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Belur, M.N.; Trentelman, H.L.

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The strict dissipativity synthesis problem and the rank of the coupling QDF

M.N. Belur[a,*], H.L. Trentelman[b]

[a]Department of Electrical Engineering, Indian Institute of Technology, Powai, Mumbai 400 076, India
[b]Institute for Mathematics and Computing Science, University of Groningen, P.O. Box 800, 9700 AV Groningen, Netherlands

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Abstract

The problem of existence of a controlled behavior that is *strictly* dissipative with respect to a quadratic supply rate is studied. The relation between strictness and the rank of a suitable coupling condition that combines the dissipativity properties of the hidden behavior and the orthogonal complement of the plant behavior is analyzed.

Keywords: Behaviors; Strict dissipativity; Quadratic differential forms; Coupling condition; Rank; Storage function

1. Introduction and notation

Recently, in [11] it was shown that, given a plant and a supply rate, the problem of designing a controller such that the interconnection is a dissipative system is equivalent to the problem of finding a behavior which satisfies the following three properties: (1) it is wedged in between the plant's hidden behavior and manifest behavior, (2) it is dissipative, and (3) its input cardinality is equal to the positive signature of the supply rate. In [11] necessary and sufficient conditions for the existence of such behavior were obtained. One of these conditions is a *coupling condition*, which requires that a certain quadratic differential form (called the *coupling QDF*), coupling the dissipativity properties of the hidden behavior and manifest behavior, is non-negative. In this short paper, we study the open problem of how thecoupling condition should be modified if, instead of a dissipative system behavior, we want to find a *strictly* dissipative behavior. We will show that in this case the coupling QDF should, in addition to being non-negative, have rank equal to the sum of the McMillan degrees of the hidden behavior and the manifest behavior.

The paper is structured as follows. In the rest of this section we introduce notations and review the most important behavioral definitions. The next section, Section 2, contains the key notions concerning quadratic differential forms with an emphasis on their rank. We also prove a theorem about the rank of a QDF. This prepares the background for the subsequent Section 3 which contains the main result of this paper. Section 4 contains a proof of the main result. In order to give the proof we need to formulate and prove some preliminary lemmas that are important in their own right. The final Section 5 contains conclusions and remarks.

* Corresponding author. Fax: +91-22-25723707.
E-mail address: m.n.belur@math.rug.nl (M.N. Belur).

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The notation that is used here is standard in most respects. We use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{C}$ to denote the complex plane. $\mathbb{R}^{\mathtt{n}}$ and $\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ are the obvious extensions to vectors and matrices of the specified dimensions. We use $\mathbb{R}^{\bullet\times\mathtt{n}_2}$ when the context does not call for a specification of the row dimension (but just the column dimension) of the concerned matrix. We typically use the superscript $^{\mathtt{w}}$ (for example, $\mathbb{R}^{\mathtt{w}}$) when a generic element w has $\mathtt{w}$ components. The ring of polynomials in the indeterminate ξ with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}[\zeta,\eta]$ is the corresponding ring in two (commutative) indeterminates, and we use $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ to denote the sets of matrices with entries from the above rings, etc. The space of infinitely often differentiable functions with domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\mathtt{n}}$ is denoted by $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{n}})$, and its subspace of compactly supported elements by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{n}})$. The operator 'col' stacks its arguments into a column and is used for improving readability of matrix equations within text. We use rowdim(M) to indicate the row dimension of a matrix M and just dim(M) if M is a vector or a square matrix.

A linear time-invariant differential system (or a behavior) is a subset $\mathfrak{B}\subseteq\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ such that, for some polynomial matrix $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, we have $\mathfrak{B}=\{w\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid R(\mathrm{d}/\mathrm{d}t)w=0\}$. We use $\mathfrak{L}^{\mathtt{w}}$ to denote the set of such behaviors. Here a behavior has been specified as the kernel of a differential operator. Hence, we speak of this as a kernel representation of $\mathfrak{B}$. But, more generally, we might encounter a behavior as follows: for $R,M\in\mathbb{R}^{\bullet\times\bullet}[\xi]$,

$$\mathfrak{B}=\left\{w\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid\exists\ell\in\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\ell})\text{ such that }R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w=M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\right\}.$$

It is as a consequence of the *elimination theorem* that the set defined above is indeed a behavior in the sense we defined. A representation like the one above is called a *latent variable representation* (with ℓ as the latent variable here). The *full behavior* $\mathfrak{B}_{\text{full}}\in\mathfrak{L}^{\mathtt{w}+\ell}$ is the set of all (w,ℓ) that satisfy the equation above.

In this paper, we restrict ourselves to controllable behaviors. Roughly speaking, controllable behaviors are defined as behaviors in which for any two of its elements there exists a third element which coincides with the first one on the past and the second one on the future (for details, see [4]). $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ (a subset of $\mathfrak{L}^{\mathtt{w}}$) denotes this set of controllable behaviors. Given a behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$, it is possible to choose some components of w as any function in $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$. The maximal number of such components that can be chosen arbitrarily is called the *input cardinality* of $\mathfrak{B}$ and is denoted as $\mathtt{m}(\mathfrak{B})$. We also need the notion of state for a behavior. We refer to [5] for a detailed exposition, with only a brief review here. A latent variable representation of $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ is called a state representation if the latent variable (denoted here by x) has the *property of state*, i.e.: if $(w_1,x_1),(w_2,x_2)\in\mathfrak{B}_{\text{full}}$ are such that $x_1(0)=x_2(0)$ then $(w_1,x_1)\wedge(w_2,x_2)$, the concatenation (at $t=0$, here), belongs to the $\mathscr{L}_1^{\text{loc}}$-closure of $\mathfrak{B}_{\text{full}}$. We call such an x a state for $\mathfrak{B}$. A state map for a $\mathfrak{B}$ is a differential operator $X(\mathrm{d}/\mathrm{d}t)$ (induced by $X\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$) such that $X(\mathrm{d}/\mathrm{d}t)w$ is a state for $\mathfrak{B}$. A state map $X\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ is minimal if every other state map has at least as many rows as those of X, and this minimal number of state variables (called the *McMillan degree*) of $\mathfrak{B}$ is denoted by $\mathtt{n}(\mathfrak{B})$. The rows of such an X are linearly independent over $\mathbb{R}$. A minimal state map $X(\mathrm{d}/\mathrm{d}t)$ has a property called *trimness*, i.e., for all $x_0\in\mathbb{R}^{\mathtt{n}(\mathfrak{B})}$, there exists a $(w,X(\mathrm{d}/\mathrm{d}t)w)\in\mathfrak{B}_{\text{full}}$ such that $(X(\mathrm{d}/\mathrm{d}t)w)(0)=x_0$. Issues concerning existence and constructive algorithms about state maps have been dealt with in [5].

2. Quadratic differential forms

This section contains a brief review of bilinear differential forms, quadratic differential forms and other necessary notions like the rank of a QDF, etc. A bilinear form (BF) on the vector spaces $(\mathbb{V}_1,\mathbb{V}_2)$ is a mapping $\ell:\mathbb{V}_1\times\mathbb{V}_2\to\mathbb{R}$ that is linear in each of its two arguments. Given such an ℓ, its *rank* is the number of independent linear functionals $\ell(\cdot,v_2)$ where v_2 ranges over $\mathbb{V}_2$, or equivalently the number of independent linear functionals $\ell(v_1,\cdot)$, where v_1 ranges over $\mathbb{V}_1$. When $\mathbb{V}_1=\mathbb{V}_2=\mathbb{V}$, a BF ℓ on $(\mathbb{V},\mathbb{V})$ is called symmetric if $\ell(v_1,v_2)=\ell(v_2,v_1)$. Also, when $\mathbb{V}_1=\mathbb{V}_2=\mathbb{V}$, we speak of the quadratic form (QF) induced by ℓ on $\mathbb{V}$, defined by $q(v) := \ell(v, v)$. The rank of a QF is the rank of the symmetric BF that induces it. A QF q on $\mathbb{V}$ can be expressed as $q(v) = \sum_{k=1}^{n_+} |f_k^+(v)|^2 - \sum_{k=1}^{n_-} |f_k^-(v)|^2$ with the f_k^+'s and f_k^-'s linear functionals on $\mathbb{V}$, if (and only if) q has finite rank. We can choose $f_1^+, f_2^+, \ldots, f_{n_+}^+, f_1^-, f_2^-, \ldots, f_{n_-}^-$ linearly independent over $\mathbb{R}$. In this case n_- and n_+ are individually minimal over all such decompositions of q as a sum and difference of squares. We call the corresponding pair of integers (n_-, n_+) the *signature* of q and denote it as $\mathrm{sign}(q) = (\sigma_-(q), \sigma_+(q))$. The rank of q equals $\sigma_-(q) + \sigma_+(q)$.

