Similarity measure computation of convex polyhedra revisited

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Published in:
EPRINTS-BOOK-TITLE

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Abstract We study the computation of rotation-invariant similarity measures of convex polyhedra, based on Minkowski's theory of mixed volumes. To compute the similarity measure, a (mixed) volume functional has to be minimized over a number of critical orientations of these polyhedra. These critical orientations are those relative configurations where faces and edges of the two polyhedra are as much as possible parallel. Two types of critical orientations exist for two polyhedra A and B. Type-1 critical orientations are those relative orientations where a face of B is parallel to a face of A, and an edge of B is parallel to a face of A, or vice versa. Type-2 critical orientations correspond to the case that three edges of A are parallel to three faces of B, or vice versa. It has been conjectured that to perform minimization of the volume functional, it is sufficient to consider Type-1 critical orientations only. Here we present experimental proof showing this conjecture to be false.

1 Introduction

Shape comparison is one of the fundamental problems of machine vision. Shape similarity may be quantified by introducing a similarity measure. The requirement of invariance under some set of shape transformations in general leads to complicated optimization problems. Therefore, one often studies shape classes and transformation sets for which a compromise between generality and efficiency can be found.

Recently, a new approach to similarity measure computation of convex polyhedra has been developed based on Minkowski addition [2, 5]. These similarity measures are based upon the Minkowski inequality and its descendants, and the central operation is the minimization of (mixed) volume functionals. An attractive property of this family of similarity measures is that they are invariant under translations and possibly under scaling, rotation, and reflection. The method may be used in any-dimensional space, but we will concentrate on the 3D case.

For computing a rotation-invariant similarity measure of two convex polyhedra, a (mixed) volume functional has to be evaluated over a number of special relative orientations of these polyhedra, the so-called critical orientations. These critical orientations are those relative configurations where faces and edges of the two polyhedra are parallel as much as possible. (Two faces are called parallel when they have the same outward normal.) Given two polyhedra A and B, the set of critical orientations can be divided in two classes, denoted by Type 1 and 2, respectively. Type 1 occurs when a face of B is parallel to a face of A, and an edge of B is parallel to a face of A, or vice versa; Type 2 occurs when three edges of A are parallel to three faces of B, or vice versa. It was proved in [5] that (i) for a given rotation axis, it is sufficient to compute the (mixed) volume functionals only for a finite number of critical angles, thus generalizing a result for the 2D case [2]; and (ii) the number of rotation axes to be checked is finite. The second result is trivial for Type-1 orientations, but the proof for Type 2 is more involved, and only establishes finiteness of the number of axes to be checked, without giving an explicit upper bound on the number of axes. Such an explicit upper bound for the number of Type-2 critical orientations was given in [1], where it was shown that the problem can be reduced to solving an algebraic equation of degree 8, which has to be solved numerically. So, given three edges of A and three faces of B, the number of critical orientation axes is at most 8.

Experiments on mixed-volume minimization were reported by Tuzikov and Sheynin in [6]. Only Type-1 critical orientations were taken into account, and the authors conjectured that to find the global minimum of the mixed-volume functional it is sufficient to consider Type-1 critical orientations only. In this paper we reconsider this issue, and present experimental proof showing this conjecture to be false. Experiments were carried out with randomly generated pairs of tetrahedra, and the minimum of the mixed volume functional was computed by taking into account either Type-1 or Type-2 critical orientations. As a result, we have found that the minimum value of the mixed volume functional for Type-2 minimization can be larger as well as smaller than that of Type-1 minimization.

The paper is organized in the following way. In Section 2 we define Minkowski addition of convex polyhedra and their slope diagram representation, and introduce a rotation-invariant similarity measure based on inequalities for the (mixed) volume. In Section 3 similarity measure computation by minimization of a mixed-volume functional is considered, and the main results from the literature are summarized. In Section 4 we give experimental results on minimization of the mixed-volume functional, by taking into account Type-1 and Type-2 critical orientations, respectively. Conclusions are summarized in Section 5.

