Output Tracking Using a Discrete-Time Sliding Mode Controller with Reduced-Order State-Error Estimation

Monsees, G.; Scherpen, J.M.A.

Published in:
Proceedings of the 1st IFAC/IEEE Symposium on System Structure and Control

Publication date:
2001

OUTPUT TRACKING USING A DISCRETE-TIME SLIDING MODE CONTROLLER WITH REDUCED-ORDER STATE-ERROR ESTIMATION

G. Monsees * **J.M.A. Scherpen** *

* *Control Laboratory,*
Faculty of Information Technology and Systems,
Delft University of Technology, P.O. Box 5031, 2600 GA Delft,
The Netherlands
G.Monsees@ITS.TUDelft.nl
J.M.A.Scherpen@ITS.TUDelft.nl

Abstract: This paper presents a novel output-based, discrete-time, sliding mode controller design methodology. In order to reproduce an output target profile, feed-forward controllers yield an excellent performance, however their robustness against disturbances and parameter variations is limited. In this paper a combination is made of a feed-forward controller for good tracking performance and an output-based sliding mode controller to increase robustness. Also the possibility of using a reduced order state-error observer is considered.

Keywords: Sliding mode control, Variable structure systems, Discrete-time systems, Observers, Output feedback.

1. INTRODUCTION

Sliding mode control is a well known robust control algorithm for linear- as well as nonlinear systems (DeCarlo *et al.*, 1988), (Edwards and Spurgeon, 1998), (Hung *et al.*, 1993), (Utkin *et al.*, 1999), (Utkin, 1992). Continuous-time sliding mode control has been extensively studied and has been used in various applications. Much less is known of discrete-time sliding mode controllers. In practice it is often assumed that the sampling frequency is sufficiently high to assume that the controller is continuous-time (Young and Özgünter, 1999). Another possibility is to design the sliding mode controller in discrete-time, based on a discrete-time model. However, to our knowledge, research in discrete-time sliding mode control has so far been focused on state-based sliding mode controllers. In this paper we introduce an output-based sliding mode controller to be used in combination with a feed-forward controller. The feed-forward controller is supposed to give perfect tracking for the obtained model. Since in reality the model never describes the system perfectly, the output-based sliding mode controller is added to obtain robustness against model uncertainty and disturbances.

In this paper we consider the system:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] + F(\mathbf{x},\mathbf{u},k) \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{u} \in \mathbb{R}^m$, $F(\mathbf{x},\mathbf{u},k) \in \mathbb{R}^n$, and consequently $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, and $\mathbf{C} \in \mathbb{R}^{p\times n}$. The vector $F(\mathbf{x},\mathbf{u},k)$ contains all model uncertainties and disturbances. It is assumed that $p \geq m$, Rank$\{\mathbf{CB}\} = m$, and that the triple $(\mathbf{A},\mathbf{B},\mathbf{C})$ is both controllable and observable. The proposed control law is the superposition of a feed-forward term $\mathbf{u}_{ff}[k]$ and a (sliding mode) feedback term $\mathbf{u}_{fb}[k]$, represented by:

$$\mathbf{u}[k] = \mathbf{u}_{ff}[k] + \mathbf{u}_{fb}[k] \tag{2}$$

[1] This work was supported by Brite-Euram under contract number BRPR-CT97-0508 and project number BE-97-4186

The feed-forward component $\mathbf{u}_{ff}[k]$ is supposed to give perfect tracking of the nominal system and hence:

$$\begin{aligned} \mathbf{x}^d[k+1] &= \mathbf{A}\mathbf{x}^d[k] + \mathbf{B}\mathbf{u}_{ff}[k] \\ \mathbf{y}^d[k] &= \mathbf{C}\mathbf{x}^d[k] \end{aligned} \quad (3)$$

(with the superscript d denoting desired). If we now define the state tracking error by $\mathbf{e}_x[x] = \mathbf{x}[k] - \mathbf{x}^d[k]$ and the output tracking error by $\mathbf{e}_y[k] = \mathbf{y}[k] - \mathbf{y}^d[k]$, we can write with equations (1) and (3) for the error dynamics:

$$\begin{aligned} \mathbf{e}_x[k+1] &= \mathbf{A}\mathbf{e}_x[k] + \mathbf{B}\mathbf{u}_{fb}[k] + F(\mathbf{x},\mathbf{u},k) \\ \mathbf{e}_y[k] &= \mathbf{C}\mathbf{e}_x[k] \end{aligned} \quad (4)$$

The topic of this paper is the design of a discrete-time, output-based, sliding mode feed-back controller, which stabilizes the above error system despite any modeling errors or disturbances. First, Section 2 introduces a procedure to design an output-based sliding surface. Then, in Section 3, an output-based controller is presented which steers the closed-loop system to the sliding surface. Section 4 extends the derived controller with a reduced order state-error observer. In Section 5 an output-based sliding mode controller with and without state-error estimation are tested on a practical trajectory tracking problem. Finally Section 6 presents the conclusions.