The QF on $\mathbb{R}^{\mathtt{n}}$ induced by the matrix $S = S^{\mathrm{T}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is defined as $q_S(x) := x^{\mathrm{T}} S x$. We shall also use $|x|_S^2$ to denote it, and when $S = I$ the subscript is often dropped. We denote the signature of S by $\mathrm{sign}(S) = (\sigma_-(S), \sigma_+(S))$ where $\sigma_-(S)$ and $\sigma_+(S)$ are, respectively, the number of negative and positive eigenvalues of S. Further, $\mathrm{sign}(S) = \mathrm{sign}(q_S)$. We have $\sigma_-(q_S) = 0 \Leftrightarrow q_S(x) \geqslant 0$ for all $x \in \mathbb{R}^{\mathtt{n}}$. We call such a q_S non-negative. Also, the usual definition of positive definiteness (of matrices) results in $\sigma_+(q_S) = \mathtt{n} \Leftrightarrow q_S(x) > 0$ for all $x \neq 0$.

We now move over to the notions of bilinear differential forms (BDFs) and quadratic differential forms (QDFs). Let $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ be written as a finite sum $\Phi(\zeta, \eta) = \Sigma_{k,\ell\in\mathbb{Z}_+} \Phi_{k\ell} \zeta^k \eta^\ell$ with $\Phi_{k\ell} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$—its coefficient matrices. Let $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}_1}$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}_2}$. Then, Φ induces the map $L_\Phi : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by

$$L_\Phi(w_1, w_2) := \sum_{k,\ell\in\mathbb{Z}_+} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w_1\right)^{\mathrm{T}} \Phi_{k\ell} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w_2\right)$$

called the BDF on $\mathfrak{B}_1 \times \mathfrak{B}_2$ induced by Φ and which is denoted by $L_\Phi|_{\mathfrak{B}_1\times\mathfrak{B}_2}$. When $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, Φ also induces the map $Q_\Phi : \mathfrak{B} \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ with $Q_\Phi(w) := L_\Phi(w, w)$. We call this map the QDF on $\mathfrak{B}$ induced by Φ and denote it by $Q_\Phi|_{\mathfrak{B}}$. Define the $*$ operator as $(\Phi^*)(\zeta, \eta) := (\Phi(\eta, \zeta))^{\mathrm{T}}$. When considering QDFs, it is sufficient to consider Φ's that are symmetric, i.e., those that satisfy $\Phi = \Phi^*$.

We are interested in non-negativity of QDFs on behaviors. For $f : A \to \mathbb{R}, f \geqslant 0$ means $f(t) \geqslant 0$ for all $t \in A$. We shall use this general definition of non-negativity for QDFs too. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. We call the QDF Q_Φ non-negative on $\mathfrak{B}$ (and denote it by $Q_\Phi|_{\mathfrak{B}} \geqslant 0$) if $Q_\Phi(w) \geqslant 0$ for all $w \in \mathfrak{B}$. Extending this notion of non-negativity of a QDF to positive definiteness the usual way, we say $Q_\Phi|_{\mathfrak{B}} > 0$ if for all $w \in \mathfrak{B}$: $Q_\Phi(w) \geqslant 0$ *and* $Q_\Phi(w) = 0$ implies that $w = 0$. Here, $\mathfrak{B}$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and in the special case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, the subscript $\mathfrak{B}$ is skipped. It is noteworthy that if the non-negativity of Q_Φ is given then, as a function, $Q_\Phi(w)$ is non-negative *pointwise* also. But the same is not true for positive definiteness of QDFs (and pointwise positivity). Roughly, this is because $Q_\Phi(w)$ usually involves only a finite number of derivatives of w, and hence for a non-zero trajectory w, $Q_\Phi(w)(t)$ can be zero for some $t \in \mathbb{R}$. It is in this context and for the purpose of the problem in this paper that the rank of a QDF plays an important role.

Let $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}_1}$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}_2}$. There is a one-to-one correspondence between the BDF L_Φ on $\mathfrak{B}_1 \times \mathfrak{B}_2$ and the BF on $\mathfrak{B}_1 \times \mathfrak{B}_2$ defined by $(w_1, w_2) \mapsto L_\Phi(w_1, w_2)(0)$. Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, there is a similar correspondence between the QDF Q_Φ on $\mathfrak{B}$ and the QF on $\mathfrak{B}$ defined by $w \mapsto Q_\Phi(w)(0)$. We define the ranks and signatures of a BDF or QDF by this correspondence. Although they act on infinite dimensional spaces, both $L_\Phi|_{\mathfrak{B}_1\times\mathfrak{B}_2}$ and $Q_\Phi|_{\mathfrak{B}}$ have finite rank. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ then Φ can be expressed as $\Phi(\zeta, \eta) = F_+^{\mathrm{T}}(\zeta) F_+(\eta) - F_-^{\mathrm{T}}(\zeta) F_-(\eta)$, with $F = \mathrm{col}(F_+, F_-) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, such that the rows of F induce (linear) functionals on $\mathfrak{B}$ that are linearly independent over $\mathbb{R}$. A factorization of Φ as the one above is called a canonical factorization on $\mathfrak{B}$. Such a factorization yields the signature and the rank of $Q_\Phi|_{\mathfrak{B}}$ by $\mathrm{sign}(Q_\Phi|_{\mathfrak{B}}) = (\mathrm{rowdim}(F_-), \mathrm{rowdim}(F_+))$ and $\mathrm{rank}(Q_\Phi|_{\mathfrak{B}}) = \mathrm{rowdim}(F)$, and $Q_\Phi|_{\mathfrak{B}}$ can be expressed canonically as $Q_\Phi(w) = |F_+(\mathrm{d}/\mathrm{d}t)w|^2 - |F_-(\mathrm{d}/\mathrm{d}t)w|^2$. A formal exposition on QDFs can be found in [10].

Note the similarity of linear independence over $\mathbb{R}$ of the rows of F and of those of a minimal state map. This similarity lies behind the very appealing result of [8]. We need a related property of a minimal state map which is also satisfied by other polynomial matrices under suitable assumptions. In this context, we have the following theorem. The proof of this theorem is fairly straightforward and can be found in [1, Theorem 5.4.7].

