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Dualities of strings and branes

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Chapter 4

Target Space Actions

In Section 3.1 we already discussed briefly the global symmetries of the low energy effective actions of the common sector, the Heterotic and Type IIA/B theories. In this chapter we will study in more detail these discrete and non-compact symmetries. We will look in Section 4.1 at the symmetries of theories after compactification over one dimension, since here most of the properties of (toroidally) compactified theories are present in a simple form. In Section 4.2 we will find the same properties in the duality relations between the six-dimensional Heterotic, Type IIA and Type IIB theory at the level of the target space actions.

This chapter contains results presented in [27] and [14].

4.1 Duality Symmetries in Ten and Nine Dimensions

In this section we look in detail to the symmetry properties that arise from the dimensional reduction of the low energy effective string action from ten to nine dimensions. First we discuss the symmetry group of the common sector and explain the origin of the different symmetry transformations. Then we look at how the group structure gets enhanced, c.q. broken, in the presence of (Abelian) vector fields in the case of the Heterotic string or R-R fields in the case of the Type IIA/B theory.

4.1.1 Symmetries of the Common Sector

As discussed in Section 3.1, dimensional reduction over a circle already shows many of the interesting features of toroidal compactification. Let us therefore look in more detail at the dimensionally reduced $D = 9$ common sector action (3.28):

$$S^{(9)} = \frac{1}{2} \int d^9 x \sqrt{|g|} e^{-2\phi} \left[-R + 4(\partial\phi)^2 - \frac{3}{4}H^2 - (\partial \log k)^2 \right]$$

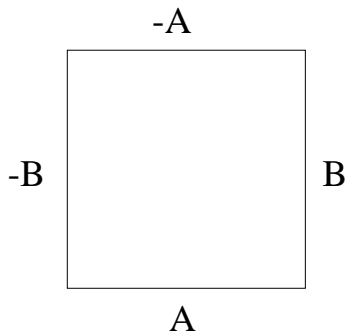


Figure 4.1: Each discrete symmetry of the square corresponds to a symmetry acting on the two vectors A and B . The four sides of the square correspond to the pairs $(A, -A)$ and $(B, -B)$.

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} + \frac{1}{2} A_{[\mu} F_{\nu\rho]}(B) + \frac{1}{2} B_{[\mu} F_{\nu\rho]}(A) + \frac{1}{4} k^2 F^2(A) + \frac{1}{4} k^{-2} F^2(B) \Big], \quad (4.1)$$

As mentioned earlier, this action has a manifest global $O(1,1)$ -symmetry, coming from the dimensional reduction, which decomposes in a subgroup of proper $O(1,1)$ transformation and a mapping class group:

$$O(1,1) = SO^\uparrow(1,1)_x \times \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}. \quad (4.2)$$

The different subgroups act on the Kaluza-Klein scalar k and the vector fields A_μ and B_μ as

$$\begin{aligned} SO^\uparrow(1,1)_x : & \quad k \rightarrow \Lambda^{-1} k, & A_\mu & \rightarrow \Lambda A_\mu, & B_\mu & \rightarrow \Lambda^{-1} B_\mu, \\ \mathbb{Z}_2^{(S)} : & \quad k \rightarrow k & A_\mu & \rightarrow -A_\mu, & B_\mu & \rightarrow -B_\mu, \\ \mathbb{Z}_2^{(T)} : & \quad k \rightarrow k^{-1}. & A_\mu & \rightarrow B_\mu, & B_\mu & \rightarrow A_\mu \end{aligned} \quad (4.3)$$

The $SO^\uparrow(1,1)_x \times \mathbb{Z}_2^{(S)}$ -symmetry comes from the fact that the action (4.1) was obtained via a dimensional reduction over x of the ten-dimensional action (3.20), and is therefore invariant under reflections and rescalings in the compact x -direction:

$$\begin{aligned} \mathbb{Z}_2^{(S)} : & \quad x' = -x, \\ SO^\uparrow(1,1)_x : & \quad x' = \Lambda x. \end{aligned} \quad (4.4)$$

It is not difficult to see that these ten-dimensional transformations act on the nine-dimensional fields as in (4.3).

The appearance of the $\mathbb{Z}_2^{(T)}$, the T -duality transformation (3.4), cannot be explained from the point of view of dimensional reduction, and is difficult to interpret, ignoring the stringy character of the action (4.1). We refer to the discussion in Section 3.1.

In addition to the two discrete groups given above, there exists yet another \mathbb{Z}_2 transformation, which we call $\mathbb{Z}_2^{(A)}$ because of the fact that it acts on the axion and the winding

Name	$g_{\mu\nu}$	$B_{\mu\nu}$	A_μ	B_μ	e^ϕ	k	$S^{(9)}$
$SO^\uparrow(1,1)_x$	0	0	1	-1	0	-1	0
$SO^\uparrow(1,1)_y$	-1	-1	0	-1	$-\frac{5}{4}$	$\frac{1}{2}$	-1
\mathbb{R}_ϕ	0	0	0	0	1	0	-2

Table 4.1: *Weights of the common sector fields under the two $SO^\uparrow(1,1)$ symmetries of the action $S^{(9)}$ and the \mathbb{R}_ϕ which scales the action in nine dimensions.*

vector:

$$\mathbb{Z}_2^{(A)} : \quad B'_{\mu\nu} = -B_{\mu\nu}, \quad B'_\mu = -B_\mu. \quad (4.5)$$

This $\mathbb{Z}_2^{(A)}$ does not commute with $\mathbb{Z}_2^{(T)}$ and therefore the three \mathbb{Z}_2 's together combine into the non-Abelian dihedral group D_4 , the group of symmetry transformations of a square with undirected sides: every D_4 -transformation on the vectors A_μ and B_μ corresponds to a transformation that leaves a square with sides $(A, -A)$ and $(B, -B)$ invariant (see Figure 4.1). The only D_4 -transformation that acts non-trivially on the scalar k is $\mathbb{Z}_2^{(T)}$.

Furthermore there are two more non-compact symmetries, $SO^\uparrow(1,1)_y$ and \mathbb{R}_ϕ ¹, which scale the action but leave the equations of motion invariant. Their interpretation will become clear later on in this section, in the context of the symmetries of the Type II theory. The weights of the fields under the various scale transformations is given in Table 4.1. The full group of symmetries the equations of motion is then given by

$$SO^\uparrow(1,1)_x \times SO^\uparrow(1,1)_y \times \mathbb{R}_\phi \times D_4. \quad (4.6)$$

In the presence of fields that do not belong to the common sector, such as vector fields in the case of the Heterotic theory, or R-R fields in the case of Type IIA/B theory, part of the symmetry gets broken. How much the symmetry gets broken depends on the situation. Let us therefore discuss each of the two cases separately.

4.1.2 Symmetries of the Heterotic Theory

In the presence of an (Abelian) vector field \hat{V}_μ , the situation changes in two ways: the extra Chern-Simons term in the axion field strength tensor will break the $\mathbb{Z}_2^{(A)}$ symmetry (and thus the D_4), while on the other hand the Abelian vector field combines with the A_μ and B_μ into the bigger reduction group $SO^\uparrow(1,2)_x$.

¹With \mathbb{R} we denote the additive group of real numbers, which is isomorphic to $SO^\uparrow(1,1)$. However we reserve the notation $SO^\uparrow(1,1)$ for groups that can combine with their mapping class group into a full $O(1,1)$.

We start our analysis from the action of the ten-dimensional Heterotic string in the presence of one Abelian vector field:

$$S = \frac{1}{2} \int d^{10}x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left[-\hat{R} + 4(\partial\hat{\phi})^2 - \frac{3}{4}\hat{H}^2 + \frac{1}{4}\hat{F}_{\hat{\mu}\hat{\nu}}(\hat{V})\hat{F}^{\hat{\mu}\hat{\nu}}(\hat{V}) \right], \quad (4.7)$$

where the three-form field strength is of the form

$$\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}]} - \frac{1}{2}\hat{V}_{[\hat{\mu}}\hat{F}_{\hat{\nu}\hat{\rho}]}(\hat{V}). \quad (4.8)$$

In principle, there is an ambiguity in the relative sign between $\partial\hat{B}$ and the Yang–Mills Chern–Simons term $\hat{V}\hat{F}(\hat{V})$. In fact, there are two theories whose only difference is this relative sign and which are related by the change of sign of $\hat{B}_{\hat{\mu}\hat{\nu}}$ ($\mathbb{Z}_2^{(A)}$), which is no longer a symmetry of each separate theory. Therefore, the group D_4 is broken to $\mathbb{Z}_2^{(T)} \times \mathbb{Z}_2^{(S)}$ in each theory. In fact $\mathbb{Z}_2^{(A)}$ is a duality transformation that brings us from one theory to the other, exactly as happens in the Type II duality (3.46) (see also [26]). From the sigma-model point of view, these theories are related by a change of the sign of $\hat{B}_{\hat{\mu}\hat{\nu}}$ and the simultaneous interchange of left- and right-movers. For the sake of definiteness, we will work with the above choice of relative sign.

