M-theory and gauged supergravities

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Chapter 6

Domain Walls

In this chapter, we will construct half-supersymmetric domain wall solutions to the massive and gauged supergravities of the previous chapter and we will discuss their physical interpretation in terms of branes. In the last section we will consider 1/4 supersymmetric intersections of domain walls with strings.

6.1 D8-brane in Massive IIA

D8-brane Solution

In section 3.4 we have discussed the different supersymmetric solutions of massless IIA supergravity. The situation for massive IIA supergravity is radically different: there is no maximally supersymmetric solution [98] and only one half-supersymmetric solution, the D8-brane solution [62]. It is carried by the metric and the dilaton, which read

$$ds^2 = H^{1/8} dx_9{}^2 + H^{9/8} dy^2 \,, \qquad e^{\phi} = H^{-5/4} \,. \tag{6.1}$$

Note that this is of the form of the generic p-brane solutions (3.37) with $d = 9$, $\tilde{d} = -1$ and $\Delta = 4$, and has $\sqrt{-g} g^{yy} = 1$. It is expressed in terms of one harmonic function $H = c + m_{\mathrm{R}} y$, where we take m_{R} positive and c is an arbitrary integration constant. This solution preserves half of supersymmetry under the supersymmetry rules (B.5) with explicit massive deformations (5.3) with Killing spinor

$$\epsilon = H^{1/32} \epsilon_0 \,, \text{ with } (1 + \Gamma^{\underline{y}}) \, \epsilon_0 = 0 \,, \tag{6.2}$$

where ϵ_0 is a constant spinor that satisfies the above linear constraint. Thus the D8-brane has 16 unbroken supersymmetries.

For later use we would also like to present the D8-brane in a different coordinate system, which is related via

$$\tilde{H} = 2m_{\rm R}\tilde{y} + \tilde{c} = H(y)^2 \,. \tag{6.3}$$

In the new transverse coordinate $\tilde{y}$ the solution reads

$$ds^2 \; = \tilde{H}^{1/16} dx_9{}^2 + \tilde{H}^{-7/16} d\tilde{y}^2 \,, \qquad e^{\phi} = \tilde{H}^{-5/8} \,. \tag{6.4}$$

Note that we now have $\sqrt{-g}g^{tt} = -1$. For the present section, we will use the first parameterisation (6.1), however.

As discussed in section 3.4, a domain wall with harmonic function $H = c + m_{\rm R} y$ is not well-defined. The zeroes in H induce singularities in the solution. To avoid these, one has to include source terms (corresponding to a thin domain wall) to modify the behaviour of the harmonic function. We will discuss such source terms for the D8-brane solution in the next subsection.

Source Terms and Piecewise Constant Parameters

To this end we introduce a number of source terms, corresponding to eight-branes. Since these couple to a nine-form potential it is necessary to dualise the mass parameter of massive IIA to a ten-form field strength:

$$m_{\rm R} = e^{-5\phi/2} \star G^{(10)} \,, \qquad G^{(10)} = dC^{(9)} \,, \tag{6.5}$$

as discussed in section 3.2. In the absence of sources, the field equation for $C^{(9)}$ implies $m_{\rm R}$ to be constant. When sources are present, however, the parameter $m_{\rm R}$ is required to be piecewise constant, i.e. it can take different (constant) values in different regions of the transverse space. This property is the reason why the corresponding solution is called a domain wall; the eight-brane sources separate physically different regions.

The eight-brane source terms are given by

$$S_8 = -\frac{2\pi}{(2\pi\ell_s)^9} \int d^9x \{ e^{-\phi} \sqrt{-g_{(9)}} + \tfrac{1}{9!} \varepsilon^{(9)} C^{(9)} \} \,, \tag{6.6}$$

with $\varepsilon^{(9)\ \mu_0 \cdots \mu_8} = \varepsilon^{(10)\ \mu_0 \cdots \mu_8 y}$ and we use the ranges $\mu, \nu = (0, \ldots, 8)$ in this section. Depending on the coefficients of S_8 in the total action, the source terms have a different interpretation in string theory:

- Objects with positive coefficients correspond to D8-branes. Passing through such a domain wall leads to a decrease of the slope of the harmonic function [62, 71, 186]. The prime example is

$$H = \begin{cases} c - m_{\rm R} y \,, & y > 0 \,, \\ c + m_{\rm R} y \,, & y < 0 \,, \end{cases} \tag{6.7}$$

with c and $m_{\rm R}$ positive. This can be written as $H = c - m_{\rm R}|y|$, where the absolute value of the transverse coordinate y can be seen as a consequence of the piecewise constant parameter $m_{\rm R}$. It follows that H will vanish for some critical value of y.

- Objects with negative coefficients correspond to so-called O8-planes. These are orientifold planes[1], which arise by dividing out by a specific $\mathbb{Z}_2$ symmetry. In this case the relevant symmetry is $I_y\Omega$, where I_y is a reflection in the transverse space and Ω is the world-sheet parity operation. Its effect on the IIA supergravity fields reads

$$\begin{aligned} y &\to -y\,, \\ \big\{\phi, g_{\mu\nu}, B_{\mu\nu}\big\} &\to \big\{\phi, g_{\mu\nu}, -B_{\mu\nu}\big\}\,, \\ \big\{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\big\} &\to (-)^{n+1}\big\{C^{(2n-1)}_{\mu_1\cdots\mu_{2n-1}}\big\}\,, \\ \big\{\psi_\mu, \lambda, \epsilon\big\} &\to \Gamma^{\underline{y}}\big\{\psi_\mu, -\lambda, \epsilon\big\}\,, \end{aligned} \tag{6.8}$$

and the parity of the fields with one or more indices in the y-direction is given by the rule that every index in the y-direction gives an extra minus sign compared to the above rules.

Due to the inclusion of such source terms, the harmonic function will be e.g. $H = c + m_{\rm R}|y|$ with c and $m_{\rm R}$ positive [71, 186]. Thus H is positive for all values of y and has a minimum at the O8-plane.

One thus finds that the introduction of D8-branes leads to zeroes in H and thus to a 'critical distance'. It forces one to include O8-planes at a smaller distance, such that the zero in H is avoided. If the transverse space is $\mathbb{R}/\mathbb{Z}_2$, we can take one O8-plane with Ramond-Ramond charge -16 (in units where a D8-brane has charge $+1$) and n D8-branes and their images with $n \leq 8$. For $n > 8$ the total tension is positive and a zero in the harmonic function will occur. On the other hand, if the transverse space is $S^1/\mathbb{Z}_2$ (i.e. the range of y is compact), the total tension has to vanish and one is led to type I$'$ string theory with two O8-planes at the two fixed points and 16 D8-branes and their images in between [62].

Type I$'$ String Theory and Supergravity

We will consider an example of the latter situation in full detail. First we choose our space-time to be $M^9 \times S^1$. All fields satisfy $\Phi(y) = \Phi(y + 2\pi R)$ with R the radius of S^1. Furthermore, the fields are either even or odd under $I_9\Omega$. Modding out this $\mathbb{Z}_2$ symmetry, the odd fields vanish on the fixed points $y = 0$ and $y = \pi R$ of the orientifold, where we will put the brane sources. However, the type IIA theory would be inconsistent under orientifold truncation unless extra gauge degrees of freedom appear in the theory. It turns out we have to place 32 D8-branes between these O8-planes [62], leading to type I$'$ string theory. It is T-dual

[1]Orientifold planes arise when modding out with a discrete symmetry that involves Ω, the string world sheet parity operation; see [187] for a nice introduction.

to type I string theory, which is obtained by modding the IIB theory with the $\mathbb{Z}_2$ symmetry Ω, as discussed in section 2.2. This also explains the origin of the 32 D8-branes: the type I gauge group $SO(32)$ can be seen to come from 32 unoriented D9-branes (filling all of space-time) and performing T-duality yields the 32 D8-branes [64].

We will consider the special situation where all D-branes coincide with either one of the O-planes. In addition, we assume that there is no matter on the branes. Thus, we are describing the vacuum solution of the D-brane system, switching off the excitations on the branes. Therefore, our total effective action is given by

$$S = 2(n-8)S_8\delta(y) + 2(8-n)S_8\delta(y-\pi R) - \frac{2\pi}{(2\pi\ell_s)^9}\int d^9x\mathcal{L}_{\text{bulk}}\,, \tag{6.9}$$

which is given by the bulk action and an O8-plane and $2n$ D8-branes at $y = 0$ and an O8-plane and $32-2n$ D8-branes at πR. For definiteness we will take $8 < n \leq 16$, i.e. the D8-branes dominate the O8-plane at $y = 0$ while the latter dominates at $y = \pi R$.

The D8-brane solution is given by (6.1) with harmonic function [71]

$$H = c + \frac{(8-n)}{2\pi\ell_s}|y|\,. \tag{6.10}$$

Thus we may identify the mass parameter as follows:

$$m_{\text{R}} = \begin{cases} \dfrac{8-n}{2\pi\ell_s}\,, & y > 0\,, \\ \dfrac{n-8}{2\pi\ell_s}\,, & y < 0\,. \end{cases} \tag{6.11}$$

The harmonic function (6.10) with piecewise constant mass parameter m_{R} will have a zero if the range of y is too large; the distance between the branes must be small enough to prevent the harmonic function from vanishing. The radius of the circle and distance between the O-planes is thus restricted to

$$R < \frac{2c\ell_s}{(n-8)}\,. \tag{6.12}$$

The saturating case is called the critical distance R_c. Thus it seems that type I′ string theory is consistent only on $M^9 \times (S^1/\mathbb{Z}_2)$ with a circle of restricted radius.

Of course we have only considered a special case of the type I′ theory with all D-branes on one of the fixed points. However, also with D-branes in between the O-planes we expect the vacuum solution to imply a critical distance: each O8-plane necessarily has 16 D8-branes in its vicinity. The same phenomenon of type I′ was found in [62] in the context of the duality between the heterotic and type I theories. Note that the maximal distance depends on the distribution of the D-branes. In the most asymmetric case ($n = 16$) it is smallest while in the most symmetric case ($n = 8$) there is no restriction on R.

