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5 Singularity theory

The space of functions – or maps – is huge. Fortunately, many of its elements may be regarded as equivalent in a natural way, for example under right- or left-right transformations. Singularity theory studies how such equivalences foliate these spaces into a more manageable family of orbits of equivalent functions or maps.

5.1 Overview

Singularity theory plays a pivotal role in this work. It provides the theorems to prove existence of normalizing transformations to the one-degree-of-freedom model systems, which is the subject of Chaps. 2 and 3. In Chap. 7 the constructive counterparts of these theorems are used to compute the normalizing transformations. For this to work, the bases for the tangent spaces involved must be brought into so-called *standard* form. The algebraic structure of these spaces depends on the equivalence class (e.g. right-transformations, left-right transformations), and motivated the search for the appropriate generalizations of Gröbner bases in Chap. 6.

This chapter collects the results on singularity theory that are used throughout. Most of the results were taken from the rather technical [Mar82, Mat68]. More accessible introductions are [BL75, BG84, GG73, Gib79, Lu76, Was74], and [GS79, PS78, Sma67, Tho72] give much motivation and philosophy. Though we do not use it here, classification of singularities touch at the core of the subject; see e.g. [Arn81, Arn93b, Sie74, Sie73]. The relation of singularity theory and symmetries has also been extensively studied; see e.g. [BF91, GS85, GSS88, Poè76, BHvNV99]

We note here that the Mather-Malgrange preparation theorem is only referred to in passing. This important theorem bridges the gap between finite-dimensional deformation theory, summarized in Sect. 5.2, and deformation theory in the smooth category. This chapter indeed deals with C^∞ functions and maps, but our interest lies in the computation of finite parts of reparametrizations. For this purpose, it makes sense to work with (truncated) formal power series, and Chaps. 6 and 7 are set in this context. Regarding the relevance of the current chapter for those, via the jet-map and Borel's theorem, it is an exercise to show that the main results of singularity theory for smooth functions and maps, immediately

imply those for the formal setting. In fact the formal results may be proven directly without much difficulty, and indeed the results of Chaps. 6 and 7 can be viewed as constructive (i.e., algorithmic) proof of the formal versions of theorems 5.5 and 5.22.

The setup of this chapter is as follows. The first section is an overview of the finite-dimensional analogue of singularity theory, giving a palatable introduction of the fundamental concepts. The next section presents singularity theory of functions, with right-transformations as the equivalence relation. This section also deals with BCKV normal forms, a specific kind of deformations geared to the application of Chap. 2. Finally, the elements of singularity theory dealing with maps under left-right transformations, necessary for analyzing the energy-momentum map in Chap. 3, are presented in Sect. 5.4.

5.2 Introduction: The finite dimensional case

Almost every concept used in the singularity theory of functions and maps, has a direct counterpart in the finite dimensional context of Lie groups acting on smooth finite-dimensional manifolds. In this finite-dimensional setting, the proofs are straightforward and only involve the implicit function theorem. Since this theory is so technically undemanding but nonetheless conceptually rich, it is a good introduction to sections 5.3 and 5.4. Our main source for this section was [Gib79].

Let M be a manifold, G a Lie group, and suppose that both are smooth, finite dimensional manifolds. Let $\zeta : G \times M \rightarrow M$ be a smooth action of G on M . Instead of $\zeta(\gamma, f)$ we sometimes simply write γf . For a given point $f \in M$, the action ζ gives rise to an orbit, in this notation given by Gf . We are interested in the tangent space to this orbit at the point f , which we denote by $T_f(Gf)$.

Let $\zeta_f : G \rightarrow M$ denote the map $\gamma \mapsto \zeta(\gamma, f)$. Its image is the orbit Gf . The tangent space to Gf at $f = \zeta_f(Id)$ is just the image under the differential $D\zeta_f$ of the tangent space $T_{Id}(G)$ to G at the identity element:

$$T_f(Gf) = D_{Id}\zeta_f(T_{Id}(G)).$$

(Here, and elsewhere in this chapter, D_x denotes the (total) differential at x , not the differential with respect to x .) The codimension of $T_f(Gf)$ in $T_f(M)$ is also called the *codimension of f* . If this codimension is 0 the inverse function theorem can be applied, with the result that for every f' in some neighborhood of f , there exist $\gamma \in G$ such that $f' = \zeta(\gamma, f) = \gamma f$. We say that f' is *equivalent* to f , and write $f' \sim f$. Concisely,

Definition 5.1. $f' \sim f \iff \exists \gamma \in G : f' = \gamma f$

An element f with the property that it is equivalent to every f' in its neighborhood is called a *stable* element. Note that the dimension of $T_f(Gf)$ is never larger than the dimension of G itself, therefore G has to be ‘large’ enough in order for stable elements to exist.