Following the standard rules for dimensional reduction in the presence of vector fields [115]

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} - k^2 A_\mu A_\nu, & \hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu} B_{\nu]} + \ell A_{[\mu} V_{\nu]}, \\ \hat{g}_{x\mu} &= -k^2 A_\mu, & \hat{B}_{x\mu} &= B_\mu + \frac{1}{2} \ell V_\mu, \\ \hat{g}_{xx} &= -k^2, & \hat{\phi} &= \phi + \frac{1}{2} \log k, \\ \hat{V}_x &= \ell, & \hat{V}_\mu &= V_\mu + \ell A_\mu, \end{aligned} \quad (4.9)$$

we obtain the nine-dimensional action, which is the generalisation of (4.1):

$$\begin{aligned} S = \frac{1}{2} \int d^9x \sqrt{|g|} e^{-2\phi} \left\{ -R + 4(\partial\phi)^2 - \frac{3}{4}H^2 - \left[(\partial \log k)^2 + \frac{1}{2k^2}(\partial\ell)^2 \right] \right. \\ \left. + \frac{1}{4} \left[\frac{(2k^2 + \ell^2)^2}{4k^2} F^2(A) + k^{-2} F^2(B) + \frac{\ell^2}{k^2} F(A)F(B) \right] \right. \\ \left. + F(V) \left[\left(\frac{2k^2\ell + \ell^3}{4k^2} \right) F(A) + \frac{\ell}{2k^2} F(B) \right] \right. \\ \left. + \frac{1}{4} \left(\frac{k^2 + \ell^2}{k^2} \right) F^2(V) \right\}. \end{aligned} \quad (4.10)$$

This can be written in a manifestly $O(1, 2)$ invariant notation (3.37) [115]:

$$\begin{aligned} S = \frac{1}{2} \int d^9x \sqrt{|g|} e^{-2\phi} \left\{ -R + 4(\partial\phi)^2 - \frac{3}{4}H^2 \right. \\ \left. + \frac{1}{8} \text{Tr} \left[\partial_\mu M^{-1} \partial^\mu M \right] - \frac{1}{4} \mathcal{F}_{\mu\nu}^i(\mathcal{A}) M_{ij} \mathcal{F}^{\mu\nu j}(\mathcal{A}) \right\}, \end{aligned} \quad (4.11)$$

Name	$g_{\mu\nu}$	$B_{\mu\nu}$	A_μ	B_μ	e^ϕ	k	ℓ	V_μ	$S^{(9)}$
$SO^\uparrow(1,1)_y$	-1	-1	0	-1	$-\frac{7}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
\mathbb{R}_ϕ	0	0	0	0	1	0	0	0	-2
$SO^\uparrow(1,1)_\alpha$	0	0	1	-1	0	-1	-1	0	0

Table 4.2: *Weights of the fields under the two $SO^\uparrow(1,1)$ duality symmetries of the action of the nine-dimensional Heterotic string and the \mathbb{R}_ϕ which scales it.*

where $H_{\mu\nu\rho}$, $\mathcal{F}_{\mu\nu}^i$ and \mathcal{A}_μ^i are as in (3.38) and M^{-1} is the $O(1,2)$ -matrix

$$M_{ij}^{-1} = \begin{pmatrix} -(2k^2 + \ell^2)^2/4k^2 & -\ell^2/2k^2 & -(2k^2\ell + \ell^3)/2k^2 \\ -\ell^2/2k^2 & -1/k^2 & -\ell/k^2 \\ -(2k^2\ell + \ell^3)/2k^2 & -\ell/k^2 & -(k^2 + \ell^2)/k^2 \end{pmatrix}. \quad (4.12)$$

Let us now analyse in detail the different symmetries of this theory. First of all, the $SO^\uparrow(1,1)_y \times \mathbb{R}_\phi$ of (4.6) can be extended straightforwardly to the action (4.10). The weights of the fields are given in Table 4.2.

The dihedral group D_4 gets broken to the mapping class group of $O(1,2)$, namely $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$, which now, due to the presence of the vector field is of the form:

$$\begin{aligned} \mathbb{Z}_2^{(S)} &: \begin{cases} A'_\mu = -A_\mu, & B'_\mu = -B_\mu, \\ (k^2)' = (k^2), & \ell' = -\ell, \end{cases} \\ \mathbb{Z}_2^{(T)} &: \begin{cases} \tilde{A}_\mu = B_\mu, & \tilde{B}_\mu = A_\mu, \\ \tilde{k}^2 = \frac{4k^2}{(\ell^2 + 2k^2)^2}, & \tilde{\ell} = \frac{2\ell}{\ell^2 + 2k^2}. \end{cases} \end{aligned} \quad (4.13)$$

The interpretation of these \mathbb{Z}_2 transformations is the same as in (4.3): $\mathbb{Z}_2^{(S)}$ corresponds to a change of sign of the compact direction and $\mathbb{Z}_2^{(T)}$ corresponds to the T -duality transformations, which now in ten dimensions appear as a generalization of (3.4):

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \hat{g}_{\mu\nu} + \left(\hat{g}_{xx} \hat{G}_{x\mu} \hat{G}_{x\nu} - 2\hat{G}_{xx} \hat{G}_{x(\mu} \hat{g}_{\nu)x} \right) / \hat{G}_{xx}^2, \\ \tilde{B}_{\hat{\mu}\hat{\nu}} &= \hat{B}_{\hat{\mu}\hat{\nu}} - \hat{G}_{x[\hat{\mu}} \hat{G}_{\hat{\nu}]x} / \hat{G}_{xx}, \\ \tilde{g}_{x\mu} &= (-\hat{g}_{x\mu} \hat{G}_{xx} + \hat{g}_{xx} \hat{G}_{x\mu}) / \hat{G}_{xx}^2, \\ \tilde{B}_{x\mu} &= (\hat{G}_{x\mu} - \hat{B}_{x\mu}) / \hat{G}_{xx}, \end{aligned}$$

$$\begin{aligned}
\tilde{\hat{g}}_{xx} &= \hat{g}_{xx}/\hat{G}_{xx}^2, \\
\tilde{\hat{\phi}} &= \hat{\phi} - \frac{1}{2} \log |\hat{G}_{xx}|, \\
\tilde{\hat{V}}_x &= -\hat{V}_x/\hat{G}_{xx}, \\
\tilde{\hat{V}}_\mu &= \hat{V}_\mu - \hat{V}_x \hat{G}_{x\mu}/\hat{G}_{xx},
\end{aligned} \tag{4.14}$$

where $\hat{G}_{\hat{\mu}\hat{\nu}}$ is an ‘‘effective metric’’

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} + \hat{B}_{\hat{\mu}\hat{\nu}} - \frac{1}{2} \hat{V}_{\hat{\mu}} \hat{V}_{\hat{\nu}}, \tag{4.15}$$

which transforms under $\mathbb{Z}_2^{(T)}$ in the following particularly simple form:

$$\begin{aligned}
\tilde{\hat{G}}_{\mu\nu} &= \hat{G}_{\mu\nu} - \hat{G}_{x\mu} \hat{G}_{\nu x} / \hat{G}_{xx}, & \tilde{\hat{G}}_{xx} &= 1/\hat{G}_{xx}, \\
\tilde{\hat{G}}_{x\mu} &= \hat{G}_{x\mu} / \hat{G}_{xx}, & \tilde{\hat{G}}_{\mu x} &= -\hat{G}_{\mu x} / \hat{G}_{xx}.
\end{aligned} \tag{4.16}$$

Note that for $\hat{V}_{\hat{\mu}} = V_\mu = \ell = 0$, (4.13) and (4.14) reduce to the known T -duality transformations (3.4) and (4.3).

We next consider the continuous $SO^\uparrow(1,2)_x$ transformations. It is convenient to first consider the $so(1,2)$ Lie algebra with generators J_3 , J_+ and J_- :

$$[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = J_3. \tag{4.17}$$

The generators J_3 , J_+ and J_- can be represented by 3×3 matrices

$$J_+ = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponentiation of J_3 , J_+ and J_- leads to the following $SO^\uparrow(1,2)$ group elements:

$$\begin{aligned}
\exp \alpha J_3 &= \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & e^{-\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
\exp \beta J_- &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}\beta^2 & 1 & -\beta \\ -\beta & 0 & 1 \end{pmatrix}, \\
\exp \gamma J_+ &= \begin{pmatrix} 1 & \frac{1}{2}\gamma^2 & -\gamma \\ 0 & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix}.
\end{aligned} \tag{4.18}$$

An arbitrary $SO^\uparrow(1,2)$ group element Ω can be written as the product of these basis elements. Using the fact that the vectors A_μ , B_μ and V_μ transform in the fundamental representation of $SO^\uparrow(1,2)$ and the scalars as $(M^{-1})' = \Omega M^{-1} \Omega^T$ (see (3.39)), one can verify that the transformations in the basis above induce three transformations in nine dimensions.

