Dualities of strings and branes

Janssen, Bert

Chapter 3

Duality

String theory is a very powerful tool in the attempt to find a unifying description of all interactions. However the theory, as it was known till the early nineties (i.e. as was briefly described in the previous chapter), has some problems. First of all, the theory is only defined at the perturbative level, as a Feynman "sum-over-histories" approach, without an understanding of the dynamical principles that form the theory and that allow one to go beyond perturbation theory. A second problem is the fact that, although techniques are known to come down from the ten-dimensional superstring world to our phenomenologically observable four-dimensional world, these techniques give rise to many degenerate ground states, parametrized by the scalars (moduli) that appear in these reductions. It is not at all clear which of these compactifications corresponds to a model that looks like something we know from experiments (the Standard Model) and why Nature chooses precisely this vacuum. But maybe the most annoying feature is that on the one hand string theory claims to be a unifying theory of gravity and quantum field theory, yet on the other hand five different versions of string theory are known: Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. So there seem to exist five different unification candidates and five different ways to formulate a theory involving quantized gravity, which is not an appealing idea for a unification theory.

In the early nineties, the second "superstring revolution"[1] introduced the concept of "dualities", which indicated the possibility to solve many of the above problems at once: it was realised that a certain theory A, compactified on a large volume, could be equivalent to a theory B, compactified on a small volume, or that a theory C at weak coupling could be mapped to a theory D at strong coupling. In this way, it was possible to regard different vacua as being equivalent, find a more unifying description for the different string theories and to get insight into the physics beyond the perturbative level.

[1] The first superstring revolution was the one in the mid eighties, when it was realized that the above mentioned string theories are the only consistent ones and that these have a well defined perturbation expansion.

In this chapter we will give an overview of the different duality symmetries in string theory. In Section 3.1 we will present a duality that acts on the target space of the string, the Target Space Duality or T-duality. In section 3.2 we discuss the duality that relates the strong and weak coupling regime of the different theories, the so-called S-duality (Strong/Weak coupling duality). In section 3.3 we will present the unifying picture as it stands at this moment.

General references for string dualities are [141, 58, 158, 67, 162, 101, 104, 56, 148] .

3.1 Target Space Duality

Target Space duality, or for short T-duality, is a symmetry transformation that relates different string backgrounds to each other. It was first introduced at the level of the bosonic sigma model in the presence of an isometry as a $\mathbb{Z}_2$-symmetry that interchanges certain components of the metric with certain components of the axion [37]. The general T-duality transformations are intimately related with the idea of dimensional reduction via the appearance of the non-compact $O(d, d+n)$ groups. Their importance lies in the fact that T-duality gives a way to divide the many degenerate ground states in T-duality classes of equivalent physics. For extensive reviews about T-duality in string theory, we refer to [75, 3].

3.1.1 T-duality in World Volume Theory

The T-duality transformation rules can be derived from the non-linear sigma model (2.27):

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}\partial_i X^{\hat\mu}\partial_j X^{\hat\nu} g_{\hat\mu\hat\nu} \\ +\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\ \partial_i X^{\hat\mu}\partial_j X^{\hat\nu} B_{\hat\mu\hat\nu}. \end{aligned} \tag{3.1}$$

Suppose that the background fields $g_{\hat\mu\hat\nu}$ and $B_{\hat\mu\hat\nu}$ are independent of one embedding coordinate X, so the D-dimensional indices $\hat\mu$ can be split into the index x of the isometry direction and the indices of the $(D-1)$ remaining directions: $\hat\mu = (x, \mu)$.

We can then consider the derivative of the isometry coordinate $\partial_i X$ to be an independent field V_i by adding a Lagrange multiplier $\tilde X$ and rewriting (3.1) as

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}\Big[\partial_i X^\mu\partial_j X^\nu g_{\mu\nu} + 2\partial_i X^\mu V_j\ g_{\mu x} + V_i V_j\ g_{xx}\Big] \\ +\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\Big[\partial_i X^\mu\partial_j X^\nu B_{\mu\nu} + 2V_i\partial_j X^\nu B_{x\nu}\Big] \\ -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\ \tilde X\partial_i V_j. \end{aligned} \tag{3.2}$$

The equation of motion of $\tilde{X}$ states that $V_i = \partial_i X$ and relates action (3.2) to action (3.1). On the other hand, solving the equation of motion of V_i and substituting in (3.2), we find the dual action, in terms of the dual coordinates $\tilde{X}^{\hat{\mu}} = (X^\mu, \tilde{X})$:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}\partial_i\tilde{X}^{\hat{\mu}}\partial_j\tilde{X}^{\hat{\nu}}\tilde{g}_{\hat{\mu}\hat{\nu}} + \frac{1}{4\pi\alpha'}\int d^2\sigma\ \varepsilon^{ij}\ \partial_i\tilde{X}^{\hat{\mu}}\partial_j\tilde{X}^{\hat{\nu}}\tilde{B}_{\hat{\mu}\hat{\nu}}\ . \tag{3.3}$$

This is again a non-linear sigma model action for a string moving in the dual background fields $\tilde{g}_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$, where the relation between the original and the dual fields is given by the so-called T-duality rules [37]:

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= g_{\mu\nu} - (g_{x\mu}g_{x\nu} - B_{x\mu}B_{x\nu})/g_{xx},\\
\tilde{B}_{\mu\nu} &= B_{\mu\nu} - (g_{x\mu}B_{x\nu} - g_{x\nu}B_{x\mu})/g_{xx},\\
\tilde{g}_{x\mu} &= B_{x\mu}/g_{xx},\\
\tilde{B}_{x\mu} &= g_{x\mu}/g_{xx},\\
\tilde{g}_{xx} &= 1/g_{xx}\,.
\end{aligned} \tag{3.4}$$

The transformation rule for the dilaton cannot be obtained via the equation of motion of V_i, but by demanding that the conformal invariance of (2.27) at order $(\alpha')^0$ can be found back in the dual action. The dilaton transforms as

$$\tilde{\phi} = \phi - \tfrac{1}{2}\log|g_{xx}|. \tag{3.5}$$

The T-duality rules relate two geometrically different, but dynamically equivalent sets of background fields: although the geometry of the space is altered, the physical properties of the model are unchanged under the duality transformation. Let us illustrate this with some simple examples for the closed and the open string in some simple backgrounds.

Suppose a closed string is moving in a flat space-time where one coordinate X is a circle of radius R. The metric is of the form $g_{\hat{\mu}\hat{\nu}} = \text{diag}[1, -1, -1, ..., -R^2/\alpha']$, all other background fields are set equal to zero.

The boundary conditions on $X^{\hat{\mu}}$ are given by

$$\begin{aligned}
X^\mu(\tau, \sigma + 2\pi) &= X^\mu(\tau, \sigma),\\
X(\tau, \sigma + 2\pi) &= X(\tau, \sigma) + 2\pi m R,
\end{aligned} \tag{3.6}$$

where m is an integer that indicates how many times the string is wound around the compact direction X. The periodicity of X forces the momentum in this direction to be quantized: e^{iPX} should be single valued for X and $X + 2\pi R$, so $P = n/R$. The solution of the equation of motion (2.7) for the string, satisfying the above boundary conditions, is

$$X^{\hat{\mu}}_\pm : \begin{cases} X^\mu_\pm & \text{as in (2.9)},\\ X_\pm = \tfrac{1}{2}x + \sqrt{\tfrac{\alpha'}{2}}P_\pm\ (\tau \pm \sigma)\ +\ \text{oscillations}\ , \end{cases} \tag{3.7}$$

where $X_+^{\hat\mu}$ and $X_-^{\hat\mu}$ are defined as in (2.8) and

$$P_\pm = \frac{1}{\sqrt{2}}\Big(\frac{\sqrt{\alpha'}}{R}\ n\ \pm\ \frac{R}{\sqrt{\alpha'}}\ m\Big). \tag{3.8}$$

We see that the expressions for $P_\pm$ are invariant under simultaneous interchange of $R \leftrightarrow \alpha'/R$ and $m \leftrightarrow n$ [103, 135], and since the on-shell mass condition is given by $M = (P^{\hat\mu}P_{\hat\mu} +$ oscillator terms), also the spectrum is invariant under this interchange. The string does not see whether it is wound m times around a circle with small radius R while having a momentum n, or n times around a circle with large radius α'/R having momentum m.

It is not difficult to show that an $R \to \alpha'/R$ transformation is in fact a T-duality transformation where the compact direction X is dualized into the dual coordinate $\tilde X = X_+ - X_-$. The inversion of the radius seems to suggest that there exists a "minimal length" $R = \sqrt{\alpha'}$, at the string scale: going beyond this "minimal length" would give the same physics as at large length scales.

For an open string, freely moving in a flat space with one compact dimension $X = 2\pi R$ (i.e. satisfying Neumann boundary conditions $\partial_\sigma X^{\hat\mu} = 0$), we can rewrite (2.13) as ($X^{\hat\mu} = X_+^{\hat\mu} + X_-^{\hat\mu}$ and $P = n/R$):

$$\left\{\begin{array}{lll} X_\pm^\mu & & \text{as in (2.13)}, \\ X_\pm & = & \tfrac12 x \pm \tfrac12 C + \tfrac12 \alpha' \tfrac{n}{R}\ (\tau \pm \sigma)\ + \tfrac12 \sum_n \tfrac1n\ \tilde a_n^\mu\ e^{in(\tau\pm\sigma)}. \end{array}\right. \tag{3.9}$$

Again we can dualize X into $\tilde X = X_+ - X_-$ and we find [129]:

$$\tilde X = C + \alpha' \tfrac{n}{R}\ \sigma\ + \tfrac12 \sum_n \tfrac1n\ \tilde a_n^\mu\ e^{in\tau} \sin n\sigma. \tag{3.10}$$

This is the solution (2.14) for the equations of motion of a string satisfying Dirichlet boundary conditions. It turns out that T-duality has interchanged the Neumann boundary conditions $\partial_\sigma X|_{\sigma=0}^{\sigma=\pi} = 0$ for Dirichlet conditions $\partial_\tau \tilde X|_{\sigma=0}^{\sigma=\pi} = 0$ in the dualized direction: where for the freely moving string (2.13) the zero-modes of the string were independent of σ, here the zero-modes in the $\tilde X$ direction are independent of τ. This means that the endpoints of the string are fixed in the $\tilde X$-direction:

$$\left\{\begin{array}{l} \tilde X(0) = C, \\ \tilde X(\pi) = C\ +\ 2n\pi \tilde R\ . \end{array}\right. \tag{3.11}$$

The string is attached to a $(D-2)$-dimensional hypersurface $\tilde X = C$, while it can wind n times around the the compact dimension $\tilde X$ of radius $\tilde R = \alpha'/R$. This hypersurface is in fact the D-brane we encountered as a solution of the equations of motion in Section 2.3.

