Dualities of strings and branes

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Chapter 2

String Theory

In this chapter we will give a general introduction to various aspects of string theory. We review in section 2.1 the basic string dynamics, introducing the sigma models of the classical bosonic string and the superstring. In section 2.2 we will look at the low energy effective actions of the various types of superstring theories, and in section 2.3 attention will be paid to the different solutions that arise in these theories.

A general introduction to the different aspects of string theory can be found in [78, 95, 105, 114, 127], for a review on string solutions and p-branes we refer to [61, 151].

2.1 World Volume Theory

Let us consider a classical bosonic string, moving in a D-dimensional Minkowski space, represented by the coordinates X^μ and the flat metric $\eta_{\mu\nu} = \text{diag}[1, -1, -1, ..., -1]$.

While moving through space, the string sweeps out a two-dimensional surface Σ which we call the world sheet of the string, and which can be parametrised by the two-tuple $\sigma^i = (\tau, \sigma)$, where τ is a time-like parameter of the string and σ parametrises the length.

In analogy with the point particle, we can write down an action which describes the dynamics of the string, that is proportional to the surface of the world sheet:

$$S = -T \int_\Sigma d^2\sigma \sqrt{|\det(\partial_i X^\mu(\sigma^k) \partial_j X^\nu(\sigma^k) \eta_{\mu\nu})|}. \qquad (2.1)$$

The action (2.1) is called the Nambu-Goto action for the bosonic string.

The constant T is the string tension and has the dimension of $(\text{mass})^2$. Note that the X^μ are functions of τ and σ, and give the embedding of the string in the D-dimensional space-time. They are described by a two-dimensional field theory on the world sheet. They induce a metric g_{ij} on Σ via the expression $g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}$, so we see that (2.1) is indeed proportional to the surface of Σ.

There exists also another action which is, at least classically, equivalent to (2.1), but does not have the non-linearity caused by the square root:

$$S = -\frac{T}{2}\int d^2\sigma \sqrt{|\gamma|}\; \gamma^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}. \tag{2.2}$$

This action, called the Polyakov action [131] (though first introduced in [55, 35]), makes use of the metric γ_{ij} on the world sheet as an independent but non-dynamical variable. We will see later that it can be gauged away completely. Its equation of motion defines the energy-momentum tensor

$$T_{ij} = -\frac{1}{T}\frac{1}{\sqrt{|\gamma|}}\;\frac{\delta S}{\delta\gamma^{ij}} = \tfrac{1}{2}\partial_i X^\mu \partial_j X_\mu - \tfrac{1}{4}\gamma_{ij}\gamma^{kl}\partial_k X^\mu \partial_l X_\mu = 0. \tag{2.3}$$

Taking the determinant of the matrix equation $T_{ij} = 0$ and taking the square root, we find

$$\sqrt{|\det(\partial_i X^\mu \partial_j X_\mu)|} = \tfrac{1}{2}\sqrt{|\gamma|}\gamma^{kl}\partial_k X^\mu \partial_l X_\mu, \tag{2.4}$$

which gives the relation between the Nambu-Goto and the Polyakov action. Let us now discuss the symmetries of the Polyakov action. First of all, Eqn (2.2) is, just as (2.1), invariant under reparametrisations of the world sheet $(\tau,\sigma) \rightarrow (f_1(\tau,\sigma), f_2(\tau,\sigma))$, as it should be. Since parametrisations (τ,σ) of the world sheet do not have a physical meaning and are in principle arbitrary, no physical result can depend on them. Furthermore, Eqn (2.2) has an extra symmetry which is intrinsically related to the fact that we are dealing with strings, one dimensional objects: the Weyl-rescaling. Only on a two-dimensional world sheet, is $\sqrt{|\gamma|}\;\gamma^{ij}$ invariant under

$$\gamma_{ij} \rightarrow \Lambda(\sigma)\;\gamma_{ij}. \tag{2.5}$$

We can use these local symmetries to gauge away the world sheet metric and write (2.2) in a simpler form. Making use of the reparametrisation invariance, we can write locally $\gamma_{ij} = \Omega(\sigma)\;\eta_{ij}$, the flat world sheet metric times a conformal factor, and scale away this conformal factor via the Weyl invariance. We then end up with the action of the free bosonic string.

$$S = -\frac{T}{2}\int d^2\sigma\;\eta^{ij}\;\partial_i X^\mu \partial_j X_\mu, \tag{2.6}$$

for which we can easily calculate the equation of motion of X^μ. This turns out to be the two-dimensional free wave equation

$$(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0, \tag{2.7}$$

with the well-known solution

$$X^\mu(\tau,\sigma) = X_+^\mu(\tau+\sigma) + X_-^\mu(\tau-\sigma), \tag{2.8}$$

$X_+^\mu(\tau+\sigma)$ and $X_-^\mu(\tau-\sigma)$ being arbitrary functions for the left and right moving modes on the string.

We still have to impose boundary conditions on Eqn (2.7). At this point, we have to distinguish between two topologically different types of strings: the open string, which is a string with free endpoints, and the closed string, which has no ends[1]. For closed strings we impose periodic boundary conditions $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$. The Fourier expansion of Eqn. (2.8) for the closed string, satisfying these periodic boundary conditions, is then given by

$$\begin{aligned} X^\mu_-(\tau-\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2\pi T}\, p^\mu\ (\tau-\sigma) + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\ a^\mu_n\ e^{-in(\tau-\sigma)}, \\ X^\mu_+(\tau+\sigma) &= \tfrac{1}{2}x^\mu + \tfrac{1}{2\pi T}\, p^\mu\ (\tau+\sigma) + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\ \tilde{a}^\mu_n\ e^{-in(\tau+\sigma)}. \end{aligned} \tag{2.9}$$

x^μ and p^μ are the position and momentum of the center of mass and the a^μ_n and $\tilde{a}^\mu_n$ the Fourier coefficients of the oscillation modes of the string. Reality of X^μ requires that $(a^\mu_n)^\dagger = a^\mu_{-n}$ and $(\tilde{a}^\mu_n)^\dagger = \tilde{a}^\mu_{-n}$. The oscillation modes provide the string with extra dynamical degrees of freedom which distinguish the string from a point particle.

For the open string the boundary conditions come from the surface term in the variation of (2.6) between τ_i and τ_f (where we took $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$):

$$-T\int d\tau\ \delta X^\mu \partial_\sigma X_\mu\ \Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.10}$$

This condition can be satisfied in two ways. The most obvious one is the Neumann boundary condition

$$\text{Neumann}: \quad \partial_\sigma X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0, \tag{2.11}$$

because of its $SO(D-1,1)$ Poincaré invariance. Its physical meaning is that there is no momentum flow out of the string at both endpoints.