Note that the identification (6.11) implies a quantisation of the mass parameter of massive IIA supergravity. Upon dimensional reduction, this should coincide with the special case of the $SL(2,\mathbb{R})$ mass parameters

$$\vec{m} = (0, \frac{\tilde{n}}{2\pi R}, \frac{\tilde{n}}{2\pi R}) \,, \qquad \tilde{n} \in \mathbb{Z} \,, \tag{6.13}$$

as can be seen from (5.28) and (5.33). At first sight, these quantisation conditions do not seem to match. The resolution can be found in [73], where factors of g_s are properly taken into account. Being related to the mass of D-branes, both quantised masses (6.11) and (6.13) are inversely proportional to g_s of IIA and IIB, respectively (see the discussion below (3.49)):

$$m_A = \frac{\pm(n-8)}{2\pi \ell_s g_A} \,, \qquad m_B = \frac{\tilde{n}}{2\pi R_B g_B} \,, \tag{6.14}$$

where we have included the A and B labels and omitted the s subscript of g_s. The T-duality relations between the IIA and IIB parameters

$$R_A R_B = {\ell_s}^2 \,, \qquad g_A \ell_s = g_B R_B \,, \tag{6.15}$$

then exactly relate the two expressions for the quantised mass with $\tilde{n} = \pm(n-8)$.

It is clear that the D8-O8 system can be generalized further. To start with, placing D-branes in any compact transverse space requires the presence of oppositely charged branes that need to have opposite tensions in order to be in supersymmetric equilibrium [71]. If all the negative-tension branes are identified with orientifold planes, as we have suggested here, then the compact transverse spaces must be orbifolds with the orientifold planes placed at the orbifold points. The $\mathbb{Z}_2$ reflection symmetries associated to the orientifold planes can be part of more general orbifold groups ($\mathbb{Z}_n$ etc.). It would be interesting to realize these bulk & brane configurations explicitly.

6.2 Domain Walls in CSO Gaugings and their Uplift

The DW/QFT Correspondence

Due to the AdS/CFT correspondence [20], it has been realized that there is an intimate relationship between certain branes of string or M-theory and corresponding lower-dimensional $SO(n)$ gauged supergravities. The relation is established via a maximally supersymmetric vacuum configuration of string or M-theory, which is the direct product of an AdS space and a sphere (see section 3.4): for a p-brane with n transverse directions, we are dealing with an $AdS_{p+2} \times S^{n-1}$ vacuum configuration. On the one hand, this vacuum configuration arises as the near-horizon limit of an M2-, D3- or M5-brane; on the other hand, the coset reduction over the spherical part leads to the related $SO(n)$ gauged supergravity in $p+2$ dimensions, which allows for a maximally supersymmetric AdS_{p+2} vacuum configuration (see section 5.5). The gauge theory of the AdS/CFT correspondence can be taken at the boundary of this AdS_{p+2} space. All dilatons are constant for this vacuum configuration (with no extra dilaton present, i.e. $a = 0$). This is related to the conformal invariance of the gauge theory.

There are two ways to depart from conformal invariance, which both involve exciting some of the dilatons in the vacuum configuration. The first deformation can be introduced via the $n - 1$ dilatons of the AdS supergravities. By exciting some of these dilatons one obtains a deformed Anti-de Sitter configuration. In the AdS/CFT correspondence this corresponds to considering the (non-conformal) Coulomb branch of the gauge theory [20].

Alternatively, one can obtain a non-conformal theory by considering the other branes of string and M-theory, for which there is an extra dilaton present in the scalar potential of the gauged supergravities (corresponding to the $\surd$ in table 6.1). This leads to DW supergravities, where the maximally supersymmetric AdS vacuum is replaced by a non-conformal and half-supersymmetric domain wall solution. This situation is encountered when one generalises the AdS/CFT correspondence to a DW/QFT correspondence [33, 34].

D	n	ϕ	Brane
10	1	$\surd$	D8
9	2	$\surd$	D7
8	3	$\surd$	D6
7	5 / 4	$-$ / $\surd$	M5 / NS5A
6	5	$\surd$	D4
5	6	$-$	D3
4	8 / 7	$-$ / $\surd$	M2 / D2

Table 6.1: *The domain walls and gauged supergravities in D dimensions with n mass parameters are related to M- or D-branes with n transverse directions.*

A natural generalisation is to excite some of the $n - 1$ dilatons describing the Coulomb branch of the CFT and the extra dilaton that leads to a non-conformal QFT at the same time. This leads to domain wall solutions of $SO(n)$ gauged DW supergravities [188] that describe the Coulomb branch of the (non-conformal) QFT. The uplift of these multiple domain walls leads to (the near-horizon-limit of) brane distributions in string or M-theory, as we will see in the next subsections. This is based on results from [189].

A p-brane can be reduced in two ways: via a double dimensional reduction (leading to a $(p-1)$-brane in one dimension lower) or a direct dimensional reduction (leading to a p-brane in one dimension lower). It has been pointed out [34] that direct dimensional reduction leads from $SO(n)$ gauged supergravities to the generalised CSO gauged supergravities of

[176, 177]. Thus, direct dimensional reduction corresponds to a group contraction of the gauged supergravity (see section 5.5). In contrast, double dimensional reduction of a brane corresponds to a dimensional reduction of the gauged supergravity, which was discussed in the same section. Recapitulating, we have the following correspondences between the brane and gauged supergravity points of views:

Brane		Gauged supergravity
direct dimensional reduction	$\Leftrightarrow$	group contraction
double dimensional reduction	$\Leftrightarrow$	toroidal reduction

Note that not all branes of string or M-theory are present in table 6.1. The missing cases of the D0 and F1A can rather easily be included, as has been done in [189]. The corresponding $D = 1, 2$ gauged supergravities have $n = 9, 8$, respectively [190, 191]. The remaining cases are the IIB doublets of NS5B/D5 and F1B/D1 branes. The associated theories are the reduction of IIB over S^3 or S^7 with an electric or magnetic flux of the NS-NS/R-R three form field strength [124,129]. For the $D = 3$ $SO(8)$ theories corresponding to the IIB strings (which are different from the F1A result), see [191]. The five-brane cases are supposed to lead to new $D = 7$ $SO(4)$ gauged supergravities, which might be related to the theories constructed in [180].

Domain Walls

In this subsection, we give a unified description of a class of domain wall solutions for the CSO gauged supergravities in various dimensions, which is of particular relevance to the DW/QFT correspondence.

We consider the following Ansatz for the domain wall with $D - 1$ world-volume coordinates $\vec{x}$ and one transverse coordinate y:

$$ds^2 = g(y)^2 d\vec{x}^2 + f(y)^2 dy^2 \,, \qquad M = M(y) \,, \qquad \phi = \phi(y) \,. \tag{6.16}$$

The idea is to substitute this Ansatz into the action, consisting of the Einstein-Hilbert term, scalar kinetic terms and the scalar potential (5.92), and write this as a sum of squares [192]. Using (5.93), the reduced one-dimensional action can be written as

$$\begin{aligned} S = \int dy\, g^{D-1} f \Big[& \frac{D-1}{4(D-2)} \Big(\frac{2(D-2)}{fg} \frac{dg}{dy} - W \Big)^2 - \frac{1}{2} \Big(\frac{1}{f} \frac{d\vec{\phi}}{dy} + \vec{\partial} W \Big)^2 + \\ & - \frac{1}{2} \Big(\frac{1}{f} \frac{d\phi}{dy} + \partial_\phi W \Big)^2 + \frac{1}{f} \frac{dW}{dy} + (D-1) \frac{1}{fg} \frac{dg}{dy} W \Big] \,, \end{aligned} \tag{6.17}$$

which is a sum of squares, up to a boundary term. Minimalisation of this action therefore corresponds to the vanishing of the squared terms. This gives rise to the first-order Bogomol'nyi equations

$$\frac{1}{f}\frac{d\vec{\phi}}{dy} = -\vec{\partial}W\,, \quad \frac{1}{f}\frac{d\phi}{dy} = -\partial_\phi W\,, \quad \frac{2(D-2)}{fg}\frac{dg}{dy} = W\,, \tag{6.18}$$

Note that one should not expect a Bogomol'nyi equation associated to f since it can be absorbed in a reparameterisation of the transverse coordinate y.

The Bogomol'nyi equations can be solved by the elegant domain wall solution, generalising [188, 193],

$$\begin{aligned} ds^2 &= h^{1/(2D-4)} d\vec{x}^2 + h^{(3-D)/(2D-4)} dy^2\,, \\ M &= h^{1/n} \text{diag}(1/h_1, \ldots, 1/h_n)\,, \qquad e^{\phi} = h^{-a/4}\,, \end{aligned} \tag{6.19}$$

written in terms of n harmonic functions h_i and their product h:

$$h_i = 2q_i y + l_i^2\,, \qquad h = h_1 \cdots h_n\,. \tag{6.20}$$

Note that this transverse coordinate basis[2] has $\sqrt{-g}g^{tt} = -1$. The functions h_i are necessarily positive since the entries of M are positive. For all $q_i \geq 0$, this implies that y can range from 0 to ∞; if there is at least one $q_i < 0$, the range of y is bounded from above.

The solution is parameterised by n integration constants[3] l_i. However, if a charge q_i happens to be vanishing, the corresponding l_i can always be set equal to one (by $SL(n,\mathbb{R})$ transformations that leave the scalar potential invariant). In addition, one can eliminate one of the remaining l_i's by a redefinition of the variable y. Therefore we effectively end up with $p+q-1$ independent constants, parameterising the $p+q$ harmonics. We define m to be the number of linearly independent harmonics h_i with $q_i \neq 0$ and call the corresponding solution (6.19) an m-tuple domain wall. For different values of the constants l_i, one finds different numbers m of linearly independent harmonics. For examples of truncations to single domain walls, see tables 6.2 and 6.3.