3.1.2 Dimensional Reduction

Before we study the effect of T-duality on the low energy effective action of string theories, let us first make a small intermezzo about dimensional reduction and compactification.

In Chapter 2 we mentioned that superstring theory only can be quantized consistently if the string lives in a ten-dimensional space-time. However, if we want our theory to be "realistic", we have to be able to make contact with the phenomenologically observable world, which is four-dimensional: we have to find a way to hide away six dimensions and to rewrite high-dimensional results in terms of low-dimensional ones. This can be done through compactification. Suppose that six of the ten dimensions are compactified over a very small volume, with length scales of the order of the Planck-scale, such that they are invisible at low energies or at large length scales. The ten-dimensional manifold is a product of a four-dimensional space-time times a six-dimensional compact space: $\mathcal{M}^{10} = M^4 \times K^6$. We can translate the ten-dimensional theory to an effective theory in four dimensions, where the precise form of the effective theory depends on the geometry of the compact manifold. This idea is sometimes called Kaluza-Klein compactification, because Kaluza and Klein tried to write electromagnetism and general relativity in four dimensions as a single theory of pure gravity in five dimensions [97, 107].

There exist an infinite number of compact manifolds K over which we can compactify, but only a limited number of these give useful results for string theory[2]. The compactification we will study in this section and mostly use in the rest of this work, is the most simple case, namely the compactification over a d-dimensional torus T^d. Compactifications over more complicated manifolds, such as $K3$ or Calabi-Yau manifolds, may give phenomenologically more relevant results (chiral fermions, a Minimal Supersymmetric Standard Model, ...), but torus compactification will already be sufficient for the features we are interested in, namely the symmetry groups of compactified theories and the relations between different supergravity actions. Properties of a more complicated compactification will be discussed in Chapter 4, when we study the symmetries of Type IIA/B, compactified on $K3$.

If the fields of the uncompactified theory depend on the compact coordinates, then extra massive states appear in the lower-dimensional theory. This can be seen in a simple example: suppose a field $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ in a flat D-dimensional space-time with one compact dimension x obeys the equations of motion[3]

$$\partial_{\hat{\mu}}\partial_{\hat{\mu}}\hat{\Phi}(\hat{x}^{\hat{\nu}}) \;=\; \partial_{\mu}\partial_{\mu}\hat{\Phi}(\hat{x}^{\hat{\nu}}) \;-\; \partial_x\partial_x\hat{\Phi}(\hat{x}^{\hat{\nu}}) = 0. \tag{3.12}$$

Since the field $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ depends on all coordinates $\hat{x}^{\hat{\mu}}$, in particular also on the compact coordinate x, we can perform a separation of variables and do a Fourier expansion of field in the x-coordinate. This corresponds to an expansion of $\hat{\Phi}(\hat{x}^{\hat{\mu}})$ in modes of the quantised momentum in the compact direction:

$$\hat{\Phi}(\hat{x}^{\hat{\mu}}) = \sum_n \Phi_n(x^{\mu})\; e^{\frac{inx}{R}}, \tag{3.13}$$

where R is the radius of the compact dimension. The equation of motion for the

[2]It turns out that only the compactifications that preserve some amount of supersymmetry are consistent. Compactifications that break all supersymmetry give rise to theories that do not have a well defined perturbation theory.

[3]From now on we will use the notation that hatted fields and indices are higher-dimensional ones and unhatted ones lower-dimensional. It should be clear from the context in which dimension each field (hatted or unhatted) lives.

coefficient $\Phi_n(x^\mu)$ of the n-th mode is now of the form

$$\partial_\mu\partial_\mu\ \Phi_n(x^\mu)\ +\ \tfrac{n^2}{R^2}\ \Phi_n(x^\mu) = 0, \tag{3.14}$$

which is the Klein-Gordon equation $(\Box + M_n^2)\Phi_n = 0$ for a field with mass $M_n = n/R$. So the different modes of the field $\hat{\Phi}$ manifest themselves in lower dimensions as an infinite tower of states with masses equal to the quantised momentum. The proportionality constant is the inverse radius of the compact direction $1/R$. These modes are called the Kaluza-Klein modes of $\hat{\Phi}$. If $R \to \infty$, so upon decompactification, the massive states become massless and form a continuous spectrum. For small R (comparable to the Planck-length) however, the states with $n \neq 0$ are very massive, with masses of the order of the Planck-mass.

At low energies, or equivalently at length scales much bigger then the size of the compact dimension, only the massless lowest mode can be detected. Since in the low energy effective actions of string theory the massive string modes have already been integrated out, it is therefore consistent to exclude also the massive Kaluza-Klein modes from the theory. This is the same as removing the dependence of the D-dimensional fields on the compactified coordinates. Throughout this section we will suppose that this is the case.

The precise way the higher-dimensional fields reduce to lower dimensions is determined by gauge invariance: a general coordinate transformation in higher dimensions will manifest itself as a lower-dimensional general coordinate transformation and gauge symmetries. The reduction rules are given in [136, 46, 115]: let us derive them for some typical examples.

Suppose a D-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ is independent of d coordinates. Coordinate transformations of the metric give:

$$\delta\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\lambda}}\ \partial_{\hat{\lambda}}\hat{g}_{\hat{\mu}\hat{\nu}}\ +\ \partial_{\hat{\mu}}\hat{\xi}^{\hat{\lambda}}\hat{g}_{\hat{\lambda}\hat{\nu}}\ +\ \partial_{\hat{\nu}}\hat{\xi}^{\hat{\lambda}}\hat{g}_{\hat{\mu}\hat{\lambda}}. \tag{3.15}$$

We can split the D-dimensional indices $\hat{\mu}$ in $\hat{\mu} = (\mu, a)$ with $0 \leq \mu \leq D-d-1$ and $1 \leq a \leq d$ and compactify over the coordinates x^a. The different components of $\hat{g}_{\hat{\mu}\hat{\nu}}$ then transform as:

$$\begin{array}{ll}
\delta\hat{g}_{ab} = & \xi^\lambda\ \partial_\lambda\hat{g}_{ab}\ , \\
\delta\hat{g}_{\mu a} = & \xi^\lambda\ \partial_\lambda\hat{g}_{\mu a} + \partial_\mu\xi^\lambda\hat{g}_{\lambda a} + \partial_\mu\xi^b\hat{g}_{ba}\ , \\
\delta\hat{g}_{\mu\nu} = & \underbrace{\xi^\lambda\ \partial_\lambda\hat{g}_{\mu\nu} + \partial_\mu\xi^\lambda\hat{g}_{\lambda\nu} + \partial_\nu\xi^\lambda\hat{g}_{\mu\lambda}}_{\text{g.c.t in } (D-d)} + \underbrace{\partial_\mu\xi^a\hat{g}_{a\nu} + \partial_\nu\xi^a\hat{g}_{\mu a}}_{\text{“internal”}}\ ,
\end{array} \tag{3.16}$$

where we also took the $\hat{\xi}^{\hat{\mu}}$ independent of x^a. The variations look like the transformation rules for a set of scalars, vectors and a metric under $(D-d)$-dimensional general coordinate transformations plus some extra variations coming from the internal components ξ^a.

In order to get rid of these extra terms, we define the $(D-d)$-dimensional quantities G_{ab}, A_μ^a and $g_{\mu\nu}$ as:

$$G_{ab} \ = \ \hat{g}_{ab},$$

$$
\begin{aligned}
A_\mu^a &= \hat{g}^{ab}\,\hat{g}_{\mu b}, \\
g_{\mu\nu} &= \hat{g}_{\mu\nu} - \hat{g}^{ab}\hat{g}_{\mu a}\hat{g}_{\nu b},
\end{aligned} \tag{3.17}
$$

where $\hat{g}^{ab}$ are the components of the inverse metric. It is easy to see that these fields transform in the correct way as a set of $\frac{1}{2}d(d+1)$ scalars, d vectors and one metric under the $(D-d)$-dimensional general coordinate transformations. Furthermore the D-dimensional transformations induce a $U(1)$-gauge transformation on the vector fields: $\delta A_\mu^a = \partial_\mu \xi^a$. The vectors A_μ^a are usually called Kaluza-Klein vectors, and the scalars G_{ab} the moduli of the compactification, since they parametrise the internal space.

