,3,1,4,0,0,0\}$ $[2^4,5^3]\{1,3,4,1,0,0,0\}$	$[5^7]\{0,0,7,0,0,0,2\}$ $[5^7]\{0,0,0,7,0,0,1\}$ $[2^3,5^4]\{0,0,6,2,0,0,0\}$	$[2^1,5^6]\{1,0,4,3,0,0,1\}$	
N=6	**(2,4,2)**	**(0,3,4)**	**{1,2,4,1,0,1}**	
	$[5^6]\{1,2,2,2,1,1\}$ $[2^2,5^4]\{1,4,2,1,1,0\}$ $[2^2,5^4]\{2,2,2,3,0,0\}$ $[2^3,5^3]\{2,3,3,1,0,0\}$ $[2^4,5^2]\{2,5,2,0,0,0\}$	$[5^6]\{0,0,4,3,0,1\}$ $[5^6]\{0,3,4,0,0,2\}$ $[2^2,5^4]\{0,2,4,2,0,0\}$ $[2^3,5^3]\{0,3,5,0,0,0\}$	$[2^1,5^5]\{1,2,4,1,0,1\}$	
N=5	**(4,4)**	**(1,6)**	**{2,3,3,0,1}**	
	$[5^5]\{2,2,2,2,1\}$ $[2^1,5^4]\{3,1,3,2,0\}$ $[2^2,5^3]\{3,3,2,1,0\}$ $[2^3,5^2]\{4,3,2,0,0\}$ $[2^4,5^1]\{5,4,0,0,0\}$	$[5^5]\{1,4,2,0,2\}$ $[5^5]\{0,2,4,1,1\}$ $[2^1,5^4]\{0,4,2,2,0\}$ $[2^1,5^4]\{1,6,0,1,1\}$ $[2^2,5^3]\{1,3,4,0,0\}$ $[2^3,5^2]\{1,6,1,0,0\}$	$[2^1,5^4]\{2,3,3,0,1\}$	
N=4	**(8,0)**	**(4,3)**	**(0,6)**	**{1,6,0,1}**
	$[2^2,5^2]\{6,1,2,0\}$ $[2^4]\{8,0,0,0\}$ $[5^4]\{4,0,4,1\}$	$[5^4]\{3,3,1,2\}$ $[5^4]\{1,3,3,1\}$ $[2^1,5^3]\{4,3,1,1\}$ $[2^1,5^3]\{2,3,3,0\}$ $[2^2,5^2]\{3,4,1,0\}$ $[2^3,5^1]\{5,3,0,0\}$	$[2^2,5^2]\{0,7,0,0\}$ $[5^4]\{0,6,0,2\}$	$[2^1,5^3]\{1,6,0,1\}$
N=3	**(6)**			
	$[5^3]\{6,0,3\}$ $[5^3]\{0,6,1\}$	$[5^3]\{3,3,2\}$ $[2^1,5^2]\{5,2,1\}$	$[2^1,5^2]\{2,5,0\}$ $[2^2,5^1]\{5,2,0\}$ $[2^3]\{6,0,0\}$	
N=2	**(4)**			
	$[5^2]\{4,3\}$	$[2^1,5^1]\{5,1\}$	$[2^2]\{4,0\}$	

Table 5.4: *Table of M-brane intersections in D=11: the number N indicates the number of independent harmonics. The boldface labels $(n_1,\ldots,n_{[N/2]})$ correspond to the $D=10$ D-brane intersection to which the $D=11$ solutions reduce (when applicable). The numbers between square brackets indicate the number of $M2$-branes and $M5$-branes involved in the intersection. The labels $\{n_1,\ldots,n_N\}$ specify the structure of the $D=11$ metric as explained in the text.*

by the requirement of partially unbroken supersymmetry [54]. Since the procedure is identical to the one used in $D = 10$ we will not give the details. The amount of unbroken supersymmetry for the different solutions is indicated in Figure 5.2.

Having determined the "no-force" condition between the basic eleven-dimensional solutions in Subsection 5.1.2, we next consider multiple intersections that also involve gravitational waves and Kaluza-Klein monopoles. We will again restrict ourselves to the configurations that can be reduced to intersections with only D-branes in $D = 10$. Looking back at Table 5.2, we see that all pairs involving monopoles should then be of the form $(2|M2, \mathcal{KK})$, $(3|M5, \mathcal{KK})$ or $(4|\mathcal{KK}, \mathcal{KK})^a$, and that with a wave only $(1|M5, \mathcal{W})$ may be used.

Our strategy will be to take Table 5.4 as our starting point and then to consider to which M-brane intersections waves and/or monopoles can be added. The rule for adding a wave is known [160, 134]: to each intersection involving at least a common string a wave can be added in such a way that the z-isometry direction of the wave lies in the space-like common string direction. Furthermore, at most one wave can be added to any given intersection.