2 Preliminaries

In this section the Minkowski sum, mixed volumes, a similarity measure based on the Minkowski like inequality, and the slope diagram representation of convex polyhedra are introduced. The compact convex subsets of $\mathbb{R}^3$ are denoted by $\mathcal{C} = \mathcal{C}(\mathbb{R}^3)$. Two shapes A and B are said to be *equivalent* if they differ only by translation; we denote this as $A \equiv B$.

2.1 Minkowski sum and mixed volumes

The Minkowski sum of two sets $A, B \subseteq \mathbb{R}^3$ is defined as

$$A \oplus B = \{a + b | a \in A, b \in B\}. \tag{1}$$

It is well known [4] that every convex set A is uniquely determined by its *support function*, given by:

$$h(A,u) = \sup\{\langle a,u\rangle \mid a \in A\}, \qquad u \in S^2.$$

Here $\langle a,u\rangle$ is the inner product of vectors a and u, and S^2 denotes the unit sphere in $\mathbb{R}^3$. Also [4]:

$$h(A \oplus B, u) = h(A,u) + h(B,u), \qquad u \in S^2, \tag{2}$$

for $A, B \in \mathcal{C}$.

Denote by $V(A)$ the volume of the set $A \subset \mathbb{R}^3$. Given convex sets $A, B \subset \mathbb{R}^3$ and $\alpha, \beta \geq 0$, the following holds:

$$V(\alpha A \oplus \beta B) = \alpha^3 V(A) + 3\alpha^2\beta V(A,A,B) + 3\alpha\beta^2 V(A,B,B) + \beta^3 V(B). \tag{3}$$

Here $V(A,A,B)$ and $V(A,B,B)$ are called *mixed volumes*.

The Minkowski inequality for convex sets $A, B \in \mathcal{C}(\mathbb{R}^3)$ reads [4]

$$V(A,A,B)^3 \geq V(A)^2 V(B), \tag{4}$$

where equality holds if and only if $B \equiv \lambda A$ for some $\lambda > 0$.

2.2 Similarity measure

Using the Minkowski inequality (4), a similarity measure σ may be defined as follows:

$$\sigma(A,B) = \sup_{R \in \mathcal{R}} \frac{V(B)^{\frac{2}{3}} V(A)^{\frac{1}{3}}}{V(B,B,R(A))} = \sup_{R \in \mathcal{R}} \frac{V(B)^{\frac{2}{3}} V(A)^{\frac{1}{3}}}{V(A,R(B),R(B))} \tag{5}$$

where $\mathcal{R}$ denotes the set of all spatial rotations, and where $R(B)$ denotes a rotation of B by $R \in \mathcal{R}$. The second equality follows from the fact that $V(B,B,R(A)) = V(R(A),B,B) = V(A,R^{-1}(B),R^{-1}(B))$. Obviously, $0 \leq \sigma(A,B) \leq 1$, where $\sigma(A,B) = 1$ when $B \equiv \lambda\, R(A)$ for some rotation R and some $\lambda > 0$. The similarity measure σ is invariant under rotations and scalar multiplications. It is not symmetric in its arguments. Symmetric versions may be defined in various ways; an example is the measure $\sigma'(A,B) = \frac{1}{2}(\sigma(A,B) + \sigma(B,A))$.

To find the maximum in (5), the mixed volume $V(A,R(B),R(B))$ has to be minimized over all orientations of A.

If B is a convex polyhedron with faces F_i and corresponding outward unit normal vectors $u_i,\ i = 1,\ldots,k$, then [4]

$$V(A,B,B) = V(B,B,A) = \frac{1}{3}\sum_{i=1}^{k} h(A,u_i) S(F_i), \tag{6}$$

where $S(F_i)$ is the area of the face F_i of B and $h(A,u_i)$ is the value of the support function of A for the normal vector u_i.