The anti-symmetric tensor field $\hat{B}_{\hat{\mu}\hat{\nu}}$ transforms, besides under general coordinate transformations as in (3.15), also under the ten-dimensional gauge transformation $\delta\hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}}\hat{\Sigma}_{\hat{\nu}]}$. The variations of the different components yield:

$$
\begin{aligned}
\delta\hat{B}_{ab} &= \delta_L \hat{B}_{ab}, \\
\delta\hat{B}_{a\mu} &= \delta_L \hat{B}_{a\mu} + \partial_\mu \xi^b \hat{B}_{ab} - \partial_\mu \Sigma_a, \\
\delta\hat{B}_{\mu\nu} &= \delta_L \hat{B}_{\mu\nu} + \partial_{[\mu}\xi^a \hat{B}_{a\nu]} + \partial_{[\mu}\Sigma_{\nu]},
\end{aligned} \tag{3.18}
$$

where with δ_L we mean the variation under $(D-d)$-dimensional general coordinate transformations. In $(D-d)$ dimensions we therefore obtain a set of $\frac{1}{2}d(d-1)$ scalars B_{ab}, d vectors $B_{a\mu}$ and a rank-two anti-symmetric tensor $B_{\mu\nu}$, given as functions of the D-dimensional fields by:

$$
\begin{aligned}
B_{ab} &= \hat{B}_{ab}, \\
B_{a\mu} &= \hat{B}_{a\mu} - \hat{g}^{cb}\hat{g}_{c\mu}\hat{B}_{ab}, \\
B_{\mu\nu} &= \hat{B}_{\mu\nu} + \hat{g}^{ab}\hat{g}_{a[\mu}\hat{B}_{\nu]b} - 2\,\hat{g}^{ab}\hat{g}^{cd}\hat{g}_{a[\mu}\,\hat{B}_{bc}\,\hat{g}_{\nu]d}\ .
\end{aligned} \tag{3.19}
$$

Again all these fields transform in the proper way and $B_{a\mu}$ behaves like a $U(1)$-gauge field under the remnant gauge transformation of $\hat{B}_{\hat{\mu}\hat{\nu}}$: $\delta B_{a\mu} = \partial_\mu \Sigma_a$. $B_{a\mu}$ is usually called the winding vector, since one can show that it couples to string states that are wound a number of times around the compact dimension x^a. The scalars B_{ab} span, together with the G_{ab}, the moduli space of toroidal compactifications.

Let us now look at the reduction of the action of the common sector (2.42). For simplicity, we only reduce from ten to nine dimensions, since all typical and interesting features can already be found in this example. Later on we will study more extensively the reductions of the various superstring actions over more dimensions.

Suppose all fields in the ten-dimensional action (2.42)

$$
S = \tfrac{1}{2}\int d^{10}x\sqrt{|\hat{g}|}\; e^{-2\hat{\phi}}\Big[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}^2\Big] \tag{3.20}
$$

are independent of the coordinate x, over which we are going the reduce.

At this point it is convenient to write the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ locally as a flat metric:

$$
\hat{g}_{\hat{\mu}\hat{\nu}} \;=\; \hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}\;\hat{e}_{\hat{\nu}}{}^{\hat{\beta}}\;\hat{\eta}_{\hat{\alpha}\hat{\beta}}, \tag{3.21}
$$

where $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}$ is the ten-dimensional vielbein, which relates the curved indices $\hat{\mu}$ to the flat ones $\hat{\alpha}$. The vielbein transforms under Lorentz transformations, therefore we can choose a gauge in which the vielbein is of the form

$$\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} = \begin{pmatrix} e_\mu{}^\alpha & kA_\mu \\ 0 & k \end{pmatrix}, \tag{3.22}$$

where $e_\mu{}^\alpha$ is the nine-dimensional vielbein. This corresponds to a choice for the reduction rules

$$\begin{array}{rclrcl} \hat{g}_{\mu\nu} & = & g_{\mu\nu} - k^2 A_\mu A_\nu\,, & \hat{B}_{\mu\nu} & = & B_{\mu\nu} + A_{[\mu}B_{\nu]}\,, \\ \hat{g}_{x\mu} & = & -k^2 A_\mu\,, & \hat{B}_{x\mu} & = & B_\mu\,, \\ \hat{g}_{xx} & = & -k^2\,. & & & \end{array} \tag{3.23}$$

For the gauge choice (3.22), we have that

$$\sqrt{|\hat{g}|} = \det(\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}) = k\,\det(e_\mu{}^\alpha) = k\,\sqrt{|g|}. \tag{3.24}$$

So if we take for the reduction rule of the dilaton[4]

$$\hat{\phi} = \phi + \tfrac{1}{2}\log k, \tag{3.25}$$

we see that $\sqrt{|\hat{g}|}\,e^{-2\hat{\phi}} = \sqrt{|g|}\,e^{-2\phi}$. It can be shown that the first two terms of (3.20) reduce like

$$-\hat{R} + 4(\partial\hat{\phi})^2 = -R + 4(\partial\phi)^2 - (\partial\log k)^2 + \tfrac{1}{4}k^2 F_{\mu\nu}(A)F^{\mu\nu}(A), \tag{3.26}$$

while the axion field strength $\hat{H}_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = \hat{e}_{\hat{\alpha}}{}^{\hat{\mu}}\,\hat{e}_{\hat{\beta}}{}^{\hat{\nu}}\,\hat{e}_{\hat{\gamma}}{}^{\hat{\rho}}\,\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ decomposes as:

$$\begin{array}{rcl} \hat{H}_{\alpha\beta x} & = & \frac{1}{3k}\,e_\alpha{}^\mu\,e_\beta{}^\nu F_{\mu\nu}(B)\,, \\ \hat{H}_{\alpha\beta\gamma} & = & e_\alpha{}^\mu\,e_\beta{}^\nu e_\gamma{}^\rho\left[\partial_{[\mu}B_{\nu\rho]} + \frac{1}{2}A_{[\mu}F_{\nu\rho]}(B) + \frac{1}{2}B_{[\mu}F_{\nu\rho]}(A)\right], \\ & = & e_\alpha{}^\mu\,e_\beta{}^\nu e_\gamma{}^\rho\,H_{\mu\nu\rho} \;=\; H_{\alpha\beta\gamma}. \end{array} \tag{3.27}$$

So after dimensional reduction, (3.20) takes the form

$$\begin{aligned} S = \tfrac{1}{2}\int d^9x\,\sqrt{|g|}\,e^{-2\phi}\Big[-R + 4(\partial\phi)^2 - \tfrac{3}{4}H^2 - (\partial\log k)^2 \\ + \tfrac{1}{4}k^2F^2(A) + \tfrac{1}{4}k^{-2}F^2(B)\Big]\,. \end{aligned} \tag{3.28}$$

Note that the reduced action is invariant under nine-dimensional general coordinate transformations, as it should be, and under the $U(1)$-symmetries $\delta A_\mu = \partial_\mu\xi$, $\delta B_\mu = \partial_\mu\Sigma$, provided that the reduced axion transforms as:

$$\delta B_{\mu\nu} = \partial_{[\mu}\Sigma_{\nu]} + \partial_{[\mu}\xi B_{\nu]} - A_{[\mu}\partial_{\nu]}\Sigma. \tag{3.29}$$

Furthermore the action has an $O(1,1)$-symmetry, which is a direct product: $O(1,1) = SO^\uparrow(1,1)\times\mathbb{Z}_2^{(S)}\times\mathbb{Z}_2^{(T)}$. Just as in the case of the Lorentz group, this non-compact

[4]The higher dimensional analogue for this reduction rule is $\hat{\phi} = \phi + \frac{1}{2}\log|G_{ab}|$.

group consists of four disconnected parts, of which only the subgroup of proper, time-orientation preserving transformations $SO^{\uparrow}(1,1)$ is continuously connected to the identity. The other parts (the improper and/or non-orthochronous transformations) are connected via the mapping class group $\mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(T)}$.

The continuous scale transformation $SO^{\uparrow}(1,1)$ scales the fields A_μ, B_μ and k with a factor $\Lambda > 0$ according to their weight under this scale transformation:

$$A_\mu \to \Lambda\ A_\mu, \qquad B_\mu \to \Lambda^{-1} B_\mu, \qquad k \to \Lambda^{-1} k. \tag{3.30}$$

The discrete subgroup $\mathbb{Z}_2^{(S)}$ flips the sign of the vector fields, while the $\mathbb{Z}_2^{(T)}$-symmetry is generated by an interchange of the vector fields and an inversion of k:

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = k^{-1}. \tag{3.31}$$

Using (the inverse of) the reduction rules (3.23), we can easily see that this symmetry (3.31) corresponds in ten dimensions to the T-duality transformation (3.4). The $O(1,1)$ is therefore called the T-duality group, which parametrises the moduli space of compactifications: the modulus k is directly related to the size of the compact dimension

$$k = \sqrt{|\hat{g}_{xx}|} = \frac{R}{\sqrt{\alpha'}}. \tag{3.32}$$

Different values of k label different compactifications, which are related via the $O(1,1)$ transformations. However T-duality ($\mathbb{Z}_2^{(T)}$) states that compactification over a radius R is equivalent to compactification over a radius $1/R$, so the points k and k^{-1} in moduli space are equivalent. Also the sign of the vector fields is irrelevant ($\mathbb{Z}_2^{(S)}$), so the moduli space of *inequivalent* compactifications is given by

$$\frac{O(1,1)}{\mathbb{Z}_2^{(T)} \times \mathbb{Z}_2^{(S)}} \;=\; SO^{\uparrow}(1,1). \tag{3.33}$$

In this simple example of compactification over one dimension, all generic features of toroidal compactification are present. In the next subsection we will study more general compactifications over d dimensions in the presence of n Abelian vector fields. This will give rise to bigger $O(d, d+n)$ groups and more complicated coset structures, but the same features will reappear. For an extensive study of the symmetry transformations of the dimensionally reduced action (3.28), we refer to [27].