It should not be a surprise that all scalar potentials of table satisfy the relation (5.93) since these are embedded in a supergravity theory, whose Lagrangian "is the sum of the supersymmetry transformations" and therefore always yields first-order differential equations. For this reason, domain wall solutions to the separate terms in (6.17) will always preserve half of supersymmetry. The corresponding Killing spinor is given by

$$\epsilon = h^{1/(8D-16)}\epsilon_0\,, \qquad (1+\Gamma_{\underline{y}})\epsilon_0 = 0\,, \tag{6.21}$$

where the projection constraint eliminates half of the components of ϵ_0. An exception is $a = 0$, $q_i = 1$ and $l_i = 0$, in which case the domain wall solution (6.19) becomes a maximally (super-)symmetric Anti-De Sitter space-time in horospherical coordinates. Then the singularity at $y = 0$ is a coordinate artifact and there is an extra Killing spinor, yielding fully unbroken supersymmetry.

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[2] For $D = 10$ and $n = 1$, the solution (6.19) coincides with the D8-brane in the $\tilde{y}$-coordinate (6.4).

[3] Strictly speaking, it is ${l_i}^2$ rather than l_i that appears as integration constant, allowing for positive and negative ${l_i}^2$. However, one can always take these positive by shifting y, in which case the crucial distinction between l_i and ${l_i}^2$ disappears.

Higher-dimensional Origin and Harmonics

Upon uplifting these domain walls, one obtains higher-dimensional solutions, which are related to the 1/2 supersymmetric brane solutions of 11D, IIA and IIB supergravity, as given in table 6.1. Note that the number of mass parameters (and therefore the number of harmonic functions h_i of the transverse coordinate) always equals the transverse dimension of the brane. Thus, in D dimensions, the number n of mass parameters is given by the co-dimension of the half-supersymmetric $(D-2)$-brane of IIA, IIB or M-theory.

The metric of the uplifted solution can in all cases be written in the form

$$ds^2 = H_n^{(2-n)/(D+n-3)}\, dx_{D-1}^2 + H_n^{(D-1)/(D-n-3)}\, ds_n^2\,, \tag{6.22}$$

where H_n is a harmonic function on the transverse space, whose powers are appropriate for the corresponding D-brane solution in ten dimensions or M-brane solution in eleven dimensions (as can be checked from section 3.4). From the form of the metric one infers that the solution corresponds to some brane distribution. For all $q_i = 1$, these solutions were found in [192–194] for the D3-, M2- and M5-branes and in [188, 195] for other branes.

The harmonic function takes the form

$$H_n(y, \mu_i) = h^{-1/2}\Big(\sum_{i=1}^{n} \frac{q_i^2 \mu_i^2}{h_i}\Big)^{-1}\,, \tag{6.23}$$

where μ_i are Cartesian coordinates, fulfilling (5.83). The transverse part of the metric is given by [193]

$$ds_n^2 = H_n^{-1} h^{-1/2} dy^2 + \sum_{i=1}^{n} h_i\, d\mu_i^2\,, \tag{6.24}$$

With a change of coordinates, it can be seen that the n-dimensional transverse space is flat[4] [193, 196]

$$z_i = \sqrt{h_i}\mu_i\,, \quad ds_n^2 = \sum_{i=1}^{n} dz_i dz_i\,. \tag{6.25}$$

The above is easily verified

$$dz_i = h_i^{-1/2} q_i \mu_i\, dy + h_i^{1/2}\, d\mu_i\,, \quad \sum_{i=1}^{n} dz_i dz_i = \sum_{i=1}^{n} \frac{q_i^2 \mu_i^2}{h_i}\, dy^2 + \sum_{i=1}^{n} h_i\, d\mu_i^2\,, \tag{6.26}$$

where we have used $\sum_{i=1}^{n} q_i\, \mu_i\, d\mu_i = 0$, which follows from (5.83). Note that one has $\sqrt{-g} g^{ii} = 1$ after the coordinate change to z_i.

[4] For $D = 10$ and $n = 1$, this coordinate transformation coincides with (6.3). Indeed, the brane solution (6.22) is identical to the D8-brane with transverse y-coordinate (6.1).

The harmonic function H_n specifies the dependence on the n transverse coordinates z_i. The constants l_i parameterise the possible harmonics that are consistent with the reduction Ansatz. The mass parameters q_i specify this reduction Ansatz. Thus, changing a mass parameter q_i changes both the reduction Ansatz and the harmonic function that is compatible with that Ansatz. Sending a mass parameter to zero, e.g. $q_n \to 0$, corresponds to truncating the harmonic function on n-dimensional flat space to

$$H_n(q_n = 0, l_n = 1) = H_{n-1}\,, \tag{6.27}$$

i.e. a harmonic function on $(n-1)$-dimensional flat space.

It is difficult to obtain the explicit expression for the harmonic function H_n in terms of the Cartesian coordinates z_i (the example of $n = 2$ will be given in the next section). Nevertheless, one can show that H_n is indeed harmonic on $\mathbb{R}^n$ for all values of q_i, thus extending the analysis of [194] where $q_i = 1$. The calculation is facilitated by the following definitions

$$A_m = \sum_{i=1}^{n} \frac{q_i^m z_i^2}{h_i^m}\,, \quad B_m = \sum_{i=1}^{n} \frac{q_i^m}{h_i^m}\,. \tag{6.28}$$

In terms of A_m and B_m we calculate

$$\partial_i H_n = h^{-1/2}\Big(-\frac{q_i z_i}{h_i}\frac{B_1}{A_2^2} + 4\,\frac{q_i z_i}{h_i}\frac{A_3}{A_2^3} - 2\frac{q_i^2 z_i}{h_i^2}\frac{1}{A_2^2}\Big)\,, \tag{6.29}$$

from which we finally get

$$\begin{aligned}\sum_{i=1}^{n} \partial_i\partial_i H_n &= h^{-1/2}\Big(2\frac{B_2}{A_2^2} - 2\frac{B_1 A_3}{A_2^3} - 16\frac{A_4}{A_2^3} + 16\frac{A_3^2}{A_2^4} - 2\frac{B_2}{A_2^2} + 16\frac{A_4}{A_2^3} + 2\frac{B_1 A_3}{A_2^3} - 16\frac{A_3^2}{A_2^4}\Big)\\ &= 0\,,\end{aligned} \tag{6.30}$$

which proves the harmonicity of H_n on $\mathbb{R}^n$.

Brane Distributions for $SO(n)$ Harmonics

Since the harmonic function H_n depends on the angular variables in addition to the radial, the uplifted solution will in general correspond to a distribution of branes rather than a single brane. For $D < 9$ and all $q_i = 1$ (i.e. the $SO(n)$ cases[5] with $n \geq 3$) this means that the harmonic function can be written in terms of a charge distribution σ as follows [188, 193]

$$H_n(\vec{z}) = \int d^n z' \frac{\sigma(\vec{z}\,')}{|\vec{z} - \vec{z}\,'|^{n-2}}\,, \tag{6.31}$$

[5] Note that we can also include the cases where some $q_i = 0$, using (6.27).

and since H_n appears without an integration constant, the distributions will actually be a near-horizon limit of the brane distribution.

It turns out that the distributions are given in terms of higher dimensional ellipsoids [193, 197]. The dimensions of these ellipsoids are given in terms of the number m of independent harmonics h_i or, equivalently, the number $m-1$ of non-vanishing constants l_i. It is convenient to define

$$x_{m-1} = 1 - \sum_{i=1}^{m-1} \frac{z_i^2}{l_i^2}\,, \qquad \vec{l} = (l_1, \dots l_{m-1}, 0, \dots, 0)\,, \tag{6.32}$$

where the last $n - m + 1$ constants l_i are vanishing. Starting with the case $m = n$, i.e. all harmonics h_i independent and only l_n equal to zero, we have a negative charge distributed inside the ellipsoid and a positive charge distributed on the boundary:

$$\sigma_n \sim \frac{1}{l_1 \cdots l_{n-1}} \Big(- x_{n-1}^{-3/2} \Theta(x_{n-1}) + 2\, x_{n-1}^{-1/2} \delta(x_{n-1}) \Big)\, \delta^{(1)}(z_n)\,, \tag{6.33}$$

where $\Theta(x)$ is the Heaviside step function: $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$. Upon sending l_{n-1} to zero, the charges in the interior of the ellipsoid cancel, leaving one with a positive charge on the boundary of a lower dimensional ellipsoid:

$$\sigma_{n-1} \sim \frac{1}{l_1 \cdots l_{n-2}}\, \delta(x_{n-2})\, \delta^{(2)}(z_{n-1}, z_n)\,, \tag{6.34}$$

which is the brane distribution corresponding to $n - 1$ independent harmonics since h_{n-1} and h_n are linearly dependent. Next, the contraction of more constants will yield brane distributions over the inside of an ellipsoid. The distribution $\sigma(z_i)$ is then a product of a delta-function and a theta-function and the branes are localised along $n - m + 1$ coordinates and distributed within an $m-1$-dimensional ellipsoid, defined by $x_{m-1} = 0$. For $2 \leq m \leq n-2$ non-zero constants, one has

$$\sigma_m \sim \frac{1}{l_1 \cdots l_{m-1}}\, x_{m-1}^{(n-m-3)/2}\, \Theta(x_{m-1})\, \delta^{(n-m+1)}(z_m, \dots, z_n)\,. \tag{6.35}$$

Finally, one is left with all constant l_i vanishing, in which case the distribution has collapsed to a point and generically reads

$$\sigma_1 = \delta^{(n)}(z_1, \dots, z_n)\,, \tag{6.36}$$

i.e. we are left with a single brane with all harmonics h_i linearly dependent. All these distributions satisfy

$$\sigma_{m-1} = \delta(z_{m-1}) \int \sigma_m\,, \tag{6.37}$$

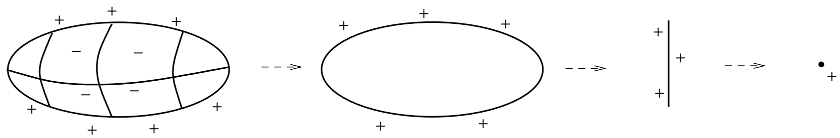

Figure 6.1: *The distributions of NS5A-branes corresponding to the uplift of the 7D $ISO(4)$ domain walls with three, two, one and zero non-vanishing l_i's, respectively.*

consistent with the picture of distributions that collapse the z_{m-1}-coordinate upon sending l_{m-1} to 0. The case of NS5A-branes is illustrated in figure 6.1.