From the intersection (5.26) we see that the world volume of the $M2$-brane must lie in the world volume directions of the monopole. Two intersecting $M2$-branes have distinct (space-like) world volume directions and since the monopole has six (space-like) world volume directions we conclude that monopoles may be added to configurations that contain at most three $M2$-branes [43]:

$$\left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \\ \times & - & - & - & - & \times & \times & - & - & - & - \\ \times & \times & \times & \times & \times & \times & \times & z & A_8 & A_9 & A_{10} \end{array} \right. \tag{5.54}$$

We next consider the $M5$-branes. Using only the pair $(3|M5, \mathcal{KK})$ we see that the z-isometry direction of the monopole should lie in a common world volume direction of the $M5$-branes. One finds that to a single monopole one can add at most four $M5$-branes. An example of such a configuration is:

$$\left\{ \begin{array}{c|cccccccccc} \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & - & \times & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \\ \times & \times & \times & \times & \times & \times & \times & A_7 & A_8 & z & A_{10} \end{array} \right. \tag{5.55}$$

The harmonic functions depend only on the coordinate x_{10}. However, one may add more than one monopole to the four five-branes. From (5.55) it is clear that the monopole could also have been placed with two components of the vector field in the $(x_1, x_2), (x_3, x_4)$ or (x_5, x_6) directions. In fact, in this way one can combine four monopoles with the four $M5$-branes:

$$\left\{ \begin{array}{c|cccccccccc} \times & - & \times & \times & - & \times & - & \times & - & \times & - \\ \times & \times & - & - & \times & \times & - & \times & - & \times & - \\ \times & \times & - & \times & - & - & \times & \times & - & \times & - \\ \times & \times & - & \times & - & \times & - & - & \times & \times & - \\ \times & \times & \times & \times & \times & \times & \times & A_7 & A_8 & z & A_{10} \\ \times & \times & \times & \times & \times & B_5 & B_6 & \times & \times & z & B_{10} \\ \times & \times & \times & C_3 & C_4 & \times & \times & \times & \times & z & C_{10} \\ \times & D_1 & D_2 & \times & \times & \times & \times & \times & \times & z & D_{10} \end{array} \right. \tag{5.56}$$

N=8	**(0,4,0,4)** $_{SUSY=1/32}$	**(1,0,7,0)** $_{SUSY=1/32}$	**(0,0,0,7)** $_{SUSY=1/16}$
	$[2^4,5^4]$\{*0,4,0,5,0,0,0*\}	$[2^4,5^4]$\{*1,0,6,1,1,0,0,0*\}	$[2^3,5^4]$\{*0,0,6,2,0,0,0*\}+$\mathcal{KK}$
	$[2^3,5^4]$\{*1,2,4,1,1,0,0*\}+$\mathcal{KK}$	$[2^3,5^4]$\{*1,3,1,4,0,0,0*\}+ $\mathcal{KK}$	$[2^1,5^4]$\{*1,6,0,1,1*\}+3$\mathcal{KK}$
	$[2^2,5^4]$\{*2,2,2,3,0,0*\}+2$\mathcal{KK}$	$[2^2,5^4]$\{*1,4,2,1,1,0,0*\}+2$\mathcal{KK}$	$[5^7]$\{*0,0,0,7,0,0,1*\}+$\mathcal{W}$
	$[2^1,5^4]$\{*0,4,2,2,0*\}+3$\mathcal{KK}$	$[2^2,5^4]$\{*0,2,4,2,0,0*\}+2$\mathcal{KK}$	
	$[5^4]$\{*4,0,4,1*\}+4$\mathcal{KK}$	$[2^1,5^4]$\{*3,1,3,2,0*\}+3$\mathcal{KK}$	
	$[5^7]$\{*0,3,0,4,0,1,1*\}+ $\mathcal{W}$	$[5^4]$\{*0,6,0,2*\}+4$\mathcal{KK}$	
		$[5^7]$\{*0,0,7,0,0,0,2*\}+ $\mathcal{W}$	
		$[5^7]$\{*1,0,4,0,3,0,1*\}+ $\mathcal{W}$	

Table 5.5: *N=8 intersections that reduce to pure D-brane intersections: The boldface numbers indicate the ten dimensional T-duality class. The notation* $[2^k,5^l]+n\mathcal{KK}$ *indicates that the intersections contain k* $M2$*-branes, l* $M5$*-branes and n monopoles. An additional wave is indicated by* $+\mathcal{W}$.