First of all, the transformation generated by J_3 in nine dimensions is just the scale transformation $SO^\dagger(1,1)_x$ of previous sections and corresponds to a general coordinate transformation (g.c.t.) $x \rightarrow e^\alpha x$. The weights of the various fields under this transformation are given in Table 4.2.

We next consider the transformation generated by J_- . The nine-dimensional rules are given by

$$\begin{aligned} A'_\mu &= A_\mu, & (k^2)' &= k^2, \\ B'_\mu &= B_\mu - \beta V_\mu + \frac{1}{2}\beta^2 A_\mu, & \ell' &= \ell + \beta, \\ V'_\mu &= V_\mu - \beta A_\mu. \end{aligned} \quad (4.19)$$

The corresponding transformation of the ten-dimensional fields is

$$\begin{aligned} \hat{V}'_x &= \hat{V}_x + \beta, \\ \hat{B}'_{x\mu} &= \hat{B}_{x\mu} - \frac{1}{2}\beta \hat{V}_\mu. \end{aligned} \quad (4.20)$$

All other fields are invariant. It turns out that this transformation is a particular finite $U(1)$ gauge transformation, under which also the axion transforms (cfr. (3.35)):

$$\begin{aligned} \hat{V}'_{\hat{\mu}} &= \hat{V}_{\hat{\mu}} + \partial_{\hat{\mu}} \Lambda, \\ \hat{B}'_{\hat{\mu}\hat{\nu}} &= \hat{B}_{\hat{\mu}\hat{\nu}} + \hat{V}_{[\hat{\mu}} \partial_{\hat{\nu}]} \Lambda, \end{aligned} \quad (4.21)$$

with the parameter Λ given by $\Lambda = \beta x$.

Finally, we consider the transformation generated by J_+ . The transformation rules in nine dimensions are given by:

$$\begin{aligned} A'_\mu &= A_\mu + \frac{1}{2}\gamma^2 B_\mu - \gamma V_\mu, & (k^2)' &= \left(\frac{4k}{4+4\gamma\ell+(\ell^2+2k^2)\gamma^2} \right)^2, \\ B'_\mu &= B_\mu, & \ell' &= \frac{4\ell+2(\ell^2+2k^2)\gamma}{4+4\gamma\ell+(\ell^2+2k^2)\gamma^2}, \\ V'_\mu &= -\gamma B_\mu + V_\mu. \end{aligned} \quad (4.22)$$

For the (complicated) expression for these transformations in ten dimensions we refer to [27]. This transformation cannot be interpreted as a g.c.t. or a gauge transformation in ten dimensions. Together with the T -duality transformation $\mathbb{Z}_2^{(T)}$, it forms the $O(1,2)$ -subgroup $\mathbb{Z}_2^{(T)} \times O(2)$ of solution generating transformations [144, 83]. Note that the subgroup $\mathbb{Z}_2^{(T)} \times O(2) = O(1) \times O(2)$ corresponds exactly to the subgroup that is factored out in the Narain coset $O(1,2)/(O(1) \times O(2))$ of inequivalent compactifications [119] we encountered in Section 3.1. Therefore, the $\mathbb{Z}_2^{(T)} \times O(2)$ transformations parametrise the elements within each class of equivalent compactifications of (4.7). Acting with these transformations on a given solution, generates all other solution within the same equivalence class.

More generally, one can show [144, 83] that for a dimensionally reduced action with an $O(d, d+n)$ symmetry, the transformations belonging to the $O(d) \times O(d+n)$ subgroup are non-trivial solution generating transformations, while the coset $O(d, d+n)/(O(d) \times O(d+n))$ corresponds to the coset of gauge transformations.

Name	$C_{\mu\nu\rho}$	$g_{\mu\nu}$	$B_{\mu\nu}^{(1)}$	$B_{\mu\nu}^{(2)}$	$A_\mu^{(1)}$	A_μ	B_μ	k	ℓ	e^ϕ	S
$\mathbb{R}_{\text{brane}}$	1	1	1	1	0	0	1	$\frac{1}{2}$	0	$\frac{1}{4}$	3
$SO^\uparrow(1,1)_{x-y}$	0	1	1	-1	-1	1	0	$-\frac{1}{2}$	-2	$\frac{7}{4}$	0
$SO^\uparrow(1,1)_{x+y}$	0	1	1	1	-1	-1	2	$\frac{3}{2}$	0	$\frac{3}{4}$	2
$\mathbb{Z}_2^{(S)}$	-	+	-	+	+	-	+	+	-	+	+

Table 4.3: *Weights of the $D = 9$ Type II supergravity fields and action under $SL(2, \mathbb{R}) \times SO^\uparrow(1,1)_{x+y} \times \mathbb{R}_{\text{brane}} \times \mathbb{Z}_2^{(S)}$.*

4.1.3 Symmetries of Type IIA/B

As we have seen in section 3.1, Type IIA and Type IIB theory in the presence of an isometry are related via the Type II T -duality rules (3.46) [26]. Therefore they also have the same symmetry group [27]:

$$SL(2; \mathbb{R}) \times SO^\uparrow(1,1)_{x+y} \times \mathbb{R}_{\text{brane}} \times \mathbb{Z}_2^{(S)}. \quad (4.23)$$

The $SL(2, \mathbb{R})$ group is a symmetry of the action. From the Type IIB point of view it is the manifest $SL(2, \mathbb{R})$ symmetry (3.69)-(3.70) of the original theory [92, 26, 17], while from the point of view of the Type IIA it is a part of the symmetry group coming from the dimensional reduction of the eleven dimensional supergravity theory: the group of two-dimensional general coordinate transformations $GL(2, \mathbb{R}) = SL(2, \mathbb{R}) \times SO^\uparrow(1,1) \times \mathbb{Z}_2$.²

The $SL(2, \mathbb{R})$ contains one particular subgroup of scalings: $SO^\uparrow(1,1)_{x-y}$, corresponding to the eleven-dimensional g.c.t. $x \rightarrow e^\alpha x, y \rightarrow e^{-\alpha} y$. This is of course the particular combination of the scaling symmetries $SO^\uparrow(1,1)_x$ and $SO^\uparrow(1,1)_y$ of the previous sections. Another (linearly independent) combination is the $SO^\uparrow(1,1)_{x+y}$, which scales the fields and the action and corresponds to the eleven-dimensional g.c.t. $x \rightarrow e^\alpha x, y \rightarrow e^\alpha y$. The $\mathbb{R}_{\text{brane}}$ is a symmetry that can already be found back in eleven dimensions and that scales the action, giving each field a weight according to its mass dimension [27]. Finally, $\mathbb{Z}_2^{(S)}$ corresponds to improper g.c.t.s in the internal space, for instance $x \rightarrow -x$ (up to $SL(2, \mathbb{R})$ rotations) [85]. The weights of the different nine-dimensional fields are summarized in Table 4.3.

The discrete $\mathbb{Z}_2^{(S)}$ is the only part that remains from the dihedral group D_4 in (4.6). The $\mathbb{Z}_2^{(A)}$ -symmetry is broken by the topological term in (3.42), and the $\mathbb{Z}_2^{(T)}$ is the Type II T -duality (3.46), which is not a symmetry of the nine-dimensional Type II

²Schwarz [140] and Aspinwall [7] interpreted the presence of the $SL(2, \mathbb{Z})$ symmetry of the Type IIB string as the $SL(2, \mathbb{Z})$ modular invariance of the torus on which the $D = 11$ supergravity is compactified in order to relate this to the Type IIB theory.

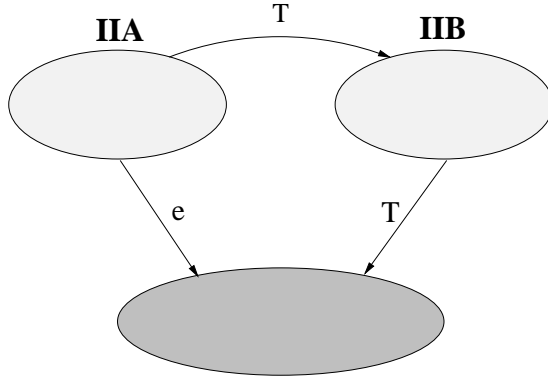


Figure 4.2: *The Type II T-duality in ten dimensions describes a map between the Type IIA and Type IIB theory. The reduction to $D = 9$ of the Type IIA (Type IIB) is indicated with \mathbf{e} (\mathbf{T})*

action (3.42), but relates the ten-dimensional Type IIA and Type IIB with each other. Instead of being a symmetry of a single theory, it is a map between two different theories, which can be constructed relating the two different reduction schemes (3.43) and (3.44) to each other (see Figure 4.2)

We will call these two reduction schemes \mathbf{e} and \mathbf{T} respectively, the reason for this being that the reduction scheme \mathbf{T} is the T -dual formulation (3.31) of the reduction scheme \mathbf{e} , when restricted to the common sector³.