2. SLIDING SURFACE DESIGN

As opposed to continuous-time sliding mode control, true sliding mode is in discrete-time no longer achievable (Young and Özgünter, 1999). Except in the case of perfect model and disturbance knowledge, the closed-loop system cannot be maintained on the sliding surface. The goal in discrete-time sliding mode control is to bring the system as close as possible to the sliding surface. If the system is close enough to the sliding surface it is acceptable to approximate the sliding variable σ_{e_y} by zero. This approximation is used in the design of the sliding surface, for which the same procedure can be used as in continuous-time sliding mode control (Young and Özgünter, 1999). The design procedure for the output based sliding mode controller presented in this section is based on the design procedure given by Edwards and Spurgeon in (Edwards and Spurgeon, 1998). Without any proof we repeat their procedure, which is now to be used in discrete-time for the nominal error system (system (4) without the disturbance vector $F(\mathbf{x},\mathbf{u},k)$):

$$\begin{aligned} \mathbf{e}_x[k+1] &= \mathbf{A}\mathbf{e}_x[k] + \mathbf{B}\mathbf{u}_{fb}[k] \\ \mathbf{e}_y[k] &= \mathbf{C}\mathbf{e}_x[k] \end{aligned} \quad (5)$$

The switching function is defined by:

$$\sigma_{e_y}[k] = \mathbf{S}\mathbf{e}_y[k] \quad (6)$$

where $\sigma_{e_y} \in \mathbb{R}^m$ and $\mathbf{S} \in \mathbb{R}^{m\times p}$. By a nonsingular transformation the system (5) can be transformed to:

$$\begin{bmatrix} \mathbf{e}_{x_0}[k+1] \\ \mathbf{e}_{x_1}[k+1] \\ \mathbf{e}_{y_1}[k+1] \\ \mathbf{e}_{y_2}[k+1] \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ 0 & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ 0 & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{x_0}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \\ \mathbf{e}_{y_2}[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_{fb}[k] \quad (7)$$

$$y[k] = [0_{p\times(n-p)} \quad \mathbf{T}] \begin{bmatrix} \mathbf{e}_{x_0}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \\ \mathbf{e}_{y_2}[k] \end{bmatrix} \quad (8)$$

where $\mathbf{e}_{x_0} \in \mathbb{R}^r$, $\mathbf{e}_{x_1} \in \mathbb{R}^{(n-p-r)}$, $\mathbf{e}_{y_1} \in \mathbb{R}^{(p-m)}$, $\mathbf{e}_{y_2} \in \mathbb{R}^m$, $\mathbf{T} \in \mathbb{R}^{p\times p}$ is invertible, and rank($\mathbf{B}_2$) = m. Defining $[\mathbf{S}_1 \quad \mathbf{S}_2] = ST \quad (\mathbf{S}_1 \in \mathbb{R}^{p\times(p-m)}$ and $\mathbf{S}_2 \in \mathbb{R}^{p\times m})$, leads to:

$$\sigma_{e_y}[k] = \mathbf{S}_1\mathbf{e}_{y_1}[k] + \mathbf{S}_2\mathbf{e}_{y_2}[k] \quad (9)$$

The dynamics in sliding mode can be obtained by setting the above equation to zero, making $\mathbf{e}_{y_2}[k]$ explicit and substituting this in the equations for $\mathbf{e}_{x_0}[k+1]$, $\mathbf{e}_{x_1}[k+1]$, and $\mathbf{e}_{y_1}[k+1]$ (equation (7)) resulting in:

$$\begin{bmatrix} \mathbf{e}_{x_0}[k+1] \\ \mathbf{e}_{x_1}[k+1] \\ \mathbf{e}_{y_1}[k+1] \end{bmatrix} = \mathbf{A}_{sm} \begin{bmatrix} \mathbf{e}_{x_0}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \end{bmatrix} \quad (10)$$

with:

$$\mathbf{A}_{sm} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \left(\mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\ 0 & \mathbf{A}_{22} & \left(\mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\ 0 & \mathbf{A}_{32} & \left(\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \end{bmatrix} \quad (11)$$

The poles of the above system are given by:

$$\lambda(\mathbf{A}_{sm}) = \lambda(\mathbf{A}_{11}) \cup \lambda\left(\begin{bmatrix} \mathbf{A}_{22} & \left(\mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\ \mathbf{A}_{32} & \left(\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \end{bmatrix}\right) \quad (12)$$

(where $\lambda(\mathbf{A})$ returns the eigenvalues of the matrix $\mathbf{A}$) It is well known that the eigenvalues of the sub-matrix $\mathbf{A}_{11}$ contains the open-loop zeros of the system (5) (Edwards and Spurgeon, 1998). Therefore, in order to stabilize the closed-loop system, the open-loop system should be minimum-phase which is assumed to be the case in the remainder of this paper.