5.2.1 Deformations

Stable elements, with codimension 0, form the simplest case. Now let us proceed, and suppose that the codimension of f is nonzero, say equal to d . Small changes to f along orbits of G will *not* change f 's equivalence class; however changes *transversal* to its G -orbit will. A catalog of representatives of all equivalence classes that occur in a neighborhood of f is given by a transversal section of the orbit Gf at f . Such a transversal section, which is a submanifold in M of dimension d , can be parametrized as a d -parameter family of elements in M . Families depending on parameters are called *deformations*:

Definition 5.2. A d -parameter deformation of f in M is a map $F(u) : \mathbb{R}^d \rightarrow M$ such that $F(0) = f$,

and the infinitesimal directions in which $F(0) = f$ is deformed are called “deformation directions”:

Definition 5.3. Let $F(u) : \mathbb{R}^d \rightarrow M$ be a deformation of $F(0) = f$. The elements $\frac{\partial F}{\partial u_i} \Big|_{u=0} \in T_f(Gf)$ are called the deformation directions of F .

Note that this definition is coordinate-dependent.

Instead of *deformation direction* the synonym *initial speed* is also used; and instead of *deformations* the term *unfoldings* is used often. The latter are sometimes defined a little differently, see e.g. [Mar82], the difference being mainly notational, see e.g. [Mon94].

Now, the deformation F is a transversal section if its deformation directions complement the tangent space, or symbolically:

$$(5.1) \quad T_f(Gf) \oplus D_0F(\mathbb{R}^d) = T_f(M).$$

Deformations for which (5.1) hold are called *transversal deformations*. Now let $G(v) : \mathbb{R}^k \rightarrow M$ be some deformation of f . It is said to be *induced* from F if there exists a reparametrization $h : \mathbb{R}^k \rightarrow \mathbb{R}^d$ with $h(0) = 0$, and a deformation $I : \mathbb{R}^k \rightarrow G$ of the identity element in G , such that

$$G(v) = I(v)F(h(v)).$$

If it happens that *every* deformation G of f can be induced from F in this way, then F is called a *versal* deformation. Again by the inverse function theorem it can be shown that a deformation is versal if it is transversal, see e.g. [Gib79, p. 90]. The “only if” direction is trivial. Sometimes the notion of *universal deformation* is used; this is a versal deformation with a minimal number of parameters.

It thus turns out that the infinitesimal data of the tangent space T_f is enough to write down a versal deformation, which captures all possible behavior of the singularity f in a full neighborhood of it, modulo the equivalences of the Lie group.

5.3 Functions and right-transformations

We now turn to the case of smooth functions on \mathbb{R}^n acted upon by the group of right-transformations. Neither the manifold of smooth functions nor the group is finite-dimensional, and the previous section does not apply as it stands. It is a remarkable fact that the results do continue to hold true for this case. The proofs are much more difficult, however, since the inverse function theorem cannot be used; see e.g. [Mar82, BL75] for proofs and more details. Additional complications arise from the necessity of truncating when doing actual computations. Lots of results exist that guarantee sufficiency of certain truncation-orders in several special cases. These can be regarded as generalizations or extensions of the Morse lemma, which states¹ that if a germ has a nondegenerate quadratic part, it is equivalent to the germ obtained by truncating at degree 2. Of this class of results we apply only two, namely propositions 5.7 and 5.8.

In contrast to context of smooth functions, for formal power series the proof of the main theorem (theorem 5.5, the equivalence of transversal and versal deformations) is relatively straightforward. In fact the results of Chap. 7 can be regarded as constructive proof. (See also remark 5.23 below.) Another straightforward way of proving the results in the formal context is to proceed directly from the smooth results, using the jet-map and Borel's theorem. We shall not make this explicit, however, and be satisfied with the algorithms of Chap. 7.

Let us introduce the main players in more detail. The manifold M is the set of Γ -invariant C^∞ functions on \mathbb{R}^n , symbolically $M = \mathcal{E}_n^\Gamma$. Here Γ is some compact (usually finite) Lie group with a linear action on \mathbb{R}^n . The group of transformations G that act on M is the group of origin preserving Γ -equivariant C^∞ maps on \mathbb{R}^n , and they act on elements of M by composition on the right. The results of this section are well-known for the case that $\Gamma = \{Id\}$, and straightforward generalizations otherwise; see [Was75, BHLV98].