2.3 Slope diagram representation

Denote face i of polyhedron A by $F_i(A)$, edge j by $E_j(A)$, and vertex k by $V_k(A)$. The slope diagram representation (SDR) of polyhedron A, denoted by SDR(A), is a unit sphere covered with spherical polygons. A vertex of A is represented by the interior of a polygon on SDR(A), an edge by a spherical arc on SDR(A), and a face by a vertex of some polygon on SDR(A). To be more precise:

- *Face representation.* $F_i(A)$ is represented on the sphere by a point SDR$(F_i(A))$, located at the intersection of the outward unit normal vector u_i on $F_i(A)$ with the unit sphere.
- *Edge representation.* An edge $E_j(A)$ is represented by the arc of the great circle connecting the two points corresponding to the two adjacent faces of $E_j(A)$.
- *Vertex representation.* A vertex $V_k(A)$ is represented by the interior of the polygon bounded by the arcs corresponding to the edges of A meeting at $V_k(A)$.

In Fig. 1 an example of a polyhedron and its SDR is given.

It is easily verified that the faces $F_i(A)$ and $F_j(B)$ are parallel (that is, have the same outward normal) when SDR($F_i(A)$) coincides with SDR($F_j(B)$). Also, an edge $E_i(A)$ is parallel to $F_j(B)$ when SDR($F_j(B)$) lies on SDR($E_i(A)$). Therefore, the maximum in (5) is obtained when points of SDR(A) coincide with points or edges of SDR(B).

3 Similarity measure computation

In this section, we consider the problem of computing the similarity measure (5).

Let ℓ be an axis passing through the coordinate origin and $r_{\ell,\alpha}$ be the rotation in $\mathbb{R}^3$ about ℓ by an angle α in a counter-clockwise direction. The problem to be considered is the minimization of the functional $V(A, r_{\ell,\alpha}(B), r_{\ell,\alpha}(B))$. Given a fixed axis ℓ and angle α, (6) can be used to compute this functional.

While rotating the slope diagram of polyhedron A, situations arise when spherical points of the rotated SDR of A intersect spherical arcs or points of the SDR of B. Such relative configurations of A w.r.t. B are *critical* in the sense that they are candidates for (local) minima of the objective functional to be minimized. For more precise definitions we refer to [5].

3.1 Fixed rotation axis

Let ℓ be a fixed rotation axis. The ℓ-critical angles of B with respect to A for mixed volume $V(A, r_{\ell,\alpha}(B), r_{\ell,\alpha}(B))$ are the angles $\{\alpha'_j\}$, $0 \leq \alpha'_1 < \alpha'_2 < \ldots < \alpha'_N < 2\pi$, for which spherical points of the rotated slope diagram SDR($r_{\ell,\alpha'_j}(B)$) intersect spherical points or arcs of SDR(A). The following result was proved in [5].

Proposition 1. *Given an axis of rotation ℓ, the mixed volume of the convex polyhedra A and B, i.e. $V(A, r_{\ell,\alpha}(B), r_{\ell,\alpha}(B))$, is a function of α which is piecewise concave on $[0, 2\pi)$, i.e., concave on every interval $(\alpha'_k, \alpha'_{k+1})$, for $k = 1, 2, \ldots, N$ and $\alpha'_{N+1} = \alpha'_1$.*

This result implies that in order to minimize the mixed volume for any *fixed* rotation axis ℓ, it is sufficient to compute it for all ℓ-critical angles (which are clearly finite in number), and take the minimum of the values thus obtained.