Theorem 1. *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, $F \in \mathbb{R}^{\mathtt{q}\times\mathtt{w}}[\xi]$ *and* $K = K^{\mathrm{T}} \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}$ *and define* $\Phi(\zeta,\eta) := F^{\mathrm{T}}(\zeta)KF(\eta)$. *Assume for* $\eta \in \mathbb{R}^{\mathtt{q}}$: $\eta^{\mathrm{T}}F(\mathrm{d}/\mathrm{d}t)\mathfrak{B} = 0 \Rightarrow \eta = 0$. *Then we have*

$$K > 0 \quad \Leftrightarrow \quad \begin{array}{l} (1)\ Q_\Phi|_{\mathfrak{B}} \geqslant 0 \quad \textit{and} \\ (2)\ \mathrm{rank}(Q_\Phi|_{\mathfrak{B}}) = \mathtt{q}. \end{array}$$

Remark. A close connection exists with the assumption in the theorem above and the notion of trimness. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *trim* if for all $a \in \mathbb{R}^{\mathtt{w}}$, there exists $w \in \mathfrak{B}$ such that $w(0) = a$. It is possible to show that the property that for $\eta \in \mathbb{R}^{\mathtt{q}}$: $\eta^{\mathrm{T}}F(\mathrm{d}/\mathrm{d}t)\mathfrak{B} = 0 \Rightarrow \eta = 0$ is equivalent to the trimness of the behavior $F(\mathrm{d}/\mathrm{d}t)\mathfrak{B}$ (here $F(\mathrm{d}/\mathrm{d}t)\mathfrak{B}$ is an element of $\mathfrak{L}^{\mathtt{q}}$). In Theorem 1 above, F need not be a state map. However, as mentioned above, if F is a minimal state map of $\mathfrak{B}$ then the behavior $F(\mathrm{d}/\mathrm{d}t)\mathfrak{B}$ is always trim.

3. Synthesis of strictly dissipative systems

The notions of non-negativity, etc. are in a sense 'local' properties of a QDF. In this section, we discuss properties like dissipativity which are 'global'. Let $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. $\mathfrak{B}$ is said to be *dissipative* with respect to Q_Σ (or briefly, Σ-dissipative) if $\int_{-\infty}^{+\infty} Q_\Sigma(w)\,\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. (In this case $Q_\Sigma(w)$ equals $w^{\mathrm{T}}\Sigma w$.) Further, it is said to be dissipative on $\mathbb{R}_-$ with respect to Q_Σ (or briefly, Σ-dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Sigma(w)\,\mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We also use the analogous definition of dissipativity on $\mathbb{R}_+$. A controllable behavior $\mathfrak{B}$ is said to be *strictly dissipative* with respect to Q_Σ (or briefly, strictly Σ-dissipative) if there exists an $\varepsilon > 0$ such that $\mathfrak{B}$ is dissipative with respect to $Q_{\Sigma-\varepsilon I}$. We have the obvious definitions for strict dissipativity on $\mathbb{R}_-$ and on $\mathbb{R}_+$. Equipped with these definitions, we state below the problem that we solve in this short paper.

Strict dissipativity synthesis problem formulation: Let $\mathscr{N}$ and $\mathscr{P} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$, and let $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be non-singular. The problem is to find $\mathscr{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ such that

1. $\mathscr{N} \subseteq \mathscr{K} \subseteq \mathscr{P}$,
2. $\mathscr{K}$ is strictly Σ-dissipative on $\mathbb{R}_-$,
3. $\mathtt{m}(\mathscr{K}) = \sigma_+(Q_\Sigma)$.

The constraints that $\mathscr{K}$ has to satisfy have important control-theoretic interpretations. We call $\mathscr{P}$ the plant behavior, $\mathscr{N}$ the hidden behavior and $\mathscr{K}$ the controlled behavior. The condition $\mathscr{N} \subseteq \mathscr{K} \subseteq \mathscr{P}$ is equivalent to implementability of the controlled behavior through a restricted set of variables called control variables. The third condition formalizes the requirement that the controlled behavior should be *live* enough to accept sufficiently many exogenous inputs (which can be interpreted as disturbances). The strict Σ-dissipativity condition combines various control design specifications depending on Σ, for example, disturbance attenuation. The dissipativity on $\mathbb{R}_-$ implies stability. We refer to [11] for details and for additional material on strictly dissipative systems, see [3].

For a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and a $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, we say that $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces a storage function Q_Ψ for $\mathfrak{B}$ with respect to Q_Σ if the dissipation inequality $(\mathrm{d}/\mathrm{d}t)Q_\Psi(w) \leqslant Q_\Sigma(w)$ is satisfied for all $w \in \mathfrak{B}$. It has been shown that such a storage function exists if and only if $\mathfrak{B}$ is Σ-dissipative. Moreover, $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if there exists a storage function Q_Ψ such that $Q_\Psi|_{\mathfrak{B}} \geqslant 0$. Analogously, $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_+$ if and only if there exists a storage function Q_Ψ such that $Q_\Psi|_{\mathfrak{B}} \leqslant 0$. It is also known (see for instance, [8]) that such a storage function is always a state function, i.e., if $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ induces a state map for $\mathfrak{B}$, then associated with this Ψ there exists a $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $Q_\Psi(w) = |X(\mathrm{d}/\mathrm{d}t)w|^2_K$ for all $w \in \mathfrak{B}$. Thus we often speak of the matrix associated with a storage function (and a state map). A storage function is not unique. However, there exists a maximal and a minimal one between which every other storage function lies. We denote the largest and the smallest storage functions by Ψ^+ and Ψ^-, and their associated matrices by K^+ and K^-, respectively. Further, corresponding to each storage function, we have a *dissipation function* which is the QDF Q_Δ defined by $Q_\Delta(w) := Q_\Sigma(w) - (\mathrm{d}/\mathrm{d}t)Q_\Psi(w)$ for all $w \in \mathfrak{B}$.

Given a BDF induced by a constant matrix we have a notion of the orthogonal complement of a controllable behavior with respect to this BDF. Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$; $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be *orthogonal with respect to* L_Σ (*briefly*, Σ-*orthogonal*) if $\int_{-\infty}^{+\infty} L_\Sigma(w_1, w_2)\,\mathrm{d}t = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}$ and $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}$. This orthogonality relation between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is denoted by $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$. For $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ we define the Σ-orthogonal complement $\mathfrak{B}^{\perp_\Sigma}$ of $\mathfrak{B}$ as

$$\mathfrak{B}^{\perp_\Sigma} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,\middle|\, \int_{-\infty}^{+\infty} L_\Sigma(w, w')\,\mathrm{d}t = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathfrak{D} \right\}.$$

When $\Sigma = I$, we use $\perp$ instead of $\perp_\Sigma$. The following identities are easily verified: $\mathfrak{B}^{\perp_\Sigma} = (\Sigma\mathfrak{B})^\perp = ((\Sigma^{\mathrm{T}})^{-1})\mathfrak{B}^\perp$. (Here $^{-1}$ denotes set-theoretic inverse.) Further, if Σ is non-singular, $\mathfrak{B} = (\mathfrak{B}^{\perp_\Sigma})^{\perp_\Sigma}$. In the context of behaviors that are Σ-orthogonal we have the following result.

Proposition 2. *Let* $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *and* $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. *There exists a* $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *such that* $(\mathrm{d}/\mathrm{d}t)L_\Psi(w_1, w_2) = w_1^{\mathrm{T}}\Sigma w_2$ *for all* $(w_1, w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$ *if and only if* $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$. *Moreover*, Ψ *is essentially unique*, *i.e.*, *if* $\Psi_1, \Psi_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ *both satisfy the above equality*, *then* $L_{\Psi_1}(w_1, w_2) = L_{\Psi_2}(w_1, w_2)$ *for all* $(w_1, w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$.

We call this BDF L_Ψ on $\mathfrak{B}_1 \times \mathfrak{B}_2$, the $[(\mathfrak{B}_1, \mathfrak{B}_2); \Sigma]$-adapted bilinear differential form. Here also L_Ψ can be written as a function of the states of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, i.e., given X_1 and X_2 that induce minimal state maps for $\mathfrak{B}_1$ and for $\mathfrak{B}_2$, respectively, there exists a matrix $L \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}_1)\times\mathtt{n}(\mathfrak{B}_2)}$ such that $L_\Psi(w_1, w_2) = (X_1(\mathrm{d}/\mathrm{d}t)w_1)^{\mathrm{T}} L X_2(\mathrm{d}/\mathrm{d}t)w_2$ for all $w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$. For the case of $\Sigma = I$ and for behaviors $\mathfrak{B}$ and $\mathfrak{B}^\perp$, L happens to be invertible and we can modify one of the two (minimal) state maps to obtain a *matched pair* of state maps. (X, Z) is said to be a matched pair of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^\perp)$ if $(\mathrm{d}/\mathrm{d}t)(X(\mathrm{d}/\mathrm{d}t)w_1)^{\mathrm{T}} Z(\mathrm{d}/\mathrm{d}t)w_2 = w_1^{\mathrm{T}} w_2$ for all $(w_1, w_2) \in \mathfrak{B} \times \mathfrak{B}^\perp$. More on this can be found in [10, Section 10].