One may verify that this intersection is consistent with the $M5-\mathcal{KK}$ intersection rule (5.29) and the $\mathcal{KK}-\mathcal{KK}$ rule (5.31).

Having established the rule of how to add waves and monopoles to an intersection of $M2$-branes and $M5$-branes or a mixture thereof, we are able to list all intersections involving $M2$-branes, $M5$-branes, waves and monopoles. It is enough to give only the intersection with the largest number of independent harmonics. All other intersections can be obtained from these by setting one or more of the harmonic functions equal to one.

The result is given in Table 5.5 [21]. The maximum number of intersecting objects N equals eight if we restrict ourselves to configurations which can be reduced to pure D-brane intersections in $D=10$. This is not surprising, since the maximum number of intersecting D-branes is also $N=8$. To label the different configurations we use the M-brane notation for the intersecting brane part and indicate with $+n\mathcal{KK}$ and $+\mathcal{W}$ the waves and monopoles added to the solution. Furthermore we have divided the different solutions in classes, corresponding to the T-duality classes of the D-branes in six dimensions. In Table 5.5 we have also indicated the unbroken supersymmetry which directly follows from the unbroken supersymmetry of the corresponding D-brane intersection. Note the the solution $[5^7]$\{*0,0,7,0,0,0,2*\}+ $\mathcal{W}$ correspond to the uplifting of the $N=8$ D-brane intersection (5.48), the one that could not be described in eleven dimensions by $M2$ and $M5$-branes only.

It is instructive to consider also the pair $(1|M2,\mathcal{W})$. The reduction to $D=10$ will then necessarily include also NS-NS branes and will therefore go beyond our original motivation to find the intersections that reduce to strictly D-branes. However, this extra pair will allow us the complete the classes that are indicated by the grey colour in table 5.2 in terms of waves and monopoles. It turns out that there are three such maximum intersections. All other intersections follow by truncation of these ones. We find one intersection with $N=8$ and two intersections with $N=9$ independent harmonics:

$$\begin{aligned} N=8: &\quad [2^1,5^6]\{\mathit{1},\mathit{0},\mathit{4},\mathit{3},\mathit{0},\mathit{0},\mathit{1}\}+\mathcal{W}\,,\\ N=9: &\quad [2^1,5^7]\{\mathit{1},\mathit{0},\mathit{0},\mathit{7},\mathit{0},\mathit{0},\mathit{0},\mathit{1}\}+\mathcal{W}\,,\\ &\quad [2^1,5^4]\{\mathit{1},\mathit{6},\mathit{0},\mathit{1},\mathit{1}\}+3\mathcal{KK}+\mathcal{W}\,. \end{aligned} \tag{5.57}$$

All three solutions have 1/32 unbroken supersymmetry. Interestingly enough we find intersections with nine independent harmonics. The two intersections with $N = 9$ are extensions of $N = 8$ intersections with 1/16 supersymmetry in Table 5.5.

The remaining intersection of the class with 1/16 supersymmetry in Table 5.5, $[2^3, 5^4] + \mathcal{KK}$, can also be extended to $N = 9$, 1/32 supersymmetric solutions but this necessarily requires the use of a pair from Table 5.3. For example, an additional five-brane can be added, giving 1/32 supersymmetry.

5.3 Dimensional Reductions of Intersections

A natural application of our results is the reduction of the M-brane and D-brane intersections we found in the previous two sections to p-branes in lower dimensions. This will lead to dilatonic p-brane solutions which can be understood as D- and/or M-brane bound states in $D = 10, 11$. The interpretation of lower-dimensional solutions in terms of bound states of D- and/or M-branes in $D = 10, 11$ is a useful tool for understanding the properties of these lower dimensional solutions, especially in the case of (extremal) black holes where it has opened up the possibility for a microscopic explanation of the Bekenstein-Hawking entropy in terms of p-branes and p-brane bound states [154].

The (Einstein frame) form of our reduced action (upon truncating the scalars coming from the reduction and identifying many of the gauge fields) for $D > 2$ will always be in the class of Lagrangians of the form [8]

$$\mathcal{L}_D = \sqrt{|g^{\mathrm{E}}|}\,\Big[R_{\mathrm{E}} + \tfrac{1}{2}(\partial\phi)^2 + \frac{(-)^{p+1}}{2(p+2)!}e^{a\phi}F^2_{(p+2)}\Big]. \tag{5.58}$$