5.3.1 Equivalence and versal deformations

Definition 5.4. \mathcal{E}_n^Γ is the set of germs of smooth Γ -invariant functions on \mathbb{R}^n .

\mathbf{V}_n^Γ is the \mathcal{E}_n^Γ -module of germs of smooth Γ -equivariant vector fields vanishing at the origin.

The tangent space to the group of right-transformations G at the identity mapping is (isomorphic to) the module of vector fields \mathbf{V}_n^Γ . Now the tangent space $T_f(Gf)$ to the orbit of f , again abbreviated to T_f , is:

$$T_f := T_f(Gf) = D_{Id}\zeta_f(T_{Id}(G)) = \{Xf | X \in \mathbf{V}_n^\Gamma\}.$$

The notion of *deformation* of a d -parameter function (or, more precisely, of a *germ* of a function²) f is the straightforward analogue of a deformation in the

¹ Actually it implies a little more, see e.g. [GG73, Thm. 6.9]

² Note the difference between germs of deformations, and deformations of germs: Transformations between the former may shift the origin as a function of the parameters. We use deformations of germs, requiring transformations to keep the origin fixed.

finite-dimensional context: a map from \mathbb{R}^d to the space of functions $M = \mathcal{E}_n^\Gamma$. Again, F is called a *transversal deformation* if

$$T_f + D_0F(\mathbb{R}^d) = T_f(M).$$

In particular, f has a transversal deformation if and only if $T_f \subseteq T_f(M)$ has finite codimension. Secondly, a deformation $F(u) : \mathbb{R}^d \rightarrow M$ is called a *versal deformation* of $f = F(0)$ if for any deformation $G(v) : \mathbb{R}^q \rightarrow M$ there exist maps ϕ and h , with

$$\phi : \mathbb{R}^n \oplus \mathbb{R}^q \rightarrow \mathbb{R}^n, \quad h : \mathbb{R}^q \rightarrow \mathbb{R}^d,$$

such that $G(x, v) = F(\phi(x, v), h(v))$. It is easy to show that a versal deformation is transversal. The converse is much more difficult to prove:

Theorem 5.5. *Suppose $f \in M$ is a germ of a Γ -invariant C^∞ function, and $F : \mathbb{R}^d \rightarrow M$ is a transversal deformation of f . Then F is a versal deformation of f*

For a proof, see [Was74, Thm 3.19]. Note that existence of a transversal deformation implies that f is finitely determined, by [Was74, Thm. 2.6] or [Mat68, Thm. 3.5]. For more information see also [Sie74].

We conclude this section by giving a few effective conditions for equivalence of functions. The adjective ‘effective’ refers to the fact that these conditions can be checked by doing calculations in finite-dimensional vector spaces. The propositions below are used in Sect. 5.3.2.

Definition 5.6. a) j^k is the jet map of order k , mapping germs in \mathcal{E}_n^Γ to their Taylor polynomial up to and including order k .
b) $\mathfrak{m}_k := \{f \in \mathcal{E}_n^\Gamma \mid j^{k-1}(f) = 0\}$

In particular $j^0(f)$ is the constant part of f , and $\mathfrak{m}_1 = \mathfrak{m}$, the maximal ideal in \mathcal{E}_n^Γ . If $\Gamma = \{Id\}$ then $\mathfrak{m}_k = \mathfrak{m}^k$. We stick to the case that $\Gamma = \{Id\}$ a little longer, and drop the Γ from the notation. Note that \mathbf{V}_n is generated, as an \mathcal{E}_n -module, by $x_i \frac{\partial}{\partial x_j}$ with $1 \leq i, j \leq n$. This means that the tangent space T_f is just $\mathfrak{m} \cdot J(f)$, where $J(f)$ is the Jacobian ideal of f :

$$J(f) := \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

The following proposition gives conditions under which (non-symmetric) germs are equivalent:

Proposition 5.7. [Mar82, III.4.2] *Let $f, g \in \mathcal{E}_n$, and assume that $g - f \in \mathfrak{m}_k$, i.e., $j^{k-1}(g - f) = 0$.*

- a) If $T_f \supset \mathfrak{m}_k$ then $g \sim f$ provided that $j^k(g - f)$ is small enough.
 b) If $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ then $g \sim f$.