Defining the matrix $\mathbf{M} = \mathbf{S}_2^{-1}\mathbf{S}_1$, the problem of designing a sliding surface reduces to placing the eigenvalues of the following matrix within the unit circle:

$$\begin{bmatrix} \mathbf{A}_{22} & (\mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{M}) \\ \mathbf{A}_{32} & (\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{M}) \end{bmatrix}$$

We choose $\mathbf{S}_2$ such that $\mathbf{S}_2\mathbf{B}_2 = \mathbf{I}_m$. The switching function in the original coordinates can then be found from:

$$S = [\mathbf{B}_2^{-1}\mathbf{M} \qquad \mathbf{B}_2^{-1}]\mathbf{T}^{-1} \quad (13)$$

Now, we have a design procedure for a stable sliding surface. In the next section we continue with the introduction of a controller which steers the closed-loop system inside the so called "Quasi Sliding Mode Band", which is a preferably small region around the sliding surface.

3. CONTROLLER DESIGN

In this section we will derive an output-based, discrete-time, sliding mode controller which steers the closed-loop system to the sliding surface developed in the previous section. Applying the same transformation as was used to obtain equation (7) brings the system (4) in following form:

$$\begin{bmatrix} \mathbf{e}_{x_o}[k+1] \\ \mathbf{e}_{x_1}[k+1] \\ \mathbf{e}_{y_1}[k+1] \\ \mathbf{e}_{y_2}[k+1] \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} & \mathbf{A}_{14} \\ 0 & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{A}_{24} \\ 0 & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{A}_{41} & \mathbf{A}_{42} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{x_o}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \\ \mathbf{e}_{y_2}[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}_{fb}[k] + \begin{bmatrix} F_{x_o}(\mathbf{x},\mathbf{u},k) \\ F_{x_1}(\mathbf{x},\mathbf{u},k) \\ F_{y_1}(\mathbf{x},\mathbf{u},k) \\ F_{y_2}(\mathbf{x},\mathbf{u},k) \end{bmatrix} \quad (14)$$

The switching function in these new coordinates was defined in the previous section as (equation (6)):

$$\sigma_{e_y}[k] = \mathbf{S}_1\mathbf{e}_{y_1}[k] + \mathbf{S}_2\mathbf{e}_{y_2}[k]$$

In order to make $\sigma_{e_y}[k+1]$ explicit, we define the transformation:

$$\mathbf{T}_\sigma = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & S_1 & \mathbf{S}_2 \end{bmatrix} \quad (15)$$

which brings the system (14) into the following form:

$$\begin{bmatrix} \mathbf{e}_{x_o}[k+1] \\ \mathbf{e}_{x_1}[k+1] \\ \mathbf{e}_{y_1}[k+1] \\ \sigma_{e_y}[k+1] \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ 0 & \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ 0 & \bar{\mathbf{A}}_{32} & \bar{\mathbf{A}}_{33} & \bar{\mathbf{A}}_{34} \\ \bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{42} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{x_o}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \\ \sigma_{e_y}[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{I}_m \end{bmatrix} \mathbf{u}_{fb}[k] + \begin{bmatrix} F_{x_o}(\mathbf{x},\mathbf{u},k) \\ F_{x_1}(\mathbf{x},\mathbf{u},k) \\ F_{y_1}(\mathbf{x},\mathbf{u},k) \\ \mathbf{S}_1F_{y_1}(\mathbf{x},\mathbf{u},k) + \mathbf{S}_2F_{y_2}(\mathbf{x},\mathbf{u},k) \end{bmatrix} \quad (16)$$

where:

$$\begin{aligned}
&\bar{\mathbf{A}}_{11} = \mathbf{A}_{11} && \bar{\mathbf{A}}_{13} = \left(\mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\
&\bar{\mathbf{A}}_{12} = \mathbf{A}_{12} && \bar{\mathbf{A}}_{14} = \mathbf{A}_{14}\mathbf{S}_2^{-1} \\
&\bar{\mathbf{A}}_{22} = \mathbf{A}_{22} && \bar{\mathbf{A}}_{23} = \left(\mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\
&\bar{\mathbf{A}}_{24} = \mathbf{A}_{24}\mathbf{S}_2^{-1} && \bar{\mathbf{A}}_{33} = \left(\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1\right) \\
&\bar{\mathbf{A}}_{32} = \mathbf{A}_{32} && \bar{\mathbf{A}}_{34} = \mathbf{A}_{34}\mathbf{S}_2^{-1} \\
&\bar{\mathbf{A}}_{41} = \mathbf{S}_2\mathbf{A}_{41} && \bar{\mathbf{A}}_{42} = (\mathbf{S}_1\mathbf{A}_{32} + \mathbf{S}_2\mathbf{A}_{42}) \\
&\bar{\mathbf{A}}_{44} = \left(\mathbf{S}_1\mathbf{A}_{34}\mathbf{S}_2^{-1} + \mathbf{A}_{44}\right) \\
&\bar{\mathbf{A}}_{43} = \left(\mathbf{S}_1\mathbf{A}_{33} + \mathbf{S}_2\mathbf{A}_{43} - \mathbf{S}_1\mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1 - \mathbf{A}_{44}\mathbf{S}_1\right),
\end{aligned}$$