The general lesson to be drawn from this section is that the domain wall solutions uplift to branes with harmonic functions given by (6.23). For the cases with all $q_i \geq 0$, these harmonic functions correspond to the near-horizon limit of the brane distributions (6.33). In the simplest case, with all relevant $l_i = 0$ and therefore $m = 1$, this distribution collapses to a point (6.36) and the harmonic function stems from the near-horizon limit of a single brane. In the next sections we will see whether these findings also hold for the special cases $n = 2$ and $n = 3$.

6.3 Domain Walls in 9D and Uplift to IIB

In this section, we will consider domain wall solutions to the 9D gauged supergravities that were constructed in section 5.3. We will first focus on the $SL(2, \mathbb{R})$ gauged theories and later comment on the possibility of domain walls in the other 9D gauged theories.

Domain Walls in $SL(2, \mathbb{R})$ Gauged Supergravities

We first consider domain walls in the $SL(2, \mathbb{R})$ gauged supergravities in 9D, which are specified by three mass parameters. We will take $m_1 = 0$, which can be obtained by an $SL(2, \mathbb{R})$ transformation. Thus, we are left with a symmetric mass matrix Q with diagonal entries q_1 and q_2, see (5.85). By choosing appropriate values for q_1 and q_2, one can still cover each of the three conjugacy classes of $SL(2, \mathbb{R})$, corresponding to $q_1 q_2$ positive, negative or vanishing.

For the present purpose of domain walls, it suffices to consider a truncation to gravity and the scalars. The supersymmetry transformations of the fermions, which are given in (B.16) in full generality, then reduce to (see (B.16) and (5.20))

$$\begin{aligned}
\delta\psi_\mu &= (\partial_\mu + \omega_\mu + \tfrac{i}{4} e^{\phi} \partial_\mu \chi + \tfrac{1}{28} \gamma_\mu W) \epsilon \,, \\
\delta_0 \lambda &= i(\partial\!\!\!/ \phi + \delta_\phi W)\, \epsilon^* - e^{\phi} (\partial\!\!\!/ \chi + \delta_\chi W)\, \epsilon^* \,, \\
\delta_0 \tilde{\lambda} &= i(\partial\!\!\!/ \varphi + \delta_\varphi W)\, \epsilon^* \,,
\end{aligned} \tag{6.38}$$

with the superpotential W given by (5.19). The projector corresponding to a domain wall Ansatz is

$$\Pi_{\text{DW}} = \tfrac{1}{2}(1 + \gamma^{\underline{y}}) \,, \tag{6.39}$$

where $\underline{y}$ indicates a tangent space direction, see appendix A.

Half-supersymmetric domain walls correspond to configurations satisfying the Killing spinor equations, which are obtained by requiring (6.38) subject to the projection (6.39) to vanish. The most general solutions were first classified in [150] (for other discussions of 9D domain walls, see [198, 199]) and read[6]

$$\begin{aligned} ds^2 &= h^{1/14}(-dt^2 + dx_7^2) + h^{-3/7}dy^2 \,, \\ e^{\phi} &= h^{-1/2}h_1 \,, \quad e^{\sqrt{7}\varphi} = h^{-1} \,, \qquad \chi = c_1 {h_1}^{-1} \,, \end{aligned} \tag{6.40}$$

with the functions

$$h = h_1 h_2 - c_1^2 \,, \qquad h_1 = 2q_1 y + {l_1}^2 \,, \qquad h_2 = 2q_2 y + {l_2}^2 \,. \tag{6.41}$$

This is the most general half-supersymmetric domain wall solution.

The general domain walls are parameterised by three constants. However, as also explained for the general case in section 6.2, one can always do a coordinate transformation $y \to y + c$ to shift either l_1 or l_2 to zero. The third parameter c_1 can be understood as corresponding to the gauge symmetry with constant parameters: by performing $SL(2,\mathbb{R})$ transformations of the form (5.26) one shifts c_1. For this reason, one can always choose a gauge in which it vanishes. In this case the Killing spinor reads

$$\epsilon = h^{1/56}\,\epsilon_0 \,, \tag{6.42}$$

while in general it depends on c_1. Since the transformation to shift c_1 to zero is a gauge transformation with constant parameter, it does not affect the gauge potentials. Note that the most general domain walls therefore are $SL(2,\mathbb{R})$ orbits of the prime example (6.19) and is expressed in terms of two harmonic functions and one constant. In the $SL(2,\mathbb{R})$ frame with $c_1 = 0$, it can be seen as a harmonic superposition of the domain walls with harmonics h_1 and h_2. Due to the two independent harmonic functions, we call this the double domain wall.

In certain truncations, the general solution (6.40) becomes a single domain wall with only one independent harmonic function. This can happen either due to the vanishing of a mass parameter q_i or due to special values of the constants l_i. In table 6.2 we give the two possible truncations leading to single domain walls and the corresponding value of Δ as defined in [90]. Note that the $SO(2)$ case cannot be assigned a Δ-value since it has vanishing potential, as already noted in [150]. The domain wall is carried by the non-vanishing massive contributions to the BPS equations. In other words, the potential is zero but there is a non-vanishing superpotential.

[6] In [150] a different transverse coordinate $\tilde{y}$ was used, which is related via $h(y) = \tilde{h}(\tilde{y})^2$, where the function $\tilde{h}(\tilde{y})$ appears in the metrics of [150] and is not necessarily harmonic. Each different conjugacy class has a different function $\tilde{h}(\tilde{y})$ and therefore requires a different coordinate transformation.

Gauge group	(q_1, q_2)	h_1	h_2	Δ
$\mathbb{R}$	$(0, q)$	1	$2qy$	4
$SO(2)$	(q, q)	$2qy$	$2qy$	$\times$

Table 6.2: *The single domain walls as truncations of the 9D double domain wall solution. We give the two possible truncations and the corresponding value of Δ. Note that there does not exist a Δ-value in the $SO(2)$ case due to the vanishing of the potential.*

Seven-branes and Orientifold Planes

As discussed in the section 6.1, the occurrence of domain walls with positive tension leads to a harmonic function that vanishes at a point in the transverse space. To avoid this, one has to include orientifold planes with negative tension as well, which can be introduced by modding out the theory with a $\mathbb{Z}_2$-transformation. In 10D IIA the relevant symmetry is $I_y\Omega$ (6.8) which introduces (in the case of y compact) 16 D8-branes and their images and two O8-planes. In 9D the relevant $\mathbb{Z}_2$-symmetry can be obtained from the IIA transformation $I_y\Omega$ (6.8) by the reduction in a direction other than y. Alternatively, one could reduce the IIB transformation $(-)^{F_L} I_{xy}\Omega$ in the x direction. Upon reduction these give the same transformation and therefore are T-dual [71]. In particular, the 9D $\mathbb{Z}_2$-symmetry acts on the mass parameters as $Q \to -Q$. Thus all three mass parameters flip sign. However, one can always use an $SL(2,\mathbb{R})$-transformation to set $m_1 = 0$. Then one is left with q_1 and q_2 and since both mass parameters flip sign, one introduces orientifold planes which carry a charge of -16 with respect to both q_1 and q_2. Taking y compact (for a non-compact transverse space the discussion is analogous), one also has to introduce a number of positive tension branes to cancel the total charges. For the q_2-charge this correspond to 32 D7-branes. The cancellation of q_1-charge requires 32 Q7-branes, which are defined as S-duals of the D7-branes. Thus the following picture seems to emerge:

- Two orientifold planes, one at each of the fixed points of the S^1, each carrying a charge of $(-16, -16)$ with respect to the two mass parameters (q_1, q_2).
- Sixteen D7-branes and their images, located at arbitrary points between the two O7-planes and each carrying a charge of $(0, 1)$.
- Sixteen Q7-branes and their images, defined as S-duals of the D7-branes, also distributed between the two O7-planes and each carrying a charge of $(1, 0)$.

Depending on the positioning of the various 7-branes, the mass parameters can take different values. Note that the gauge group can change when passing through a 7-brane, since it can affect only q_1 or q_2 and thus $\det(Q)$ need not be invariant. The reduction of the type I′ theory would correspond to a special case of this general set-up, in which eight of the Q7-branes and their images are positioned at each O7-plane, thereby cancelling the $(-16, 0)$ charge and inducing $q_1 = 0$ everywhere in the bulk[7].

Uplift to IIB and D7-branes

Instead of the nine-dimensional discussion of source terms above, one can also uplift the domain walls to solutions of IIB supergravity. The general formula (6.23) applied to the 9D case yields the harmonic function

$$H_2 = \left(\sqrt{h_2}{\mu_1}^2 + \sqrt{h_1}{\mu_2}^2\right)^{-1}, \tag{6.43}$$

with the identifications (5.86) to make contact with the explicit twisted reduction Ansatz (B.14) with reduction coordinate y. In this case, it is straightforward (though perhaps not very insightful) to perform the coordinate transformation to z_i, which yields:

$$H_2 = \left(\frac{\alpha {z_1}^2 + \beta {z_2}^2 + \gamma({z_1}^2 + {z_2}^2)}{2\gamma^2}\right)^{1/2}, \tag{6.44}$$

with the definitions

$$\begin{aligned} \alpha &= q_1 q_2({z_1}^2 + {z_2}^2) + q_1 l_2^2 - q_2 l_1^2 \,, \\ \beta &= q_1 q_2({z_1}^2 + {z_2}^2) - q_1 l_2^2 + q_2 l_1^2 \,, \end{aligned} \qquad \gamma = \sqrt{\tfrac{1}{2}(\alpha^2 + \beta^2) + q_1 q_2({z_1}^2 - {z_2}^2)(\alpha - \beta)} \,. \tag{6.45}$$

Indeed, it can be checked that this function is harmonic with respect to flat (z_1, z_2)-space for all values of q_i and l_i.