An advantage of this notation is that one can easily see the \mathbb{Z}_2 group structure:

$$\begin{aligned} T(IIB \rightarrow IIA) \times T(IIA \rightarrow IIB) &= \mathbb{1}(IIA \rightarrow IIA), \\ T(IIA \rightarrow IIB) \times T(IIB \rightarrow IIA) &= \mathbb{1}(IIB \rightarrow IIB). \end{aligned} \quad (4.24)$$

This is due to our notation of the reduction formulae, which is such that, when restricted to the common sector, each reduction scheme (and its inverse) is in one-to-one correspondence with a specific \mathbb{Z}_2 -symmetry of the action (4.1).

The above analysis can also be repeated for the more complicated case of $D = 5, 6$. The six-dimensional Type IIA/B theories compactified on $K3$ are in the same way T -dual to each other upon reduction to five dimensions. Furthermore they can be related to the Heterotic theory compactified on a four-torus T^4 , which will give rise to bigger discrete duality groups.

³Upon truncation of the R-R fields, the Type II theories reduce to the common sector (4.1) and $\mathbb{Z}_2^{(T)}$ becomes a symmetry of the action.

4.2 Duality Symmetries in Six and Five Dimensions

In this section we will discuss the duality symmetries between the Heterotic, Type IIA and Type IIB theory in six and five dimensions. We will see that all three are related to each other via a string/string/string triality structure [62, 100]. Just as in the previous section we will start with the symmetries of the common sector, then present the six-dimensional form of each of the theories and reduce them to the same five-dimensional theory. In the end we will construct duality maps between the various theories.

4.2.1 The Common Sector

The common sector of the Heterotic, Type IIA and Type IIB theory in six dimensions is given by

$$S^{(6)} = \frac{1}{2} \int d^6x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left[-\hat{R} + 4(\partial\hat{\phi})^2 - \frac{3}{4}\hat{H}^2 \right]. \quad (4.25)$$

The special thing about six dimensions is that the equations of motion corresponding to the common sector are invariant under so-called string/string duality transformations [60, 61, 163]. These transformations are easiest formulated in the (6-dimensional) Einstein-frame metric

$$\hat{g}_{\mu\nu}^E = e^{-\hat{\phi}} \hat{g}_{\mu\nu}, \quad (4.26)$$

which is invariant under the string/string duality transformations. The action for the common sector in the Einstein-frame metric is given by:

$$S^{(6)} = \frac{1}{2} \int d^6x \sqrt{|\hat{g}^E|} \left[-\hat{R} - (\partial\hat{\phi})^2 - \frac{3}{4}e^{-2\hat{\phi}} \hat{H}^2 \right]. \quad (4.27)$$

It is not difficult to see that the equations of motion of the above action are invariant under:

$$\hat{\phi}' = -\hat{\phi}, \quad \hat{H}' = e^{-2\hat{\phi}} * \hat{H}, \quad (4.28)$$

where $*\hat{H}$ is the Poincaré dual (2.56) of the axion field strength \hat{H} :

$$*\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv \frac{1}{3! \sqrt{|\hat{g}|}} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}\hat{\sigma}\hat{\tau}} \hat{H}^{\hat{\lambda}\hat{\sigma}\hat{\tau}}. \quad (4.29)$$

This string/string duality is the six-dimensional analogue of the strong/weak coupling duality (3.64), or equivalently the string/five-brane duality in ten dimensions. It states that in the strong coupling limit of the six-dimensional common sector the fundamental string gets related to the solitonic string, the direct reduction of the solitonic five-brane.

We now discuss the reduction to five dimensions, assuming there is an isometry in the x -direction. Using both in $D = 6$ as well as $D = 5$ the string-frame metric, the 6-dimensional fields are expressed in terms of the five-dimensional ones as follows:

$$\begin{aligned} \hat{g}_{xx} &= -e^{-4\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} &= -e^{-4\sigma/\sqrt{3}} A_\mu, \end{aligned}$$

$$\begin{aligned}
\hat{g}_{\mu\nu} &= g_{\mu\nu} - e^{-4\sigma/\sqrt{3}} A_\mu A_\nu, \\
\hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu} B_{\nu]}, \\
\hat{B}_{x\mu} &= B_\mu, \\
\hat{\phi} &= \phi - \frac{1}{\sqrt{3}} \sigma.
\end{aligned} \tag{4.30}$$

Note that for later convenience we have renamed the Kaluza-Klein scalar $k^2 = e^{-4\sigma/\sqrt{3}}$. The reduced action in the (five-dimensional) string frame metric is given by

$$\begin{aligned}
S^{(5)} = \frac{1}{2} \int d^5 x \sqrt{|g|} e^{-2\phi} &\left[-R + 4(\partial\phi)^2 - \frac{3}{4} H^2 - \frac{4}{3} (\partial\sigma)^2 \right. \\
&\left. + e^{-4\sigma/\sqrt{3}} F(A)^2 + e^{4\sigma/\sqrt{3}} F(B)^2 \right],
\end{aligned} \tag{4.31}$$

with $H_{\mu\nu\rho}$ as in (4.1).

We next use the fact that five dimensions is special in the sense that in this dimension the antisymmetric tensor $B_{\mu\nu}$ is Poincaré dual to a vector C_μ [163, 164]:

$$H_{\mu\nu\rho} \equiv \frac{1}{3\sqrt{|g|}} e^{2\phi} \varepsilon_{\mu\nu\rho\lambda\sigma} F(C)^{\sigma\tau} \tag{4.32}$$

In terms of this vector C_μ the action is given by:

$$\begin{aligned}
S^{(5)} = \frac{1}{2} \int d^5 x \sqrt{|g|} e^{-2\phi} &\left[-R + 4(\partial\phi)^2 - \frac{4}{3} (\partial\sigma)^2 + e^{4\phi} F(C)^2 \right. \\
&\left. + e^{-4\sigma/\sqrt{3}} F(A)^2 + e^{4\sigma/\sqrt{3}} F(B)^2 \right] \\
&- \frac{1}{2} \int d^5 x \varepsilon_{(5)} A \partial B \partial C.
\end{aligned} \tag{4.33}$$

To study the symmetries of the dimensionally reduced action it is convenient to use the (five-dimensional) Einstein frame metric

$$g_{\mu\nu}^E = e^{-4\phi/3} g_{\mu\nu}, \tag{4.34}$$

so that the action becomes

$$\begin{aligned}
S^{(5)} = \frac{1}{2} \int d^5 x \sqrt{|g^E|} &\left[-R - \frac{4}{3} (\partial\phi)^2 - \frac{4}{3} (\partial\sigma)^2 + e^{-4\vec{Q}_C \cdot \vec{\Phi}/3} F(C)^2 \right. \\
&\left. + e^{-4\vec{Q}_A \cdot \vec{\Phi}/3} F(A)^2 + e^{-4\vec{Q}_B \cdot \vec{\Phi}/3} F(B)^2 \right] \\
&- \frac{1}{2} \int d^5 x \varepsilon_{(5)} A \partial B \partial C,
\end{aligned} \tag{4.35}$$

where $\vec{\Phi} = \not\phi, \phi$ and

$$\begin{aligned}
\vec{Q}_A &= \not\sqrt{3}, 1), \\
\vec{Q}_B &= \not{-\sqrt{3}, 1), \\
\vec{Q}_C &= \not{0, -2).
\end{aligned} \tag{4.36}$$

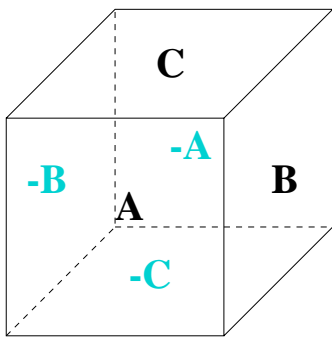


Figure 4.3: Each proper discrete symmetry of the cube corresponds to a symmetry acting on the three vectors. The six faces of the cube correspond to the pairs $(A, -A)$, $(B, -B)$ and $(C, -C)$.