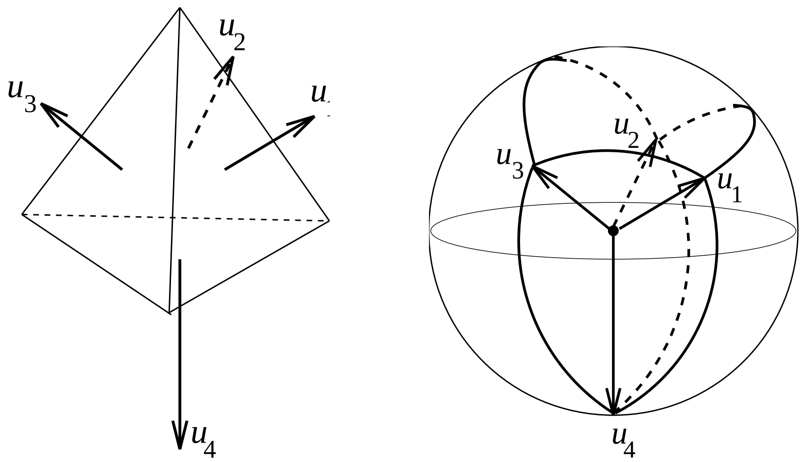


Figure1. (a): A tetrahedron with unit normal vectors on its faces. (b): Its slope diagram representation.

3.2 Minimization over all rotation axes

An extensive analysis in [5] showed that two types of critical orientations have to be considered for obtaining the global minimum of the mixed volume $V(A, R(B), R(B))$:

Type 1 A face of A is parallel to a face of B, and an edge of A is parallel to a face of B.

Type 2 Three edges of A are parallel to three faces of B.

To find the orientations of Type 1 is trivial. When a face $F_j(B)$ is parallel to a face $F_i(A)$, B has only one degree of freedom left, being a rotation around an axis through the origin and the spherical point SDR($F_j(B)$). Using the slope diagram representations of A and B, it is easy to find those rotations of B around this axis that make the slope diagram representations of faces of B coincide with the slope diagram representations of edges of A. The problem can be restated as solving a quadratic equation in one variable [6].

To find the orientations of Type 2 means looking for those orientations of B where three points on SDR(B) (representing three faces of B) lie on three spherical arcs of SDR(A). (Notice that no more than three points have to be checked, since a rotation is uniquely determined by three parameters.) This problem can be reformulated as follows: given two triples of 3D vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$, find the rotation R that transforms the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that $R(\boldsymbol{a})$ is perpendicular to $\boldsymbol{k}$, $R(\boldsymbol{b})$ is perpendicular to $\boldsymbol{l}$, $R(\boldsymbol{c})$ is perpendicular to $\boldsymbol{m}$. That is, the following system of equations has to be solved for R:

$$\begin{aligned} \langle \boldsymbol{k}, R(\boldsymbol{a}) \rangle &= 0 \\ \langle \boldsymbol{l}, R(\boldsymbol{b}) \rangle &= 0 \\ \langle \boldsymbol{m}, R(\boldsymbol{c}) \rangle &= 0 \end{aligned}$$

Using the computer-algebra program MAPLE©, this system was reduced in [1] to the solution of an algebraic equation in one variable of degree 8, whose coefficients are lengthy expressions in the elements of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$. This equation can be solved numerically using Laguerre's method [3].

In [6] the minimization of the mixed volume was carried out by taking into account Type-1 critical orientations only. The authors conjectured that this is actually sufficient to find the global minimum.

In the next section we report on a number of experiments we did in order to verify this conjecture.

4 Experimental results

In this section, we give results for minimization of the mixed volume $V(A, R(B), R(B))$ of convex polyhedra A and B when R runs over the set of all spatial rotations. Both polyhedra were chosen to be tetrahedra, whose edge sizes varied randomly. For each pair A and B, we performed minimization in two ways, depending on the set of rotations taken into account:

Type-1 minimization All critical rotations are of Type 1: a face of A is parallel to a face of B, and an edge of A is parallel to a face of B.

Type-2 minimization All critical rotations are of Type 2: three edges of A are parallel to three faces of B.

To verify the conjecture, we checked for each pair of tetrahedra A and B whether the result for Type-2 minimization was larger than that of Type-1 minimization.