and by design choice (as presented in the previous section) $I_m = \mathbf{S}_2\mathbf{B}_2$. Defining the reaching law as:

$$\sigma_{e_y}[k+1] = \mathbf{\Phi}\sigma_{e_y}[k] \quad (17)$$

where $\mathbf{\Phi} \in \mathbb{R}^{m\times m}$ is, for simplicity, chosen as a diagonal matrix with all entries $0 \le \phi_i < 1$. From equation (16) and (17) the feed-back control term can be concluded to be:

$$\begin{aligned} \mathbf{u}_{fb}[k] = \left(\mathbf{\Phi} - \bar{\mathbf{A}}_{44}\right)\sigma_{e_y}[k] - \bar{\mathbf{A}}_{43}\mathbf{e}_{y_1}[k] \\ - \bar{\mathbf{A}}_{41}\mathbf{e}_{x_o}[k] - \bar{\mathbf{A}}_{42}\mathbf{e}_{x_1}[k] \\ - \mathbf{S}_1F_{y_1}(x,\mathbf{u},k) - \mathbf{S}_2F_{y_2}(\mathbf{x},\mathbf{u},k) \end{aligned} \quad (18)$$

Obviously the previous control law is not implementable. The disturbance and modeling error components $F_{y_1}(\mathbf{x},\mathbf{u},k)$ and $F_{y_2}(\mathbf{x},\mathbf{u},k)$ where assumed to be unknown, but also the state-components $\mathbf{e}_{x_o}[k]$ and $\mathbf{e}_{x_1}[k]$ are not known. Therefore, the unknown parts are omitted resulting in the control law:

$$\mathbf{u}_{fb}[k] = \left(\mathbf{\Phi} - \bar{\mathbf{A}}_{44}\right)\sigma_{e_y}[k] - \bar{\mathbf{A}}_{43}\mathbf{e}_{y_1}[k] \quad (19)$$

The following lemma explores the stability of the closed-loop system.

Lemma 3.1. The closed-loop system formed by the minimum-phase system (5), a stable feed-forward controller $\mathbf{u}_{ff}[k]$, and the feed-back controller (19), will be stable if the matrix:

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \left(\mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{S}_2^{-1}\mathbf{S}_1\right) & \mathbf{A}_{14}\mathbf{S}_2^{-1} \\ 0 & \mathbf{A}_{22} & \left(\mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1\right) & \mathbf{A}_{24}\mathbf{S}_2^{-1} \\ 0 & \mathbf{A}_{32} & \left(\mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1\right) & \mathbf{A}_{34}\mathbf{S}_2^{-1} \\ \mathbf{A}_{41} & (\mathbf{S}_1\mathbf{A}_{32} + \mathbf{S}_2\mathbf{A}_{42}) & 0 & \mathbf{\Phi} \end{bmatrix}$$

has all its eigenvalues within the unit circle.

Proof: Since the stability properties of a system are not depending on the disturbances nor on the (stable) feed-forward signal, we set them to zero. Plugging in equation (19) in (16) then results in:

$$\begin{bmatrix} \mathbf{e}_{x_o}[k+1] \\ \mathbf{e}_{x_1}[k+1] \\ \mathbf{e}_{y_1}[k+1] \\ \sigma_{e_y}[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ 0 & \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ 0 & \bar{\mathbf{A}}_{32} & \bar{\mathbf{A}}_{33} & \bar{\mathbf{A}}_{34} \\ \bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{42} & 0 & \mathbf{\Phi} \end{bmatrix}}_{=\mathbf{A}_{cl}} \begin{bmatrix} \mathbf{e}_{x_o}[k] \\ \mathbf{e}_{x_1}[k] \\ \mathbf{e}_{y_1}[k] \\ \sigma_{e_y}[k] \end{bmatrix}$$

Substituting all expressions for $\bar{\mathbf{A}}_{ij}$ $(i,j = 1...4))$ results in the mentioned closed-loop system matrix. Obviously, the closed-loop system will be stable if all eigenvalues of $\mathbf{A}_{cl}$ are within the unit circle. □

From Lemma 3.1 it may be concluded that designing a stable sliding surface according to the procedure given in Section 2 and choosing the stable feedback matrix $\mathbf{\Phi}$, may not automatically lead to a stable closed-loop system. And thus, after designing the sliding surface S and the feedback matrix $\mathbf{\Phi}$, the closed loop stability has to be checked and possibly the design procedure has to be repeated.