The harmonic function H_2 generically depends on both ${z_1}^2 + {z_2}^2$ and ${z_1}^2 - {z_2}^2$. Note that the dependence on the latter only disappears if $\alpha = \beta = \gamma$, in which case the harmonic function reads

$$H_2 = (q_1 q_2)^{-1/2} \,. \tag{6.46}$$

This requires the relation $q_1 l_2^2 = q_2 l_1^2$. This cannot be satisfied with the charges $(q_1, q_2) = (1, -1)$ and $(1, 0)$ while keeping both $h_i > 0$. Therefore, the only possibility is charges $(1, 1)$, implying that the two constants l_i need to be equal, yielding a harmonic function given by $H_2 = 1$. Another case with a manifest isometry is provided by the charges $(1, 0)$, where the harmonic function becomes

$$H_2(q_2 = 0) = |z_1|/l_2 \,, \tag{6.47}$$

[7] Toroidal compactifications of type I′ string theory have been considered in [200] from a somewhat different point of view. It would be interesting to link its results to our analysis here.

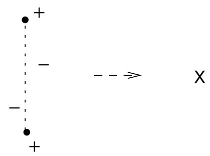

Figure 6.2: *The distributions of D7-branes corresponding to the uplift of the 9D $SO(2)$ domain walls with one and zero non-vanishing l_i's, respectively. The cross indicates the conical singularity of the locally flat space-time.*

which is a harmonic function in a one-dimensional transverse space, in agreement with (6.27).

For the $SO(2)$ case, in which we take $q_1 = q_2 = 1$, the IIB solution can be understood as a distribution of D7-branes. Without loss of generality we take $l_2 = 0$. Then one has

$$H = 1 + \int dz_1' dz_2' \sigma(z_1', z_2'; l_1) \log((z_1 - z_1')^2 + (z_2 - z_2')^2) \,, \tag{6.48}$$

with the D7-brane distribution

$$\sigma(z_1', z_2';_1) = \frac{1}{2\pi l_1}\Big[-\big(1 - \frac{{z_1'}^2}{{l_1}^2}\big)^{-3/2}\Theta\big(1 - \frac{{z_1'}^2}{{l_1}^2}\big) + 2\big(1 - \frac{{z_1'}^2}{{l_1}^2}\big)^{-1/2}\delta\big(1 - \frac{{z_1'}^2}{{l_1}^2}\big)\Big] \,, \tag{6.49}$$

for the case $m = 2$ (implying that $l_1 \neq 0$). Note that this distribution consists of a line interval of negative D7-brane density with a positive contribution at both ends of the interval. Both positive and negative contributions diverge but these cancel exactly:

$$\int dz_1' dz_2' \sigma(z_1', z_2') = 0 \,, \tag{6.50}$$

i.e. the total charge in the distribution (6.49) vanishes.

The parameter l_1 of the general $SO(2)$ solution can be set to zero, truncating to only one independent harmonic function: $m = 1$. This corresponds to a collapse of the line interval to a point, as can be seen from (6.49). However, due to the fact that the total charge vanishes, this leaves us without any D7-brane density:

$$\sigma(z_1', z_2'; l_1 = 0) = 0 \,, \tag{6.51}$$

Indeed, the general harmonic function (6.44) equals one for the $SO(2)$ case with $l_1 = l_2 = 0$. The D7-brane distributions are shown in figure 6.2.

Therefore, the two-dimensional $SO(2)$ harmonic function (6.48) of the D7-brane differs in two important ways from the generic $SO(n)$ harmonic function with $n > 2$. Firstly, the total charge distribution of D7-branes vanishes, while it adds up to a finite and positive number in the other cases. Secondly, but not unrelated, one needs to include a constant in the harmonic function (6.48) in terms of the distribution. In the generic cases this constant was absent, corresponding to the near-horizon limit of these branes. In this respect, the D7-brane is special, as can also be seen from the following observation.

As discussed in section 3.4, the near-horizon limit of D-branes (3.40) yields a metric that is conformally[8] AdS $\times$ S. To absorb the conformal factor, one needs to go to the so-called dual frame, in which the tension of the brane is independent of the dilaton:

$$g_{\mu\nu}^{\text{dual}} = \exp\left(\frac{(3-p)}{2(p-7)}\,\phi\right) g_{\mu\nu}^{\text{Einstein}}\,. \tag{6.52}$$

In the dual frame, the near-horizon geometry of all D-branes with $p \leq 6$ reads AdS$_{p+2} \times S^{n-1}$. Clearly, this formula does not hold for the D7-brane; a related complication is the fact that the dual object is the D-instanton, which lives on a Euclidean space.

Having found the general $SL(2,\mathbb{R})$ domain walls, we would like to impose the different quantisation conditions on $\vec{m}$ of section 5.3. For the $SO(2)$ case with $q_1 = q_2 = q$, these translate in a deficit angle: since the argument of the trigonometric functions of μ_i (5.86) is $\sqrt{q_1 q_2}y$ and the variable y is identified up to $2\pi R$, our $SO(2)$ angular variable has a range of $2\pi q R$. For this reason, the locally flat space-times with $H_2 = 1$ are conical space-times with a deficit angle $2\pi(1-qR)$. Let us go through the three quantisation possibilities for $SO(2)$ [150], giving the result for $l_1 = l_2$:

- The first quantisation condition (5.34) has $q = 1/(4R)$ and thus yields a deficit angle of $3\pi/2$. In other words, this is a half-supersymmetric Mink$_8 \times \mathbb{C}/\mathbb{Z}_4$ space-time with non-trivial monodromy, the bosonic part of which was also mentioned in [95].

- The second quantisation condition (5.35) cannot be applied to q but only to an $SL(2,\mathbb{R})$ related partner of our uplifted domain wall, since it requires an off-diagonal matrix Q. It gives rise to a deficit angle of $5\pi/3$, leading to a half-supersymmetric Mink$_8 \times \mathbb{C}/\mathbb{Z}_6$ space-time with non-trivial monodromy.

- The third quantisation condition (5.36) has $q = 1/R$ and thus leads to fully supersymmetric Mink$_{10}$ space-time. The monodromy is trivial and there is a second Killing spinor with opposite chirality. For the previous two quantisation conditions this second Killing spinor had a different monodromy and was therefore not a globally consistent solution of the Killing spinor equations.

Domain Walls for other Gauged Supergravities

In the previous subsections, we have constructed and discussed the most general domain wall solution to the three $SL(2,\mathbb{R})$ gauged supergravities. In this subsection we would like to address the possibility of domain walls for the other 9D gauged supergravities of section 5.3.

[8] Except for the case $p = 5$, which has Minkowski$_7$ rather than AdS$_7$ [91].

Since we are looking for 1/2 BPS solutions, we have to solve the Killing spinor equations. These are obtained by setting the supersymmetry variation of the gravitino and dilatini to zero, while the supersymmetry parameter is subject to a certain projection. The projector for a domain wall is given by $\frac{1}{2}(1 \pm \gamma^{\underline{y}})$, where y denotes the transverse direction.

In this way we solve first order equations instead of second order equations, which we would encounter if we would solve the field equations directly. For static configurations, a solution to the Killing spinor equation is also a solution to the field equations. From an analysis of the massive supersymmetry transformations $\delta_0 + \delta_m$ of the gravitino and the dilatino, it was found [145] that to solve the Killing spinor equations, one has to set all mass parameters to zero except for $\vec{m}$. Therefore, there are not more half-supersymmetric domain wall solutions than the ones given in (6.40) with mass parameters q_1 and q_2.

By analysing the possibilities for other projectors in nine dimensions (i.e. demanding that the projector squares to itself and that its trace is half of the spinor dimension), we find that there is another projector given by $\frac{1}{2}(1 \pm i\gamma^{\underline{t}})$. This projector would give a time-dependent solution, which can be seen as a Euclidean domain wall having time as a transverse direction. See [201] for an example.

6.4 Domain Walls in 8D and Uplift to IIA and 11D

In this section, we will construct the most general half-supersymmetric domain wall solution to the 8D gauged maximal supergravities of section 5.4. We will start with the class A theories and only comment on class B at the end.

Domain Walls of Class A Theories

In section 5.4 we have obtained the bosonic action (5.68) and supersymmetry transformations (5.63) of the $D = 8$ gauged maximal supergravities with gauge groups of class A. We now look for domain wall solutions that preserve half of the supersymmetry, following [136]. For an earlier discussion of a subset of these solutions, see [202].

We consider the following domain wall Ansatz:

$$ds^2 = g(y)^2 dx_7{}^2 + f(y)^2 dy^2\,, \qquad \mathcal{M} = \mathcal{M}(y)\,, \qquad \varphi = \varphi(y)\,, \qquad \epsilon = \epsilon(y)\,. \quad (6.53)$$

Our Ansatz only includes the metric and the scalars. All other fields are vanishing except the $SL(2,\mathbb{R})/SO(2)$ scalar ℓ which we have set constant. It turns out that there are no half-supersymmetric domain walls for non-constant ℓ. We need to satisfy the Killing spinor equations (which are a truncation of (5.63) to the fields of the Ansatz (6.53))

$$\begin{aligned}
\delta\psi_\mu &= 2\partial_\mu\epsilon - \tfrac{1}{2}\not{\psi}_\mu\epsilon + \tfrac{1}{2}\not{Q}_\mu\epsilon + \tfrac{1}{24}e^{-\varphi/2}f_{ijk}\Gamma^{ijk}\Gamma_\mu\epsilon = 0\,,\\
\delta\lambda_i &= -\not{P}_{ij}\Gamma^j\epsilon - \tfrac{1}{3}\not{\partial}\varphi\Gamma_i\epsilon - \tfrac{1}{4}e^{-\varphi/2}(2f_{ijk} - f_{jki})\Gamma^{jk}\epsilon = 0\,,
\end{aligned}$$

where the Killing spinor satisfies the condition

$$(1 + \Gamma_{\underline{y123}})\epsilon = 0\,. \tag{6.54}$$

The indices $1, 2, 3$ refer to the internal group manifold directions.