3.1.3 T-duality in the Target Space Action

Let us now look at T-duality in the full low energy effective string theory actions. We will start with the Heterotic string theories. Their action is given by (2.36)

$$S = \tfrac{1}{2}\int d^{10}x\sqrt{|\hat{g}|}\ e^{-2\hat{\phi}}\Big[-\hat{R} + 4(\partial\hat{\phi})^2 - \tfrac{3}{4}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} + \tfrac{1}{4}\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}_I^{\hat{\mu}\hat{\nu}}\Big]. \tag{3.34}$$

The $\hat{F}^I_{\hat{\mu}\hat{\nu}}$ are the field strengths of the $SO(32)$ or $E_8 \times E_8$ gauge fields $\hat{V}^I_{\hat{\mu}}$; the axion field strength $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ contains a Chern-Simons term:

$$\begin{aligned} \hat{F}^I_{\hat{\mu}\hat{\nu}} &= \partial_{\hat{\mu}} \hat{V}^I_{\hat{\nu}} - \partial_{\hat{\nu}} \hat{V}^I_{\hat{\mu}} - f_{KL}{}^I \hat{V}^K_{\hat{\mu}} \hat{V}^L_{\hat{\nu}}, \\ \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} &= \partial_{[\hat{\mu}} \hat{B}_{\hat{\nu}\hat{\rho}]} - \tfrac{1}{2}\Big[\hat{V}^I_{[\hat{\mu}} \hat{F}_{\hat{\nu}\hat{\rho}]I} + \tfrac{1}{3} f_{IKL} \hat{V}^I_{[\hat{\mu}} \hat{V}^K_{\hat{\nu}} \hat{V}^L_{\hat{\rho}]} \Big]. \end{aligned} \tag{3.35}$$

The gauge transformations of the Yang-Mills groups are given by:

$$\begin{aligned} \delta \hat{V}^I_{\hat{\mu}} &= \partial_{\hat{\mu}} \Lambda^I + f_{KL}{}^I \Lambda^K \hat{V}^L_{\hat{\mu}}, \\ \delta \hat{B}_{\hat{\mu}\hat{\nu}} &= \hat{V}^I_{[\hat{\mu}} \partial_{\hat{\nu}]} \Lambda_I. \end{aligned} \tag{3.36}$$

Dimensional reduction over T^d yields an action with a $(10-d)$-dimensional metric, axion and dilaton, d Kaluza-Klein vectors A^a_μ, d winding vectors $B_{\mu a}$, Yang-Mills vectors V^I_μ and moduli G_{ab}, B_{ab} and ℓ^I_a coming from the reduction of the metric, the axion and the Yang-Mills fields in 10 dimensions. The precise reduction rules will be given in (4.9).

In a generic point in the moduli space, the ℓ^I_a have a non-zero expectation value and via a Higgs mechanism they will give masses to the vector fields in the Yang-Mills group. Only the Abelian fields V^m_μ in the Cartan sub-algebra will remain massless after reduction. For both $SO(32)$ and $E_8 \times E_8$ this Cartan sub-algebra is 16-dimensional, so both groups break to $U(1)^{16}$. The low energy effective action therefore contains $(2d+16)$ Abelian vector fields, which form a $U(1)^{(2d+16)}$ gauge group. Furthermore these Abelian fields fit into a global $O(d, d+16)$-group representation such that the action can be written as [136, 115]:

$$\begin{aligned} S = \tfrac{1}{2} \int d^{10-d}x \sqrt{|g|}\, e^{-2\phi} \Big[& -R + 4(\partial\phi)^2 - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & + \tfrac{1}{8} \mathrm{Tr}(\partial_\mu M \partial^\mu M^{-1}) - \tfrac{1}{4} \mathcal{F}^i_{\mu\nu} M^{-1}_{ij} \mathcal{F}^{\mu\nu j} \Big], \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} & \mathcal{F}^i_{\mu\nu}(\mathcal{A}) = \partial_\mu \mathcal{A}^i_\nu - \partial_\nu \mathcal{A}^i_\mu, \\ & H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} + \tfrac{1}{2} \mathcal{A}^i_{[\mu} \mathcal{F}^j_{\nu\rho]}(\mathcal{A}) L_{ij}, \\ & \mathcal{A}^i_\mu = \begin{pmatrix} A^a_\mu \\ B_{\mu a} \\ V^m_\mu \end{pmatrix}, \qquad L_{ij} = \begin{pmatrix} 0 & \mathbb{1}_d & 0 \\ \mathbb{1}_d & 0 & 0 \\ 0 & 0 & -\mathbb{1}_{16} \end{pmatrix}. \end{aligned} \tag{3.38}$$

The $d(d+16)$ moduli $G_{ab}, B_{ab}, \ell_a{}^m$ are combined into the symmetric $(2d+16) \times (2d+16)$ matrix M^{-1}, satisfying $M^{-1} L M^{-1} = L$, where L is the invariant metric on $O(d, d+16)$.

Different values of the moduli correspond to different radii of the torus and therefore to different compactifications. The moduli parametrise the $d(d+16)$-dimensional coset space $O(d, d+16)/(O(d) \times O(d+16))$ of different compactifications [119].

It is easy to see that (3.37) is invariant under general $O(d, d+16)$ transformations

$$\mathcal{A}'_\mu = \Omega \mathcal{A}_\mu, \qquad (M^{-1})' = \Omega M^{-1} \Omega^T, \qquad \Omega^T L \Omega = L. \tag{3.39}$$

This $O(d,d+16)$ is not a symmetry of the full string theory, since quantum corrections will break the group structure. An analysis at the level of the sigma model shows [119, 120] that the allowed $(2d+16)$ vector fields charges of the string states form a $(2d+16)$-dimensional, even self-dual lattice and the symmetry group of the full theory should leave this lattice invariant. The transformations that preserve this lattice form the discrete $O(d,d+16;\mathbb{Z})$-group, the sub-group of $O(d,d+16)$-transformations with integer parameters which is conjectured to be a symmetry of the full string theory. In fact this $O(d,d+16;\mathbb{Z})$ is the generalization of the T-duality transformations (3.31) and (3.4) and is usually called the T-duality group. It relates compactifications over different tori as equivalent ones. The moduli space of inequivalent toroidal compactifications is therefore given by the coset

$$\frac{O(d,d+16)}{O(d)\times O(d+16)}\Big/ O(d,d+16;\mathbb{Z}). \tag{3.40}$$

Note that this is the moduli space for both the $SO(32)$ as $E_8\times E_8$ theory. In fact the two theories correspond to two distinct points in this moduli space and can be continuously connected [73]. This means that they are two manifestations of one and the same Heterotic theory and can be mapped one into the other via T-duality.

A similar thing happens for the $N=2$ theories Type IIA and Type IIB: although the two theories look very different in ten dimensions, upon reduction over a circle the massless spectrum of the two theories precisely coincides: besides the NS-NS sector (3.28), they both have a scalar, a vector, a two-form and a three-form gauge field in their R-R sector[5]:

$$\begin{array}{ll}\text{Type IIA}: & \{\hat{A}^{(1)}_x,\quad \hat{A}^{(1)}_\mu,\quad \hat{C}_{\mu\nu x},\quad \hat{C}_{\mu\nu\rho}\ \}\\ \text{Type IIB}: & \{\ \hat{\ell},\quad \hat{B}^{(2)}_{x\mu},\quad \hat{B}^{(2)}_{\mu\nu},\quad \hat{D}^{+}_{\mu\nu\rho x}\}.\end{array} \tag{3.41}$$

Furthermore their low energy effective actions can be mapped on to one and the same Type II action in nine dimensions [26],

$$\begin{aligned} S = \tfrac{1}{2}\int d^9x\ \sqrt{|g|}\ \Big\{ & e^{-2\phi}\Big[-R\ +\ 4(\partial\phi)^2 -\ \tfrac{3}{4}(H^{(1)})^2 -\ (\partial\log k)^2 \\ & \qquad +\ \tfrac{1}{4}k^2F^2(A)\ +\ \tfrac{1}{4}k^{-2}F^2(B)\Big] \\ & +\tfrac{1}{4}\Big(F^{(1)}+\ell F^{(2)}\Big)^2 - \tfrac{1}{2}\ k^{-1}(\partial\ell)^2 + \tfrac{3}{4}\ k\ G^2 \\ & -34\ k^{-1}\Big(H^{(1)}+\ell H^{(2)}\Big)^2\Big\} \\ -\tfrac{1}{64}\int d^9x\ \varepsilon_{(10)}\Big(&\partial C\partial CB + \partial C\partial B^{(a)}\partial B^{(b)}\varepsilon^{ab} + 2\ \partial CA^{(a)}\partial B^{(a)}B \\ & -\partial CA^{(a)}A^{(b)}\partial BB\varepsilon^{ab}\Big), \end{aligned} \tag{3.42}$$

[5]The four-form gauge field $\hat{D}^{+}_{\mu\nu\rho\lambda}$ is not an independent field in nine dimensions, but is completely determined by the self-duality condition (2.35) and can therefore be ignored.