Remark In fact, in our implementation we use two routines, one which performs Type-1 minimization, and another one which minimizes over the combined set of both Type-1 and Type-2 critical rotations. If we find that the minimum in the second case is smaller than that in the first case, then we know that the conjecture is false. Also, instead of fixing A and rotating B, we may as well fix B and rotate A in view of the identity $V(A, R(B), R(B)) = V(B, B, R^{-1}(A))$.

In the experiments, we found cases among the randomly generated pairs of tetrahedra for which the result for Type-1 minimization was actually larger than that of Type-2 minimization, although the differences were often small. An example where the difference is substantial is shown in in Fig. 2. In this case the mixed volume for Type-1 minimization equals 1.81213e+006, and that for Type-2 minimization equals 1.59156e+006, which is significantly smaller. The corresponding tetrahedra are shown in Fig. 2.

It is interesting to look at the Minkowski sum of $B \oplus R^*(A)$, with R^* the rotation which realizes the minimum of the mixed volume for Type-1 and Type-2 minimization, respectively, see Fig. 3. The corresponding slope diagrams are shown in Fig. 4, with the spherical arcs of A shown in bold. From the pictures, one can verify that indeed for Type-1 minimization, a spherical point of B coincides with a spherical point of A, and another spherical point of B is on a spherical arc of A, whereas for Type-2 minimization, three spherical points of B are on spherical arcs of A.

5 Conclusion

We have studied the computation of a rotation-invariant similarity measure of convex polyhedra A and B, involving the minimization of a mixed-volume functional $V(A, R(B), R(B))$ with R running over the set of critical rotations. Two types of critical orientations were distinguished: for Type-1 critical orientations a face of A is parallel to a face of B, and an edge of A is parallel to a face of B; for Type-2 critical orientations three edges of A are parallel to three faces of B. We performed experiments with randomly generated tetrahedra, and computed the minimum of the volume functional by taking into account either Type-1 or Type-2 critical orientations. We found that the result for Type-2 minimization can be larger as well as smaller than that of Type-1 minimization. Therefore, in contrast to what has been conjectured in [6], one has in general to take both Type-1 and Type-2 critical orientations into account to compute the global minimum of the mixed volume.

References

1. Bekker, H., and Roerdink, J. B. T. M. Calculating critical orientations of polyhedra for similarity measure evaluation. In *Proc. 2nd Annual IASTED International Conference on Computer Graphics and Imaging, Palm Springs, California USA, Oct. 25-27* (1999), pp. 106–111.
2. Heijmans, H. J. A. M., and Tuzikov, A. Similarity and symmetry measures for convex shapes using Minkowski addition. *IEEE Trans. Patt. Anal. Mach. Intell. 20*, 9 (1998), 980–993.
3. Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T. *Numerical Recipes, the Art of Scientific Computing*. Cambr. Univ. Press, New York, 1986.
4. Schneider, R. *Convex Bodies. The Brunn-Minkowski Theory*. Cambridge University Press, 1993.
5. Tuzikov, A. V., Roerdink, J. B. T. M., and Heijmans, H. J. A. M. Similarity measures for convex polyhedra based on Minkowski addition. *Pattern Recognition 33*, 6 (2000), 979–995.
6. Tuzikov, A. V., and Sheynin, S. A. Minkowski sum volume minimization for convex polyhedra. In *Mathematical Morphology and its Applications to Image and Signal Processing*, J. Goutsias, L. Vincent, and D. S. Bloomberg, Eds. Kluwer Acad. Publ., Dordrecht, 2000, pp. 33–40.