The most general class A domain wall solution reads

$$\begin{aligned}
&ds^2 = h^{\frac{1}{12}} dx_7^2 + h^{-\frac{5}{12}} dy^2\,,\\
&e^{\varphi} = h^{\frac{1}{4}}, \quad e^{\sigma} = h^{-\frac{1}{2\sqrt{3}}} h_1^{\frac{\sqrt{3}}{2}}, \quad e^{\phi} = h^{-\frac{1}{2}} h_1^{-\frac{1}{2}} (h_1 h_2 - c_1^2)\,,\\
&\chi_1 = c_1 h_1^{-1}, \quad \chi_2 = \chi_1\chi_3 + c_2 h_1^{-1}, \quad \chi_3 = (c_1 c_2 + c_3 h_1)\left(h_1 h_2 - c_1^2\right)^{-1}, \end{aligned} \tag{6.55}$$

where the dependence on the transverse coordinate y is governed by

$$\begin{aligned}
&h(y) = h_1 h_2 h_3 - c_3^2 h_1 - c_2^2 h_2 - c_1^2 h_3 - 2c_1 c_2 c_3\,,\\
&h_1 = 2q_1 y + {l_1}^2, \quad h_2 = 2q_2 y + {l_2}^2, \quad h_3 = 2q_3 y + {l_3}^2\,. \end{aligned} \tag{6.56}$$

The corresponding Killing spinor is quite intricate so we will not give it here. Note that the solution is given by three harmonic function h_i. For this reason we call the general solution a triple domain wall.

The general solution has six integration constants c_i and l_i. As before, one can eliminate one of the constants l_i by a redefinition of the variable y. The other three constants c_1, c_2 and c_3 can be understood to come from the following symmetry. The mass deformations do not break the full global $SL(3,\mathbb{R})$; indeed, they gauge the three-dimensional subgroup of $SL(3,\mathbb{R})$ that leaves the mass matrix Q invariant. Thus one can use the unbroken global subgroup to transform any solution[9], introducing three constants. In our solution these correspond to c_1, c_2 and c_3, which can therefore be set to zero by fixing the $SL(3,\mathbb{R})$ frame. From now on we will always assume the frame choice $c_1 = c_2 = c_3 = 0$ unless explicitly stated otherwise. This results in

$$\chi_1 = \chi_2 = \chi_3 = 0\,, \qquad \mathcal{M} = h^{-2/3}\text{diag}(h_2 h_3, h_1 h_3, h_1 h_2)\,, \qquad h = h_1 h_2 h_3\,. \tag{6.57}$$

In this $SL(3,\mathbb{R})$ frame the expression for the Killing spinor simplifies considerably and reads $\epsilon = h^{1/48}\epsilon_0$. Thus, analogously to 9D, we find that the most general domain wall solution to these gauged supergravities is given by the $SL(3,\mathbb{R})$ orbits of the generic solution (6.19).

The triple domain wall can be truncated to double or single domain walls when restricting the constants l_1, l_2 and l_3. In table 6.3 we give the three possible truncations leading to single domain walls and the corresponding value of Δ as defined in [90]. The Bianchi II case was given in [202] and the Bianchi IX case in [34] (up to coordinate transformations). Note that the Bianchi VII_0 case cannot be assigned a Δ-value since it has vanishing potential. The domain wall is carried by the non-vanishing massive contributions to the BPS equations. The same mechanism occurs in $SO(2)$ gauged $D = 9$ supergravity [150], see section 6.3.

[9] Note that one cannot use the unbroken local subgroup of $SL(3,\mathbb{R})$ (the gauge transformations) since this would induce non-vanishing gauge vectors and thus would be inconsistent with our Ansatz (6.53).

Bianchi	$Q = \text{diag}$	h_1	h_2	h_3	Δ	Uplift
II	$(0,0,q)$	1	1	$2qy$	4	(6.70)
VII_0	$(0,q,q)$	1	$2qy$	$2qy$	$\times$	(6.68)
IX	(q,q,q)	$2qy$	$2qy$	$2qy$	$-\frac{4}{3}$	(6.65)

Table 6.3: *The single domain walls as truncations of the 8D triple domain wall solution. Note that there exists no Δ-value in the Bianchi VII_0 case due to the vanishing of the potential. In the last column we indicate where the uplifted solution to 11D is given.*

Uplift to IIA and D6-branes

The special case of the $SO(3)$ D6-brane distributions (with all $q_i = 1$) was first discussed in [188]. This splits up in three separate possibilities, with $m = 3, 2$ or 1. The first distribution σ_3 consists of positive and negative densities and is given by the general formula (6.33). Upon sending l_2 to zero, this collapses to

$$\sigma_2 \sim \frac{1}{l_1}\, \delta\big(1 - \frac{{z_1}^2}{{l_1}^2}\big)\, \delta^{(2)}(z_2, z_3)\,. \tag{6.58}$$

This is a distribution at the boundary of a one-dimensional ellipse, i.e. it is localised at the points $z_1 = \pm l_1$. For this reason, the corresponding harmonic function is given by

$$H_3(\vec{z}, l_1) = \frac{1}{2((z_1 - l_1)^2 + {z_2}^2 + {z_3}^2)^{1/2}} + \frac{1}{2((z_1 + l_1)^2 + {z_2}^2 + {z_3}^2)^{1/2}}\,, \tag{6.59}$$

i.e. the near-horizon limit of the double-centered harmonic. Upon sending l_1 to zero, the brane distribution σ_1 collapses to a point, as given in (6.36). Indeed, the harmonic function becomes

$$H_3 = \frac{1}{|\vec{z}|}\,, \tag{6.60}$$

i.e. the near-horizon limit of the single-centered harmonic with $SO(3)$ isometry. The different distributions of D6-branes are shown in figure 6.3.

Uplift to 11D and KK-monopoles

The D6-brane solution is different from the other branes in table 6.1 in the following sense: they can be uplifted to a purely gravitational solution in 11D, the Kaluza-Klein monopole

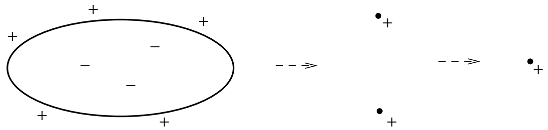

Figure 6.3: *The distributions of D6-branes corresponding to the uplift of the 8D $SO(3)$ domain walls with two, one and zero non-vanishing l_i's, respectively.*

(see also section 3.4). In the z_i coordinates, the higher-dimensional metric reads

$$ds^2 = dx_7^2 + H^{-1}(dy^3 + \sum_{i=1}^{3} A_i dz_i)^2 + \sum_{i=1}^{3} H dz_i{}^2 \, , \tag{6.61}$$

where y^3 is the isometry direction of the KK-monopole. The function $H = H(z_i)$ is given implicitly in (6.23) with $n = 3$ and $A_i = A_i(z_j)$ is subject to the condition (3.54).

However, since we do not have the harmonic function $H = H(z_i)$ explicitly, we will rather use the coordinates y and μ_i. The μ_i are related to the coordinates y_1 and y_2 of the group manifold via (5.89). In these coordinates and using the $SL(3,\mathbb{R})$ frame of (6.57), the triple domain wall solutions becomes a purely gravitational solutions with a metric of the form $\hat{ds}^2 = dx_7{}^2 + ds_4{}^2$, where

$$ds_4{}^2 = h^{-\frac{1}{2}} dy^2 + h^{\frac{1}{2}} \left(\frac{\sigma_1^2}{h_1} + \frac{\sigma_2^2}{h_2} + \frac{\sigma_3^2}{h_3} \right) . \tag{6.62}$$

Here σ_1, σ_2 and σ_3 are the Maurer-Cartan 1-forms defined in (4.46) (with y_i instead of z_i), $h = h_1 h_2 h_3$ and the three harmonics h_i are defined in (6.56). The uplifted solutions are all half-supersymmetric.

The solutions (6.62) are cohomogeneity one solutions of different Bianchi types. The $SO(3)$ expression of this four-dimensional metric was obtained previously in the context of gravitational instanton solutions as self-dual metrics of Bianchi type IX with all directions unequal [203]. More recently, the Heisenberg, $ISO(1,1)$ and $ISO(2)$ cases and their relations to domain wall solutions were considered in [133, 204], whose results are related to ours via coordinate transformations. In the following discussion we will focus on the four-dimensional part of the eleven-dimensional metric since it characterises the uplifted domain walls.

Without loss of generality, we take $q_3 = 2$ in this subsection. The coordinate transformation $2y = \frac{1}{2}r^4 - l_3{}^2$ then eliminates the constant l_3 and results in the metric

$$ds_4{}^2 = (k_1 k_2 k_3)^{-1/2} \left[dr^2 + r^2 (k_2 k_3 \sigma_1^2 + k_1 k_3 \sigma_2^2 + k_1 k_2 \sigma_3^2) \right] , \tag{6.63}$$

where $k_j = q_j/2 + s_j r^{-4}$ with $s_j = l_j{}^2 - q_j l_3{}^2$ for $j = 1, 2$, and $k_3 = 1$. As anticipated, the metric (6.63) depends only on two constant parameters s_1 and s_2, which are restricted by the (gauge group dependent) condition

$$s_j > -q_j r^4/2\,, \tag{6.64}$$

in order to satisfy the requirements $h_j > 0$.