4. STATE-ERROR ESTIMATION

The controller obtained in the previous section (equation (19)), estimates the unknown terms by zero. This choice is based on the fact that the error is supposed to converge to zero (or at least to a small region around zero). However, a state-error observer may give more accurate results. This section explores this idea, it was inspired by the sliding mode observer design methodology given in (Haskara *et al.*, 1998). A reduced order observer (in fact the order of the observer is $n-p$) is used to reconstruct the error states $\mathbf{e}_{x_o}[k]$ and $\mathbf{e}_{x_1}[k]$. The observed error-states will be represented by $\hat{\mathbf{e}}_{x_o}[k]$ and $\hat{\mathbf{e}}_{x_1}[k]$. The control law resulting from these considerations is given by:

$$\begin{aligned}\mathbf{u}_{fb}[k] = \left(\mathbf{\Phi} - \bar{\mathbf{A}}_{44}\right)\sigma_{e_y}[k] - \bar{\mathbf{A}}_{43}\mathbf{e}_{y_1}[k] \\ - \bar{\mathbf{A}}_{41}\hat{\mathbf{e}}_{x_o}[k] - \bar{\mathbf{A}}_{42}\hat{\mathbf{e}}_{x_1}[k]\end{aligned} \tag{20}$$

We will now define the reduced order state error observer as:

$$\begin{aligned}\begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k+1] \\ \hat{\mathbf{e}}_{x_1}[k+1]\end{bmatrix} &= \begin{bmatrix}\bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{22}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k] \\ \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix} \\ &+ \begin{bmatrix}\bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{y_1}[k] \\ \sigma_{e_y}[k]\end{bmatrix} \\ &- \mathbf{L}(\sigma_{e_y}[k+1] - \mathbf{\Phi}\sigma_{e_y}[k])\end{aligned} \tag{21}$$

where $\mathbf{L} \in \mathbb{R}^{(n-p)\times(n-p)}$ is a design matrix. The following lemma gives the conditions for the above observer to converge to the error states $\mathbf{e}_{x_o}$ and $\mathbf{e}_{x1}$.

Lemma 4.1. A stabilizing matrix L for observer (21) can be found for stabilizable pairs $(\mathbf{A}_{obs}, \mathbf{B}_{obs})$ where $\mathbf{A}_{obs} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $\mathbf{B}_{obs} \in \mathbb{R}^{(n-p)\times p}$ are given by:

$$\mathbf{A}_{obs} = \begin{bmatrix}\bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{22}\end{bmatrix} \qquad \mathbf{B}_{obs} = \begin{bmatrix}\bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{42}\end{bmatrix}$$

The observer error $(\mathbf{x}_{obs} - [\mathbf{e}_{x_o}[k] \quad \mathbf{e}_{x_1}[k]]^T)$ will then converge to zero.

Proof : From equation (16) we know that:

$$\begin{aligned}\sigma_{e_y}[k+1] = \bar{\mathbf{A}}_{41}\mathbf{e}_{x_o}[k] + \bar{\mathbf{A}}_{42}\mathbf{e}_{x1}[k] \\ + \bar{\mathbf{A}}_{43}\mathbf{e}_{y_1}[k] + \bar{\mathbf{A}}_{44}\sigma_{e_y}[k] + \mathbf{u}_{fb}[k]\end{aligned}$$

Substituting control law (20) in the above equation and rearranging the terms leads to:

$$\begin{bmatrix}\bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{41}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{x_o}[k] - \hat{\mathbf{e}}_{x_o}[k] \\ \mathbf{e}_{x_1}[k] - \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix} = \sigma_{e_y}[k+1] - \mathbf{\Phi}\sigma_{e_y}[k]$$

Substituting the previous equation in the observer equation (21) results in:

$$\begin{aligned}\begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k+1] \\ \hat{\mathbf{e}}_{x_1}[k+1]\end{bmatrix} &= \begin{bmatrix}\bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{22}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k] \\ \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix} \\ &+ \begin{bmatrix}\bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{y_1}[k] \\ \sigma_{e_y}[k]\end{bmatrix} \\ &- \mathbf{L}\begin{bmatrix}\bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{42}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{x_o}[k] - \hat{\mathbf{e}}_{x_o}[k] \\ \mathbf{e}_{x_1}[k] - \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix}\end{aligned}$$

From equation (16) we know that:

$$\begin{aligned}\begin{bmatrix}\mathbf{e}_{x_o}[k+1] \\ \mathbf{e}_{x_1}[k+1]\end{bmatrix} &= \begin{bmatrix}\bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{22}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k] \\ \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix} \\ &+ \begin{bmatrix}\bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{y_1}[k] \\ \sigma_{\mathbf{e}_y}[k]\end{bmatrix}\end{aligned}$$