We are now ready to state the main result of the paper, which is a solution to the strict dissipativity synthesis problem described above. Since $\mathcal{N} \subseteq \mathcal{P}$, we have that $\mathcal{N} \perp_\Sigma \mathcal{P}^{\perp_\Sigma}$. Let $\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce the $[(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}); \Sigma]$-adapted BDF. It turns out that the existence of a controlled behavior $\mathcal{K}$ as described in our problem formulation involves, in addition to a non-negativity requirement, a rank condition on the coupling QDF.

Theorem 3. *A controlled behavior* $\mathcal{K} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ *as described in the problem formulation exists if and only if the following conditions are satisfied*:

1. $\mathcal{N}$ *is strictly* Σ-*dissipative*,
2. $\mathcal{P}^{\perp_\Sigma}$ *is strictly* $(-\Sigma)$-*dissipative*,
3. *the* coupling QDF Q_{cpl} *on* $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$ *defined by*

$$Q_{\mathrm{cpl}}(v_1, v_2) := Q_{\Psi^+_{\mathcal{N}}}(v_1) - Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}(v_2) + 2L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}(v_1, v_2) \tag{1}$$

satisfies the following two properties:
(i) $Q_{\mathrm{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}} \geqslant 0$ *and*
(ii) $\boxed{\mathrm{rank}(Q_{\mathrm{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P}).}$
Here, $\Psi^+_{\mathcal{N}}$ *induces the largest storage function for* $\mathcal{N}$ *as a* Σ-*dissipative system and* $\Psi^-_{\mathcal{P}^{\perp_\Sigma}}$ *induces the smallest storage function for* $\mathcal{P}^{\perp_\Sigma}$ *as a* $(-\Sigma)$-*dissipative system.*

We note here the importance of the last statement in the theorem above. Since Q_{cpl} is a sum of three terms that are themselves functions of the states of the behaviors concerned, it cannot have rank *more than* $\mathtt{n}(\mathscr{N}) + \mathtt{n}(\mathscr{P})$. So the existence of a strictly dissipative controlled behavior as in the problem formulation requires the existence of a non-negative coupling QDF of *maximal rank*. It is in this way that the strictness of the dissipativity in the problem formulation affects the theorem. But unlike here, the McMillan degrees of the hidden behavior and the plant behavior played no role in the non-strict synthesis result of [11].

Remark. Conditions 1 and 2 in the above theorem formulation can, in fact, be replaced by

1′. $\mathscr{N}$ is strictly Σ-dissipative on $\mathbb{R}_-$,
2′. $\mathscr{P}^{\perp_\Sigma}$ is strictly $(-\Sigma)$-dissipative on $\mathbb{R}_+$.

Indeed, 1′ follows from condition 1 in the theorem above together with the non-negativity and the rank condition on the coupling QDF (condition 3). The non-negativity of this QDF gives the dissipativity on $\mathbb{R}_-$ while the strictness of this dissipativity on $\mathbb{R}_-$ is implied by the rank condition. This auxiliary result is the subject of Lemma 8 below. Similarly, 2′ can be inferred from condition 2 in the theorem together with the non-negativity and rank condition on the coupling QDF.

4. Proof of Theorem 3

In order to give a proof of Theorem 3, we need some preliminary results on strictly dissipative systems. We state and prove these lemmas before we move over to the proof of the main result.

Lemma 4. *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ and let $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be non-singular. Assume $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ if and only if $\mathfrak{B}^{\perp_\Sigma}$ is strictly $(-\Sigma)$-dissipative on $\mathbb{R}_+$.*

Proof. For the proof of this lemma we need two results from matrix theory. We state them in the following lemma but skip the straightforward proof.

Lemma 5. *Let $R = R^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be non-singular. For $\varepsilon_1, \varepsilon_2 > 0$ sufficiently small, there exist $\delta_1, \delta_2 > 0$ such that*

1. $(R^{-1} - \varepsilon_1 I)^{-1} - (R + \delta_1 R^2) \geqslant 0$, *and*
2. $R + \varepsilon_2 R^2 - (R^{-1} - \delta_2 I)^{-1} \geqslant 0$.

We now continue with proving Lemma 4. Note that for $P_1, P_2 \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ such that $P_1 \geqslant P_2$, if $\mathfrak{B}$ is P_2-dissipative then $\mathfrak{B}$ is P_1-dissipative as well. The analogous statements are true for dissipativity on $\mathbb{R}_-$ or on $\mathbb{R}_+$ also.

Only if part: Suppose $\mathfrak{B}$ is strictly dissipative with respect to Σ on $\mathbb{R}_-$. Then $\mathfrak{B}$ is dissipative with respect to $\Sigma - \varepsilon I$ on $\mathbb{R}_-$ for some $\varepsilon > 0$. We use Proposition 12 of Willems and Trentelman [11], which states that $\mathfrak{B}^{\perp}$ is dissipative on $\mathbb{R}_+$ with respect to $-(\Sigma - \varepsilon I)^{-1}$. We use statement 1 of Lemma 5 (with $R = \Sigma^{-1}$) to infer the existence of a $\delta > 0$ such that $\mathfrak{B}^{\perp}$ is dissipative on $\mathbb{R}_+$ with respect to $-(\Sigma^{-1} + \delta\Sigma^{-2})$ also. By definition of $\mathfrak{B}^{\perp_\Sigma}$, this means that $\mathfrak{B}^{\perp_\Sigma}$ is strictly dissipative with respect to $-\Sigma$ on $\mathbb{R}_+$.

If part: The proof of this part is similar to that of the 'only if part' except that we use statement 2 of the previous lemma now. This completes the proof of Lemma 4. □

Lemma 6. *Let $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ be non-singular, and let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$. Suppose $\mathfrak{B}$ is strictly Σ-dissipative. Let $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{v}}[\xi]$ induce a minimal state map for $\mathfrak{B}$. Let K^+, $K^- \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ be symmetric matrices such* *that* $|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^+}$ *and* $|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^-}$ *are, respectively, the largest and the smallest storage functions for* $\mathfrak{B}$ *as a* Σ*-dissipative system. Then,* $K^+ > K^-$*. Moreover,*

- *if* $\mathfrak{B}$ *is strictly dissipative on* $\mathbb{R}_-$, *then* $K^+ > 0$, *and*
- *if* $\mathfrak{B}$ *is strictly dissipative on* $\mathbb{R}_+$, *then* $K^- < 0$.

Proof. The proof of the part that $K^+ > K^-$ is contained in the proof of Theorem 5.7 of [10, p. 1739]. The part that $K^+ > 0$ follows from Theorem 10.2(iv) of [10], and the part $K^- < 0$ is similar to this proof. □

Lemma 7. *Let* $\Sigma = \Sigma^{\mathrm{T}} \in \mathbb{R}^{\mathtt{v}\times\mathtt{v}}$ *be non-singular, and let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{v}}_{\mathrm{cont}}$ *be strictly* Σ*-dissipative. Let* $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{v}}[\xi]$ *induce a minimal state map for* $\mathfrak{B}$*. Let* $K^+, K^- \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ *be such that* $|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^+}$ *and* $|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^-}$ *are the largest and the smallest storage functions for* $\mathfrak{B}$ *as a* Σ*-dissipative system. Further, for* $\varepsilon > 0$ *sufficiently small* $\mathfrak{B}$ *is strictly dissipative with respect to* $(\Sigma - \varepsilon I)$ *also. Let* K^+_ε *and* K^-_ε *be the corresponding matrices for the largest and the smallest storage functions for* $\mathfrak{B}$ *as a* $(\Sigma - \varepsilon I)$*-dissipative system. Then* $K^+_\varepsilon \leqslant K^+$ *for all* $\varepsilon > 0$ *and* $K^+_\varepsilon \uparrow K^+$ *as* $\varepsilon \downarrow 0$*. Analogously,* $K^-_\varepsilon \geqslant K^-$ *for all* $\varepsilon > 0$ *and* $K^-_\varepsilon \downarrow K^-$ *as* $\varepsilon \downarrow 0$.