(Here ‘ $j^k(g - f)$ small enough’ means that the coefficients of the k th order Taylor polynomial of $g - f$ at the origin are sufficiently small.) The analogous result for germs with symmetry is this:

Proposition 5.8. *Let $f, g \in \mathcal{E}_n^\Gamma$, and suppose that $j^{k-1}(g - f) = 0$. Let M denote the finite-dimensional vector space $\mathfrak{m}_k/(\mathfrak{m} \cdot \mathfrak{m}_k)$, and set*

$$M_m := M \cap (\mathfrak{m}_m/\mathfrak{m}_{m+1}) = \{h \in M : h \text{ is homogeneous of degree } m\}.$$

- a) Suppose that $T_f \supset \mathfrak{m}_k$ then $g \sim f$ provided that the projection of $g - f$ into M is sufficiently small.
 b) Suppose that $\mathfrak{m} \cdot T_f \supset \mathfrak{m}_k$ then $g \sim f$.
 c) Suppose that $T_f \supset \mathfrak{m}_k$. Suppose further that the projection of f into M is an element of M_k . Then $g \sim f$ provided that $j^k(g - f)$ is sufficiently small.

See appendix A.2 for the proof.

5.3.2 Applications

The following examples are used in Chaps. 2 and 3, and use the propositions of the previous section.

The Γ -invariant Morse lemma As a first application of proposition 5.8, we extend the Morse lemma (see e.g. [Mar82, GG73]) to the Γ -invariant case.

Proposition 5.9. *Let $f \in \mathcal{E}_n^\Gamma$ be a germ without linear part, and a nondegenerate quadratic part. Then, for any $g \in \mathcal{E}_n^\Gamma$ with $j^2(g - f)$ small enough, $f \sim g$.*

Proof: Let f_2 be the quadratic part of f . Since f_2 is nondegenerate, we have $T_{f_2} = \mathfrak{m}_2$. As $j^2(f - f_2) = 0$ by definition, proposition 5.8 (c) implies that $f \sim f_2$. Since $j^2(g - f_2) = j^2(g - f)$ is small, applying the same proposition again yields $g \sim f_2$. Together this implies $f \sim g$. ■

Remark 5.10. This result also follows from the symmetric splitting lemma, which is given in e.g. [BF91, App. A]

The hyperbolic umbilic The next example is used in the 1 : 2 planar reduction, to produce a normal form of a singular \mathbb{Z}_2 -symmetric germ with vanishing quadratic, but non-vanishing third order part. This result is used in section 2.3.2 in the proof of proposition 2.12.

Proposition 5.11. *Let \mathbb{Z}_2 be a group with \mathbb{R}^2 -action $(x, y) \mapsto (x, -y)$. Let f be the \mathbb{Z}_2 -symmetric germ $f = x(\alpha x^2 + \beta y^2) + h.o.t.$, and assume $\alpha \neq 0, \beta \neq 0$. Then f is isomorphic to $\pm x(x^2 + y^2)$.*

Proof: By a linear transformation we may assume that α and β are in fact both $+1$ or -1 . The tangent space T_g with $g = \pm x(x^2 + y^2)$ is generated by $x \frac{\partial g}{\partial x} = 3x^3 + xy^2$, $y^2 \frac{\partial g}{\partial x} = 3x^2y^2 + y^4$ and $y \frac{\partial g}{\partial y} = 2xy^2$. Equivalently, $T_g = \langle x^3, xy^2, y^4 \rangle_{\mathcal{E}_2^{\mathbb{Z}_2}} = \mathfrak{m}_3$. Now apply proposition 5.8(c) with $k = 3$ to g . Since g is homogeneous, $g \in M_3$, so we conclude that $f = g + h.o.t. \sim g$ for arbitrary higher order terms. ■

Symmetric deformation of the hyperbolic umbilic We conclude this section by giving a versal deformation of the normal form $x(x^2 + y^2)$ found in proposition 5.11 above. See also Sect. 7.2.4.