Subtracting the previous two equations from each other according to:

$$\mathbf{e}_{obs}[k] = \begin{bmatrix}\hat{\mathbf{e}}_{x_o}[k] \\ \hat{\mathbf{e}}_{x_1}[k]\end{bmatrix} - \begin{bmatrix}\mathbf{e}_{x_o}[k] \\ \mathbf{e}_{x_1}[k]\end{bmatrix}$$

leads to:

$$\mathbf{e}_{obs}[k+1] = \left(\underbrace{\begin{bmatrix}\bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{22}\end{bmatrix}}_{=\ \mathbf{A}_{obs}} - L\underbrace{\begin{bmatrix}\bar{\mathbf{A}}_{41} & \bar{\mathbf{A}}_{42}\end{bmatrix}}_{=\ \mathbf{B}_{obs}}\right)\mathbf{e}_{obs}[k]$$

where $\mathbf{e}_{obs}[k]$ represents the observer error. From the above it can be concluded that the observer error converges to zero if the eigenvalues of $\mathbf{A}_{obs} - \mathbf{L}\mathbf{B}_{obs}$ are within the unit circle. Constructing a stable observer is possible for all stabilizable pairs $(\mathbf{A}_{obs}, \mathbf{B}_{obs})$. □

The above lemma shows that in open-loop configuration the observer error converges to zero if the design matrix $\mathbf{L}$ is properly chosen. However closed-loop stability has not yet been proven. The following lemma studies the closed-loop stability.

Lemma 4.2. The closed-loop system, formed by then system (14) with controller (20) and reduced order observer (21), is stable if the matrix:

$$\begin{bmatrix}
\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{S}_2^{-1}\mathbf{S}_1 & \mathbf{A}_{14}\mathbf{S}_2^{-1} & 0 & 0 \\
0 & \mathbf{A}_{22} & \mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1 & \mathbf{A}_{24}\mathbf{S}_2^{-1} & 0 & 0 \\
0 & \mathbf{A}_{32} & \mathbf{A}_{33} - \mathbf{A}_{34}\mathbf{S}_2^{-1}\mathbf{S}_1 & \mathbf{A}_{34}\mathbf{S}_2^{-1} & 0 & 0 \\
\mathbf{S}_2\mathbf{A}_{41} & \mathbf{S}_1\mathbf{A}_{32} + \mathbf{S}_2\mathbf{A}_{42} & 0 & \mathbf{\Phi} & -\mathbf{S}_2\mathbf{A}_{41} & -\mathbf{S}_1\mathbf{A}_{32} - \mathbf{S}_2\mathbf{A}_{42} \\
-\mathbf{L}_1\mathbf{S}_2\mathbf{A}_{41} & -\mathbf{L}_1\mathbf{S}_1\mathbf{A}_{32} - \mathbf{L}_1\mathbf{S}_2\mathbf{A}_{42} & \mathbf{A}_{13} - \mathbf{A}_{14}\mathbf{S}_2^{-1}\mathbf{S}_1 & \mathbf{A}_{14}\mathbf{S}_2^{-1} & \mathbf{A}_{11} + \mathbf{L}_1\mathbf{S}_2\mathbf{A}_{41} & \mathbf{A}_{12} + \mathbf{L}_1\mathbf{S}_1\mathbf{A}_{32} + \mathbf{L}_1\mathbf{S}_2\mathbf{A}_{42} \\
-\mathbf{L}_2\mathbf{S}_2\mathbf{A}_{41} & -\mathbf{L}_2\mathbf{S}_1\mathbf{A}_{32} - \mathbf{L}_2\mathbf{S}_2\mathbf{A}_{42} & \mathbf{A}_{23} - \mathbf{A}_{24}\mathbf{S}_2^{-1}\mathbf{S}_1 & \mathbf{A}_{24}\mathbf{S}_2^{-1} & \mathbf{L}_2\mathbf{S}_2\mathbf{A}_{41} & \mathbf{A}_{22} + \mathbf{L}_2\mathbf{S}_1\mathbf{A}_{32} + \mathbf{L}_2\mathbf{S}_2\mathbf{A}_{42}
\end{bmatrix}$$

has all its eigenvalues within the unit circle, where L has been partitioned into $\mathbf{L}_1 \in \mathbb{R}^{r\times(n-p)}$ and $\mathbf{L}_2 \in \mathbb{R}^{(n-p-r)\times(n-p)}$ according to:

$$\mathbf{L} = \begin{bmatrix}\mathbf{L}_1 \\ \mathbf{L}_2\end{bmatrix}$$

Proof : The states of the closed-loop system are formed by the n states of the system and the $n-p$ states of the reduced order observer. We therefore augment the system states with the observer states leading to the total state $\mathbf{x}_{tot} \in \mathbb{R}^{(2n-p)}$:

$$\begin{aligned}\mathbf{x}_{tot}[k] = [\mathbf{e}_{x_o}[k] \quad \mathbf{e}_{x_1}[k] \quad \mathbf{e}_{y_1}[k] \quad \sigma_y[k] \\ \hat{\mathbf{e}}_{x_o}[k] \quad \hat{\mathbf{e}}_{x_1}[k]]^T\end{aligned}$$