Given the above form of the dimensionally reduced action, it is not difficult to analyse its discrete duality symmetries. It turns out that on the 3 vectors one can realize the 24-element finite group \mathcal{C}/\mathbb{Z}_2 where \mathcal{C} is the so-called cubic group. The easiest way to see how this group is realized is to write a cube, like in Figure 4.3, with faces $(A, -A)$, $(B, -B)$ and $(C, -C)$.

The reason that we only consider the 24 proper symmetries and not the full 48-element cubic group is that only the proper elements leave the last (topological) term in the action (4.35) invariant. The proper cubic group has elements of order 2 and 3.⁴ An example of a 2-order element is the reflection around the diagonal vertical plane that connects the right-front of the cube to the left-back of the cube. An example of a 3-order element is given by a (counter-clockwise) rotation of 120 degrees with axis the line going from the upper right corner at the front to the lower left corner at the back of the cube. Each of the 24 proper discrete symmetries of the cube naturally leads to a discrete symmetry acting on the 3 vectors. For instance, the 2- and 3-order elements given above induce the following discrete symmetries acting on the vectors, respectively:

$$\begin{aligned} A' &= B, & B' &= A, & C' &= C, \\ A' &= B, & B' &= C, & C' &= A. \end{aligned} \tag{4.37}$$

To see which discrete group is realized on the 2 scalars, it is easiest to write the 3 vectors \vec{Q}_A, \vec{Q}_B and \vec{Q}_C as the corners of an equilateral triangle, like in Figure 4.4. It was pointed out by Kaloper [96] that on the scalars one can realize the 6-element dihedral group

$$D_3 = \mathcal{C} / (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2), \tag{4.38}$$

i.e., to every 4 symmetries acting on the vectors one relates a single symmetry acting on the scalars. The action of the 6 elements of D_3 on the scalars is given by:

$$\mathbf{e} : \quad \sigma' = \sigma,$$

⁴The full cubic group also has elements of order 4. An example of such a 4-order element is a rotation of 90 degrees with axis the line going from the center of the lower face to the center of the upper face of the cube.

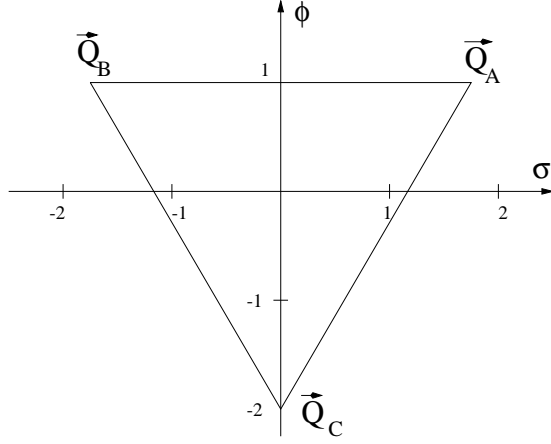


Figure 4.4: *Each symmetry of the equilateral triangle corresponds to a symmetry acting on the two scalars. The three corners of the triangle are given by the three vectors \vec{Q}_A, \vec{Q}_B and \vec{Q}_C defined in eq. (4.36).*

$$\begin{aligned}
& \phi' = \phi, \\
\mathbf{T} : & \sigma' = -\sigma, \\
& \phi' = \phi, \\
\mathbf{S} : & \sigma' = \frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi, \\
& \phi' = \frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
\mathbf{TS} : & \sigma' = -\frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi, \\
& \phi' = -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
\mathbf{ST} : & \sigma' = -\frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi, \\
& \phi' = +\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi, \\
\mathbf{TST} : & \sigma' = \frac{1}{2}\sigma - \frac{1}{2}\sqrt{3}\phi, \\
& \phi' = -\frac{1}{2}\sqrt{3}\sigma - \frac{1}{2}\phi.
\end{aligned} \tag{4.39}$$

Note that all D_3 -transformations can be obtained as products of two elements, \mathbf{T} and \mathbf{S} , where the \mathbf{T} -element corresponds to the usual T -duality transformation (3.31) and the \mathbf{S} -element corresponds to the string/string duality (4.28). By \mathbf{TS} we mean the symmetry that is obtained by a composition of \mathbf{T} and \mathbf{S} as follows:

$$\sigma'' = \frac{1}{2}\sigma' + \frac{1}{2}\sqrt{3}\phi' = -\frac{1}{2}\sigma + \frac{1}{2}\sqrt{3}\phi. \tag{4.40}$$

To every element of D_3 corresponds 4 elements of \mathcal{C}/\mathbb{Z}_2 acting on the 3 vectors. The

	e	T	S	ST	TS	TST
e	e	T	S	ST	TS	TST
T	T	e	TS	TST	S	ST
S	S	ST	e	T	TST	TS
ST	ST	S	TST	TS	e	T
TS	TS	TST	T	e	ST	S
TST	TST	TS	ST	S	T	e

Table 4.4: *Group multiplication table of the 6-element dihedral group D_3 .*

specific transformations of the vectors are given by⁵

$$\begin{aligned}
\mathbf{e} : & \quad A' = A, \quad B' = B, \quad C' = C, \\
\mathbf{T} : & \quad A' = B, \quad B' = A, \quad C' = C, \\
\mathbf{S} : & \quad A' = A, \quad B' = C, \quad C' = B, \\
\mathbf{TS} : & \quad A' = B, \quad B' = C, \quad C' = A, \\
\mathbf{ST} : & \quad A' = C, \quad B' = A, \quad C' = B, \\
\mathbf{TST} : & \quad A' = C, \quad B' = B, \quad C' = A.
\end{aligned} \tag{4.41}$$

Finally, for the sake of completeness we give the complete group multiplication table of D_3 in Table 4.4.

4.2.2 $D = 6$ Heterotic, Type IIA and Type IIB Theory

In this Subsection we describe the actions and symmetries of the six-dimensional Heterotic compactified on T^4 and Type IIA and Type IIB theory compactified on $K3$.

The toroidally compactified Heterotic theory was already discussed in subsection 3.1.3: its field content consists of the usual metric, axion and dilaton, 24 Abelian vector fields and 80 scalars parametrising an $O(4, 20)/(O(4) \times O(20))$ coset. They can be combined into a $O(4, 20)$ -matrix \hat{M}^{-1} , satisfying $\hat{M}^{-1}L\hat{M}^{-1} = L$, where L is the $O(4, 20)$ -metric (3.38). The Heterotic action has six-dimensional $N = 2$ supersymmetry and can be written in the string-frame in an manifest $O(4, 20)$ -invariant way:

$$\begin{aligned}
S_{\text{Het}} = \frac{1}{2} \int d^6x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} & \left[-\hat{R} + 4(\partial\hat{\phi})^2 - \frac{3}{4}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right. \\
& \left. + \frac{1}{8}\text{Tr} \left[\hat{g}_{\hat{\mu}\hat{\nu}}\hat{M}^{\hat{\mu}}\partial^{\hat{\mu}}\hat{M}^{-1} \right] - \hat{F}(\hat{V})^i_{\hat{\mu}\hat{\nu}}\hat{M}^{-1}_{ij}\hat{F}(\hat{V})^{\hat{\mu}\hat{\nu}j} \right],
\end{aligned} \tag{4.42}$$

where \hat{H} is defined as in (3.38).

⁵We only give 6 elements of C/\mathbb{Z}_2 . To every element below one can associate 3 more elements by changing (in 3 possible ways) *two* signs in the given transformation rules.

In order to rewrite this theory in $D = 5$, we make the following Ansatz for the reduction scheme:

$$\mathbf{e} : \begin{cases} \hat{g}_{xx} &= -e^{-4\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} &= -e^{-4\sigma/\sqrt{3}}A_\mu, \\ \hat{g}_{\mu\nu} &= g_{\mu\nu} - e^{-4\sigma/\sqrt{3}}A_\mu A_\nu, \\ \hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu}B_{\nu]} + \ell^i V_{[\mu}^j A_{\nu]} L_{ij}, \\ \hat{B}_{x\mu} &= B_\mu - \frac{1}{2}\ell^i V_\mu^j L_{ij}, \\ \hat{\phi} &= \phi - \frac{1}{\sqrt{3}}\sigma, \\ \hat{V}_\mu^i &= V_\mu^i + \ell^i A_\mu, \\ \hat{V}_x^i &= \ell^i, \\ \hat{M} &= M. \end{cases} \quad (4.43)$$

Just as in the previous subsection the axion $B_{\mu\nu}$ can be dualized to a vector C_μ via the formula:

$$H^{\mu\nu\rho} = \frac{1}{3\sqrt{|g|}} e^{2\phi} \varepsilon^{\mu\nu\rho\lambda\sigma} F(C)_{\lambda\sigma}. \quad (4.44)$$