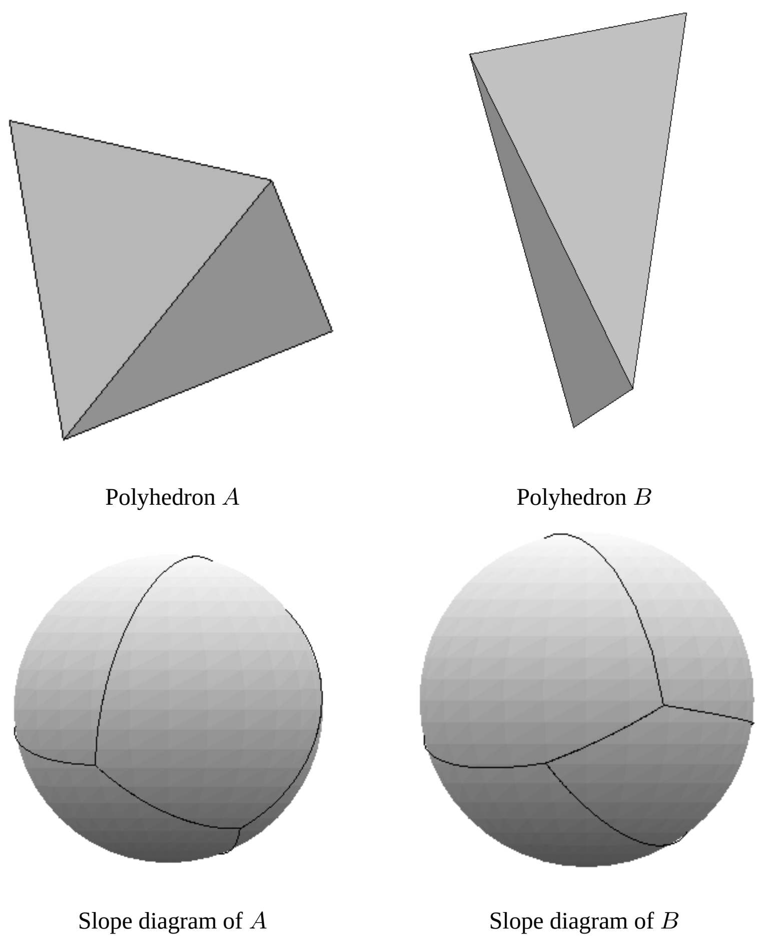


Figure2. Polyhedra A and B as used in the experiment, for which $V(B, B, R_2^*(A))$, with R_2^* the rotation which realizes the minimum of Type 2, is smaller than $V(B, B, R_1^*(A))$, with R_1^* the rotation which realizes the minimum of Type 1.