Proposition 5.12. *Let \mathbb{Z}_2 be a group acting on \mathbb{R}^2 by $(x, y) \mapsto (x, -y)$. Then $F(x, y; u_0, u_1, u_2) := x(x^2 + y^2) + u_0 + u_1x + u_2y^2$ is a versal deformation of $x(x^2 + y^2)$ in the space of \mathbb{Z}_2 -invariant functions.*

Proof: The module of \mathbb{Z}_2 -equivariant vector fields is generated by $\frac{\partial}{\partial x}$ and $y \frac{\partial}{\partial y}$, so that $T_f = \langle 3x^2 + y^2, 2xy^2 \rangle_{\mathcal{E}_2^{\mathbb{Z}_2}} = \langle 3x^2 + y^2, 2xy^2, x^3, y^4 \rangle_{\mathcal{E}_2^{\mathbb{Z}_2}}$. This shows that $T_f + \text{span}_{\mathbb{R}}\{1, x, y^2\} = \mathcal{E}_2^{\mathbb{Z}_2}$. By definition this means that F is a transversal deformation of $x(x^2 + y^2)$. An application of theorem 5.5 completes the proof. ■

Remark 5.13. These examples are the tip of an iceberg. For more on classification of singularities see e.g. [Arn81, BF91, BL75, BG84, GS85, GSS88, Mar82, Sie73, Sie74].

5.3.3 BCKV normal form

The deformations appearing in Sect. 5.3.1, see for example Proposition 5.12, have two kinds of variables: the ordinary ‘phase space’ variables like x and y , and the deformation parameters u_1, \dots, u_d . The different roles played by these variables is clearly seen in the transformations (i.e., equivalence relation) between deformations: Transformations of the phase variables may depend on both kinds of variables, whereas transformations of the deformation parameters must not depend on the phase variables.

In some applications, more than two classes of variables appear in a natural way. For example, there are problems where time, space and parameters appear and should be distinguished; see [Was75]. This section deals with yet another situation, where we have two kinds of parameters, namely *ordinary* and *distinguished* ones, the dependence between these variables in transformations of deformations is according to the following table:

Variable \rightarrow	phase distinguished ordinary		
May depend on \downarrow	-----		
phase	yes	no	no
distinguished	yes	yes	no
ordinary	yes	yes	yes

In Chap. 2 this situation is met. There the distinguished parameters are phase variables which are no part of the (reduced) dynamical system. Since they are phase variables, the (ordinary) parameters of the model may not depend on them. Coordinate transformations that respect this structure are called BCKV transformations (see [BCKV93, BCKV95]).

Since BCKV normal form puts restrictions on the reparametrizations, it is reasonable to expect the number of (equivalence classes of) normal forms to increase. This indeed happens, even to the extent that normal forms of finite codimension cease to exist. This problem is tackled using the *path formulation* [GS85, GS79, Mon94]. In effect this amounts to fixing a number of deformation parameters in the deformation, suitable to the problem at hand, in other words fixing a *path* through the infinite dimensional space of deformation parameters; see also [BCKV93].

The normal form and transformations are supposed to be equivariant under the action of some compact symmetry group Γ .

BCKV theory – definitions and main theorem The main point was made above: reparametrizations may not depend on the distinguished parameters. Secondly, in the intended application, the distinguished parameter is an *angular momentum*, and therefore nonnegative. Hence, we also require the reparametrization of the distinguished parameter to respect the zero level. These ingredients lead to definition 5.15 below.

Remark 5.14. (*Notation*) With BCKV deformations, there are three levels of variables: *phase* variables $x \in \mathbb{R}^n$, *distinguished* parameters $\lambda \in \mathbb{R}^r$ and *ordinary* parameters $u \in \mathbb{R}^s$. We abbreviate $\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^s$ by \mathbb{R}^{n+r+s} , and similarly for subspaces.

Definition 5.15. (*BCKV-restricted morphisms:*) Let two deformations $F \in \mathcal{E}_{n+r+s}^\Gamma$ and $G \in \mathcal{E}_{n+r+t}^\Gamma$ of $f = F(\cdot, \cdot, 0) \in \mathcal{E}_{n+r}^\Gamma$ be given, such that $f(0, \lambda) = 0$. F is said to be induced from G by Γ -equivariant BCKV-restricted morphisms if there exist germs of Γ -equivariant mappings $\Psi : \mathbb{R}^{n+r+s} \rightarrow \mathbb{R}^{n+r+t}$, $\Phi : \mathbb{R}^{r+s} \rightarrow \mathbb{R}^{r+t}$ and $\Theta : \mathbb{R}^s \rightarrow \mathbb{R}^t$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 \mathbb{R}^{n+r} \times \{0\}^s & \hookrightarrow & \mathbb{R}^{n+r+s} & \xrightarrow{\pi_1} & \mathbb{R}^{r+s} & \xrightarrow{\pi_2} & \{0\}^r \times \mathbb{R}^s & \hookrightarrow & \mathbb{R}^{r+s} \\
 \text{Id} \downarrow & & \Psi \downarrow & \searrow^F & \Phi \downarrow & & \Theta \downarrow & & \Phi \downarrow \\
 & & & \mathbb{R} & & & & & \\
 & & & \nearrow_G & & & & & \\
 \mathbb{R}^{n+r} \times \{0\}^t & \hookrightarrow & \mathbb{R}^{n+r+t} & \xrightarrow{\tilde{\pi}_1} & \mathbb{R}^{r+t} & \xrightarrow{\tilde{\pi}_2} & \{0\}^r \times \mathbb{R}^t & \hookrightarrow & \mathbb{R}^{r+t}
 \end{array}$$