With the Ansatz

$$\begin{aligned} ds^2_{\mathrm{E},D} &= H^{\alpha}ds^2_{p+1} - H^{\beta}ds^2_{d-p-1}, \\ e^{2\phi} &= H^{\gamma}, \\ F_{0..pi} &= \delta\;\partial_i H^{-1}, \end{aligned} \tag{5.59}$$

one finds the general p-brane solution ($D > 2$) [112]:

$$\begin{aligned} \alpha &= -\frac{4(D-p-3)}{\Delta(D-2)} \quad , \quad \beta = \frac{4(p+1)}{\Delta(D-2)}, \\ \gamma &= \frac{4a}{\Delta} \quad , \quad \delta^2 = \frac{4}{\Delta}, \end{aligned} \tag{5.60}$$

with

$$\Delta = a^2 + 2\frac{(p+1)(D-p-3)}{D-2}. \tag{5.61}$$

[8]For $D = 2$ there does not exist a transformation to go from the string frame to the Einstein frame. Therefore the calculations should be done in the string frame. For the details and the precise form of the solutions, we refer to [20].

The lower dimensional p-brane solutions which follow from the reduced D-brane and M-brane intersections (now containing only one independent harmonic function) must fall inside this class of solutions. A property of supersymmetric solutions is that [112, 111]:

$$\Delta = 4/N\,, \tag{5.62}$$

where N is an integer labeling the number of participating field strengths, or equivalently, the number of intersecting branes.

Any toroidal Kaluza-Klein reduction of the $D = 10, 11$ intersections will be a supersymmetry preserving p-brane solution in a lower dimension. Because the number of participating field strengths is equal to the number of intersecting branes we can immediately read off the dilatonic p-brane solution from (5.60) and (5.61).

For example, combining (5.62) and (5.61), we find that for the $D = 4$ black hole ($p = 0$) the possible dilaton couplings are [124, 106]

$$a = \sqrt{4/N - 1}. \tag{5.63}$$

We find four types of $D = 4$ (extremal) dilaton black holes preserving half of the supersymmetry with different values for a. These can therefore be interpreted as bound states of D-branes (M-branes) compactified on a six-torus (seven-torus) [124, 160, 106, 70, 9, 13]:

1. $a = \sqrt{3}$: compactification of a single D-brane
2. $a = 1$: compactification of two intersecting D-branes
3. $a = 1/\sqrt{3}$: compactification of three intersecting D-branes
4. $a = 0$: compactification of four intersecting D-branes

More precisely this corresponds to the compactification of the $N = 4$ (0,6) class of solutions (see Figure 5.1), upon identifying the different harmonic functions, and its truncations to intersections with lower N.

As another illustration, consider the $N = 8$ D-brane intersections (see Figure 5.1). We see that one of them, labeled by (0,0,0,7), can be naturally reduced to 0-branes in $D = 3$ by reducing over all relative transverse directions. Every truncation of this solution can of course also be reduced to 0-branes, giving rise to 8 different supersymmetry preserving solutions in $D = 3$. Doing the explicit Kaluza-Klein reduction we find that the different values of a representing the different solutions (the explicit solution can be determined using (5.60)) are given by

$$a = \sqrt{4/N}, \tag{5.64}$$

which is just (5.61) with $p = 0$, $D = 3$ and N running from 1 to 8. So we find eight supersymmetry preserving 0-branes in $D = 3$ (in contrast to the four 0-branes in $D = 4$) with the dilaton coupling given by (5.64) [113].

To see how many a-values correspond to a particular p-brane solution in D dimensions, one has to find the highest N intersection in the D- or M-brane intersections that

	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
$D=6$	2	2	2	2	2
$D=5$	3	3	4	3	–
$D=4$	4	7	7	–	–
$D=3$	8	8	–	–	–
$D=2$	9	–	–	–	–

Table 5.6: *Bound state interpretation of dilatonic p-branes in $D \leq 6$ dimensions: the numbers in the table give the number of dilatonic p-brane solutions in D dimensions with different values for the dilaton coupling, coming from different intersections in higher dimensions.*

can be reduced to a single p-brane in a lower dimension. The p-brane solutions in the lower dimension are given by (5.60) and (5.61) with $\Delta = 4/N$. Note that N is the only parameter, and that therefore different configurations of intersecting D- or M-branes with the same N, will all reduce to the same p-brane in lower dimensions upon identification of the harmonic functions (even if the $D = 10, 11$ intersecting solutions preserve different amounts of supersymmetry).

All p-brane solutions in lower dimensions preserve half of the maximal (lower-dimensional) supersymmetry in contrast to the intersecting D- or M-intersections in $D = 10, 11$. This gain in supersymmetry is a result of the identification of the different harmonics (equal charges). For an overview of the number of dilatonic p-brane solutions in lower dimensions ($D \leq 6$) with an interpretation as bound states of Table 5.5 or their truncations, we refer to Table 5.6. Many of the solutions that arise in the reduction and are listed in Table 5.6 were given in [112, 111, 113, 124, 160, 106, 24, 108, 21].