With equations (14), (20), and (21) we can find the closed-loop dynamics $\mathbf{x}_{tot}[k+1] + \mathbf{A}_{cl}\mathbf{x}_{tot}[k]$,

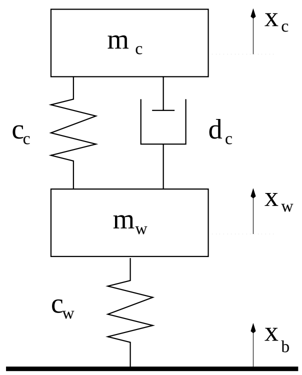

Fig. 1. Mechanical diagram of the Quarter Car Model.

which is obviously stable if all eigenvalues of $\mathbf{A}_{cl}$ are within the unit circle. Substituting all expressions for $\bar{\mathbf{A}}_{ij}$ $(i, j = 1 \ldots 4)$ results in the mentioned closed-loop system matrix. □

5. SIMULATION EXAMPLE

As a simulation example we have chosen the so called Quarter Car. It represents one quarter of a vehicle placed on a moving base. To improve reproducibility of test procedures for cars as well as durability tests of new developed cars, one would like to reproduce predefined road-profiles exactly. Therefore the goal of the controller is to reproduce a measured road profile and hence give the car on the base exactly the same accelerations in every successive test. A schematic test setup is presented in Figure 1 for which we can obtain the continuous-time linear model:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}$$

The system state, input and output represent the following physical quantities:

$$\mathbf{x} = [x_c \quad x_w \quad \dot{x}_c \quad \dot{x}_w]^T \qquad \mathbf{y} = [\dot{x}_c \quad \ddot{x}_c] \qquad u = x_b$$

The variables x_c, x_w, and x_b represent the car, wheel and base displacement respectively. The system matrices are given by:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{c_c}{m_c} & \dfrac{c_c}{m_c} & -\dfrac{d_c}{m_c} & \dfrac{d_c}{m_c} \\ \dfrac{c_c}{m_w} & -\dfrac{c_c + c_w}{m_w} & \dfrac{d_c}{m_w} & -\dfrac{d_c}{m_w} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{c_w}{m_w} \end{bmatrix}^T$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\dfrac{c_c}{m_c} & \dfrac{c_c}{m_c} & -\dfrac{d_c}{m_c} & \dfrac{d_c}{m_c} \end{bmatrix}$$

where m_w is the mass of the wheel, m_c is one quarter of the mass of the car, c_w is the wheel stiffness, c_c the suspension stiffness and d_c the suspension damping. The model is used for the controller design. The designed controllers are tested on the "system" for which the parameters differ considerably from the nominal system, as can be seen in Table 1. The discrete-time model is obtained by a sample and hold dicretization with sampling time $T_s = 5ms$.

Variable	Model	System	unit
m_c	200	300	[kg]
m_w	33	30	[kg]
c_c	9000	7000	[N/m]
c_w	20.000	22.000	[N/m]
d_c	1200	1100	[Nsec/m]

Table 1. Variable values of the model and the system.

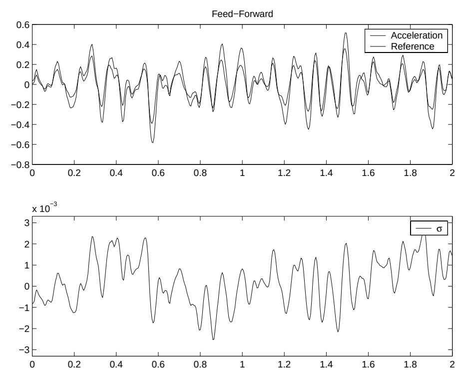

Fig. 2. Simulation results with feed-forward controller only.

The total control action $u[k]$ is taken as the sum of a feed-forward term $u_{ff}[k]$ and a sliding mode feed-back term $u_{fb}[k]$, hence:

$$u[k] = u_{ff}[k] + u_{fb}[k]$$

The feed-forward term is computed such that it gives optimal tracking for the nominal system. The feed-back part will be used to compensate for modeling errors, therefore the switching function is defined as:

$$\sigma_{e_y}[k] = S\left(\mathbf{y}[k] - \mathbf{r}[k]\right)$$

where the signal $r\mathbf{r}[k]$ is the reference, or target, signal. Simulations are presented for the feed-forward controller only (Figure 2), the output based sliding mode controller without observed error state (Figure 3), and the output-based sliding mode controller with a state-error observer (4). For the sliding mode controller $\Phi = 0$ and $\mathbf{S} = [0.123 \quad 0.00058]$ are used which gives the poles in sliding mode $p_1 = 0.96$ and $p_2 = 0.90$. The closed-loop (nominal) system of the controller without the reduced order state-error observer can be shown to be stable with the aid of Lemma 3.1. The reduced order observer gain L is given by: $\mathbf{L} = [-20.28 \quad -190.25]$ which results in a dead-beat observer (all observer poles at zero). The poles of the closed-loop system with the reduced