On \mathbb{R}^{n+r} , \mathbb{R}^{n+r+s} and \mathbb{R}^{n+r+t} the action of Γ is defined by trivially extending it on \mathbb{R}^n .

Commutativity of the diagram amounts to: There exist $\phi : \mathbb{R}^{r+s} \rightarrow \mathbb{R}^r$, $\psi : \mathbb{R}^{n+r+s} \rightarrow \mathbb{R}^n$ such that $\Phi = (\phi, \Theta)$, $\Psi = (\psi, \phi, \Theta)$, $\psi(x, \lambda, 0) = x$, $\phi(\lambda, 0) = \lambda$, $\phi(0, u) = 0$, $\Theta(0) = 0$, and $F(x, \lambda, u) = G(\psi(x, \lambda, u), \phi(\lambda, u), \Theta(u))$. The ϕ , ψ and

Θ are the analogs of ϕ and ρ_i of (2.12), but obey more restrictions. Morphisms (Ψ, Φ, Θ) as above are called *BCKV-restricted morphisms*.

Again, *versal* deformations are those deformations from which every other can be induced, and a *universal* deformation is a versal deformation with the minimal possible number of parameters. In [BCKV93, thm. 11], versal deformations with respect to BCKV-restricted morphisms are characterized. Next we prove the Γ -equivariant version of this:

Theorem 5.16. (*BCKV-restricted versal deformations:*) *Let $f \in \mathcal{E}_{n+r}^\Gamma$ be a family of germs of Γ -equivariant germs depending on a distinguished parameter $\lambda \in \mathbb{R}^r$. Let $f_0 \in \mathcal{E}_n^\Gamma : x \mapsto f(x, 0)$ have codimension c . Then*

1. *f has a universal deformation with respect to Γ -equivariant BCKV-restricted morphisms if and only if f , considered as a deformation of f_0 , is versal with respect to ordinary Γ -equivariant morphisms.*
2. *If $F(x, \lambda, u)$ is a (uni)versal deformation of f with respect to Γ -equivariant BCKV-restricted morphisms, then $F(x, 0, u)$ is a (uni)versal deformation of f_0 with respect to ordinary Γ -equivariant morphisms.*
3. *If $f(x, \lambda)$ is a universal deformation of f_0 with respect to ordinary Γ -equivariant morphisms, then $r = c$ and $F : \mathbb{R}^{n+c+c} \rightarrow \mathbb{R}$ defined by*

$$F(x, \lambda, u) = f(x, \lambda) + \sum_{j=1}^c u_j \frac{\partial f}{\partial \lambda_j}(x, 0)$$

is a universal deformation of f with respect to Γ -equivariant BCKV-restricted morphisms.

Proof: The proof for the non-equivariant case can be carried over to the present setting with obvious changes. See [BCKV93]. ■

Path formulation As the number of distinguished parameters is fixed, theorem 5.16 implies that if the central singularity f_0 has a high codimension, there are no versal deformations with respect to Γ -equivariant BCKV-restricted morphisms. However, we can view the system as a *subfamily* of a versally deformed system. The normal form then includes functions that describe the submanifold, embedded in the versal system's parameter space, that the system traces out. Bifurcations of the intersection of this submanifold with the bifurcation set yields additional information. This description is usually called the path formulation, see [GS85, GS79, Mon94, BF91]. For this final reduction, we need the following:

Definition 5.17. *A BCKV-restricted reparametrization is a mapping (ϕ, θ) with $\phi : \mathbb{R}^{r+s} \rightarrow \mathbb{R}^r$, $\theta : \mathbb{R}^s \rightarrow \mathbb{R}^s$ such that $\phi(0, u) = 0$, $\theta(0) = 0$.*

Note that it is not required that $\phi(\lambda, 0) = \lambda$. The following lemma is a slightly stronger version of [BCKV93, lemma 7], and is used in the proof of proposition 5.19 below.