The Dirichlet boundary condition

$$\text{Dirichlet}: \quad \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = 0 \iff X^\mu\Big|_{\sigma=0}^{\sigma=\pi} = C^\mu, \tag{2.12}$$

with C^μ a constant vector, looks a bit strange at first sight, since it implies that the endpoints of the open string are fixed in space. However it will turn out that this is indeed a physically relevant boundary condition.

Suppose an open string satisfies Neumann boundary conditions in all but one direction, and Dirichlet boundary conditions in one direction X^1. This means that there is a $(D-2)$-dimensional hyperplane $X^1 = C$ in the Minkowski space to which the endpoints of the string are attached. This hyperplane is called a "Dirichlet-brane" or D-brane, because of the Dirichlet boundary conditions on the string. The interactions with open strings make the D-brane a dynamical object that, as we will see later, will play an important role in non-perturbative string theory.

[1] A string theory with open strings also contains closed strings, since the joining and splitting of open strings can lead to closed ones. The reverse is not true. For the open string we will choose the parametrisation $\sigma = [0,\pi]$, while for closed strings $\sigma = [0, 2\pi]$.

The Fourier expansion of the open string solution to (2.7), satisfying Neumann or Dirichlet conditions is given by

$$X_N^\mu(\tau,\sigma) \;=\; x^\mu + \tfrac{1}{2\pi T}\; p^\mu\; \tau + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\; a_n^\mu\; e^{-in\tau} \cos n\sigma, \tag{2.13}$$

$$X_D^\mu(\tau,\sigma) \;=\; x^\mu + \tfrac{1}{2\pi T}\; p^\mu\; \sigma + \tfrac{i}{2}\tfrac{1}{\sqrt{\pi T}} \sum_{n\neq 0} \tfrac{1}{n}\; \tilde{a}_n^\mu\; e^{-in\tau} \sin n\sigma, \tag{2.14}$$

where X_N^μ satisfies the Neumann conditions and X_D^μ the Dirichlet conditions.

At this point it would be logical to go beyond the purely classical analysis and try to quantize the bosonic string. Making use of techniques as conformal invariance and BRST-quantisation, one can compute the physical spectrum of this string theory and do string scattering amplitude calculations. However, these calculations go beyond the aim of this introduction. For a discussion of conformal symmetry and the BRST-formalism to compute string spectra, we refer to [74, 84]. Let us make some remarks though, which are worth mentioning because of their later relevance or because they complete the general picture.

First of all, a calculation of the spectrum of the bosonic string reveals that this string theory can only consistently be quantised in a 26-dimensional space-time. $D = 26$ is therefore called the critical dimension for the bosonic string and strings that live in other then the critical dimension are called non-critical strings. The fact that the dimensionality of the space-time is not a free parameter, but given by the theory is one of the nice surprises of string theory. Since string theory pretends to be the final, unifying theory, it also should be able to determine the precise value of quantities that entered as free parameters in other theories. It might be worrisome, however, that the number of dimensions, predicted by the bosonic string, differs so much from our "real", four-dimensional world. We will see that for other types of string theories, the number of dimensions will be lower, and that there exist techniques to make contact with the familiar $D = 4$ world.

A more worrying problem is the fact that in the spectrum of the bosonic string a tachyon appears, a particle with an imaginary mass, that moves faster than the speed of light. This will mess up the causality structure of the theory and is therefore an undesired feature. The problem is due to the fact that we are dealing with the bosonic string. Introducing the fermions in the right way will eliminate the tachyon from the spectrum.

Let us therefore make our string model a bit more realistic by also introducing fermions in the theory. We do this by allowing fermionic fields in the two-dimensional field theory on the world sheet, which will get the interpretation of "fermionic modes" of the string. As it turns out, these fermionic modes give rise to fermion fields in space-time. Let us consider the action:

$$S = -\frac{T}{2}\int d^2\sigma \Big[\partial_i X^\mu \partial^i X_\mu + i\bar{\psi}^\mu \rho^i \partial_i \psi_\mu\Big]. \tag{2.15}$$

Here, ψ^μ is a Majorana spinor on the world sheet that transforms as a vector under the $SO(D-1,1)$-Lorentz group of the Minkowski space. The ρ^i are the two-dimensional Dirac matrices.

The action (2.15) is invariant under a symmetry transformation that interchanges the bosonic and fermionic fields in the theory, the supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= i\bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= \rho^i \partial_i X^\mu \epsilon, \end{aligned} \tag{2.16}$$

where ϵ is a constant spinor. Because of the invariance under these supersymmetry transformations, the string model we are considering is called the superstring.

Note that we wrote the action (2.15) in the so-called conformal gauge, where the world sheet metric is already gauged away (compare with (2.6)). Therefore the fields in (2.15) have to obey certain constraints, such as the vanishing of the energy momentum tensor (as in (2.3)) and the conserved supersymmetry current. Though important in the general formulation of superstring theory, these constraints do not enter in the rest of our discussion, so we will not consider them.

The equations of motion and the dynamics of the bosonic part of (2.15) are the same as for the bosonic string. So let us concentrate on the fermionic part. Varying (2.15) with respect to $\bar{\psi}^\mu$ gives the equations of motion

$$\rho^i \partial_i \psi^\mu = 0, \tag{2.17}$$

and the boundary conditions

$$\bar{\psi}^\mu \rho^1 \delta\psi_\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0. \tag{2.18}$$

In order to solve these equations it is convenient to choose a basis in which the Dirac matrices ρ^i are real:

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.19}$$

and to decompose ψ^μ into two real valued components

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}. \tag{2.20}$$

ψ^μ_+ and ψ^μ_- are the left and right moving fermionic modes on the world sheet. The equations of motion can then be rewritten as:

$$\begin{aligned} (\partial_\tau - \partial_\sigma)\psi^\mu_+ &= 0 \ , \\ (\partial_\tau + \partial_\sigma)\psi^\mu_- &= 0 \ . \end{aligned} \tag{2.21}$$

Let us first look at the solution of these equations for the case of the open string. We see that the boundary condition

$$\psi^\mu_- \ \delta\psi_{-\mu} - \psi^\mu_+ \ \delta\psi_{+\mu} \Big|_0^\pi = 0 \tag{2.22}$$

is satisfied if $\psi^\mu_+ = \pm\psi^\mu_-$ and $\delta\psi^\mu_+ = \pm\delta\psi^\mu_-$ at $\sigma = 0, \pi$. Since an overall sign in the boundary conditions in irrelevant, we can set without loss of generality $\psi^\mu_+(0) = \psi^\mu_-(0)$. What remains to be fixed is the boundary condition at $\sigma = \pi$. There are two possibilities:

1. Ramond (R) boundary conditions: $\psi_+^\mu(\pi) = \psi_-^\mu(\pi)$. The solution of (2.21, 2.22) then yields

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_n b_n^\mu \, e^{-in(\tau\pm\sigma)}, \qquad n \in \mathbb{Z}. \tag{2.23}$$

2. Neveu-Schwarz (NS) boundary conditions: $\psi_+^\mu(\pi) = -\psi_-^\mu(\pi)$. Eqn (2.21, 2.22) is then solved by

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_r c_r^\mu \, e^{-ir(\tau\pm\sigma)}, \qquad r + \tfrac{1}{2} \in \mathbb{Z}. \tag{2.24}$$

String excitations coming from world sheet fields satisfying R-boundary conditions, will manifest themselves as fermionic fields from the space-time point of view, while excitations of fields satisfying the NS-boundary condition will appear as bosonic fields.