For the proof we need to introduce a few more concepts and we do that here. An image representation of a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is a latent variable representation of the form: $w = M(\mathrm{d}/\mathrm{d}t)\ell$ with $M \in \mathbb{R}^{\mathtt{w}\times\ell}[\xi]$. Such a representation exists if and only if $\mathfrak{B}$ is controllable. The latent variable ℓ is said to be observable from the manifest variable w if for any $(w,\ell_1),(w,\ell_2) \in \mathfrak{B}_{\mathrm{full}}$ implies $\ell_1 = \ell_2$. Hence, we speak of an observable image representation of a behavior.

We also need the notion of a symmetric factorization of a para-Hermitian (polynomial) matrix. We briefly introduce it here. $P \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is called para-Hermitian if $P(-\xi)^{\mathrm{T}} = P(\xi)$. $A \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is said to induce a symmetric factorization of P if $A(-\xi)^{\mathrm{T}}A(\xi) = P(\xi)$. We call such a factorization *anti-Hurwitz* if A is anti-Hurwitz, i.e., it has full rank on the closed left half complex plane. It is called *almost* anti-Hurwitz if the above rank condition holds on just the *open* left half complex plane.

Proof. Since K^+_ε is associated with the largest storage functions for $\mathfrak{B}$ as a $(\Sigma - \varepsilon I)$-dissipative system, we have $(\mathrm{d}/\mathrm{d}t)|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^+_\varepsilon} \leqslant v^{\mathrm{T}}(\Sigma - \varepsilon I)v \leqslant v^{\mathrm{T}}\Sigma v$. This means that $|X(\mathrm{d}/\mathrm{d}t)v|^2_{K^+_\varepsilon}$ is a storage function for $\mathfrak{B}$ as a Σ-dissipative system also. By definition of K^+ we have $K^+_\varepsilon \leqslant K^+$ for all $\varepsilon > 0$. Using a similar argument we also have $K^+_{\varepsilon_1} \leqslant K^+_{\varepsilon_2}$ for $\varepsilon_2 < \varepsilon_1$. Let $K^+_0 := \lim_{\varepsilon\downarrow 0} K^+_{\varepsilon_2}$. Then $K^+_0 \leqslant K^+$. We shall show the equality of K^+_0 and K^+. Let $w = M(\mathrm{d}/\mathrm{d}t)\ell$ be an observable image representation of $\mathfrak{B}$. For $\varepsilon > 0$ we define $\Psi^+_\varepsilon \in \mathbb{R}^{\ell\times\ell}[\zeta,\eta]$ by $\Psi^+_\varepsilon(\zeta,\eta) := M^{\mathrm{T}}(\zeta)X^{\mathrm{T}}(\zeta)K^+_\varepsilon X(\eta)M(\eta)$. We have

$$(\zeta+\eta)\Psi^+_\varepsilon(\zeta,\eta) = M^{\mathrm{T}}(\zeta)(\Sigma - \varepsilon I)M(\eta) - A^{\mathrm{T}}_\varepsilon(\zeta)A_\varepsilon(\eta), \tag{2}$$

where $A_\varepsilon \in \mathbb{R}^{\ell\times\ell}[\xi]$ induces a symmetric anti-Hurwitz factorization of $M^{\mathrm{T}}(-\xi)(\Sigma - \varepsilon I)M(\xi)$. Also, define $\Psi^+(\zeta,\eta) := M^{\mathrm{T}}(\zeta)X^{\mathrm{T}}(\zeta)K^+X(\eta)M(\eta)$. Now, in Eq. (2), let $\varepsilon \downarrow 0$. Since K^+_ε converges to K^+_0, $\Psi^+_\varepsilon(\zeta,\eta)$ converges to $M^{\mathrm{T}}(\zeta)X^{\mathrm{T}}(\zeta)K^+_0X(\eta)M(\eta) =: \Psi^+_0(\zeta,\eta)$. This implies that $A^{\mathrm{T}}_\varepsilon(\zeta)A_\varepsilon(\eta)$ converges as $\varepsilon \downarrow 0$. By a standard argument, there exists a sequence $(\varepsilon_n)_{n=1}^{\infty}$ with $\varepsilon_n \to 0$ $(n \to \infty)$, such that $A_{\varepsilon_n}(\xi)$ converges to, say, $A(\xi)$. Clearly $A \in \mathbb{R}^{\ell\times\ell}[\xi]$ is almost anti-Hurwitz. For such an $A(\xi)$ we have

$$(\zeta_\eta)\Psi^+_0(\zeta,\eta) = M^{\mathrm{T}}(\zeta)\Sigma M(\eta) - A^{\mathrm{T}}(\zeta)A(\eta).$$

By $A(\xi)$ being almost anti-Hurwitz, this finally implies that, in fact, $\Psi^+_0(\zeta,\eta) = \Psi^+(\zeta,\eta)$, which by trimness of X, implies $K^+_0 = K^+$. The proof of K^- is treated similarly. □

The above lemma is used in the proof of Theorem 3. In addition to that, the above lemma is useful for proving the following lemma which relates strict dissipativity and dissipativity on $\mathbb{R}_-$.

Lemma 8. *Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be Φ-dissipative on $\mathbb{R}_-$. Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ induce a minimal state map for $\mathfrak{B}$. Let $K^+ \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be such that $|X(\mathrm{d}/\mathrm{d}t)w|^2_{K^+}$ is the largest storage function for $\mathfrak{B}$ as a Φ-dissipative system. Then,*

$$\left.\begin{array}{l} K^+ > 0 \\ \mathfrak{B} \textit{ is strictly dissipative } (\textit{on } \mathbb{R}) \end{array}\right\} \Rightarrow \mathfrak{B} \textit{ is strictly dissipative on } \mathbb{R}_-.$$

Proof. Since $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}$, there exists a storage function for $\mathfrak{B}$ as a $(\Phi - \varepsilon I)$-dissipative system for some $\varepsilon > 0$ sufficiently small. Let K_ε^+ be the matrix associated with the largest such storage function. By Lemma 7 we have that $K_\varepsilon^+ \uparrow K^+$ as $\varepsilon \to 0$. Hence for ε sufficiently small, $K_\varepsilon^+ > 0$. This yields that $\mathfrak{B}$ is $(\Phi - \varepsilon I)$-dissipative on, in fact, $\mathbb{R}_-$. Hence $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}_-$, as required. □

An analogous statement relating negative definiteness and strict dissipativity on $\mathbb{R}_+$ is also true. We now continue with the proof of Theorem 3.

Proof of Theorem 3. The proof of the theorem follows the lines of the proof of the theorem about the existence of a *non-strictly* dissipative $\mathscr{K}$ [11, Theorem 5]. Hence, we skip the obvious similarities and emphasize on the role strictness plays in the proof. Let $(X_{\mathscr{N}}, Z_{\mathscr{N}})$ be a matched pair of minimal state maps for $(\mathscr{N}, \mathscr{N}^\perp)$. Also let $(X_{\mathscr{K}}, Z_{\mathscr{K}})$ and $(X_{\mathscr{P}}, Z_{\mathscr{P}})$ be matched pairs of minimal state maps for $(\mathscr{K}, \mathscr{K}^\perp)$ and $(\mathscr{P}, \mathscr{P}^\perp)$, respectively. Let $L \in \mathbb{R}^{\mathtt{n}(\mathscr{N})\times\mathtt{n}(\mathscr{P})}$ be associated with the adapted BLDF $L_{\Psi_{(\mathscr{N},\mathscr{P}^{\perp_\Sigma})}}$ corresponding to the state maps $X_{\mathscr{N}}$ of $\mathscr{N}$ and $Z_{\mathscr{P}}$ of $\mathscr{P}^\perp$.