provided that one uses two different reduction schemes for each theory. For the Type IIA theory the relation between the action (2.33) and the above action is given by the reduction rules

$$\begin{aligned}
&\hat{g}_{\mu\nu} = g_{\mu\nu} - k^2 A_\mu A_\nu, && \hat{B}_{\mu\nu} = B^{(1)}_{\mu\nu} + A_{[\mu}B_{\nu]}\,,\\
&\hat{g}_{x\mu} = -k^2 A_\mu\,, && \hat{B}_{x\mu} = B_\mu,\\
&\hat{g}_{xx} = -k^2, && \hat{\phi}^A = \phi + \tfrac{1}{2}\log k,\\
&\hat{A}^{(1)}_x = \ell, && \hat{A}^{(1)}_\mu = A^{(1)}_\mu + \ell A_\mu,\\
&\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}} = C_{\mu\nu\rho}, && \hat{C}_{\mu\nu x} = \tfrac{2}{3}(B^{(2)}_{\mu\nu} - A^{(1)}_{[\mu}B_{\nu]})\ ,
\end{aligned} \tag{3.43}$$

while for the Type IIB theory the relation between the ten and the nine-dimensional fields is given by

$$\begin{aligned}
&\hat{G}_{\mu\nu} = g_{\mu\nu} - k^{-2}B_\mu B_\nu, && \hat{\mathcal{B}}^{(1)}_{\mu\nu} = B^{(1)}_{\mu\nu} - A_{[\mu}B_{\nu]},\\
&\hat{G}_{x\mu} = -k^{-2}B_\mu\,, && \hat{\mathcal{B}}^{(1)}_{x\mu} = A_\mu,\\
&\hat{G}_{xx} = -k^{-2}, && \hat{\phi}^B = \phi - \tfrac{1}{2}\log k,\\
&\hat{\mathcal{B}}^{(2)}_{\mu\nu} = B^{(2)}_{\mu\nu} + A^{(1)}_{[\mu}B_{\nu]}, && \hat{\mathcal{B}}^{(2)}_{x\mu} = A^{(1)}_\mu,\\
&\hat{D}_{\mu\nu\rho x} = \tfrac{3}{8}(C_{\mu\nu\rho} - A^{(a)}_{[\mu}B^{(a)}_{\nu\rho]} - \varepsilon^{ab}A^{(a)}_{[\mu}A^{(b)}_{\nu}B_{\rho]}), && \hat{\ell} = \ell.
\end{aligned} \tag{3.44}$$

The fact that Type IIA and Type IIB can be mapped on to the same Type II theory means that in ten dimensions they are different embeddings of one and the same theory which become equivalent after compactification on circles S^1_A and S^1_B, where the relation between the two compactification radii is given by:

$$\frac{R_A}{\sqrt{\alpha'}} = \sqrt{|\hat{g}_{xx}|} = k = \frac{1}{\sqrt{|\hat{G}_{xx}|}} = \frac{\sqrt{\alpha'}}{R_B}. \tag{3.45}$$

In other words the limits, $k \to \infty$ (Type IIA) and $k \to 0$ (Type IIB) are different limits in the moduli space of the Type II theory in nine dimensions. Furthermore, a careful analysis [57, 51] of the fermionic part of the action reveals a change in chirality of the fermions, which is necessary to relate the non-chiral Type IIA to the chiral Type IIB theory.

The relation between the fields of both theories in ten dimensions can be read off from (3.43) and (3.44):

$$\begin{aligned}
\hat{C}_{x\mu\nu} &= \tfrac{2}{3}\left[\hat{\mathcal{B}}^{(2)}_{\mu\nu} + 2\hat{\mathcal{B}}^{(2)}_{x[\mu}\hat{G}_{\nu]x}/\hat{G}_{xx}\right],\\
\hat{C}_{\mu\nu\rho} &= \tfrac{8}{3}\hat{D}_{x\mu\nu\rho} + \varepsilon^{ab}\hat{\mathcal{B}}^{(a)}_{x[\mu}\hat{\mathcal{B}}^{(b)}_{\nu\rho]} + \varepsilon^{ab}\hat{\mathcal{B}}^{(a)}_{x[\mu}\hat{\mathcal{B}}^{(b)}_{|x|\nu}\hat{G}_{\rho]x}/\hat{G}_{xx}\,,\\
\hat{g}_{\mu\nu} &= \hat{G}_{\mu\nu} - \left(\hat{G}_{x\mu}\hat{G}_{x\nu} - \hat{\mathcal{B}}^{(1)}_{x\mu}\hat{\mathcal{B}}^{(1)}_{x\nu}\right)/\hat{G}_{xx}\,,\\
\hat{B}^{(1)}_{\mu\nu} &= \hat{\mathcal{B}}^{(1)}_{\mu\nu} + 2\hat{\mathcal{B}}^{(1)}_{x[\mu}\hat{G}_{\nu]x}/\hat{G}_{xx}\,,
\end{aligned}$$

$$
\begin{array}{rclrcl}
\hat{g}_{x\mu} & = & \hat{\mathcal{B}}^{(1)}_{x\mu}/\hat{G}_{xx}\,, & \hat{B}^{(1)}_{x\mu} & = & \hat{G}_{x\mu}/\hat{G}_{xx} \\
\hat{A}^{(1)}_{\mu} & = & -\hat{\mathcal{B}}^{(2)}_{x\mu} + \hat{\ell}\hat{\mathcal{B}}^{(1)}_{x\mu}\,, & \hat{g}_{xx} & = & 1/\hat{G}_{xx}\,, \\
\hat{\phi}^{A} & = & \hat{\phi}^{B} - \frac{1}{2}\log\left(-\hat{G}_{xx}\right), & \hat{A}^{(1)}_{x} & = & \hat{\ell}\,.
\end{array}
\tag{3.46}
$$

These transformation rules look very similar to the T-duality rules (3.4), though this time the T-duality transformation is not a symmetry of the action, but a transformation that takes us from the Type IIB to the Type IIA action [26]. The inverse transformation from the Type IIA to the Type IIB action can easily be constructed in the same way.

We see that from the $O(1,1)$-symmetry group of the common sector (3.28), only the $SO^{\uparrow}(1,1) \times \mathbb{Z}_2^{(S)}$ survives as a symmetry of the action (3.42), while the $\mathbb{Z}_2^{(T)}$, which corresponds to (3.46), is a map from Type IIA to Type IIB and vice versa.

In a generalization to reduction over d dimensions, the T-duality group is $O(d,d;\mathbb{Z})$ and the moduli parametrise the coset $O(d,d)/(O(d) \times O(d))$. The moduli space of inequivalent compactifications is given by

$$
\frac{O(d,d)}{O(d) \times O(d)} \Bigg/ O(d,d;\mathbb{Z}). \tag{3.47}
$$

3.1.4 T-duality between Solutions

In the previous subsection we have seen that some of the string theories may be connected via T-duality, at least at the level of the string effective action. This implies that also T-duality transformations should exist between the solutions of these actions.

However, in the derivation of the T-duality rules (3.4) we intrinsically made use of the fact we were doing a duality transformation on a string-like solution: only on a two-dimensional world volume can a scalar X be dualized to another scalar $\tilde{X}$. From this procedure it is not clear how to generalize these rules to the extended objects we encountered in section 2.3.

Nevertheless there exists another, even more general way of deriving the T-duality rules, which in fact we already used, when we showed the T-duality between Type IIA and Type IIB theory: if we can map two actions (solutions) via different ways of dimensional reduction (one over a circle with radius R and the other over a circle with radius $1/R$) on to the same action (solution) one dimension lower, then we can say that the two actions (solutions) are connected via T-duality and the T-duality rules can be read off in the same way that we derived the Type II rules (3.46). In fact this procedure is more general, since not only do we find the transformation rules for all the participating fields (besides the NS-NS fields that enter in (3.4) also the rules for the R-R fields), but also this allows us to make T-duality transformations between objects of different spatial extension, while before we could in principle only go from string-like solutions to string-like solutions. The technique of performing T-duality via dimensional reduction will be studied more accurately in Chapter 6, where we will use it to prove the duality relations between the different world volume actions of solutions connected via T-duality.

Using the reduction rules (3.23) and (3.25), one easily sees that the reduction of the fundamental string solution (2.44)

$$F1 = \begin{cases} ds^2 = H^{-1}(dt^2 - dx_1^2) - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H \\ B_{01} = H^{-1} \end{cases} \tag{3.48}$$

over the world volume direction x^1 gives rise to a nine-dimensional point particle solution

$$m0 = \begin{cases} ds^2 = H^{-1}dt^2 - (dx_2^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{1}{2}} \\ k = H^{-\frac{1}{2}} \\ B_0 = -H^{-1} \\ B_{\mu\nu} = A_\mu = 0 \ , \end{cases} \tag{3.49}$$

while the reduction of the ten-dimensional gravitational wave (2.64)

$$\mathcal{W}_{10} \ : \ ds^2 = (2-H)dt^2 - Hdz^2 + 2(1-H)dtdz - (dx_2^2 + ... + dx_9^2) \tag{3.50}$$

over the propagation direction z of the wave gives

$$m\tilde{0} = \begin{cases} d\tilde{s}^2 = H^{-1}dt^2 - (dx_2^2 + ... + dx_9^2) \\ e^{-2\tilde{\phi}} = H^{\frac{1}{2}} \\ \tilde{A}_0 = -H^{-1} \\ \tilde{k} = H^{\frac{1}{2}} \\ \tilde{B}_{\mu\nu} = \tilde{B}_\mu = 0 \ . \end{cases} \tag{3.51}$$

We see that these two point particle solutions are actually the same if one identifies

$$\tilde{A}_\mu = B_\mu, \qquad \tilde{B}_\mu = A_\mu, \qquad \tilde{k} = 1/k. \tag{3.52}$$

Note that this is precisely the nine-dimensional T-duality transformation (3.31). Also direct application of the ten-dimensional rules (3.4) maps the $F1$ to the $\mathcal{W}_{10}$ and vice versa. Note that a T-duality transformation in a transverse direction leaves the string and the wave solution invariant.

The same procedure can be followed for the solitonic five-brane (2.54) and the Kaluza-Klein monopole (2.65): reduction of the $S5$ over a transverse direction gives a new five-brane solution in nine dimensions

$$mS5 = \begin{cases} ds^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_8^2) \\ e^{-2\phi} = H^{-\frac{1}{2}} \\ F_{mn}(B) = \varepsilon_{mnp}\ \partial_p H \\ k = H^{\frac{1}{2}} \\ B_{\mu\nu} = A_\mu = 0 \ , \end{cases} \tag{3.53}$$

while the reduction of the $\mathcal{KK}_{10}$ over the isometry direction z yields

$$m\tilde{S}5 = \begin{cases} d\tilde{s}^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_8^2) \\ e^{-2\tilde{\phi}} = H^{-\frac{1}{2}} \\ F_{mn}(\tilde{A}) = \varepsilon_{mnp}\ \partial_p H \\ \tilde{k} = H^{-\frac{1}{2}} \\ \tilde{B}_{\mu\nu} = \tilde{B}_\mu = 0\ . \end{cases} \tag{3.54}$$

Again the two solutions can be identified, using (3.52), which proves the T-duality between the $S5$ and the $\mathcal{KK}_{10}$. A T-duality transformation in a world volume direction leaves both solutions invariant.