For closed strings we can impose either periodic or anti-periodic boundary conditions on each component ψ_+^μ and ψ_-^μ separately:

1. Periodic boundary conditions (R) $\psi_\pm^\mu(0) = \psi_\pm^\mu(\pi)$:

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_n d_n^\mu \, e^{-in(\tau\pm\sigma)}, \qquad n \in \mathbb{Z} \tag{2.25}$$

2. Anti-periodic boundary conditions (NS) $\psi_\pm^\mu(0) = -\psi_\pm^\mu(\pi)$:

$$\psi_\pm^\mu = \frac{1}{\sqrt{2\pi T}} \sum_r f_r^\mu \, e^{-ir(\tau\pm\sigma)}, \qquad r + \tfrac{1}{2} \in \mathbb{Z}. \tag{2.26}$$

So in total there are four possible combinations of left and right movers, each satisfying either one of the above boundary conditions: NS-NS, NS-R, R-NS and R-R. Excitations of the ψ^μ for which the different components satisfy NS-NS or R-R conditions, appear in the space-time as bosonic fields, whereas the ones that have NS-R or R-NS conditions manifest themselves as fermions.

The supersymmetry on the world sheet also induces supersymmetry transformations between the fermion and the boson fields in the space-time. For open strings this is $N = 1$ (so supersymmetry with one space-time supersymmetry generator) and for closed strings $N = 2$ supersymmetry (except for some special cases, as we will see in the next section).

The supersymmetry transformations (2.16) enable us to remove the tachyon we found in the spectrum of the bosonic string. Furthermore the number of space-time dimensions for the superstring is reduced to $D = 10$. From a phenomenological point of view, this is still a very high dimensional space, but as we will see in the section 3.1.2, there exist techniques to compactify over a number of dimensions to make contact with our $D = 4$ world.

Until now we have only considered strings moving in a Minkowski space, but in the end we are interested in strings moving in spaces with more general background fields, for example some curved space-time characterized by a metric $g_{\mu\nu}$. In section 2.2 we will give the most general covariant two-derivative action. These more general backgrounds complicate considerably the theory.

To perform string calculations one often uses perturbation expansions. One such is an expansion in α', a parameter with dimension of (length)^2, which is related to the string tension via $\alpha' = \frac{1}{2\pi T}$. It introduces a fundamental length scale $\sqrt{\alpha'}$, which is the string scale, where stringy effects become important. Most of the time, we will work in the so-called "zero-slope limit"[2] $\alpha' \to 0$, unless mentioned differently. This corresponds to the string tension $T \to \infty$, so the size of the string shrinks to zero and it can be approximated by a point particle.

A second perturbation expansion is the expansion in the string coupling constant (given by the expectation value of the dilaton field e^{ϕ}, which we will introduce in the next section). This expansion counts the number of loops in string scattering processes, and thus the genus of the world sheet Σ. In fact this is the string generalisation of the Feynman diagrams in quantum field theory.

2.2 Target Space Action

Let us now for a moment go back to the bosonic string and try to write down a string moving in a more general space-time than the Minkowski space we have considered in the previous section. The most general covariant action we can write down with two world sheet derivatives is the non-linear sigma model action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\Big\{\Big(\sqrt{|\gamma|}\gamma^{ij}g_{\mu\nu}(X) - \varepsilon^{ij}B_{\mu\nu}(X)\Big)\,\partial_i X^\mu \partial_j X^\nu - \alpha'\sqrt{|\gamma|}\;\phi(X)\;\mathcal{R}^{(2)}\Big\}. \tag{2.27}$$

This is the action of a string moving through a background characterized by a metric $g_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$, called the axion, and a scalar field ϕ called the dilaton. $\mathcal{R}^{(2)}$ is the Ricci scalar of the two-dimensional world sheet metric γ_{ij} and ε^{ij} the fully antisymmetric tensor in two dimensions.

For a constant mode of the dilaton ϕ_0, the last term in (2.27) is a topological term which is proportional to the Euler characteristic

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{|\gamma|}\;\mathcal{R}^{(2)} = 2 - 2g, \tag{2.28}$$

where g is the genus (number of holes) of the surface Σ. In other words, the last term in (2.27) counts the number of loops in the string scattering diagrams. A g-loop diagram in the (Euclidean) path integral gets weighted by a factor $(e^{\phi})^{2-2g}$ and the string coupling constant can be identified with the expectation value of e^{ϕ}.

[2]The name zero-slope limit comes from the fact that α' is the proportionality constant between the angular momentum J of a rotating string with energy E and the square of the energy, so the slope of the plot $J(E^2)$.

The difference between the actions (2.2) and (2.27) is that the latter does not turn into the action (2.6) in the conformal gauge $\gamma_{ij} = \Omega(\sigma)\ \eta_{ij}$, which makes it a non-trivial two-dimensional field theory and forces us to a perturbation expansion in α', if we want to do quantum calculations.

The first two terms of (2.27) are invariant under Weyl rescaling (2.5) at the classical level, but the demand that Weyl invariance should hold at the quantum level forces the β-functions of the fields to vanish. This is because the β-functions give the scale dependence of the couplings of the various fields, so Weyl invariance (and therefore scale invariance) implies $\beta = 0$.

The conditions for Weyl invariance to hold are then, at first non-trivial order in α' and at tree level in the loop expansion [38]:

$$\begin{aligned}
\beta^g_{\mu\nu} &= R_{\mu\nu} - 2\nabla_\mu\partial_\nu\phi + \tfrac{9}{4}H_{\mu\rho\lambda}H_\nu{}^{\rho\lambda} + O(\alpha') = 0, \\
\beta^B_{\mu\nu} &= \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu}\partial_\rho\phi\ + O(\alpha')\ = 0, \\
\tfrac{1}{\alpha'}\beta^\phi &= \tfrac{1}{\alpha'}(D-26) + 3\Big(R + 4(\partial\phi)^2 - 4\nabla^2\phi + \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\Big)\ + O(\alpha')\ = 0.
\end{aligned} \tag{2.29}$$

Here $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar for the background metric $g_{\mu\nu}$ and ∇_μ the covariant derivative on the space-time. $H_{\mu\nu\rho}$ is the rank three field strength tensor of $B_{\mu\nu}$:

$$H_{\mu\nu\rho}\ =\ \tfrac{1}{3}(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\ =\ \partial_{[\mu}B_{\nu\rho]}, \tag{2.30}$$

and is invariant under the gauge transformations $\delta B_{\mu\nu} = \partial_{[\mu}\Sigma_{\nu]}$.