Only if part: Since $\mathscr{N} \subseteq \mathscr{K}$ we have that $\mathscr{N}$ is also strictly Σ-dissipative on $\mathbb{R}_-$. By Lemma 4, $\mathscr{K}$ being strictly Σ dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathscr{K}) = \sigma_+(\Sigma)$, together imply that $\mathscr{K}^{\perp_\Sigma}$ is strictly $(-\Sigma)$-dissipative on $\mathbb{R}_+$. Also, $\mathscr{K} \subseteq \mathscr{P}$ ensures that $\mathscr{P}^{\perp_\Sigma} \subseteq \mathscr{K}^{\perp_\Sigma}$ and hence $\mathscr{P}^{\perp_\Sigma}$ is also strictly $(-\Sigma)$-dissipative on $\mathbb{R}_+$. Further, we have that $L_{\Psi_{(\mathscr{N},\mathscr{P}^{\perp_\Sigma})}}(v_1, v_2) = (X_{\mathscr{K}}(\mathrm{d}/\mathrm{d}t)v_1)^{\mathrm{T}} Z_{\mathscr{K}}(\mathrm{d}/\mathrm{d}t)\Sigma v_2$ for $(v_1, v_2) \in \mathscr{N} \times \mathscr{P}^{\perp_\Sigma}$. We now show the existence of a suitable coupling QDF on $\mathscr{N} \times \mathscr{P}^{\perp_\Sigma}$.

We consider the largest and the smallest storage functions of $\mathscr{K}$ as a Σ-dissipative system. Let K_+, $K_- \in \mathbb{R}^{\mathtt{n}(\mathscr{K})\times\mathtt{n}(\mathscr{K})}$ be matrices such that these extremum storage functions are expressed as $|X_{\mathscr{K}}(\mathrm{d}/\mathrm{d}t)v|^2_{K_+}$ and $|X_{\mathscr{K}}(\mathrm{d}/\mathrm{d}t)v|^2_{K_-}$, respectively. Further, since $\mathscr{K}$ is strictly dissipative, by Lemma 6, $K_+ > K_-$. By [11, Proposition 12, statement 5], K_- also satisfies $K_- > 0$. Thus we have $K_+ > K_- > 0$ which is equivalent to K_- being non-singular and $\left[\begin{smallmatrix} K_+ & I \\ I & K_-^{-1} \end{smallmatrix}\right] > 0$. Further, by the same proposition, (statement 3) we have that $-|Z_{\mathscr{K}}(\mathrm{d}/\mathrm{d}t)\Sigma v|_{K_-^{-1}}$ is a storage function for $\mathscr{K}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system. Consequently, the QDF Q_{cpl0} on $\mathscr{K} \times \mathscr{K}^{\perp_\Sigma}$ defined by

$$Q_{\mathrm{cpl0}}(v_1, v_2) = \begin{bmatrix} X_{\mathscr{K}}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right) v_1 \\ Z_{\mathscr{K}}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right) \Sigma v_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} K_+ & I \\ I & K_-^{-1} \end{bmatrix} \begin{bmatrix} X_{\mathscr{K}}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right) v_1 \\ Z_{\mathscr{K}}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right) \Sigma v_2 \end{bmatrix} \tag{3}$$

is non-negative for all $(v_1, v_2) \in \mathscr{K} \times \mathscr{K}^{\perp_\Sigma}$.

The QDF $Q_{\mathrm{cpl0}}(v_1, 0)$ with $v_1 \in \mathscr{N}$ is a storage function for $\mathscr{N}$ as a Σ-dissipative system and the QDF $-Q_{\mathrm{cpl0}}(0, v_2)$ with $v_2 \in \mathscr{P}^{\perp_\Sigma}$ is a storage function for $\mathscr{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system. Because $Q_{\mathrm{cpl0}}|_{\mathscr{N}\times\mathscr{P}^{\perp_\Sigma}} \geqslant 0$ all the conditions in the theorem are satisfied by these storage functions *except* the rank condition. We shall modify these storage functions suitably to meet the rank condition as well. We first note that

$$\left(X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1\right)^{\mathrm{T}} LZ_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2 = \left(X_{\mathcal{K}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1\right)^{\mathrm{T}} Z_{\mathcal{K}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2$$

for $(v_1, v_2) \in \mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$.

Let $K^+_{\mathcal{N}} \in \mathbb{R}^{\mathrm{n}(\mathcal{N})\times \mathrm{n}(\mathcal{N})}$ and $K^-_{\mathcal{P}^{\perp_\Sigma}} \in \mathbb{R}^{\mathrm{n}(\mathcal{P})\times \mathrm{n}(\mathcal{P})}$ be such that $|X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)v_1|^2_{K^+_{\mathcal{N}}}$ and $|Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma v_2|^2_{K^-_{\mathcal{P}^{\perp_\Sigma}}}$ are the largest and the smallest storage functions for $\mathcal{N}$ as a Σ-dissipative system and for $\mathcal{P}^{\perp_\Sigma}$ as a $(-\Sigma)$-dissipative system, respectively. Then we have that $Q_{\mathrm{cpl0}}(v_1, 0) \leqslant |X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)v_1|^2_{K^+_{\mathcal{N}}}$ for all $v_1 \in \mathcal{N}$ and $Q_{\mathrm{cpl0}}(0, v_2) \leqslant -|Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma v_2|^2_{K^-_{\mathcal{P}^{\perp_\Sigma}}}$ for all $v_2 \in \mathcal{P}^{\perp_\Sigma}$.

Now consider the QDF Q_{cpl} on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$ defined in the theorem (Eq. (1)). It can be expressed as in the equation:

$$Q_{\mathrm{cpl}}(v_1, v_2) = \begin{bmatrix} X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1 \\ Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix} \begin{bmatrix} X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1 \\ Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2 \end{bmatrix}. \tag{4}$$

We have $0 \leqslant Q_{\mathrm{cpl0}} \leqslant Q_{\mathrm{cpl}}$ on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$. Further, since $X_{\mathcal{N}}$ and $Z_{\mathcal{P}}$ are minimal and hence trim, we already obtain that $\left[\begin{smallmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{smallmatrix}\right]$ is non-negative. Also, because of the strictness of the dissipativity of $\mathcal{N}$ and of $\mathcal{P}^{\perp_\Sigma}$, using Lemma 6, we get that $K^+_{\mathcal{N}}$ and $(-K^-_{\mathcal{P}^{\perp_\Sigma}})$ are both positive definite. We now show $\left[\begin{smallmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{smallmatrix}\right]$ is non-singular. Let

$$\begin{bmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{5}$$

Because of the trimness of $X_{\mathcal{N}}$ and $Z_{\mathcal{P}}$, there exist $\bar{v}_1 \in \mathcal{N}$ and $\bar{v}_2 \in \mathcal{P}^{\perp_\Sigma}$ such that $X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)\bar{v}_1(0) = a$ and $Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma\bar{v}_2(0) = b$. For this $\bar{v}_1$ and $\bar{v}_2$ let $X_{\mathcal{K}}(\mathrm{d}/\mathrm{d}t)\bar{v}_1(0) = p$ and $Z_{\mathcal{K}}(\mathrm{d}/\mathrm{d}t)\Sigma\bar{v}_2(0) = q$. Now, $Q_{\mathrm{cpl}}(\bar{v}_1, \bar{v}_2)(0) = 0$ implies $Q_{\mathrm{cpl0}}(\bar{v}_1, \bar{v}_2)(0) = 0$ which in turn implies $p = 0$ and $q = 0$ (because of positive definiteness of the matrix in Eq. (3)). This results in $0 = p^{\mathrm{T}}q = a^{\mathrm{T}}Lb$. This and Eq. (5) (where we have that $Lb = -K^+_{\mathcal{N}}a$) together with the positive definiteness of $K^+_{\mathcal{N}}$ yields $a = 0$. Using negative definiteness of $K^-_{\mathcal{P}^{\perp_\Sigma}}$ and from Eq. (5) we have $b = 0$ too. This implies that the matrix in Eq. (4) is non-singular. In order to apply Theorem 1, one needs to check whether the behavior $\mathrm{col}(X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)\mathcal{N}, Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma\mathcal{P}^{\perp_\Sigma})$ is trim. Since both $X_{\mathcal{N}}$ and $Z_{\mathcal{P}}$ are minimal state maps for $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$, respectively, the behaviors $X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)\mathcal{N}$ and $Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma\mathcal{P}^{\perp_\Sigma}$ are trim and hence their Cartesian product is trim too. We use Theorem 1 to obtain the desired rank condition on Q_{cpl}. This completes the 'only if' part of the proof.