It thus turns out that the solutions of the equations of motion of the common sector are related amongst each other via T-duality. This is not so strange, since we showed in Subsection 3.1.2 that the common sector (3.20) itself is invariant under T-duality. Let us now look at how the D-brane solutions, the solutions of Type IIA/B, transform under this duality.

T-duality is an important feature in the theory of D-branes: we already saw that a T-duality transformation on a freely moving open string changes the boundary conditions of the string and attaches it to a D-brane. But also the D-branes themselves are related [129]: applying T-duality on a string attached to a Dp-brane (so satisfying $(p+1)$ Neumann conditions and $(9-p)$ Dirichlet conditions) will change one of the Neumann conditions to a Dirichlet one or vice versa, so after the transformation the string will be attached to a $D(p\pm 1)$ brane. This should of course be visible at the level of the Dp-brane solutions (2.58) of the equations of motion [18].

Indeed, a straightforward application of the duality rules (3.4) on the D-brane solution $(p: 0, ..., 8)$

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(\text{R-R})}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m: p+1, ..., 9), \end{cases} \tag{3.55}$$

inverts the metric component of the direction in which the T-duality is performed and changes a world volume direction into a transverse one and back. Also the dilaton and gauge field dependence change in the right way to obtain a $D(p\pm 1)$-brane. The exact form of the transformation rules for the R-R fields can be found in [18]. The fact that Dp-branes with p even (odd) get mapped to p-odd (even) branes corresponds to the fact that T-duality is a map from Type IIA(B) to Type IIB(A) theory.

3.2 Strong/Weak Coupling Duality

Another type of duality symmetry which has been found in string theory is the S-duality or Strong/Weak coupling duality, so called because it relates the strong and weak coupling limits of theories to each other. The importance of S-duality is that it gives a way to go beyond perturbation theory and to obtain a good picture of what string theory is like at strong coupling.

At small values of the coupling constant g, perturbative calculations give a reasonably good understanding of the theory: the weak coupling limit of the theory has a number of electrically charged, elementary states which can be handled in perturbation theory and some magnetically charged, solitonic states, which are very massive and strongly coupled (cfr. $F1$ and $S5$ in section 2.3). For large values of g these perturbative techniques break down and reliable results are much more difficult to obtain.

The idea of S-duality now is that in the large coupling limit the situation might be reversed: it is conjectured by Montonen and Olive [117] that when $g \to \infty$, the elementary, weakly coupled states are the magnetically charged ones and the strongly coupled, massive, solitonic states are electrically charged.

In other words, the Olive-Montonen conjecture states that at strong coupling the theory can be reformulated in terms of new, dual fields and a new coupling constant, such that it is again a weakly coupled theory in this dual formulation. This symmetry is believed to be exact for theories that have $N = 4$ supersymmetry [165], and to hold for some special cases with $N = 1, 2$ supersymmetry as well [143].

The interchange of electric and magnetic charge is very much connected to the interchange of strong and weak coupling through the Dirac quantization rule: the electric and magnetic charges of a state are a measure of how strongly the state interacts with other states and have therefore the role of coupling constants. Since due to the Dirac quantization rule magnetic charge is inversely proportional to electric charge, an electric/magnetic duality is equivalent to a strong/weak coupling duality.

In this section we will review some examples in string theory where S-duality is found and applied to get new results. We will start by looking at the S-duality symmetry in the Heterotic string, compactified on a six-torus. Then we will study the strong coupling limits of the different string theories and make contact with eleven dimensional supergravity.

3.2.1 The Heterotic String in Four Dimensions

The dimensional reduction over a six-torus T^6 of the low energy effective action (2.36) of the Heterotic string gives $N = 4$ supergravity coupled to Yang-Mills theory in four dimensions. The bosonic part of the four-dimensional action contains a metric, a dilaton and an axion, 28 Abelian vector fields and 132 scalars (3.37):

$$\begin{aligned} S = \tfrac{1}{2} \int d^4x \, \sqrt{|g|} \, e^{-2\phi} \Big[& -R + 4(\partial\phi)^2 - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & + \tfrac{1}{8} \mathrm{Tr}(\partial_\mu M \partial^\mu M^{-1}) - \tfrac{1}{4} \mathcal{F}^i_{\mu\nu} M^{-1}_{ij} \mathcal{F}^{\mu\nu j} \Big], \end{aligned} \tag{3.56}$$

where $\mathcal{F}^i_{\mu\nu}, H_{\mu\nu\rho}$ and M^{-1}_{ij} are defined as in (3.38). As argued in Subsection 3.1.3, the vector and scalars transform under the $O(6, 22)$ group, which is a symmetry of the action.

There is yet another symmetry, which is a symmetry of the equations of motion, not of the action (3.56). This can be seen if we rewrite the above action by introducing a scalar field ψ and a rescaled metric $g^{\rm E}_{\mu\nu}$ via

$$H^{\mu\nu\rho} = -\frac{1}{\sqrt{|g|}}\, e^{2\phi}\varepsilon^{\mu\nu\rho\lambda}\,\partial_\lambda\psi\,, \tag{3.57}$$

$$g^{\rm E}_{\mu\nu} = e^{-2\phi}\, g_{\mu\nu}. \tag{3.58}$$

The scalar ψ is the Poincaré dual of the anti-symmetric tensor $B_{\mu\nu}$, as in (2.56). The new metric $g^{\rm E}_{\mu\nu}$ is called the Einstein metric since this is the canonical metric that appears in the Einstein-Hilbert action. The metric $g_{\mu\nu}$ we have been using until now is usually called the string metric.

In terms of these new fields, the action (3.56) can be rewritten as

$$S = \tfrac{1}{2}\int d^4x\,\sqrt{|g^{\rm E}|}\,\Big[-R_{\rm E} - \frac{1}{2(\lambda_2)^2}(\partial\lambda\,\partial\bar\lambda) + \tfrac{1}{8}{\rm Tr}(\partial M\partial M^{-1}) - \tfrac{1}{4}\lambda_2\mathcal{F}^i M^{-1}_{ij}\mathcal{F}^j - \tfrac{1}{16}\lambda_1\mathcal{F}^i L_{ij}{}^*\mathcal{F}^j\Big]\,, \tag{3.59}$$

where we combined the scalars ψ and $e^{-2\phi}$ in one complex scalar

$$\lambda = \lambda_1 + i\lambda_2 = \psi + ie^{-2\phi}\,, \tag{3.60}$$

and

$${}^*\mathcal{F}^{\mu\nu i} = \frac{1}{\sqrt{|g^{\rm E}|}}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{F}^i_{\rho\sigma}. \tag{3.61}$$

It can be shown that the equations of motion of the above action are invariant under the $SL(2,\mathbb{R})$ transformation [149, 146, 139]:

$$\lambda \to \frac{a\lambda+b}{c\lambda+d}\,, \qquad ad-bc=1$$
$$\mathcal{F}^i_{\mu\nu} \to (c\lambda_1+d)\mathcal{F}^i_{\mu\nu} + c\lambda_2(ML)^i{}_j{}^*\mathcal{F}^j_{\mu\nu}. \tag{3.62}$$

More precisely, the equations of motion of the vector fields and their Bianchi identities can be written schematically as

$$\begin{array}{ll} \text{Eqns of motion:} & D_\mu[\lambda(ML\mathcal{F}+i^*\mathcal{F}) - \bar\lambda(ML\mathcal{F}-i^*\mathcal{F})] = 0 \\ \text{Bianchi identity:} & D_\mu[(ML\mathcal{F}+i^*\mathcal{F}) - (ML\mathcal{F}-i^*\mathcal{F})] = 0 \end{array} \tag{3.63}$$

and it is straightforward to calculate that under an $SL(2;\mathbb{R})$ transformation (3.62) these two equations get mapped one into another. The equations of motion of all other fields are left invariant.

If we consider the particular $SL(2;\mathbb{R})$ transformation where the group parameters have the values $a=d=0$ and $b=-c=1$, we find the transformation

$$\lambda \to -\frac{1}{\lambda}\,, \qquad \mathcal{F}^i_{\mu\nu} \to -\lambda_1\mathcal{F}^i_{\mu\nu} - \lambda_2(ML)^i{}_j{}^*\mathcal{F}^j_{\mu\nu}. \tag{3.64}$$

For $\lambda_1 = 0$ this corresponds to a strong/weak coupling symmetry, where the electric fields of $\mathcal{F}^i_{\mu\nu}$ get interchanged with the magnetic ones of ${}^*\mathcal{F}^i_{\mu\nu}$, together with an inversion of the string coupling constant $e^{\phi} \to e^{-\phi}$.

So the low energy limit of the four-dimensional Heterotic string, compactified on T^6, has a symmetry which relates the strong coupling regime of the theory with its weak coupling regime. This is an example of the Olive-Montonen conjecture embedded in the context of string theory.

It is known that quantum effects break the $SL(2;\mathbb{R})$ symmetry to the discrete subgroup $SL(2;\mathbb{Z})$ [149, 145], the group of $SL(2;\mathbb{R})$ transformation with integer parameters and in analogy with the T-duality group $O(d, d+n;\mathbb{Z})$ of Narain, this $SL(2;\mathbb{Z})$ is conjectured to be a symmetry of the full string theory [66]. This is a very bold conjecture, since $SL(2;\mathbb{Z})$ is clearly a non-perturbative symmetry, as we can see already in (3.64).