In general the metrics have curvatures that both go to zero as r^{-6} for large r and diverge at $r = 0$ and $k_j = 0$, producing incomplete metrics [203, 205]. There are two exceptions to this behaviour, which coincide with the special cases of the D6-brane discussion with $SO(3)$ isometry:

- The first one corresponds to the case with $s_1 = s_2 = 0$, which can only be obtained for the $SO(3)$ case due to (6.64). Taking $q_1 = q_2 = 2$, the metric is locally flat space:

  $$ds_4{}^2 = dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\,, \tag{6.65}$$

  where r is the radius of the three-dimensional spheres. This corresponds to the uplift of the 9D Bianchi type IX single domain wall or the D6-brane with harmonic function (6.60).

- The second exception corresponds to the $SO(3)$ gauging (taking $q_1 = q_2 = 2$) with $s_1 = s_2 = s < 0$. It is known as the Eguchi-Hanson (EH), or Eguchi-Hanson II, metric [205]:

  $$ds_4{}^2 = \left(1 + \frac{s}{r^4}\right)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \left(1 + \frac{s}{r^4}\right)\sigma_3^2\,. \tag{6.66}$$

  This metric corresponds to the uplift of the D6-brane with harmonic function (6.59).

Another case that we want to discuss, although it is singular, is obtained in the $SO(3)$ gauging (with $q_1 = q_2 = 2$) by choosing $s_1 = s \neq 0$ and $s_2 = 0$:

$$ds_4{}^2 = \left(1 + \frac{s}{r^4}\right)^{-1} (dr^2 + r^2\sigma_1^2) + \left(1 + \frac{s}{r^4}\right)(\sigma_2^2 + \sigma_3^2)\,. \tag{6.67}$$

This metric is called the Eguchi-Hanson I (EH-I) metric [205].

The uplifted metrics for the singular cases with $\det(Q) = 0$ are also given in (6.63). Among them are two special metrics, which can be obtained by contraction of the EH metrics (6.66) and (6.67). This contraction is possible because the solutions contain at least one non-zero constant s_i, and must be performed before identifying the charges q_1 and q_2 and the constants s_1 and s_2. We will take $q_1 = 0$ (which implies $s_1 > 0$ due to (6.64)) and $q_2 \neq 0$ to get the uplifted metrics for the $ISO(2)$ gaugings.

As an example, let us perform such contractions on the contracted EH metrics, leading from $SO(3)$ isometries to $ISO(2)$ isometries. After contraction, the expression for the EH-I metric is

$$ds_4{}^2 = \left(\frac{s}{r^4}\right)^{-1/2} (dr^2 + r^2\sigma_1^2) + \left(\frac{s}{r^4}\right)^{1/2} (\sigma_2^2 + \sigma_3^2)\,, \tag{6.68}$$

and the EH-II metric reads

$$
\begin{aligned}
ds_4{}^2 = & \Big(\frac{s}{r^4}(1+\frac{s}{r^4})\Big)^{-1/2} dr^2 + \Big(\frac{s}{r^4}(1+\frac{s}{r^4})\Big)^{1/2} \sigma_3^2 + \\
& + r^2\Big((1+\frac{r^4}{s})^{1/2}\sigma_1^2 + (1+\frac{r^4}{s})^{-1/2}\sigma_2^2\Big)\,,
\end{aligned}
\tag{6.69}
$$

In the EH-I contracted metric (6.68) we have taken $s_1 = s$ whereas in the EH contracted metric (6.69) we have set $s_1 = s_2 = s$, while $q_2 = q_3 = 2$ in both cases. Notice that the contracted EH-I metric with $ISO(2)$ isometry is precisely the four-dimensional part of the uplifted metric for the Bianchi type VII_0 single domain wall.

The metrics with Heisenberg isometry are obtained by a further contraction $q_2 = 0$ in the metric (6.63). Among these metrics, there is again one special case that can also be obtained by a contraction of the contracted EH metric with isometry $ISO(2)$. Notice that it is not possible to have a contracted EH-I metric with Heisenberg isometry since this would require $s_2 = 0$ and the metrics with Heisenberg isometry have $s_2 \neq 0$. The expression for the contracted EH metric with Heisenberg isometry is

$$
\begin{aligned}
ds_4{}^2 &= \Big(\frac{s}{r^4}\Big)^{-1} dr^2 + r^2(\sigma_1^2+\sigma_2^2) + \Big(\frac{s}{r^4}\Big)\,\sigma_3^2\,, \\
&= \Big(\frac{s}{r^4}\Big)^{-1} dr^2 + r^2({dz_1}^2+{dz_2}^2) + \Big(\frac{s}{r^4}\Big)\,(dz_3+2z^1dz_2)^2\,,
\end{aligned}
\tag{6.70}
$$

where $s_2 = s$ and $q_3 = 2$. This is the four-dimensional part of the uplifted metric for the Bianchi type II single domain wall. This contraction was considered previously in [206].

It is interesting to note that the uplifting of the triple domain wall solution (6.55) does not lead to the most general four-metrics. For example, there are three complete and non-singular hyper-Kähler metrics with $SO(3)$ isometry in four dimensions: the Eguchi-Hanson, Taub-NUT and Atiyah-Hitchin metric (for a useful discussion of these metrics, see [207]). The absence of the Taub-NUT and Atiyah-Hitchin metrics in our analysis stems from the fact that only the Eguchi-Hanson metric allows for a covariantly constant spinor that is independent of the $SO(3)$ isometry directions [208]. In performing the group manifold reductions, we have assumed that our spinors are independent of the internal coordinates. This is not compatible with the Taub-NUT and the Atiyah-Hitchin metrics, which therefore reduce to non-supersymmetric domain walls in 8D. It would be interesting to see whether one can alter the procedure of group manifold reductions, to allow for the Taub-NUT and Atiyah-Hitchin metrics to reduce to half-supersymmetric domain walls in $D = 8$ dimensions.

Isometries of the 3D Group Manifold

The internal three-dimensional manifolds are by definition invariant under the three-dimensional group of isometries given in (5.61) and (5.62). This holds for arbitrary values of the scalars in (5.58). However, there can be more isometries, which rotate two of the Maurer-Cartan one-forms σ^i and σ^j into each other. This is an isometry of the metric in two cases:

- $q_i = q_j = 0$: In this case one can use the automorphism group to set $l_i = l_j = 1$. Equation (6.62) shows that a rotation between σ^i and σ^j is an isometry for all solutions of this class.

- $q_i = q_j \neq 0$: In this case one must set $l_i = l_j$ by hand, after which a rotation between σ^i and σ^j is an isometry. Thus, this only holds for a truncation of the solutions of this class and since $h_i = h_j$ corresponds to decreasing m by one.

This leads to the different possibilities summarised in table 6.4. These exhaust all possible number of isometries on three-dimensional class A group manifolds [153].

Bianchi	(q_1, q_2, q_3)	$m = 0$	$m = 1$	$m = 2$	$m = 3$
I	$(0, 0, 0)$	6	-	-	-
II	$(0, 0, 1)$	-	4	-	-
VI_0	$(0, -1, 1)$	-	-	3	-
VII_0	$(0, 1, 1)$	-	6	3	-
VIII	$(1, -1, 1)$	-	-	4	3
IX	$(1, 1, 1)$	-	6	4	3

Table 6.4: *The numbers of isometries of the three-dimensional group manifold for the different multiple domain wall solutions with m independent harmonic functions h_i. For a given type one finds isometry enhancement by decreasing m, i.e. upon identifying two harmonic functions.*

The extra fourth isometry was constructed by Bianchi [159] for the types II, VIII and IX. He claimed that type VII_0 did not allow for isometry enhancement but the existence of three extra Killing vectors[10] was later shown in [209]. These three extra isometries appear upon identifying the two y-dependent harmonics. Note that the extra isometries may not be isometries of the full manifold in which the group submanifold is embedded. Indeed, this is what happens for type VII_0 where two of the extra isometries are y-dependent and therefore do not leave the full metric invariant [209]. The extra Killing vectors of the group manifold for the uplifted domain wall solutions (6.55) are explicitly given by [153]

[10] We thank Sigbjørn Hervik for a valuable discussion on this point.

- Type I with $Q = \mathrm{diag}(0,0,0)$:

$$L_4 = -z^2 \frac{\partial}{\partial z^1} + z^1 \frac{\partial}{\partial z^2}\,, \qquad L_5 = -z^3 \frac{\partial}{\partial z^1} + z^1 \frac{\partial}{\partial z^3}\,,$$
$$L_6 = -z^3 \frac{\partial}{\partial z^2} + z^2 \frac{\partial}{\partial z^3}\,. \tag{6.71}$$

- Type II with $Q = \mathrm{diag}(0,0,1)$:

$$L_4 = -z^2 \frac{\partial}{\partial z^1} + z^1 \frac{\partial}{\partial z^2} + \tfrac{1}{4}\left((z^1)^2 - (z^2)^2\right) \frac{\partial}{\partial z^3}\,. \tag{6.72}$$

- Type VII$_0$ with $Q = \mathrm{diag}(0,1,1)$ with $h(y) = h_2 = h_3$:

$$\begin{aligned} L_4 &= -h^{-1/2} z^2 \frac{\partial}{\partial z^1} + h^{1/2} z^1 \frac{\partial}{\partial z^2}\,, \\ L_5 &= -h^{-1/2} z^3 \frac{\partial}{\partial z^1} + h^{1/2} z^1 \frac{\partial}{\partial z^3}\,, \\ L_6 &= -z^3 \frac{\partial}{\partial z^2} + z^2 \frac{\partial}{\partial z^3}\,. \end{aligned} \tag{6.73}$$

- Type VIII with $Q = \mathrm{diag}(1,-1,1)$:

$$L_5 = \frac{s_{3,2,1}s_{1,3,2}}{c_{1,3,2}} \frac{\partial}{\partial z^1} + c_{3,2,1} \frac{\partial}{\partial z^2} + \frac{s_{3,2,1}}{c_{1,3,2}} \frac{\partial}{\partial z^3}\,. \tag{6.74}$$

- Type IX with $Q = \mathrm{diag}(1,1,1)$:

$$\begin{aligned} L_4 &= -\frac{c_{3,2,1}s_{1,3,2}}{c_{1,3,2}} \frac{\partial}{\partial z^1} + s_{2,3,1} \frac{\partial}{\partial z^2} - \frac{c_{3,2,1}}{c_{1,3,2}} \frac{\partial}{\partial z^3}\,, \\ L_5 &= \frac{s_{3,2,1}s_{1,3,2}}{c_{1,3,2}} \frac{\partial}{\partial z^1} + c_{3,2,1} \frac{\partial}{\partial z^2} + \frac{s_{3,2,1}}{c_{1,3,2}} \frac{\partial}{\partial z^3}\,, \\ L_6 &= -\frac{\partial}{\partial z^1}\,, \end{aligned} \tag{6.75}$$

where we have used the definitions (5.56). The extra Killing vectors L_4, L_5 and L_6 correspond to rotations between σ^1 and σ^2, σ^1 and σ^3 and σ^2 and σ^3, respectively.