The physical interpretation of these constraints is that they can be seen as the equations of motion of the action

$$S = \tfrac{1}{2}\int d^Dx\sqrt{|g|}e^{-2\phi}\left\{-\frac{(D-26)}{3\ \alpha'} - R + 4(\partial\phi)^2 - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\right\} + O(\alpha'), \tag{2.31}$$

This action is called the low-energy effective action or target space action, because it describes the massless modes of slowly varying X^μ's, as fields in the target space, the space in which the string moves. It can therefore be seen as a low energy approximation of string theory. For strings living in their critical dimensional space, the $(D-26)$-term in the third equation of (2.29) and in the action (2.31) drops out. From now on we will suppose that this is always the case.

The fact that the space-time metric $g_{\mu\nu}$ appears as a dynamical field, via the Ricci tensor, is the first indication we meet that gravity is contained in string theory. In fact (2.31) is the action for 26-dimensional gravity coupled to tensor and scalar fields. Higher orders in α' or string loop expansion will give rise to more terms in (2.31), and therefore predict corrections to general relativity. For a deeper analysis to higher order corrections, particularly for the Heterotic string, we refer to [155] and references therein.

The same procedure for computing the low-energy effective action can also be done for the supersymmetric string (2.15). It turns out that the low-energy description for the superstring is 10-dimensional supergravity, a locally supersymmetric quantum field theory. As already mentioned in the previous section, the $N=1$ world sheet supersymmetry induces $N=2$ space-time supersymmetry, i.e. a supersymmetry transformation with two space-time supersymmetry generators. The different ways these space-time supersymmetries can be introduced in the theory give rise to different types of superstring theories and different low energy effective actions:

- **Type I:** This is a theory of open strings. Closed strings however are also included in this theory because two interacting open strings can join and form a closed one. The boundary conditions for the open string eliminate one of the supersymmetries and break the original $N=2$ to $N=1$ supersymmetry. At the endpoints of the string charges can be attached, inducing a Yang-Mills gauge group in the theory. Consistency at the quantum level only allows $SO(32)$ as Yang-Mills group.

  The bosonic part of the low energy effective action of the Type I string is given by the bosonic part of $N=1, D=10$ supergravity [41, 19, 42]

$$S_I = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\,\Big[e^{-2\phi}\Big(-R+4(\partial\phi)^2\Big) - \tfrac{3}{4}H^2_{(3)} + \tfrac{1}{4}e^{-\phi}F^I_{(2)}F_{(2)I}\Big], \qquad (2.32)$$

  where we used the sub-index to indicate the rank of the field strength tensor. $F^I_{(2)}$ is the field strength of the vector field corresponding to the $SO(32)$-group and transforms under the adjoint representation of the group.

- **Type IIA:** This is a theory of closed strings only. The two space-time supersymmetries appear with opposite chirality, so the string itself is non-chiral and has $N=2$ supersymmetry. There is no freedom to introduce a Yang-Mills group, but in the bosonic field content we see, besides the metric, axion and dilaton of Type I, also a one-form $A_{(1)}$ and a three-from gauge field $C_{(3)}$ [93, 71, 40]:

$$\begin{aligned} S_{IIA} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\,\Big\{&e^{-2\phi}\Big[-R+4(\partial\phi)^2 - \tfrac{3}{4}H^2_{(3)}\Big] \\ &+\tfrac{1}{4}F^2_{(2)} + \tfrac{3}{4}G^2_{(4)} + \tfrac{1}{64}\frac{\varepsilon_{(10)}}{\sqrt{|g|}}\partial C_{(3)}\partial C_{(3)}B_{(2)}\Big\}, \end{aligned} \qquad (2.33)$$

  with $F_{(2)}$ and $G_{(4)}$ the field strengths of the gauge fields $A_{(1)}$ and $C_{(3)}$ respectively and $\varepsilon_{(10)}$ the ten-dimensional fully anti-symmetric tensor. The NS-NS fields, satisfying double anti-periodic boundary conditions (2.26) on their world sheet fermions, appear differently in the above action as the R-R fields, satisfying double periodic boundary conditions (2.25). The fields of the NS-NS sector have an explicit dilaton coupling via the factor $e^{-2\phi}$, while the R-R fields are not multiplied by this factor. The R-R fields appear in the action (2.33) as the bosonic fields necessary to extend $N=1$ to $N=2$ supersymmetry. Their different dilaton coupling means that they correspond to a higher order in string coupling constant. As we will see later, the solutions that couple to these R-R fields do not belong to the perturbative spectrum.

- **Type IIB:** This is also a theory for closed strings with $N=2$ supersymmetry, though this time with two supersymmetries that have the same chirality, so the theory is chiral. Again it is impossible to introduce Yang-Mills groups and besides the NS-NS fields that appear in the same way as in Type IIA, the R-R sector consists of a scalar ℓ, a two-form gauge field $B^{(2)}_{\mu\nu}$ and a self-dual four-form gauge field $D^{+}_{\nu\mu\rho\lambda}$. Due to the self-duality condition of the four-form, it is impossible to write down a covariant low energy effective action for this theory [3]. The field equations of Type IIB supergravity can be found in [138]. In [17] an action is given in which the self-duality condition is not used, but is put in by hand as an extra equation of motion for the four-form:

$$\begin{aligned} S^{NSD}_{IIB} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\ \Big\{ e^{-2\phi}\Big[-R+4(\partial\phi)^2 - \tfrac{3}{4}(\mathcal{H}^{(1)})^2\Big] \\ -\tfrac{1}{2}(\partial\ell)^2 - \tfrac{3}{4}\ (\mathcal{H}^{(2)}-\ell\mathcal{H}^{(1)})^2 - \tfrac{5}{6}F^2_{(5)}(D) \\ -\frac{1}{96\sqrt{|g|}}\varepsilon^{ab}\varepsilon^{(10)}D_{(4)}\mathcal{H}^{(a)}\mathcal{H}^{(b)}\Big\}, \end{aligned} \tag{2.34}$$

$$F(D^{+})_{\mu_1\ldots\mu_5} = \frac{1}{5!\sqrt{|g|}}\ \varepsilon_{\mu_1\ldots\mu_{10}}F(D^{+})^{\mu_6\ldots\mu_{10}}. \tag{2.35}$$

$F_{\mu\nu\rho\lambda\sigma}$ and $\mathcal{H}^{(2)}_{\mu\nu\rho}$ are the field strengths of $D^{+}_{\mu\nu\rho\lambda}$ and $B^{(2)}_{\mu\nu}$.