However Sen was able to present indications that this is indeed the case [147] by showing that the charge spectrum of the theory and the BPS mass formula are invariant under $SL(2;\mathbb{Z})$ transformations. Furthermore he could identify elementary string excitations and known solitons as being $SL(2;\mathbb{Z})$ transforms of each other and therefore fitting in $SL(2;\mathbb{Z})$ multiplets.

In [142, 147] a low energy effective action was presented, which has a manifest $SL(2;\mathbb{R})$ symmetry with $O(6, 22)$ as a symmetry of the equations of motion. This action is obtained by dimensional reduction of the ten-dimensional "dual" (six-form) action [63], where the $SL(2;\mathbb{Z})$ appears as the T-duality group of the reduced dual action. This hints at another type of duality, namely the string/five-brane duality [152, 64], which states that in ten dimensions string theory is equivalent to a theory of five-branes, that couple naturally to the six-form potential, which is the Poincaré dual (2.56) of the axion. In this duality the $O(6, 22;\mathbb{Z})$ and the $SL(2;\mathbb{Z})$ appear on the same footing [142]: a symmetry of the action in one theory is a symmetry of the equations of motion in the other and vice versa. Their role gets interchanged and we can talk of a "duality of dualities".

3.2.2 Strong Coupling Limits of String Theories

Let us now look at the strong coupling limits of each of the string theories presented in section 2.2 and see whether S-duality can help us to find these limits.

From (2.32) and (2.36), we see that the Type I and the Heterotic $SO(32)$ string are quite similar: they have the same (bosonic) field content, the same gauge group $SO(32)$, and the same amount of supersymmetry. However the vector fields and the two-form anti-symmetric tensor are coupled in different ways to the dilaton in the two theories. The difference becomes more clear if we rescale the string metric to go to the Einstein frame:

$$g^{\mathrm{E}}_{\mu\nu} = e^{-\phi/2}\ g_{\mu\nu}. \tag{3.65}$$

The actions (2.32) and (2.36) in this frame yield

$$\begin{aligned} S_I &= \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\rm E}|}\,\Big[-R_{\rm E} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{4}\,e^{\phi}\,H^2 + \tfrac{1}{4}\,e^{\phi/2}F^2\Big], \\ S_{Het} &= \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\rm E}|}\,\Big[-R_{\rm E} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{3}{4}\,e^{-\phi}\,H^2 + \tfrac{1}{4}\,e^{-\phi/2}F^2\Big]. \end{aligned} \tag{3.66}$$

We see that the difference between the Type I and the Heterotic $SO(32)$ low energy effective action is the sign of the dilaton: the transformation $\phi \to -\phi$ takes one action into the other. This seems to suggest that the strong coupling limit of the Heterotic string is the Type I string and vice versa [163].

There is more evidence to support this idea: the fundamental string solution in the Heterotic theory, which couples to the axion, can be shown to coincide with the D-string of Type I theory, which couples to the R-R two-form. The same goes for the Heterotic $S5$ and the Type I $D5$ [49, 90]. Furthermore after compactification to nine dimensions the points in moduli space of the Heterotic string, for which an enhancement of the gauge symmetry occurs, correspond exactly to the points where the perturbative description of Type I theory breaks down [130].

Type IIB theory is manifestly $SL(2;\mathbb{R})$ invariant [92]. This can be seen best by rewriting the Type IIB action (2.34) in the Einstein-frame metric (3.65):

$$\begin{aligned} S_{IIB} = \tfrac{1}{2}\int d^{10}x\sqrt{|g^{\rm E}|}\,\Big[&-R_{\rm E} + \tfrac{1}{4}{\rm Tr}(\partial N\partial N^{-1}) - \tfrac{3}{4}\mathcal{H}^{(a)}N_{ab}\mathcal{H}^{(b)} \\ &-\tfrac{5}{6}F_{(5)}^2 - \tfrac{1}{96\sqrt{|g^{\rm E}|}}\varepsilon^{ab}\varepsilon^{(10)}D_{(4)}\mathcal{H}^{(a)}\mathcal{H}^{(b)}\Big], \end{aligned} \tag{3.67}$$

where N_{ab} is the $SL(2;\mathbb{R})$ matrix

$$N_{ab} = \frac{1}{\lambda_2}\begin{pmatrix} |\lambda|^2 & -\lambda_1 \\ \lambda_1 & 1 \end{pmatrix} \tag{3.68}$$

and $\lambda = \lambda_1 + \lambda_2 = \ell + ie^{-\phi}$.

In this form, the action is invariant under the $SL(2;\mathbb{R})$ transformation [26, 17]

$$\begin{aligned} \mathcal{H}^{(a)} &= \omega^a{}_b\;\mathcal{H}^{(b)}, \\ N'_{ab} &= \omega\;N\;\omega^{-1}, \\ \omega &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ ad &- bc = 1. \end{aligned} \tag{3.69}$$

The transformation rule for the complex scalar λ is as in (3.62)

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}\,, \tag{3.70}$$

which implies again an S-duality symmetry $\lambda \to \lambda^{-1}$. For $a = d = \ell = 0$ and $b = -c = 1$ we have

$$e^{\phi} \to e^{-\phi}, \qquad \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}, \qquad \mathcal{B}^{(2)} \to -\mathcal{B}^{(1)}. \tag{3.71}$$

Note however that this S-duality is different from the one given in (3.62), in the sense that here the transformation does not interchange the field strength $\mathcal{H}$ with its Poincaré dual $\mathcal{H} \rightarrow {}^*\mathcal{H}$, but mixes the NS-NS form with the R-R form and vice versa. The strong/weak coupling duality can be understood as the interchange of states from the perturbative (NS-NS) sector with states from the non-perturbative (R-R) sector. In particular, the Type IIB fundamental string gets mapped to the $D1$-brane and the solitonic five-brane to the $D5$.

The behaviour of the Type IIA theory at strong coupling is rather different from the way the Heterotic, Type I or Type IIB behave. Type IIA at strong coupling does not get related to a different, previously known string theory, but it turns out that there is an intimate relation with $D = 11$ supergravity and a not yet well formulated theory, called M-theory.

At the level of the low energy effective action, the connection between the Type IIA action (2.33) and the $D = 11$ supergravity action (2.37) is that the former is a simple dimensional reduction of the latter over a circle S^1 [93, 71, 40]. Using the conventions of [26], the reduction rules between ten and eleven dimensions are:

$$\begin{array}{ll}
\hat{g}_{xx} = -e^{\frac{4}{3}\phi}, & \hat{C}_{\mu\nu x} = \frac{2}{3}B_{\mu\nu}, \\
\hat{g}_{\mu x} = -e^{\frac{4}{3}\phi}A^{(1)}_\mu, & \hat{C}_{\mu\nu\rho} = C_{\mu\nu\rho}, \\
\hat{g}_{\mu\nu} = e^{-\frac{2}{3}\phi}g_{\mu\nu} - e^{\frac{4}{3}\phi}A^{(1)}_\mu A^{(1)}_\nu . &
\end{array} \tag{3.72}$$

We see that the ten-dimensional R-R vector $A^{(1)}_\mu$ is actually the Kaluza-Klein vector from the reduction and the Kaluza-Klein scalar, the measure of the compactification radius, is given by the ten-dimensional dilaton ϕ. But in ten dimensions the dilaton is associated with the coupling constant of the theory. We therefore see that the Type IIA (perturbation) theory is nothing other than an expansion around the zero-radius limit of eleven dimensions. On the other hand, in the strong coupling limit of Type IIA theory (thus for large values of the dilaton), an eleventh dimension unfolds, which previously in perturbation theory could not be seen [163]:

$$R_{11} \;=\; e^{\frac{2}{3}\phi} \;=\; g^{\frac{2}{3}}. \tag{3.73}$$

If the idea that Type IIA is really a dimensional reduction of something eleven dimensional holds also beyond the level of the low energy effective action, then this means that the (non-perturbative) spectrum of the Type IIA theory should contain all kinds of Kaluza-Klein modes coming from the wrapping of the eleven-dimensional solutions around the compact dimension. It was shown [163] that these modes would have masses inversely proportional to the coupling constant and therefore they could be identified with the Type IIA D-branes.

In fact the whole spectrum of Type IIA fundamental objects can be given an eleven-dimensional interpretation [156]: using the reduction rules (3.72), one sees that the Type IIA fundamental string can be understood as the eleven dimensional $M2$-brane wrapped around the compact dimension (double dimensional reduction), while the $D2$-brane is the direct reduction (reduction over a transverse direction) of the same $M2$-brane. The same goes for the $D4$ and the $S5$-brane in type IIA, which turn out to be

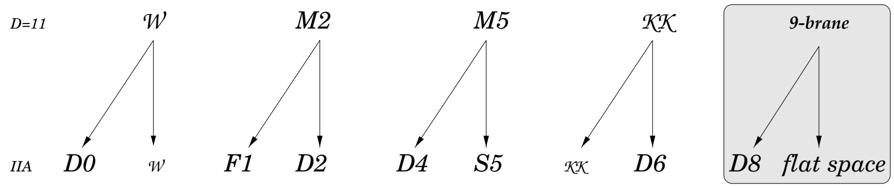

Figure 3.1: *The relation between $D = 10$ IIA and $D = 11$ solutions: Vertical lines imply direct dimensional reduction, diagonal lines double dimensional reduction. The shadowed area indicates the relationship between known ten-dimensional solutions and a conjectured 9-brane in $D = 11$.*

the double and direct reductions of the $M5$. The reduction of the eleven-dimensional gravitational wave $\mathcal{W}_{11}$ yields again a gravitational wave $\mathcal{W}_{10}$ in ten dimensions upon reduction over a transverse direction and a massive $D0$-brane if one reduces over the propagation direction of the wave. In the same way the Kaluza-Klein monopole $\mathcal{KK}_{11}$ in eleven dimensions gives rise to a ten dimensional monopole $\mathcal{KK}_{10}$ and a $D6$-brane upon reduction over a world-volume coordinate or the isometry direction z, respectively.