Lemma 5.18. [BCKV93] Let $r \leq s$, let $\pi : \mathbb{R}^s \rightarrow \mathbb{R}^r$ be a projection onto some r -dimensional subspace of \mathbb{R}^s , and let $\tilde{h} : (\lambda, u) \in \mathbb{R}^{r+s} \rightarrow \mathbb{R}^s$ be a map (a 'normal form') such that $\tilde{h}(0, 0) = 0$ and the derivatives $D_\lambda(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ and $D_u(\pi \circ \tilde{h}(\lambda, u))|_{\lambda=u=0}$ both have rank r . Then, for any $h \in \mathcal{E}(r+s, s)$ with $h(0, 0) = 0$ there exists a BCKV-restricted reparametrization $\Upsilon = (\phi, \theta)$ such that

$$\pi(h(\lambda, u)) = \pi(\tilde{h}(\Upsilon(\lambda, u))).$$

Moreover, if also $D_\lambda \pi \circ h(\lambda, u)$ and $D_u \pi \circ h(\lambda, u)$ both have rank r (at $\lambda = u = 0$), then Υ can be chosen invertible.

Proof: As $D_u \pi \circ \tilde{h}(0, u)$ has full rank, and $\pi \circ h(0, 0) = \pi \circ \tilde{h}(0, 0)$, by the inverse function theorem there exists a function $\theta(u)$ with $\theta(0) = 0$ such that $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$. Now $D_\lambda \pi \circ \tilde{h}(\lambda, \theta(u))$ has full rank, and moreover $\pi \circ h(0, u) = \pi \circ \tilde{h}(0, \theta(u))$, for all u , so, applying the inverse function theorem again, we find a function $\phi(\lambda, u)$ with $\phi(0, u) = 0$, such that $\pi \circ h(\lambda, u) = \pi \circ \tilde{h}(\phi(\lambda, u), \theta(u))$. The last remark follows by applying the lemma with the roles of h and \tilde{h} interchanged. ■

Proposition 5.19. Let $g(x, \lambda, u) : \mathbb{R}^{n+r+s} \rightarrow \mathbb{R}$ be a generic Γ -invariant germ, and assume that $f(x, \sigma_1, \dots, \sigma_s)$ is a universal deformation of $g(x, 0, 0)$ using unrestricted Γ -equivariant morphisms. Then there exists a BCKV-restricted reparametrization Υ such that for the normal form

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \dots, \lambda_r + u_r, \tilde{\sigma}_{r+1}(\lambda, u), \dots, \tilde{\sigma}_s(\lambda, u)),$$

where $\tilde{\sigma}_i$, $i = r+1, \dots, s$, are some functions, we have that g can be induced from $F \circ (\pi_x, \Upsilon)$ using BCKV-restricted Γ -equivariant morphisms. Here π_x denotes the projection $\pi_x : (x, \lambda, u) \mapsto x$.

Proof: Let $h(\lambda, u)$ be a reparametrization, and $\Phi(x, \lambda, u)$ a coordinate transformation, such that $f(\Phi(x, \lambda, u), h(\lambda, u)) = g(x, \lambda, u)$. Define $\tilde{h}_i(\lambda, u) := \lambda_i + u_i$ if $1 \leq i \leq r$ and $\tilde{h}_i(\lambda, u) := u_i$ if $r+1 \leq i \leq s$, and set $\pi(\sigma_1, \dots, \sigma_s) = (\sigma_1, \dots, \sigma_r)$. The lemma now applies. By genericity we may assume that the relevant derivatives have rank r , so we find an invertible BCKV-restricted reparametrization Υ such that $h_i(\lambda, u) = \tilde{h}_i(\Upsilon(\lambda, u))$ for $i = 1, \dots, r$, which means that for

$$F(x, \lambda, u) := f(x, \lambda_1 + u_1, \dots, \lambda_r + u_r, h_{r+1} \circ \Upsilon^{-1}(\lambda, u), \dots, h_s \circ \Upsilon^{-1}(\lambda, u)),$$

we have $g(x, \lambda, u) = F \circ (\pi_x, \Upsilon) \circ (\Phi, \pi_\lambda, \pi_u)$, where $\pi_\lambda : (x, \lambda, u) \mapsto \lambda$ and $\pi_u : (x, \lambda, u) \mapsto u$, proving the proposition. ■

5.4 Maps and left-right transformations

In Chap. 3, the object to normalize is the *energy-momentum map*, a parameter-dependent map from phase space to \mathbb{R}^2 . We are interested in its fibers. Since these are smoothly deformed by left-right transformations, we use these transformations to normalize the map. For more information see Sect. 3.1, and [Dui84].