If part: Each term in the given coupling QDF Q_{cpl} of Eq. (1) can be expressed as a function of the state of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$. Let $K^+_{\mathcal{N}}$, $K^-_{\mathcal{P}^{\perp_\Sigma}}$ be the matrices associated with the storage functions $Q_{\Psi^+_{\mathcal{N}}}$ and $Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}$, i.e., $Q_{\Psi^+_{\mathcal{N}}}(v_1) = |X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)v_1|^2_{K^+_{\mathcal{N}}}$ and $Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}(v_2) = |Z_{\mathcal{P}}(\mathrm{d}/\mathrm{d}t)\Sigma v_2|^2_{K^-_{\mathcal{P}^{\perp_\Sigma}}}$. Hence the given coupling condition can be expressed as

$$0 \leqslant Q_{\mathrm{cpl}}(v_1, v_2) = \begin{bmatrix} X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1 \\ Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix} \begin{bmatrix} X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1 \\ Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2 \end{bmatrix}.$$

Theorem 1 together with the rank condition on the above QDF implies that the matrix in the above equation is positive definite. In particular, $K^+_{\mathcal{N}}$ is positive and hence, as mentioned in the remark following this theorem, $\mathcal{N}$ is strictly dissipative on, in fact, $\mathbb{R}_-$. Hence there exists an $\varepsilon > 0$ such that $\mathcal{N}$ is dissipative on $\mathbb{R}_-$ with respect to $\Sigma - \varepsilon I$. Let $K^+_{\mathcal{N},\varepsilon}$ be the matrix associated with the largest storage function for $\mathcal{N}$ as a $(\Sigma - \varepsilon I)$-dissipative system. Using Lemma 7, we have that $K^+_{\mathcal{N},\varepsilon} \to K^+_{\mathcal{N}}$ as $\varepsilon \to 0$, where $K^+_{\mathcal{N}}$ is associated with the largest storage function of $\mathcal{N}$ as a Σ-dissipative system. Similarly, $\mathcal{P}^{\perp_\Sigma}$ is dissipative with respect to $(-\Sigma - \varepsilon I)$ for $\varepsilon > 0$ sufficiently small. If $K^-_{\mathcal{P}^{\perp_\Sigma},\varepsilon}$ and $K^-_{\mathcal{P}^{\perp_\Sigma}}$ are the matrices associated with the smallest storage functions for $\mathcal{P}^{\perp_\Sigma}$ as $(-\Sigma - \varepsilon I)$ and $(-\Sigma)$-dissipative systems, respectively, then $K^-_{\mathcal{P}^{\perp_\Sigma},\varepsilon} \to K^-_{\mathcal{P}^{\perp_\Sigma}}$ as $\varepsilon \to 0$. Hence for ε sufficiently small, the following holds:

$$0 < \begin{bmatrix} K^+_{\mathcal{N}} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix} \Rightarrow 0 < \begin{bmatrix} K^+_{\mathcal{N},\varepsilon} & L \\ L^{\mathrm{T}} & -K^-_{\mathcal{P}^{\perp_\Sigma},\varepsilon} \end{bmatrix} =: K_\varepsilon \text{ (say).}$$

Let $Q_{\Delta_{\mathcal{N},\varepsilon}}$ be the dissipation rate associated with the storage function $|X_{\mathcal{N}}(\mathrm{d}/\mathrm{d}t)v_1|^2_{K^+_{\mathcal{N},\varepsilon}}$ for $\mathcal{N}$ as a $(\Sigma - \varepsilon I)$-dissipative system, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1\right|^2_{K^+_{\mathcal{N},\varepsilon}} = |v_1|^2_\Sigma - \varepsilon|v_1|^2 - Q_{\Delta_{\mathcal{N},\varepsilon}}(v_1) \quad \text{and} \quad Q_{\Delta_{\mathcal{N},\varepsilon}}|_{\mathcal{N}} \geqslant 0.$$

Analogously, we denote the dissipation rate for $\mathcal{P}^{\perp_\Sigma}$ by $Q_{\Delta_{\mathcal{P}^{\perp_\Sigma},\varepsilon}}$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}|Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2|^2_{K^-_{\mathcal{P}^{\perp_\Sigma},\varepsilon}} = -|v_2|^2_\Sigma - \varepsilon|v_2|^2 - Q_{\Delta_{\mathcal{P}^{\perp_\Sigma},\varepsilon}}(v_2) \quad \text{and} \quad Q_{\Delta_{\mathcal{P}^{\perp_\Sigma},\varepsilon}}|_{\mathcal{P}^{\perp_\Sigma}} \geqslant 0.$$

We define the QDF's Q_{Δ_ε} on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$ and Q_{Ψ_ε} on $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. Let

$$\begin{aligned}
&Q_{\Delta_\varepsilon}(v_1, v_2) \\
&\quad := |v_1 + v_2|^2_\Sigma - \frac{\mathrm{d}}{\mathrm{d}t}\left|\mathrm{col}\left(X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1, Z_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2\right)\right|^2_{K_\varepsilon} \\
&\quad = |v_1|^2_\Sigma - \frac{\mathrm{d}}{\mathrm{d}t}\left|X_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_1\right|^2_{K^+_{\mathcal{N},\varepsilon}} + |v_2|^2_\Sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left|Z_\Sigma\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_2\right|^2_{K^-_{\mathcal{P}^{\perp_\Sigma},\varepsilon}} \\
&\quad = Q_{\Delta_{\mathcal{N},\varepsilon}}(v_1) + \varepsilon|v_1|^2 - Q_{\Delta_{\mathcal{P}^{\perp_\Sigma},\varepsilon}}(v_2) - \varepsilon|v_2|^2
\end{aligned}$$

for $(v_1, v_2) \in \mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$. Define

$$Q_{\Psi_\varepsilon}(v_3) := |v_3|^2_{\Sigma - \varepsilon I} - \frac{\mathrm{d}}{\mathrm{d}t}\left|\mathrm{col}\left(Z_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma v_3, X_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)v_3\right)\right|^2_{K_\varepsilon^{-1}}$$

for $v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$. We factor Q_{Ψ_ε} canonically on $\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$ as $Q_{\Psi_\varepsilon}(v_3) = |F^+_\varepsilon(\mathrm{d}/\mathrm{d}t)v_3|^2 - |F^-_\varepsilon(\mathrm{d}/\mathrm{d}t)v_3|^2$.