The dimensionally reduced action in the (five-dimensional) string-frame is given by

$$\begin{aligned} S = \frac{1}{2} \int d^5x \sqrt{|g|} e^{-2\phi} & \left[-R + 4(\partial\phi)^2 + \frac{1}{8} \text{Tr} \not{\partial}_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} \right. \\ & \left. + e^{4\phi} F(C)^2 - F(A)_{\mu\nu}^I \mathcal{M}_{ij}^{-1} F(A)^{\mu\nu J} \right] \\ & - \frac{1}{4} \int d^5x \varepsilon_{(5)} C \partial A^I \partial A^J \mathcal{L}_{IJ}, \end{aligned} \quad (4.45)$$

where \mathcal{L} is the invariant metric on $O(5, 21)$. The $O(5, 21)$ -vectors \mathcal{A}^I ($I = 1, \dots, 26$) are given by

$$\mathcal{A}_\mu^I = \begin{pmatrix} A_\mu \\ B_\mu \\ V_\mu^i \end{pmatrix}. \quad (4.46)$$

The explicit expression of the $O(5, 21)$ -matrix \mathcal{M} in terms of the 105 scalars σ , ℓ^i and the 80 scalars contained in the $O(4, 20)$ matrix M is given by

$$\mathcal{M}^{-1} = \begin{pmatrix} -e^{-4\sigma/\sqrt{3}} + \ell^i \ell^j M_{ij}^{-1} - \frac{1}{4} e^{4\sigma/\sqrt{3}} \ell^4 & \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 & \ell^i M_{ij}^{-1} - \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 \ell_j \\ \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 & -e^{4\sigma/\sqrt{3}} & e^{4\sigma/\sqrt{3}} \ell_j \\ \ell^i M_{ij}^{-1} - \frac{1}{2} e^{4\sigma/\sqrt{3}} \ell^2 \ell_j & e^{4\sigma/\sqrt{3}} \ell_j & M_{ij}^{-1} - e^{4\sigma/\sqrt{3}} \ell_i \ell_j \end{pmatrix}$$

where $\ell^2 \equiv \ell^i \ell^j L_{ij}$ and $\ell_i \equiv \ell^j L_{ij}$. These scalars parametrise the coset $O(5, 21)/O(5) \times O(21)$.

The action (4.45) defines the Type II theory in 5 dimensions. It clearly contains the common sector given in (4.33). This may be seen by imposing the following constraints:

$$\ell^i = V_\mu^i = 0, \quad M_{ij}^{-1} = \delta_{ij}. \quad (4.47)$$

Now, we will compare this result to the actions of the Type IIA/B theories, compactified on $K3$. $K3$ is a four-dimensional manifold that can be best seen as an orbifold of the

four-torus T^4 : it can be obtained from the T^4 after identification of the points on the torus that are mapped to each other under the \mathbb{Z}_2 transformation $x^a \rightarrow -x^a$ on the coordinates. It has 16 fix-points (points that under the \mathbb{Z}_2 are mapped to themselves) and an 80-dimensional moduli space of inequivalent string compactifications

$$\frac{SO(4, 20)}{SO(4) \times SO(20) \times SO(4, 20; \mathbb{Z})}. \quad (4.48)$$

Reduction over $K3$ breaks exactly half of the supersymmetry the theory would have if it were compactified on T^4 .

The field content of the Type IIA theory reduced over $K3$ to 6 dimensions is identical to the Heterotic theory. Furthermore the reduction over $K3$ breaks half of the supersymmetry, such that we find also here six-dimensional $N = 2$. The action, however, is different. Instead of a Chern-Simons term inside \hat{H} , the action contains an additional topological term as compared to the Heterotic action. We thus have

$$S_{\text{IIA}} = \frac{1}{2} \int d^6 x \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left[-\hat{R} + (\partial\hat{\phi})^2 - \frac{3}{4} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right. \\ \left. + \frac{1}{8} \text{Tr} \left[\hat{\partial}_{\hat{\mu}} \hat{M} \partial^{\hat{\mu}} \hat{M}^{-1} \right] - e^{2\hat{\phi}} \hat{F}(\hat{V})^i_{\hat{\mu}\hat{\nu}} \hat{M}_{ij}^{-1} \hat{F}(\hat{V})^{\hat{\mu}\hat{\nu}j} \right] \\ - \frac{1}{8} \int d^6 x \varepsilon_{(6)} \hat{B} \partial \hat{V}^i \partial \hat{V}^j L_{ij}. \quad (4.49)$$

Just as in the ten-to-nine reduction of Type IIA/B in the previous section, the six-dimensional Type IIA action can be mapped onto the same five-dimensional Type II action (4.45) as the Heterotic theory, provided we use a different reduction scheme for the Type IIA theory:

$$\mathbf{S} : \begin{cases} \hat{g}_{xx} &= -e^{-2\hat{\phi}-2\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} &= -e^{-2\hat{\phi}-2\sigma/\sqrt{3}} A_{\mu}, \\ \hat{g}_{\mu\nu} &= e^{-2\hat{\phi}+2\sigma/\sqrt{3}} g_{\mu\nu} - e^{-2\hat{\phi}-2\sigma/\sqrt{3}} A_{\mu} A_{\nu}, \\ \hat{B}_{\mu\nu} &= B_{\mu\nu} + A_{[\mu} C_{\nu]}, \\ \hat{B}_{x\mu} &= C_{\mu}, \\ \hat{\phi} &= -\phi + \frac{1}{\sqrt{3}} \sigma, \\ \hat{V}_{\mu}^i &= V_{\mu}^i + \ell^i A_{\mu}, \\ \hat{V}_x^i &= \ell^i, \\ \hat{M} &= M. \end{cases} \quad (4.50)$$

The five-dimensional antisymmetric tensor $B_{\mu\nu}$ is dualized to a vector B_{μ} via the relation

$$H^{\mu\nu\rho} = \frac{1}{3\sqrt{|g|}} e^{2\hat{\phi}+4\sigma/\sqrt{3}} \varepsilon^{\mu\nu\rho\lambda\sigma} \left[F(B)_{\lambda\sigma} + \ell^i F(V)_{\lambda\sigma}^j L_{ij} + \ell^2 F(A)_{\lambda\sigma} \right]. \quad (4.51)$$

The field content of the Type IIB theory on $K3$ is given by a metric, 5 self-dual anti-symmetric tensors, 21 anti-self-dual anti-symmetric tensors and 105 scalars. The 105

scalars parametrize an $O(5, 21)/O(5) \times O(21)$ coset and are combined into the symmetric 26×26 dimensional matrix $\hat{\mathcal{M}}$ satisfying the condition $\hat{\mathcal{M}}^{-1} \mathcal{L} \hat{\mathcal{M}}^{-1} = \mathcal{L}$ where \mathcal{L} is the invariant metric on $O(5, 21)$. The theory has $N = 2$ supersymmetry.

Due to the (anti-)self-duality of the tensor fields, a covariant action is hard to write down⁶, but omitting the self-duality constraint, a non-self-dual action can be constructed. We find that in the Einstein-frame the non-self-dual Type IIB action is given by

$$S_{\text{IIB}} = \frac{1}{2} \int d^6 x \sqrt{|\hat{g}^{\text{E}}|} \left[-\hat{R} + \frac{1}{8} \text{Tr} \left(\hat{g}^{\mu\nu} \hat{\mathcal{M}} \partial^{\hat{\mu}} \hat{\mathcal{M}}^{-1} \right) + \frac{3}{8} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}^I \hat{\mathcal{M}}_{IJ}^{-1} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}J} \right], \quad (4.52)$$

where $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}^I = \partial_{[\hat{\mu}} \hat{B}_{\hat{\nu}\hat{\rho}}^I]$. The field equations corresponding to this action lead to the correct Type IIB field equations, provided that we substitute by hand the following (anti-)self-duality conditions for the antisymmetric tensors \hat{B}^I ($I = 1, \dots, 26$):

$$\hat{H}^I = \mathcal{L}^{IJ} \hat{\mathcal{M}}_{JK}^{-1} * \hat{H}^K. \quad (4.53)$$

In order to extract the common sector out of the Type IIB theory, it is necessary to use a particular parametrisation of the matrix $\hat{\mathcal{M}}^{-1}$ in terms of the 105 scalars, thereby identifying a particular scalar as the Type IIB dilaton $\hat{\phi}$. This dilaton may then be used to define a string-frame metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ via (4.26). We use the following parametrisation:

$$\hat{\mathcal{M}}^{-1} = \begin{pmatrix} -e^{-2\hat{\phi}} + \hat{\ell}^i \hat{\ell}^j \hat{M}_{ij}^{-1} - \frac{1}{4} e^{2\hat{\phi}} \hat{\ell}^4 & \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 & \hat{\ell}^i \hat{M}_{ij}^{-1} - \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 \hat{\ell}_j \\ \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 & -e^{2\hat{\phi}} & e^{2\hat{\phi}} \hat{\ell}_j \\ \hat{\ell}^i \hat{M}_{ij}^{-1} - \frac{1}{2} e^{2\hat{\phi}} \hat{\ell}^2 \hat{\ell}_j & e^{2\hat{\phi}} \hat{\ell}_j & \hat{M}_{ij}^{-1} - e^{2\hat{\phi}} \hat{\ell}_i \hat{\ell}_j \end{pmatrix}, \quad (4.54)$$

where 80 scalars are contained in the $O(4, 20)$ matrix \hat{M}^{-1} , 24 scalars are described by the $\hat{\ell}^a$ and where $\hat{\phi}$ is identified as the Type IIB dilaton.