- **Heterotic string:** This string theory makes use of the fact that for closed strings the left and the right moving sectors are independent. The left moving sector can be taken to be the left moving modes of the purely bosonic string, while for the right moving sector we take the modes from the superstring [79]. Since only one sector is supersymmetric, the Heterotic string has $N=1$ supersymmetry. This is however enough already to remove the tachyon from the bosonic spectrum. A Yang-Mills gauge group arises from the compactification of the bosonic sector on a 16-dimensional compact space, in order for the 26-dimensional bosonic string to match up with the superstring, living in 10 dimensions. Again quantum consistency restricts the gauge group to $SO(32)$ or $E_8\times E_8$.

  The bosonic part of the low energy effective action is given by

$$S_{Het} = \tfrac{1}{2}\int d^{10}x\sqrt{|g|}\ e^{-2\phi}\Big[-R+4(\partial\phi)^2-\tfrac{3}{4}H^2_{(3)}+\tfrac{1}{4}F^{I}_{(2)}F_{(2)I}\Big]. \tag{2.36}$$

Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8\times E_8$ are the only five consistent superstring theories in ten dimensions. Note that the metric, the dilaton and the axion appear in the same way in all string theories, except in Type I. We will therefore refer to this part of the action as the *common sector*.

Although the critical dimension for superstrings to live in is $D=10$, there does exist a supergravity theory in eleven dimensions. This has always been a mysterious subtlety, since on the one hand there seems to be an intimate relation between superstrings and supergravity theories, yet on the other hand this $D=11$ supergravity does not have a string theory counterpart of which it is the low energy effective action. We do mention

[3] However, see also [52]

it here though, because of the importance it has in a unifying description of the above string theories, as we will see in the next chapter.

- $D = 11$ **Supergravity:** Eleven dimensions is the highest number of dimensions for a supergravity theory to live in[4]. $D = 11$ supergravity turns out to be a unique theory with $N = 1$ supersymmetry. In its bosonic sector it has a field content consisting of a metric and a three-form gauge field $C_{\mu\nu\rho}$ and the action can be written as [47]

$$S_{D=11} = \tfrac{1}{2}\int d^{11}x\sqrt{|g|}\left\{-R + \tfrac{3}{4}G^2(C) + \tfrac{1}{384}\frac{1}{\sqrt{|g|}}\,\epsilon^{(11)}C\partial C\partial C\right\}. \qquad (2.37)$$

In Chapter 3 and Chapter 4 we will investigate the relations between these different supergravity actions and the symmetries they have. But let us first take a look at the solutions in string theory coming from these actions.

2.3 Solutions

Before we study in detail the solutions that appear in string theory, let us first focus on a special feature that occurs for field theories that have extended supersymmetry. We will see that then there exist states with special properties, namely states whose mass is related to their charge. The importance of these states is that they do not get any quantum corrections, so the semi-classical result is already exact.

The supersymmetry generators Q^I form an algebra which is typically of the form $\{Q, Q\} = \gamma^\mu P_\mu$, but for theories with more then two generators (so $I : 1, ..., N \geq 2$), in the presence of a soliton solution, a central charge term Z^{IJ} is present besides the usual momentum term P_μ,:

$$\{Q^I_\alpha, Q^J_\beta\} = \gamma^\mu_{\alpha\beta}P_\mu\delta^{IJ} + Z^{IJ}_{\alpha\beta} \qquad (2.38)$$

The central charge term arises as a boundary term in the supersymmetry algebra and has a non-zero value of solutions with non-trivial topological charges (solitons). It can therefore be thought of as the electric or magnetic charge of the soliton solution.

The presence of the central charge puts a bound on the mass of the particles. Because of the positivity of the supersymmetry algebra, the expectation value of (2.38) becomes (schematically)

$$\langle\psi|\ \{Q, Q\}\ |\psi\rangle \ = \ \langle\psi|\ H\ |\psi\rangle + \langle\psi|\ Z\ |\psi\rangle \ \geq \ 0, \qquad (2.39)$$

with H the Hamiltonian of the system. The first term on the right-hand side of (2.39) is then the energy (or the mass) of the state $|\psi\rangle$, and the second term its charge. So (2.39) actually states that the mass of a particle is bounded from below by its charge:

$$M \geq |Z|. \qquad (2.40)$$

[4]For supergravity theories in dimensions higher then eleven, fields with spin greater then two appear [118], and it is not clear how to deal with these higher spin fields in an adequate way.

This inequality is called the Bogomol'nyi bound or BPS-bound. It was first derived in the context of 't Hooft-Polyakov monopoles by Bogomol'nyi [33] and Prasad and Sommerfield [132], and later generalized to supersymmetric theories [165].

There exist particular states that saturate the above inequality (2.39), i.e. for states that have the minimal possible mass, the above inequality turns into an equality. This happens if a state $|\psi_0\rangle$ is annihilated by some of the supersymmetry generators, $Q^{I_0}|\psi_0\rangle = 0$. The mass of such a state is completely determined by its charge:

$$M = |Z|. \tag{2.41}$$

States that saturate the BPS-bound are called BPS-states. A special feature of these states, besides their mass formula, is that they form representations of the supersymmetry algebra which are shorter (lower-dimensional) than the usual representations. This can be understood from the fact that since they get annihilated by some of the generators, fewer different states appear in each multiplet. But this also implies that they are protected by supersymmetry from quantum corrections [165]: any quantum correction (perturbative or non-perturbative) would break up the mass-charge relation (2.41) and break the multiplet structure of the BPS-states. But since states always appear in multiplets and quantum corrections cannot change a short multiplet in a long (normal) one, BPS-states have to stay in their short multiplet representation and hence do not receive quantum corrections. Their relations and properties even hold if we let the coupling constant grow strong and perturbation theory no longer holds. Therefore BPS-states will turn out to be a very important tool to investigate the behaviour of theories at strong coupling (see Chapter 3).

Let us now have a look at solutions of the equations of motion of the actions (2.32) - (2.36). Amongst the various solutions of supergravity theories, there exists the class of spatially extended objects, called p-branes, where p refers to the dimensionality of the object ($p = 0$ would be a particle, $p = 1$ a string, $p = 2$ a membrane, ...). These extended objects appear because of the fact that in string theory the central charge of the supersymmetry algebra is in general a $(p+1)$-form antisymmetric tensor gauge field $Z^{IJ}_{\mu_1 \cdots \mu_{p+1}}$, rather then a Lorentz-scalar and the BPS-state carrying the $(p+1)$-form charge is typically a p-brane or, as we will see, a $(D-p-4)$-brane. For a detailed analysis of what kind of extended solutions correspond to each central extension of the supersymmetry algebra, we refer to [91].