5.4.1 The tangent space

In the language of Sect. 5.2, the manifold M is $\mathcal{E}_n^\Gamma \times \mathcal{E}_n^\Gamma$, the Γ -invariant \mathcal{C}^∞ maps from \mathbb{R}^n to \mathbb{R}^2 . We include a symmetry group Γ , as in the application the energy–momentum map will be invariant under the circle symmetry produced by the Birkhoff normal form. The transformation group G is

$$G := \{(A, B) \in \mathcal{E}_{n,n}^\Gamma \times \mathcal{E}_{2,2} \mid A, B \text{ origin-preserving and invertible}\}.$$

The first component is the set of Γ -equivariant right transformations. The second component are the left-transformations. The group elements have the following action on M :

$$\zeta : G \times M \rightarrow M : ((A, B), \mathbf{E}) \mapsto B \circ \mathbf{E} \circ A.$$

The group operation is $(A, B) \circ (A', B') = (A' \circ A, B \circ B')$. The tangent space to G is then (isomorphic to) $\mathbf{V}_n^\Gamma \times \mathcal{E}_{2,2}$. Again, using this we can find the tangent space to the orbit of an arbitrary map $\mathbf{E} \in M$ under the action of G . This tangent space is denoted by $T_{\mathbf{E}}$:

(5.2)

$$T_{\mathbf{E}} := T_{\mathbf{E}}(G\mathbf{E}) = \{(\alpha \mathbf{E}_1 + \beta_1(\mathbf{E}_1, \mathbf{E}_2), \alpha \mathbf{E}_2 + \beta_2(\mathbf{E}_1, \mathbf{E}_2)) \mid \alpha \in \mathbf{V}_n^\Gamma, \beta \in \mathcal{E}_{2,2}\}.$$

Here \mathbf{E}_i and β_i are the components of the 2-vector-valued maps \mathbf{E} and β .

Remark 5.20. (*Fixing the origin*) In order to use germs (at 0) as left-transformation, we have to require that element of M map the origin (in \mathbb{R}^4) to the origin (in \mathbb{R}^2). This will be assumed throughout.

Remark 5.21. (*Germs and symmetry*) The group G acts on the manifold M consisting of germs at $0 \in \mathbb{R}^2$. Therefore, the action of the group must keep the origin fixed. However, when considering *germs of deformations* of maps, this is only required when the deformation parameters are 0, and the codimension of the tangent space 5.2 only gives an upper bound of the codimension of elements of M as deformations. But, if the symmetry group Γ is such that the origin is the only fixed point of the action of Γ , equivariance of the action of G implies that it fixes the origin, and the codimension of 5.2 is equal to the deformation-codimension.

From Mather's results [Mat68] we can deduce the following theorem. (In his notation we have $T_{\mathbf{E}} = (t\mathbf{E})(B) + (\omega\mathbf{E})(A)$, and $T(M) = \theta(\mathbf{E})$. Mather denotes the group G of left-right transformations by \mathcal{A} .)

Theorem 5.22. *Let $T_{\mathbf{E}} \subseteq T(M)$ have finite codimension d , and let $T(M)/T_{\mathbf{E}}$ be spanned by $t_1(x) \dots, t_d(x)$ as a real vector space. Then*

$$\mathbf{E} + \mu_1 t_1(x) + \dots + \mu_d t_d(x)$$

is a versal deformation of \mathbf{E} .

Remark 5.23. (*Relation with preparation theorem*) In fact [Mat68, Prop. 3.6] allows us to reduce the problem to a finite dimensional one, in the setting of truncated formal power series. The algorithms given in Chaps. 6 and 7 can be interpreted as constructive proof of theorem 5.22 for formal power series. Mather proves that this implies theorem 5.22 for smooth functions. It is in this step that many technical difficulties are met, the solution of which requires Malgrange and Mather's preparation theorem (see also [Mar82, Ch. X]). As our focus is on the computation of a finite piece of the transformation, we shall not digress on this subject.