Only the interpretation of the Type IIA $D8$-brane is still mysterious: it is believed to be related to the equally mysterious eleven-dimensional 9-brane upon double reduction of the latter. Direct reduction of the 9-brane would give ten-dimensional Minkowski space[6]. In Figure 3.1 the relations between the various ten and eleven-dimensional solutions is summarized.

Also the strong coupling limit of Heterotic $E_8 \times E_8$ theory (2.36) is believed to be $D = 11$ supergravity [85], though this time the Heterotic theory turns out to be a compactification of $D = 11$ supergravity on a interval with length L, or equivalently on a circle sector $S^1/\mathbb{Z}_2$. The eleven-dimensional space-time consists of two nine-dimensional hyperplanes, separated by the interval of length L. On the two boundaries, gauge fields of E_8 live and in the limit $L \to 0$, a ten-dimensional theory with $E_8 \times E_8$ gauge symmetry is recovered. As in the case of Type IIA theory, the ten-dimensional coupling constant is related to the compact dimension by $L = g^{2/3}$.

If Type IIA supergravity (2.33) and Heterotic $E_8 \times E_8$ theory (2.36) are the weak coupling limits of $D = 11$ supergravity and the low energy limit of their respective string theories, we could ask the question: "What is the strong coupling limit of Type IIA (Het $E_8 \times E_8$) string theory?" or equivalently, "Of which theory is $D = 11$ supergravity the low energy limit?". This is conjectured to be M-theory, a non-perturbative, fundamental theory, which is believed to unify the various known string theories in one picture, although little more is known about it than that it has $D = 11$ supergravity as its low energy effective theory.

In the next section we will discuss the unifying picture and the role M-theory is believed to play.

[6]Comments on the conjectured 9-brane and the relation with the $D8$ have been given in [130, 22, 86, 125, 129, 61, 21].

3.3 General Picture

In the previous sections we have encountered two kinds of duality transformations: the T-duality which relates different string compactifications with each other and the S-duality that maps the strong coupling limit of a theory to the weak coupling limit of another (or, in the case of Type IIB and $D = 4,\ N = 4$ Heterotic theory the same) theory.

These duality symmetries shed new light on the problems that arose in string theory up to the beginning of the nineteen nineties:

- The wide variety of possible compactification manifolds and the different degenerate string vacua that follow from them. It is not clear which of all these vacua corresponds to our phenomenologically observable $D = 4$ world and why precisely this vacuum is the preferred one to be picked out.
- The difficulties to extend the known string theories beyond the perturbative level at which they are formulated. Little was known about a non-perturbative formulation or the basic dynamical principles that lie at the basis of string theory.
- The fact that five different versions exist of the theory which claims to be the "final" unification of gravity and all other fundamental interactions in Nature. It was believed (hoped) that sooner or later some of these theories would turn out to be inconsistent and/or equivalent to other ones, so that in the end one final version of string theory would be left over.

The surprising fact of the duality symmetries is that they were able to solve many (though certainly not all) of these problems, or at least to make some remarkable progress.

T-duality showed that different compactifications in string theory can be considered to be equivalent: upon dimensional reduction on a d dimensional torus T^d, for example, the T-duality group $O(d, d+n; \mathbb{Z})$ maps a given point in the moduli space (i.e., a given string vacuum) to a different point in moduli space with equivalent dynamics and equivalent physics as the first one. All vacua can thus be classified in T-duality classes and the moduli space of inequivalent compactifications is given by the coset

$$\mathcal{M} = \frac{O(d, d+n)}{O(d) \times O(d+n) \times O(d, d+n; \mathbb{Z})} \ . \tag{3.74}$$

Non-toroidal compactifications will give rise to other T-duality groups and other moduli spaces, but the main principles will be the same as in the easier case of toroidal compactification.

S-duality gives insight into the strong coupling regimes of theories: the S-duality group $SL(2; \mathbb{Z})$ is intrinsically a non-perturbative symmetry, since it acts non-trivially on the coupling constant of the theory. Under this symmetry the strong coupling regime of a theory gets mapped to the weak coupling regime of another theory and vice versa.

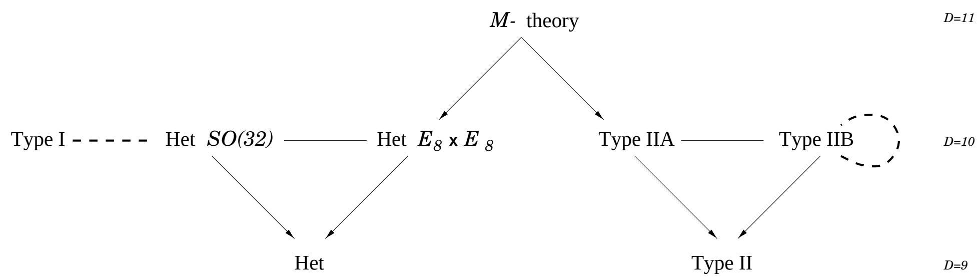

Figure 3.2: *Duality relations between the various string theories in ten dimensions and M-theory in eleven dimensions: the arrows indicate dimensional reduction from D to $D-1$ dimensions, the dotted lines represent an S-duality and a straight lines T-duality.*

Strong and weak coupling regimes therefore turn out to be different, but equivalent formulations of the same underlying theory. This yields a simple and elegant way to go beyond the level of perturbation theory: non-perturbative results in one theory can be computed in the other theory by simple perturbative calculations.

But perhaps the most striking issues of the concept of dualities is that the five, previously known string theories all turn out to be equivalent and in a rather surprising way interconnected via these dualities: T-duality relates the Type IIA and Type IIB theories in the presence of an isometry: one theory compactified on a circle of radius R gives exactly the same physics as the other theory compactified on a circle of radius $1/R$. The two theories are just different limits in moduli space of the same underlying theory. The same goes for Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$ theory. Furthermore the strong coupling limit of Heterotic $SO(32)$ coincides with the weak coupling limit of Type I strings (and vice versa), while the strong coupling of Type IIA and Heterotic $E_8 \times E_8$ both are conjectured to give a new theory, called M-theory, that has eleven dimensional supergravity as its low energy limit. Type IIB theory is believed to be S–self-dual, in the sense that its strong coupling limit is again the same Type IIB theory. A schematique picture of the relations between these theories can be seen in Figure 3.2.

The relations between the various string theories also imply connections between the solutions of their low energy effective actions: in Figure 3.1 we already showed how the Type IIA solutions were connected to the solutions of $D = 11$ supergravity, but also within ten dimensions the various solutions are related via dualities: T-duality connects all D-branes of Type IIA and Type IIB with each other, the wave with the fundamental string and the solitonic five-brane with the Kaluza-Klein monopole. S-duality connects the $F1$ with the $D1$ and the $S5$ with the $D5$ of theories that are each other's S-dual. Furthermore Poincaré duality (2.56) relates p-branes with a $(6-p)$-brane, i.e. the Dp-brane with the $D(6-p)$-brane and the $F1$ and the $S5$. These relations can be seen in Figure 3.3.

The fact that all these theories are related has led to the idea that they are not the fundamental theories we are looking for, but that all five string theories and $D = 11$ supergravity are different limits of one and the same underlying theory, called M-theory.

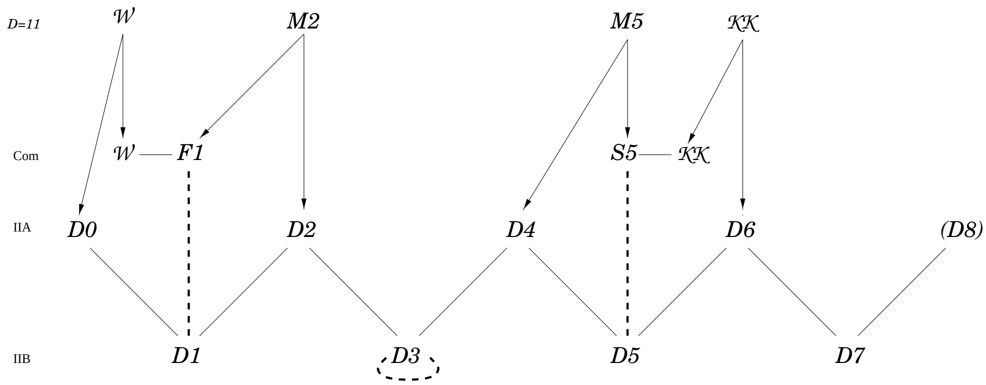

Figure 3.3: *Duality relations between the different solutions of string theory in ten dimensions and M-theory in eleven dimensions: the arrows indicate dimensional reduction from D to $D-1$ dimensions, the dotted lines represent an S-duality and a straight lines T-duality*

The different string theories and eleven dimensional supergravity can then be thought off as different perturbation expansions in different points of the moduli space of M-theory, characterized by the value of the coupling constant and the size of the compact dimensions. A full picture of what M-theory itself looks like is not yet known, though serious attempts are being made using techniques of Matrix-theory [12]. It is believed to have membrane and five-brane solutions, to be non-perturbative,... In fact, the idea of M-theory being the fundamental, underlying theory even has brought the name string theory in question, since strings no longer play a preferred role in this picture.

In the following Chapters we will apply the techniques of duality symmetries and duality transformations on the various aspects of string theory and supergravity: in Chapter 4 we will study the symmetries of the target space actions of string theories in more detail and find duality relations between them in dimensions lower than ten. In Chapter 5 we will look at the solutions of the supergravity actions and in particular the bound states they can form, and in Chapter 6 we study the duality transformations between the effective actions of the solutions, finding that also these are related in the same way as the solutions themselves.