The following equality is crucial to the proof:

$$\frac{d}{dt}\left|\left[\begin{array}{c} X_{\mathcal{N}}\left(\frac{d}{dt}\right)v_1 \\ Z_{\mathcal{P}}\left(\frac{d}{dt}\right)\Sigma v_2 \end{array}\right] + K_\varepsilon^{-1}\left[\begin{array}{c} Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3 \\ X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3 \end{array}\right]\right|^2_{K_\varepsilon}$$

$$=|v_1+v_2+v_3|^2_\Sigma - \varepsilon|v_3|^2 - Q_{\Delta_\varepsilon}(v_1,v_2) - Q_{\Psi_\varepsilon}(v_3)$$

$$=|v_1+v_2+v_3|^2_\Sigma - Q_{\Delta_{\mathcal{N},\varepsilon}}(v_1) + Q_{\Delta_{\mathcal{P}^{\perp_\Sigma},\varepsilon}}(v_2) - \left|F_\varepsilon^+\left(\frac{d}{dt}\right)v_3\right|^2$$

$$+|F_\varepsilon^-\left(\frac{d}{dt}\right)v_3|^2 - \varepsilon|v_1|^2 + \varepsilon|v_2|^2 - \varepsilon|v_3|^2. \tag{6}$$

We define the controlled behavior $\mathcal{K} := \mathcal{N} + \mathcal{F}_\varepsilon^-$ with $\mathcal{F}_\varepsilon^-$ the controllable part of the behavior $\{v \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} | F_\varepsilon^-(d/dt)v = 0\}$. For $v_1 \in \mathcal{N}, v_2 = 0$ and $v_3 \in \mathcal{F}_\varepsilon^-$, Eq. (6) yields

$$\frac{d}{dt}\left|\left[\begin{array}{c} X_{\mathcal{N}}\left(\frac{d}{dt}\right)v_1 \\ 0 \end{array}\right] + K_\varepsilon^{-1}\left[\begin{array}{c} Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3 \\ X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3 \end{array}\right]\right|^2_{K_\varepsilon}$$

$$=|v_1+v_3|^2_\Sigma - Q_{\Delta_{\mathcal{N},\varepsilon}}(v_1) - \left|F_\varepsilon^+\left(\frac{d}{dt}\right)v_3\right|^2 - \varepsilon|v_1|^2 - \varepsilon|v_3|^2$$

$$\leqslant |v_1+v_3|^2_\Sigma - \frac{\varepsilon}{2}|v_1+v_3|^2.$$

Here, in addition to the non-negativity of $Q_{\Delta_{\mathcal{N},\varepsilon}}$ on $\mathcal{N}$, we used the inequality $|v_1|^2+|v_3|^2 \geqslant \frac{1}{2}|v_1+v_3|^2$. By the definition of $\mathcal{K}$, we have $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. Also, every $v \in \mathcal{N} + \mathcal{F}_\varepsilon^-$ can be decomposed into $v_1 + v_3$ with $v_1 \in \mathcal{N}$ and $v_3 \in \mathcal{F}_\varepsilon^-$, and for every such v with compact support we have $\int_\infty^0 |v_1+v_3|^2_{(\Sigma-\varepsilon I/2)}\,dt \geqslant 0$ (because of the positive definiteness of K_ε). From this we infer that $\mathcal{K}$ is indeed strictly dissipative on $\mathbb{R}_-$ with respect to Σ. It only remains to show that $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$ for $\varepsilon > 0$ sufficiently small.

Consider the following subspaces $\mathbb{L}_\Delta$ and $\mathbb{L}_\Psi$ of $\mathbb{R}^{\mathtt{v}+2(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))}$:

$$\mathbb{L}_\Delta = \Bigg\{a \mid \exists v_1 \in \mathcal{N} \text{ and } v_2 \in \mathcal{P}^{\perp_\Sigma} \text{ such that}$$

$$a = \left(\operatorname{col}\left(v_1+v_2, X_{\mathcal{N}}\left(\frac{d}{dt}\right)v_1, Z_{\mathcal{P}}\left(\frac{d}{dt}\right)\Sigma v_2,\ \frac{d}{dt}X_{\mathcal{N}}\left(\frac{d}{dt}\right)v_1,\ \frac{d}{dt}Z_{\mathcal{P}}\left(\frac{d}{dt}\right)\Sigma v_2\right)\right)(0)\Bigg\},$$

$$\mathbb{L}_\Psi = \Bigg\{b \mid \exists v_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} \text{ such that}$$

$$b = \left(\operatorname{col}\left(\Sigma v_3, -\frac{d}{dt}Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, -\frac{d}{dt}X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3, -Z_{\mathcal{N}}\left(\frac{d}{dt}\right)\Sigma v_3, -X_{\mathcal{P}}\left(\frac{d}{dt}\right)v_3\right)\right)(0)\Bigg\}.$$

Observe that $\mathbb{L}_\Psi \subseteq \mathbb{L}_\Delta^\perp$. Consider the matrices Q_ε and (with a slight abuse of notation) Q_0, both constant matrices that induce a QF on $\mathbb{R}^{\mathtt{v}+2(\mathtt{n}(\mathcal{N})+\mathtt{n}(\mathcal{P}))}$, defined as follows:

$$Q_\varepsilon := \begin{bmatrix} \Sigma^2(\Sigma-\varepsilon I)^{-1} & 0 & 0 \\ 0 & 0 & -K_\varepsilon \\ 0 & -K_\varepsilon & 0 \end{bmatrix} \quad \text{and} \quad Q_0 := \begin{bmatrix} \Sigma & 0 & 0 \\ 0 & 0 & -K_\varepsilon \\ 0 & -K_\varepsilon & 0 \end{bmatrix}.$$

For $\varepsilon > 0$ sufficiently small, note that because $\Sigma, \Sigma-\varepsilon I$ and K_ε are invertible, so are Q_ε and Q_0. Further, by definition of Q_{Δ_ε} and Q_{Ψ_ε} we have $\operatorname{sign}(Q_{\Delta_\varepsilon}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}) = \operatorname{sign}(Q_0|_{\mathbb{L}_\Delta})$ and $\operatorname{sign}(Q_{\Psi_\varepsilon}|_{\mathcal{P}\cap\mathcal{N}^{\perp_\Sigma}}) = \operatorname{sign}(Q_\varepsilon^{-1}|_{\mathbb{L}_\Psi})$.

For ε sufficiently small, we have $\operatorname{sign}(Q_\varepsilon) = \operatorname{sign}(\Sigma) + (\operatorname{rank}(K_\varepsilon), \operatorname{rank}(K_\varepsilon))$, and $\operatorname{rank}(K_\varepsilon) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$. This is utilized in obtaining a sharp estimate of the row dimension of F_ε^- by using an argument exactly like the one in the proof of the non-strict case in [11]. We skip a repetition of the argument and conclude that $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$. □

5. Conclusions and remarks

As expected, the solution to the strictly dissipative synthesis problem differs from that of the non-strict synthesis result of [11]. We have shown that it is the coupling QDF (which was just non-negative in the non-strict case) that has to be suitably strict, namely, it should have maximal rank. In this context, the McMillan degrees of the hidden behavior and the plant behavior also come into picture. We remark that both the problem formulation and the main theorem are formulated in a representation-free manner. This makes it possible to apply Theorem 3 when the to-be-controlled plant is given by any particular representation. If, for example, $\mathcal{N}$ and $\mathcal{P}$ are the hidden behavior and plant behavior associated with the plant $\mathcal{P}_{\text{full}}$ given by the ubiquitous state space representation $(\mathrm{d}/\mathrm{d}t)x = Ax+Bu+Ed;\ y = C_1x+D_1d;\ f = C_2x+D_2u$, with to-be-controlled variable $w = (d, f)$ and control variable $c = (u, y)$, then our problem of finding a suitable behavior $\mathcal{K}$, strictly dissipative on $\mathbb{R}_-$ w.r.t. $\Sigma = \left[\begin{smallmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{smallmatrix}\right]$, amounts to finding an internally stabilizing controller that makes the closed loop transfer matrix strictly contracting. Analogously as in [9, Section 5], by applying Theorem 3 to this case we can re-obtain the well-known conditions in terms of two Riccati equations and a coupling condition that first appeared in [2], and was studied later in various forms in, for example, [6,7].

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