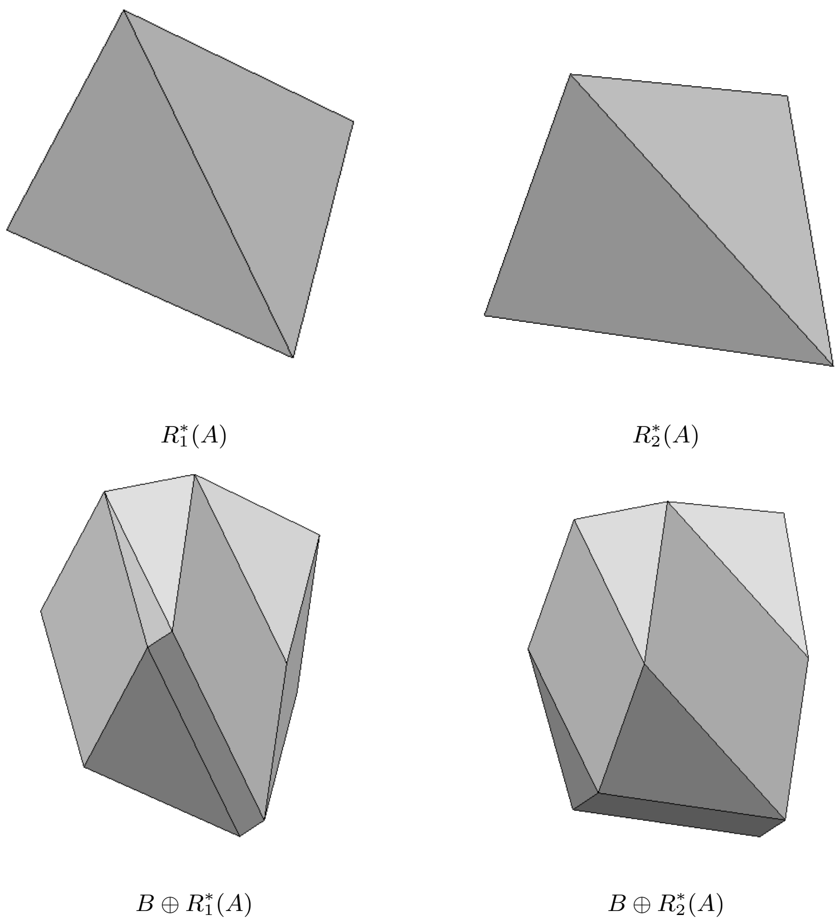
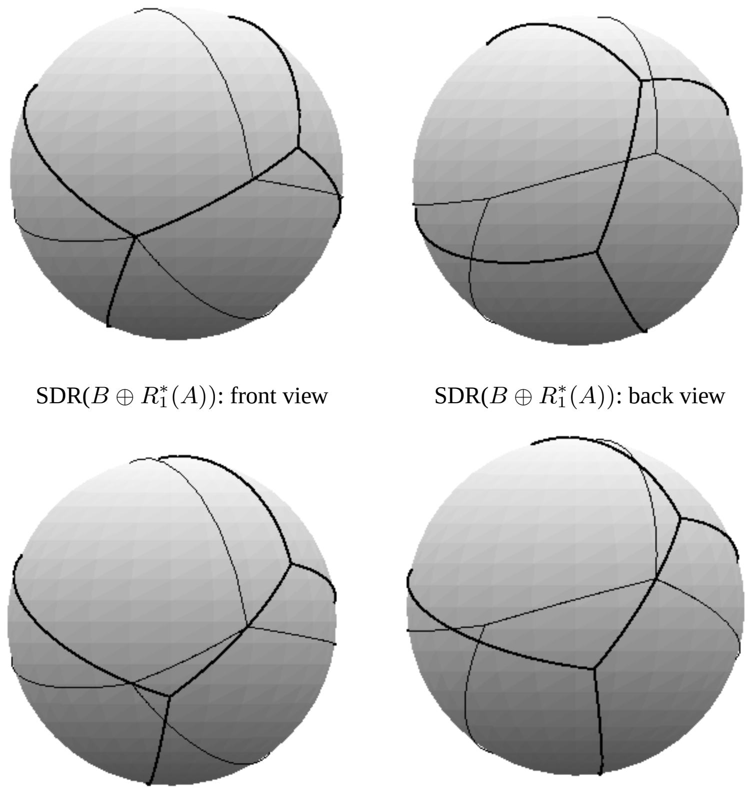


Figure3. Top row: polyhedron A in the rotated configuration which minimizes mixed volume according to Type 1 ($R_1^*(A)$) and Type 2 ($R_2^*(A)$). Bottom row: Minkowski sums of polyhedron B and rotated polyhedron $R_1^*(A)$, c.q. $R_2^*(A)$.



SDR($B \oplus R_1^*(A)$): front view SDR($B \oplus R_1^*(A)$): back view

SDR($B \oplus R_2^*(A)$): front view SDR($B \oplus R_2^*(A)$): back view

Figure4. Top row: two views of the slope diagram of the Minkowski sum of polyhedron B and rotated polyhedron $R_1^*(A)$. Bottom row: two views of the slope diagram of the Minkowski sum of polyhedron B and rotated polyhedron $R_2^*(A)$. Here R_1^* and R_2^* are the rotations which realize the minimum of Type 1 and Type 2, respectively. Bold curve segments indicate spherical arcs of polyhedron A.