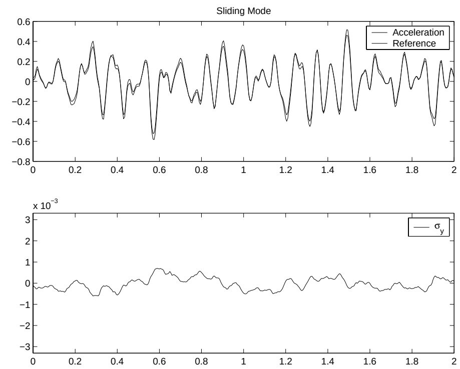

Fig. 3. Simulation results for the sliding mode controller without state-error estimation. .

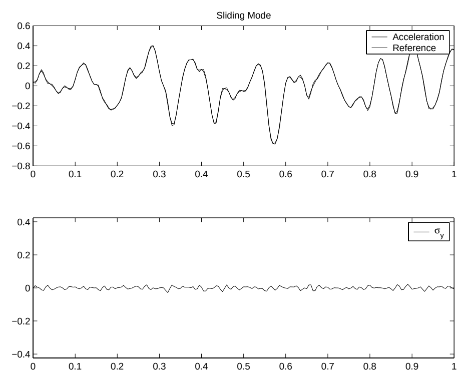

Fig. 4. Simulation results for the sliding mode controller with state-error estimation. .

order state-error observer, which can be found from Lemma 4.2, are again stable.

The Variance Accounted For (VAF) of the output, defined by:

$$\mathrm{VAF}(\mathbf{y},\mathbf{r}) = \left(1 - \frac{\mathrm{variance}(\mathbf{y}-\mathbf{r})}{\mathrm{variance}(\mathbf{y})}\right)$$

which can be computed from the simulation results, is given for each controller setup by:

$$\mathrm{VAF}_{ff}(\mathbf{y},\mathbf{r}) = \begin{bmatrix} 73.3 \\ 81.6 \end{bmatrix}$$

$$\mathrm{VAF}_{smc}(\mathbf{y},\mathbf{r}) = \begin{bmatrix} 86.8 \\ 96.1 \end{bmatrix}$$

$$\mathrm{VAF}_{smcseo}(\mathbf{y},\mathbf{r}) = \begin{bmatrix} 100.0 \\ 99.7 \end{bmatrix}$$

where the subscript $_{ff}$ stands for the feed-forward controller only, $_{smc}$ for the sliding mode controller without state error observer, and $_{smcseo}$ for the sliding mode controller with state error observer.

The figures and the computed VAF clearly demonstrate the improvements made by the (output-based) sliding mode controllers. The performance of the output-based sliding mode controller with the reduced order state-error observer is nearly perfect.

6. CONCLUSION

In this paper, a discrete-time, output-based, sliding mode controller is introduced. Together with a feed-forward tracking controller, the closed-loop system exhibits both good tracking performance as robustness against model uncertainties and disturbances. In simulation it was demonstrated that this controller setup leads to excellent performance. The discrete-time, output-based, sliding mode controller with a reduced order error-state observer achieved nearly perfect tracking, even in the presence of a severe mismatch between the model and the system.

7. REFERENCES

DeCarlo, R. A., S. H. Żak and G. P. Matthews (1988). Variable structure control of nonlinear multivariable systems: A tutorial. *Proceedings of the IEEE* **Vol. 76**, pp. 212–232.

Edwards, C. and S.K. Spurgeon (1998). *Sliding Mode Control, Theory and Applications*. Taylor & Francis Ltd.

Haskara, İ., Ü Özgüner and V. Utkin (1998). On sliding mode observers via equivalent control approach. *International Jountrol of Control* **71**(6), 1051–1067.

Hung, J.Y., W. Gao and J.C. Hung (1993). Variable structure control: A survey. *IEEE Transactions on Industrial Electronics* **Vol. 40**, pp. 2–21.

Utkin, V., J. Guldner and J. Shi (1999). *Sliding Mode Control in Electromechanical Systems*. Taylor & Francis Ltd.

Utkin, V.I. (1992). *Sliding Modes in Control Optimization*. Springer-Verlag.

Young, K. D. and Ü. Özgünter (1999). Sliding mode: Control engineering in practice. In: *Proceedings of the American Control Conference*.