We will discuss in the rest of this section some specific, "elementary" solutions that can be interpreted as the "fundamental" objects of string theory and supergravity. The fact that they can be interpreted as a single (fundamental) object is because they are all characterised by a single harmonic function $H(x)$, which determines their position in the target space. From now on we will restrict ourselves to the bosonic part only of the theories. In a first approach we will look at the solutions of the equations of motions of the common sector, since these will later reappear in the various theories. In a second step we will concentrate on solutions that occur in specific theories. For the general p-brane solution of the supergravity action, as a function of the spatial extension of the brane, the dimension of the space-time and the dilaton coupling of the gauge field, we refer to [16] and the references therein.

The variation of the action

$$S = \tfrac{1}{2} \int d^{10}x \sqrt{|g|}\; e^{-2\phi} \Big[-R + 4(\partial\phi)^2 - \tfrac{3}{4} H^2 \Big] \tag{2.42}$$

with respect to the different fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ gives

$$\begin{aligned} [g_{\mu\nu}] \quad &: R_{\mu\nu} - 2\nabla_\mu \partial_\nu \phi + \tfrac{9}{4}\; H_{\mu\rho\lambda} H_\nu{}^{\rho\lambda} = 0, \\ [\phi] \quad &: R - 4\nabla_\mu \partial^\mu \phi + 4(\partial\phi)^2 + \tfrac{3}{4} H^2 = 0, \\ [B_{\mu\nu}] \quad &: \nabla_\rho (e^{-2\phi} H^{\rho\mu\nu}) = 0. \end{aligned} \tag{2.43}$$

Since the action (2.42) is derived as a low energy effective action of a string moving in a curved space-time, it is not unreasonable to look for a string-like solution to Eqns (2.43), i.e. a solution that has an extension in one spatial and one time direction. Therefore it must have a two-dimensional Poincaré invariance times an eight-dimensional rotational symmetry: $P_2 \times SO(8)$. Such a solution, satisfying Eqns (2.43) is given in [50][5]:

$$F1 = \begin{cases} ds^2 = H^{-1}(dt^2 - dx_1^2) - (dx_2^2 + \ldots + dx_9^2) \\ e^{-2\phi} = H \\ B_{01} = H^{-1} \end{cases} \tag{2.44}$$

The function H is a harmonic function of the coordinates $(x_2, \ldots, x_9)$:

$$H = 1 + \frac{c}{r^6}, \qquad\qquad r = \sqrt{x_2^2 + \ldots + x_9^2}. \tag{2.45}$$

In particular x_1 is an isometry direction and we can indeed interpret (2.44) as a string (a one-dimensional extended object) oriented in this x_1-direction. The solution (2.44) is generally referred to as the *fundamental string* ($F1$). The sub-space spanned by the coordinates $(x_2, \ldots, x_9)$ is called the transverse space of the string and the directions (t, x_1) the world volume directions.

A closer look at the solution (2.44) and the harmonic function $H = 1 + \frac{c}{r^6}$ reveals that the $F1$ is singular for $r \to 0$. Of course one always has to be very careful with singularities in particular coordinate systems, since they can be just an artifact of the chosen coordinates. But an analysis, done in [151], reveals that the fundamental string does indeed have a (time-like) singularity[6], which invites us to put a "material" string at the singularity by adding a delta-function source term to the supergravity action (2.42). Such a source term we already encountered, namely the non-linear sigma model (2.27), which describes the dynamics of the string. So we can say that the fundamental string solution (2.44) is a solution of the equations of motion of the combined "supergravity-matter" system

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{|g|}\; e^{-2\phi} \Big[-R + 4(\partial\phi)^2 - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \Big]$$

[5] A detailed derivation of this solution and the following ones and their supersymmetry can be found in [61].

[6] Though not in the coordinates given above. In order to see the singularity, one has to use an analytic extension of these coordinates. For a detailed analysis of the space-time structure of various p-branes and their Penrose diagrams, we refer to [151].

$$
\begin{aligned}
-\frac{T}{2}\int d^2\sigma\sqrt{|\gamma|}\ \gamma^{ij}g_{\mu\nu}(X)\ \partial_i X^\mu \partial_j X^\nu \\
+\ \frac{T}{2}\int d^2\sigma\ \varepsilon^{ij}B_{\mu\nu}(X)\ \partial_i X^\mu \partial_j X^\nu .
\end{aligned}
\tag{2.46}
$$

We can choose the parametrisation of the string source to be $(X^0, X^1, X^m) = (\tau, \sigma, \vec{0})$ and $\gamma_{ij} = \eta_{ij}$, so that all equations of motion reduce to

$$
\partial_n \partial_n H(x^m) = \ \kappa^2 T\ \delta(x^m). \tag{2.47}
$$

This gives us the relation between the constant c in the harmonic function $H(x^m)$, the string tension T and the coupling constant of general relativity κ^2:

$$
c = \frac{\kappa^2\ T}{3\ \Omega_7}, \tag{2.48}
$$

where Ω_7 is the volume of the unit 7-sphere around the string.

Although (2.44) is a purely bosonic configuration, it still preserves half of the supersymmetry of the theory. This can happen if not only the fermionic fields, but also their variations under supersymmetry transformations vanish for some Killing spinor ϵ. For the $N = 1$ case we have for the dilatino λ and the gravitino ψ_μ:

$$
\begin{aligned}
\delta\psi_\mu &= D_\mu \epsilon + \tfrac{3}{8}\ H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\ \ \epsilon \ = \ 0, \\
\delta\lambda &= \gamma^\mu \partial_\mu \phi\ \epsilon + \tfrac{1}{4}\ H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\ \ \epsilon \ = \ 0.
\end{aligned}
\tag{2.49}
$$

In particular, for the $F1$ this gives a condition for ϵ:

$$
(1 + \gamma^0\gamma^1)\ \epsilon = 0. \tag{2.50}
$$

This condition defines in fact a projection operator on ϵ that breaks half of the supersymmetry and preserves the other half. This partial breaking of supersymmetry is due to the fact the the $F1$ is a BPS-state. This can be shown, comparing the mass per unit length, defined as the integral over the (00)-component of the energy-momentum tensor,

$$
M = \int_{V_8} T^{00}\ d^8x \ = \ 2\kappa^2 T \tag{2.51}
$$

to the electric charge conserved via the equations of motion of the two-form gauge field $B_{\mu\nu}$:

$$
e \ = \ \int_{V_8} \partial_m H^{01m} d^8x \ = \ \int_{S^7} H^{01i} dS_i \ = \ 2\kappa^2 T\ . \tag{2.52}
$$

There is also another way that the gauge field $B_{\mu\nu}$ can carry a conserved charge, but this time the charge is topologically conserved, not dynamically via the equations of motion.

$$
q = \int_{S_3} \varepsilon^{mnp} H_{mnp}\ d^3x. \tag{2.53}
$$

While (2.52) is the generalisation to higher dimensions and higher forms of the electric charge in Maxwell theory, (2.53) would correspond to the generalisation of the magnetic charge as it occurs in the Dirac monopole, a solitonic object in the context of electromagnetism. So also in the context of string theory, we expect the object that carries the magnetic charge as given in (2.53) to correspond to a solitonic object.