The common sector is then obtained by imposing the constraints:

$$\hat{B}_{\hat{\mu}\hat{\nu}}^i = 0, \quad (i = 3, \dots, 26), \quad \hat{\ell}^i = 0, \quad \hat{M}_{ij}^{-1} = \delta_{ij}. \quad (4.55)$$

The (anti-)self-duality conditions (4.53) reduce to

$$\hat{H}^{(2)} = -e^{-2\hat{\phi}} * \hat{H}^{(1)}. \quad (4.56)$$

Substituting the constraints (4.55) and the constraint (4.56) back into the Type IIB action (4.52) one obtains the standard form of the action for the common sector in the Einstein metric as given in (4.27). Having identified the Type IIB dilaton it is straightforward to convert this result to the string-frame metric as given in (4.25).

The above discussion for the Type IIA theory also applies to the Type IIB theory. We find that the dimensional reduction of the Type IIB theory leads to the same $D = 5$ Type II theory (4.45) as the dimensional reduction of the Heterotic and Type IIA theory

⁶However, see [52].

provided we use the following dimensional reduction formulae for the Type IIB fields:

$$\mathbf{ST} : \begin{cases} \hat{g}_{xx} &= -e^{2\phi+2\sigma/\sqrt{3}}, \\ \hat{g}_{x\mu} &= -e^{2\phi+2\sigma/\sqrt{3}}C_\mu, \\ \hat{g}_{\mu\nu} &= e^{-2\phi+2\sigma/\sqrt{3}}g_{\mu\nu} - e^{2\phi+2\sigma/\sqrt{3}}C_\mu C_\nu, \\ \hat{B}_{\mu\nu}^I &= B_{\mu\nu}^I + C_{[\mu}A_{\nu]}^I, \\ \hat{B}_{x\mu}^I &= A_\mu^I, \\ \hat{\phi} &= \frac{2}{\sqrt{3}}\sigma, \\ \hat{\ell}^i &= \ell^i, \\ \hat{M} &= M. \end{cases} \quad (4.57)$$

Note that due to the (anti-)self-duality relations (4.53) both $\hat{B}_{\mu\nu}^I$ as well as $\hat{B}_{x\mu}^I$ get related to the 5-dimensional vector fields A_μ^I . The dimensionally reduced expression for the (anti-)self-duality condition (4.53) states that the 26 anti-symmetric tensors $B_{\mu\nu}^I$ and the 26 vector A_μ^I are not independent degrees of freedom, but each other's Poincaré dual:

$$H^{\rho\sigma\lambda K} = -\frac{1}{3\sqrt{|g|}} e^{-2\phi} \varepsilon^{\mu\nu\rho\sigma\lambda} \mathcal{M}^{KI} \mathcal{L}_{IJ} F(\mathcal{A})_{\mu\nu}^J. \quad (4.58)$$

Now that we are able to map the three different six-dimensional theories, Heterotic, Type IIA and Type IIB, onto one and the same five-dimensional Type II theory, we can use these reduction formulae to construct discrete duality transformations between the different theories in six dimensions, as an analogue of the Type IIA/B T -duality in ten dimensions. This will be done in the next subsection.

First we should make a remark about the symmetries of the Type II action (4.45): the action is clearly invariant under the group

$$O(5, 21) = SO^\uparrow(5, 21) \times \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}, \quad (4.59)$$

where the mapping class group $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$ is the straightforward generalization of the nine-dimensional Heterotic case (4.13):

$$\mathbb{Z}_2^{(S)} : \quad A_\mu^I = -A_\mu^I \quad (4.60)$$

$$\mathbb{Z}_2^{(T)} : \quad \begin{cases} A_\mu^I = B_\mu^I \\ B_\mu^I = A_\mu^I \\ V_\mu^i = L_j^i V_\mu^j \\ e^{-4\sigma/\sqrt{3}} = N^{-1} e^{-4\sigma/\sqrt{3}} \\ \ell^i = N^{-1} (e^{-4\sigma/\sqrt{3}} \ell^j M_{jk}^{-1} L^{ki} - \frac{1}{2} \ell^2 \ell^i) \end{cases}$$

with $N = (e^{-8\sigma/\sqrt{3}} - e^{-4\sigma/\sqrt{3}} \ell^j \ell^k M_{jk}^{-1} + \frac{1}{4} \ell^4)$. All other fields remain invariant.

The breaking of the symmetry group of the common sector \mathcal{C}/\mathbb{Z}_2 to the above mapping class group is the five-dimensional analogue of the breaking of the dihedral group D_4 in $D = 9$ to $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$ in the presence of vector fields. If we restrict ourselves to

transformations that act non-trivially on the scalars, we see that the D_3 -group of the common sector gets broken to $\mathbb{Z}_2^{(T)}$.

4.2.3 Type II Dualities

The fact that it is possible to compactify the Heterotic, Type IIA and Type IIB action onto the same Type II action in five dimensions, means that the three theories in six dimensions are intimately related. On one hand, we have the string/string duality between Heterotic and Type IIA theory [60, 163], while on the other hand the T -duality between Type IIA and Type IIB theory on $K3$ can be constructed in the same way as in nine dimensions [26]. Together they form a web of string/string/string “trianality” transformations [62]. These transformations can now be constructed via the different reduction schemes that map the three theories onto the same one in five dimensions.

The presence of the T -duality symmetry (4.60) in five dimensions means that to each reduction formula given above we can associate a T -dual version, in the way that the Type IIB reduction scheme (3.44) was the T -dual of the Type IIA reduction scheme (3.43). Its explicit form is obtained by replacing in the original reduction formula each five-dimensional fields by its T -dual expression. The T -dual reduction formula so obtained should lead to the same action in five dimensions. This is guaranteed by the fact that the five-dimensional action is invariant under T -duality. We will indicate the T -dual versions of the reduction formulae constructed in the previous section as follows:

$$\mathbf{e} \rightarrow \mathbf{T}, \quad \mathbf{S} \rightarrow \mathbf{TS}, \quad \mathbf{ST} \rightarrow \mathbf{TST}. \quad (4.61)$$

Again we have named the different reduction schemes by the group elements of D_3 , since they are each other’s D_3 -transforms (4.39)-(4.41) when restricted to the common sector.

We thus obtain six different reduction formulae which correspond to the three down-pointing arrows in Figure 4.5. Similarly, there are six inverse reduction (decompactification) formulae which go opposite the vertical arrows in Figure 4.5. These decompactification formulae will be indicated by the inverse group elements ($\mathbf{e}^{-1}, \mathbf{T}^{-1}, \mathbf{S}^{-1}, \dots$) and can be constructed easily from the reduction formulae. The claim is now that, using these six reduction and decompactification formulae only, one is able to construct in a simple way all the discrete dualities that act within and between the Heterotic, Type IIA and Type IIB theories that are indicated in Figure 4.5.

Each discrete duality symmetry has been given a name which corresponds to the proper combination of reduction and decompactification schemes and, when restricted to the common sector, the duality becomes the corresponding D_3 duality symmetry that acts in the common sector.

To show how the dualities of Figure 4.5 may be constructed, starting from the different reduction and decompactification formulae, it is instructive to give a few examples.

1. The T -duality that acts within the Heterotic theory is obtained by first reducing the theory using the \mathbf{e} reduction formulae given in (4.43) and then using the

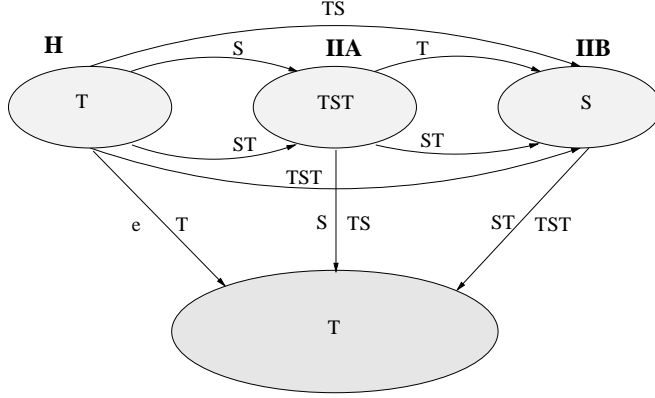


Figure 4.5: The 3 down-pointing arrows indicate the six possible ways to map the three $D = 6$ theories (Heterotic, Type IIA, Type IIB) onto the same $D = 5$ Type II theory. Each reduction formula is indicated by a (boldface) element of D_3 . As explained in the text these six reduction formula and their inverses may be used to construct the explicit form of all the discrete $D = 6$ dualities that are indicated in the figure.