Indeed, a solution of the Eqns (2.43) carrying magnetic charge is given by [39, 63]

$$S5 = \begin{cases} ds^2 = dt^2 - dx_1^2 - ... - dx_5^2 - H(dx_6^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{-1} \\ H_{mnp} = \varepsilon_{mnpr}\partial_r H \qquad\qquad (m,n,p,r : 6, ..., 9). \end{cases} \tag{2.54}$$

The harmonic function H depends this time on the coordinates $x_m = (x_6, ..., x_9)$, so we can interpret the solution as an object that has spatial extensions in the $(x_1, ..., x_5)$-directions, i.e. it has five plus one world volume directions and four transversal ones. We therefore refer to solution (2.54) as the *solitonic five-brane* ($S5$).

One can show [151] that there exist coordinate frames in which the $S5$ is completely singularity-free, so no source term is needed. The $S5$ is really a solitonic object in the sense that it corresponds to a topological defect with a large mass per unit volume, rather then with an elementary excitation of the vacuum. In fact one can show that the $S5$ is a BPS-state, so it conserves half of the supersymmetry and the Bogomol'nyi bound (2.41) between the mass and the magnetic charge is saturated.

Although the $S5$ is non-singular and a source term is not needed, we can still write down an effective action which describes the dynamics of the five-brane. Just as for the $F1$, the effective action of the $S5$ consists of two parts: a kinetic term, written in the form of a Born-Infeld (BI) term, which induces a metric on the five-brane, and a Wess-Zumino (WZ) term which gives the coupling to the gauge field. For the $N = 1$ five-brane this is:

$$\begin{aligned} S &= -\tfrac{T}{2}\int d^6\sigma\ e^{-2\phi}\sqrt{|\det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu})|} \\ &\quad + \tfrac{T}{6!}\int d^6\sigma\ \varepsilon^{i_1 \dots i_6}\ \partial_{i_1} X^{\mu_1} ...\ \partial_{i_6} X^{\mu_6} C_{\mu_1 \dots \mu_6}. \end{aligned} \tag{2.55}$$

$C_{\mu_1 \dots \mu_6}$ is the dual (magnetic) potential of $B_{\mu\nu}$. More generally, every $(p+1)$-form potential can equivalently be written as a $(D-p-3)$-form, since their field strength tensors are related via Poincaré duality

$$F_{\mu_1 \dots \mu_{(p+2)}} = \frac{1}{(D-p-2)!}\frac{1}{\sqrt{|g|}}\ \varepsilon_{\mu_1 \dots \mu_{(p+2)} \mu_{(p+3)} \dots \mu_D} F^{\mu_{(p+3)} \dots \mu_D}. \tag{2.56}$$

The factor $e^{-2\phi}$ in the kinetic term of (2.55) states that we are dealing with a solitonic object, whose mass is inversely proportional to the square of the coupling constant:

$$M_{S5} \sim \frac{1}{g^2}. \tag{2.57}$$

This means that for weak coupling, so in the perturbative regime, the five-brane becomes very massive.

Let us now look at the solutions of the Type IIA/B theories (2.33) - (2.34). Again we encounter the fundamental string and the solitonic five-brane, because the common sector is contained in both Type II strings. However, due to the presence of the R-R gauge fields, there exists a entirely new class of solutions that are charged with respect to these fields: the so-called Dirichlet-branes or D-branes [128].

Dp-branes ($0 \leq p \leq 8$) arise as hyperplanes in space-time to which the endpoints of open fundamental strings can be attached. Such a string ending on a Dp-brane satisfies Dirichlet boundary conditions in $(9-p)$ directions, constraining it to live on the world volume of the D-brane [129]. The strings attached to the D-brane describe fluctuations on the surface of the brane and make the D-branes dynamical objects, rather then static hypersurfaces. The strings can interact with each other or with strings approaching the brane and then scatter off closed strings [82]. The D-branes appear as solutions of the equations of motion of both Type II theories in the form

$$Dp = \begin{cases} ds^2 = H^{-\frac{1}{2}}(dt^2 - dx_1^2 - ... - dx_p^2) - H^{\frac{1}{2}}(dx_{p+1}^2 + ... + dx_9^2) \\ e^{-2\phi} = H^{\frac{p-3}{2}} \\ F^{(R-R)}_{012...pm} = \partial_m H^{-1} \qquad\qquad (m : p+1, ..., 9). \end{cases} \tag{2.58}$$

Again H is a harmonic function that depends on the transverse coordinates $x_m = (x_{p+1}, ..., x_9)$. $F^{(R-R)}_{012...pm}$ is the field strength of the R-R p-form gauge field that carries the R-R charge of the brane. Note that for $p \geq 3$ we have used the equivalent expression for the field strength, in terms of the magnetic (dual) potential (2.56).

Dp-branes with even p ($D0, D2, D4, D6$) couple to odd-form gauge fields and therefore occur in Type IIA theory, while p-odd branes ($D1, D3, D5, D7$), coupling to even-form gauge-fields, occur in Type IIB.

From (2.56) we see that the Dp-branes with $p < 3$ carry an electric charge, and the Dp-branes with $p > 4$ a magnetic charge. The $D3$-brane is dyonic, i.e. it has both electric and magnetic charge, due to the self-duality condition of the $D^+_{\mu\nu\rho\lambda}$ in Type IIB. These charges can be calculated in the same way as for the $F1$ and $S5$ in (2.52)-(2.53). Again the Bogomol'nyi bound is saturated

$$M_{Dp} \sim \frac{1}{g} \sim Q^{\text{R-R}}. \tag{2.59}$$

The inverse coupling constant in the mass formula indicates that the D-branes also belong to the non-perturbative spectrum, though their solitonic character is not as strong as for the $S5$.

The dynamics of the D-brane are described by a sigma model type of action [109, 68], which also plays the role of source term for the equations of motion. The BI-term describes the coupling of the NS-NS fields with a world volume vector V_i and the WZ-term gives the coupling to the R-R gauge fields [68, 77]:

$$\begin{aligned} S &= -\tfrac{T}{2} \int d^{p+1}\sigma \; e^{-\phi} \sqrt{|det(g_{ij} + \mathcal{F}_{ij})|} \\ &\quad + \tfrac{T}{(p+1)!} \int d^{p+1}\sigma \; \varepsilon^{(p+1)} \Big[C_{(p+1)} + C_{(p-1)}\mathcal{F} + C_{(p-3)}\mathcal{F}^2 + ... \Big] \end{aligned} \tag{2.60}$$

where g_{ij} is the pull-back of the metric on the world volume and $\mathcal{F}_{ij}$ the field strength of the vector field V_i:

$$
\begin{aligned}
g_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}, \\
\mathcal{F}_{ij} &= \partial_i V_j - \partial_j V_i - \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}.
\end{aligned} \tag{2.61}
$$

The $C_{(p+1)}$ are the different $(p+1)$-form R-R fields in a uniform notation. The interpretation of the world volume vector V_i is that of a $U(1)$-potential of a charged particle on the world volume of the D-brane. Charge conservation of the NS-NS two-form at the end of an open string ending on a D-brane is only maintained if there is an electric flux on the world volume coming out of the endpoint of the string. So the endpoints manifest themselves on the brane as charged particles, with a potential V_i associated to them [153].

Note that in the Type I action (2.32) the three-form field strength $H_{\mu\nu\rho}$ occurs in the same way as the R-R fields of Type IIA/B. The string and five-brane solutions of Type I should therefore be compared to the $D1$ and $D5$, rather than to the fundamental string or the solitonic five-brane.

The equations of motion of the $D=11$ supergravity action (2.37) do not contain an $F1$ or $S5$ solution (2.44, 2.54), but the three-form gauge field $C_{\mu\nu\rho}$ suggests that there has to be a two-brane and its eleven-dimensional magnetic dual, a five-brane, that couple to C. Indeed such an electrically charged membrane ($M2$) [65] and a magnetically charged five-brane ($M5$) [81] have been found[7]:

$$
M2 \quad = \left\{ \begin{array}{l} ds^2 = H^{-\frac{2}{3}}(dt^2 - dx_1^2 - dx_2^2) - H^{\frac{1}{3}}(dx_3^2 + ... + dx_{10}^2) \\ C_{012} = H^{-1} \end{array} \right. \tag{2.62}
$$

$$
M5 \quad = \left\{ \begin{array}{l} ds^2 = H^{-\frac{1}{3}}(dt^2 - dx_1^2 - ... - dx_5^2) - H^{\frac{2}{3}}(dx_6^2 + ... + dx_{10}^2) \\ G(C)_{mnpr} = \varepsilon_{mnprs}\partial_s H \qquad (m,n,p,r,s: 6,...,10), \end{array} \right. \tag{2.63}
$$

In many aspects these M-branes are much the same as their ten-dimensional counterparts (in fact in the next chapter we will see how they are related): the harmonic function H depends on the transversal coordinates x_m, they saturate the Bogomol'nyi bound and break half of the supersymmetry. The $M2$ is singular and needs a source term [30], while the $M5$ is a solitonic object that is very heavy at weak coupling.

Besides the above mentioned p-brane solutions, there exist two more solutions to both string theory and $D=11$ supergravity that are characterized by a single harmonic function and can therefore also be considered as fundamental objects of string theory and supergravity. We will encounter them often in the following chapters. They are special in the sense that they already occur as solutions of pure gravity, so they only consist of a non-trivial metric. Furthermore they do not have the typical two-block structure of world volume and transverse directions of p-branes. Therefore they can not be interpreted as "brane"-like solutions.

[7] The names $M2$ and $M5$ come from the fact that $D=11$ supergravity sometimes is called M-theory. Thus the p-branes that arise in M-theory are called M-branes.

dim	0	$\mathcal{W}_D$	1	2	3	4	5	6	$\mathcal{KK}_D$	7	8
$D=11$		$\mathcal{W}_{11}$		$M2$			$M5$		$\mathcal{KK}_{11}$		
IIA	$D0$	$\mathcal{W}_{10}$	$F1$	$D2$		$D4$	$S5$	$D6$	$\mathcal{KK}_{10}$		$D8$
IIB		$\mathcal{W}_{10}$	$F1/D1$		$D3$		$S5/D5$		$\mathcal{KK}_{10}$	$D7$	
Het		$\mathcal{W}_{10}$	$F1$				$S5$		$\mathcal{KK}_{10}$		
I		$\mathcal{W}_{10}$	$D1$				$D5$		$\mathcal{KK}_{10}$		

Table 2.1: *The solutions of the various string theories and $D=11$ supergravity.*

The first one is the D-dimensional gravitational wave or Brinkmann wave ($\mathcal{W}_D$) [36], propagating in the $z=x_1$ direction:

$$\mathcal{W}_D \ : \ ds^2 = (2-H)dt^2 - Hdz^2 + 2(1-H)dtdz - (dx_2^2 + ... + dx_{(D-1)}^2), \tag{2.64}$$

and the second the Kaluza-Klein monopole in D-dimensions ($\mathcal{KK}_D$) [150, 80]:

$$\mathcal{KK}_D \ : \ ds^2 = dt^2 - dx_1^2 - ... - dx_{(D-5)}^2 - H^{-1}(dz + A_m dx_m)^2 - H dx_m^2. \tag{2.65}$$

H is a harmonic function that depends in the case of the wave on the coordinates $t+z, x_2, ..., x_{D-1}$ and in the case of the monopole on x_m ($m = D-3, D-2, D-1$) and not on z. The z-direction is a compact isometry direction in order for the monopole to be non-singular. After a Kaluza-Klein compactification in this z-direction, one ends up with a $(D-5)$-brane, with a magnetic charge, which in the case of a five-dimensional monopole $\mathcal{KK}_5$ corresponds to a Dirac-monopole type particle. This explains its name.

Also A_i depends on x_m and the relation with H is given by:

$$F_{mn} = \partial_m A_n - \partial_n A_m = \varepsilon_{mnp}\partial_p H. \tag{2.66}$$

As mentioned above these solutions do not have a two-block structure due to off-diagonal terms in the metric, which makes it difficult to distinguish between world volume and transverse directions. We will choose, for later convenience, the z-directions in the case of the wave to be a world volume direction, but in the case of the Kaluza-Klein monopole a transverse direction.

Table 2.1 gives an overview of the different solutions we encountered in the various theories. In Chapter 3 we will see that these theories are related to each other via duality transformations. This means that there also must exist duality relations between the different solutions and the world volume actions that describe their dynamics. We will investigate in more detail these duality relations in Chapter 5 and see that under certain conditions different solutions can be superposed in a kind of "bound state". The relations between the world volume actions will be studied in Chapter 6.