T -dual decompactification formulae, defined in (4.61), i.e.

$$T(H \rightarrow H) = \mathbf{T}^{-1} \times \mathbf{e} = \mathbf{T} . \quad (4.62)$$

The duality rules are the uplifted form of (4.60).

2. The S -duality that maps the Heterotic onto the Type IIA theory is obtained by first reducing the Heterotic theory via \mathbf{e} and next decompactifying the $D = 5$ theory via \mathbf{S}^{-1} . As Figure 4.5 shows there are three other possibilities, one of them gives the same answer while the other two are related to the ST map indicated in Figure 4.5:

$$\begin{aligned} S(H \rightarrow IIA) &= \mathbf{S}^{-1} \times \mathbf{e} = \mathbf{S}^{-1} = \mathbf{S} , \\ S(H \rightarrow IIA) &= (\mathbf{TS})^{-1} \times \mathbf{T} = \mathbf{ST} \times \mathbf{T} = \mathbf{S} , \\ (ST)(H \rightarrow IIA) &= \mathbf{S}^{-1} \times \mathbf{T} = \mathbf{S} \times \mathbf{T} = \mathbf{ST} , \\ (ST)(H \rightarrow IIA) &= (\mathbf{TS})^{-1} \times \mathbf{e} = \mathbf{ST} \times \mathbf{e} = \mathbf{ST} . \end{aligned} \quad (4.63)$$

The S -duality map corresponds to the known $D = 6$ string/string duality rule [60, 163]. We find that the S -duality rules are given by (using the string-frame metric):

$$\begin{aligned} \hat{G}_{\hat{\mu}\hat{\nu}} &= e^{-2\hat{\phi}} \hat{g}_{\hat{\mu}\hat{\nu}} , \\ \hat{\Phi} &= -\hat{\phi} , \\ \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= e^{-2\hat{\phi}} * \hat{h}_{\hat{\mu}\hat{\nu}\hat{\rho}} , \end{aligned} \quad (4.64)$$

where the other fields are invariant. The capital fields are Type IIA and the small-script fields Heterotic. To derive this string/string duality rule one must

also use the two dualization formulae (4.44) and (4.51). Note that one may only derive a string/string duality rule for \hat{H} and not \hat{B} . This is of course related to the fact that from the six-dimensional point of view the string/string duality is a symmetry of the equations of motion only.

3. The S -duality that acts within the Type IIB theory is obtained by first reducing the Type IIB theory with **ST** and then decompactifying with **(TST)**⁻¹. The other way round gives the same answer:

$$\begin{aligned}
S(IIB \rightarrow IIB) &= (\mathbf{TST})^{-1} \times \mathbf{ST} = \mathbf{TST} \times \mathbf{ST} \\
&= \mathbf{STS} \times \mathbf{ST} = \mathbf{S}, \\
S(IIB \rightarrow IIB) &= (\mathbf{ST})^{-1} \times \mathbf{TST} = \mathbf{TS} \times \mathbf{TST} \\
&= \mathbf{TS} \times \mathbf{STS} = \mathbf{S},
\end{aligned} \tag{4.65}$$

where we have used the multiplication properties of the group D_3 .

The S -duality rules can be written covariantly in a six-dimensional way, i.e., in terms of the $\hat{\mu}$ -indices, in contrast to the T -duality, whose presence requires the existence of a special isometry direction. We find that the S -duality is given by a particular $O(5, 21)$ -transformation with parameter Ω given by $\Omega = \mathcal{L}$, where \mathcal{L} is the flat $O(5, 21)$ metric given in (3.38). In components, its action on the antisymmetric tensors and scalars is given by

$$\begin{aligned}
\hat{H}'^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\
\hat{H}'^{(2)}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\
\hat{H}'^a_{\hat{\mu}\hat{\nu}\hat{\rho}} &= L^a_b \hat{H}^b_{\hat{\mu}\hat{\nu}\hat{\rho}}, \\
e^{-2\hat{\phi}'} &= e^{-2\hat{\phi}} \left(e^{-4\hat{\phi}} - e^{-2\hat{\phi}'} \hat{\ell}^a \hat{\ell}^b \hat{M}_{ab}^{-1} + \frac{1}{4} \hat{\ell}^4 \right)^{-1}, \\
\hat{\ell}'^a &= \frac{e^{-2\hat{\phi}} \hat{\ell}^c \hat{M}_{cd}^{-1} L^{da} - \frac{1}{2} \hat{\ell}^2 \hat{\ell}^a}{e^{-4\hat{\phi}} - e^{-2\hat{\phi}} \hat{\ell}^a \hat{\ell}^b \hat{M}_{ab}^{-1} + \frac{1}{4} \hat{\ell}^4}, \\
\hat{M} &= \hat{M}^{-1}.
\end{aligned} \tag{4.66}$$

Note that, when restricted to the common sector, this duality transformation indeed reduces to the standard S -duality rule given in (4.39)-(4.41).

4. We deduce from Figure 4.5 that there is not only a T -duality that acts within the Heterotic theory but also a T -duality that maps the Type IIA theory onto the Type IIB theory. This is then the analogue of the Type IIA/B T -duality in ten dimensions [26]. It may be obtained in the following two ways from the reduction/decompactification formulae:

$$\begin{aligned}
T(IIA \rightarrow IIB) &= (\mathbf{ST})^{-1} \times \mathbf{S} = \mathbf{TS} \times \mathbf{S} = \mathbf{T}, \\
T(IIA \rightarrow IIB) &= (\mathbf{TST})^{-1} \times \mathbf{TS} = \mathbf{TST} \times \mathbf{TS} = \mathbf{T}.
\end{aligned} \tag{4.67}$$

Following our method described above we find the following expression for this duality transformation:

$$\begin{aligned}
\hat{\Phi} &= \hat{\phi} - \frac{1}{2} \log(-\hat{g}_{xx}), \\
\hat{G}_{xx} &= 1/\hat{g}_{xx}, \\
\hat{G}_{x\mu} &= \hat{b}_{x\mu}/\hat{g}_{xx}, \\
\hat{G}_{\mu\nu} &= \hat{g}_{\mu\nu} - \left(\hat{g}_{x\mu}\hat{g}_{x\nu} - \hat{b}_{x\mu}\hat{b}_{x\nu} \right) / \hat{g}_{xx}, \\
\hat{B}_{x\mu}^{(1)} &= \hat{g}_{x\mu}/\hat{g}_{xx}, \\
\hat{B}_{\mu\nu}^{(1)} &= \hat{b}_{\mu\nu} - \left(\hat{g}_{x\mu}\hat{b}_{x\nu} - \hat{g}_{x\nu}\hat{b}_{x\mu} \right) / \hat{g}_{xx}, \\
\hat{B}_{x\mu}^i &= \hat{v}_\mu^i - \hat{v}_x^i \hat{g}_{x\mu} / \hat{g}_{xx}, \\
\hat{\ell}^i &= \hat{v}_x^i, \\
\hat{M}_{ij} &= \hat{m}_{ij},
\end{aligned} \tag{4.68}$$

where the capital fields are Type IIB and the small-script fields are Type IIA fields, respectively. Note that the duality transformations of $\hat{B}_{\mu\nu}^{(2)}$ and $\hat{B}_{\mu\nu}^i$ are not given. Their transformation rules follow from the ones given above via the the self-duality conditions (4.53).

5. Finally, we observe that ST is a 3-order element of D_3 . This means that starting with the Heterotic theory and applying the ST -duality three times we should get back the Heterotic theory. In the diagram of Figure 4.5 this is seen as follows: The first ST duality brings us to the Type IIA theory, the second one brings us from the Type IIA to the type IIB theory. Finally, to perform the last ST -duality we observe that $ST = (TS)^{-1}$, i.e., this duality brings us back from the Type IIB theory to the Heterotic theory via the opposite direction of the oriented arrow at the top of the diagram.

Clearly, the above given examples are not all the D_3 string/string/string triality transformations. The other transformations can be constructed in the same way as the transformations above.

These six-dimensional duality relations are, just as the duality map in ten dimensions, an indication that the various string theories are different manifestations of the underlying M -theory (section 3.3). Different compactifications of different limits become equivalent and can be related via duality transformations. Figure 4.5 can therefore be compared to Figure 3.2 in section 3.3.