As we have mentioned above, two of the class A solutions uplift to flat space-time in $D = 11$: the Bianchi type IX solutions with $m = 1$ and all Bianchi type I solutions (having $m = 0$). In view of the discussion above, we can now understand why this happens. One can check that the only way to embed three-dimensional submanifolds of zero (for type I) or constant positive (for type IX) curvature in four Euclidean Ricci-flat dimensions is to embed them in four-dimensional flat space. Indeed, this is exactly what we find: the two solutions both have a maximally symmetric group manifold with six isometries and hence constant curvature and uplift to flat $D = 11$ space-time.

The type VII_0 group manifold can also have six isometries and zero curvature. For the domain wall solutions above, this cannot be embedded in four-dimensional flat space due to the y-dependence of two of its isometries. Note, however, that there is another type VII_0 solution with flat geometry and vanishing scalars that coincides with the type I solution (6.55) given above[11]. The corresponding group manifold can be embedded in four-dimensional flat space and indeed this solution uplifts to 11-dimensional Minkowski just as the type I solution. However, unlike its type I counterpart, the eight-dimensional type VII_0 solution with flat geometry and vanishing scalars breaks all supersymmetry.

Domain Walls for Class B Theories

We would like to see whether there are also supersymmetric domain wall solutions to the class B supergravities of section 5.4. It turns out that for this case there are no domain wall solutions preserving any fraction of supersymmetry, much like the situation in nine dimensions. This can be seen as follows.

The structure of the BPS equations requires the projector for the Killing spinor of a half-supersymmetric domain wall solution to be the same as (6.54). The presence of the extra term in $\delta\psi_\mu$ (see (5.63)), depending on the trace of the structure constants, implies that there are no domain wall solutions with this type of Killing spinor, since the structure of Γ-matrices of this term cannot be combined with other terms. To get a solution, one is forced to put $f_{ij}{}^j = 0$, thus leading back to the class A case. This also follows from $\delta\lambda_i$, since the resulting equation is symmetric in two indices, except for a single antisymmetric term, containing $f_{ij}{}^j$.

6.5 Domain Walls with Strings Attached

In this section, we will consider an extension of the domain walls of the previous sections where strings and particles are included. The resulting solutions will be 1/4-supersymmetric. The analysis will only be performed in massive IIA supergravity and $SL(2,\mathbb{R})$ gauged supergravity in $D = 9$, but we expect that some generalisation exists for all CSO gauged supergravities of section 5.5.

D8-F1-D0 Solution

As explained in chapter 2, it was found in the mid 1990's that D-branes can be understood as hyperplanes on which a fundamental string, or F-string, can end [24]. The endpoint of an F-string appears as an electrically charged particle on the world-volume of the D-brane. An

[11] This solution coincides, after an $SO(2)$ rotation of 90 degrees, with the Kaluza-Klein reduction of the Mink$_9$ solution [145, 148] of the $SO(2)$ gauged supergravity in $D = 9$.

exception to this generic phenomenon is the D-particle, on which a single F-string cannot end due to charge conservation[12] [210, 211].

The situation changes in the presence of a domain wall in which case charge conservation no longer forbids an F-string to end on a D-particle [212]. In fact, when a D-particle crosses a D8-brane, a stretched fundamental string with endpoints on the D0- and D8-brane is created [213–215]. This process is, via duality, related to the Hanany-Witten effect in which a stretched D3-brane is created if a D5-brane crosses an NS5-brane [216]. The intersecting configuration for this case is given by[13]

$$\begin{array}{ll|l} \mathrm{D5}: & \times & \times\times---\times\times\times- \\ \mathrm{NS5}: & \times & \times\times\times\times\times---- \\ \mathrm{D3}: & \times & \times\times-------\times \end{array} \tag{6.76}$$

The intersecting configuration of [213–215] is obtained by first applying T-duality in the directions 1 and 2, next applying an S-duality and, finally, applying a T-duality in the directions 6,7 and 8:

$$\begin{array}{ll|l} \mathrm{D0}: & \times & --------- \\ \mathrm{D8}: & \times & \times\times\times\times\times\times\times\times- \\ \mathrm{F1}: & \times & --------\times \end{array} \tag{6.77}$$

In this subsection, we consider the massive IIA supergravity background of the F1-string that is stretched between the D8-domain wall and the D0-particle. It is given by the following solution[14] [219] to the equations of motion of the $D = 10$ Romans' massive IIA supergravity theory [74]:

$$\begin{aligned} ds^2 &= -H^{1/8}\tilde{H}^{-13/8}dt^2 + H^{9/8}\tilde{H}^{-5/8}dy^2 + H^{1/8}\tilde{H}^{3/8}dx_8^2\,, \\ B_{ty} &= -\tilde{H}^{-1}\,, \qquad C_t^{(1)} = H\tilde{H}^{-1}\,, \qquad e^{\phi} = H^{-5/4}\tilde{H}^{1/4}\,, \end{aligned} \tag{6.78}$$

where the harmonic functions H and $\tilde{H}$ are defined as

$$H = c + m_{\mathrm{R}}y\,, \quad \tilde{H} = 1 + \frac{Q}{r^6}\,, \tag{6.79}$$

and the radial coordinate is given in terms of the coordinates longitudinal to the D8-brane, $r^2 = x_1^2 + \ldots + x_8^2$. The solution preserves 1/4 of supersymmetry and the Killing spinor is annihilated by the following projectors

$$\Pi_{\mathrm{D0}} = \tfrac{1}{2}(1 + \Gamma^{\underline{0}}\Gamma_{11})\,, \qquad \Pi_{\mathrm{F1}} = \tfrac{1}{2}(1 + \Gamma^{\underline{0y}})\,, \qquad \Pi_{\mathrm{D8}} = \tfrac{1}{2}(1 + \Gamma^{\underline{y}})\,, \tag{6.80}$$

where any of the three projectors can be obtained from the other two. The solution is a harmonic superposition of two elements, which can be obtained by taking different limits:

[12] Indeed, the Born-Infeld vector, which carries the corresponding degrees of freedom on the D-brane world-volume, is not present on the world-line of the D-particle.

[13] Each horizontal entry indicates one of the 10 directions $0, 1, \ldots, 9$ in space-time. A $\times(-)$ means that the corresponding direction is in the world-volume or (transverse to) the brane.

[14] There are also other string-like solutions to massive IIA supergravity [217, 218], which we will not consider.

- The limit $Q \to 0$ leads to the single D8-brane solution (6.1) which preserves 1/2 supersymmetry.
- The limit $m_{\rm R} \to 0$ leads to an (infinite) F-string with D-particles smeared in the string direction, preserving 1/4 supersymmetry. The F1- and D0-brane charges are related and therefore it is not possible to obtain these as single objects from the above solution.

More precisely, the flux distributions of the F-string and D-particle described by the solution (6.78) are given by

$$\begin{aligned} \mathcal{Q}_1 &= e^{-\phi} \star (dB) = -QH \, d\Omega_7 \,, \\ \mathcal{Q}_0 &= e^{3\phi/2} \star (dC^{(1)} + m_{\rm R} B) = Q \, dy \wedge d\Omega_7 \,, \end{aligned} \tag{6.81}$$

with $d\Omega_7$ the volume form of S^7. To obtain the corresponding charges these are to be integrated over S^7 and $S^7 \times \mathbb{R}$, respectively, where S^7 together with the 8D radius r spans $\mathbb{R}^8$ and $\mathbb{R}$ covers the y-direction transverse to the domain wall. The flux distributions are related by

$$d\mathcal{Q}_1 = -m_{\rm R} \mathcal{Q}_0 \,, \tag{6.82}$$

as required by the field equation for B. This relation shows that in the presence of a domain wall ($m_{\rm R} \neq 0$), the D-particle ($\mathcal{Q}_0 \neq 0$) leads to the creation of a fundamental string ($d\mathcal{Q}_1 \neq 0$). A similar result was obtained in [218] for the NS5-D6-D8 system, i.e. when a NS5-brane passes through a D8-brane a D6-brane, stretched between the NS5-brane and the D8-brane, is created. Both processes are related to the Hanany-Witten effect via duality.

We now return to the distribution of D-particles and F-strings described by (6.78). First of all we note that all non-zero tensor components of (6.78) are even under the $\mathbb{Z}_2$ orientifold symmetry $I_y\Omega$ (6.8). If this were not the case, one would be forced to include a source term, corresponding to the non-zero tensor components, which is smeared over a 9D hyperplane. The only odd field that we allow for is the mass parameter and the corresponding source terms are the domain walls. The supergravity solution (6.78) that we consider only has even non-zero tensor components and the inclusion of source terms for this solution was discussed in [220][15], resulting in a globally well-defined solution on $S^1/\mathbb{Z}_2$. We will now discuss its physical implications.

We note that the distribution of F-strings is linear in H, see (6.81). When we are dealing with a D8-brane, we have $H = c - m_{\rm R}|y|$ which is a linearly increasing function when going towards the domain wall. This is in agreement with the idea of creation of strings when passing through a D8-brane [213–215]. It is pictorially given in figure 6.4, where we have taken all D8-branes to coincide with one of the orientifolds. The strings are unoriented due to the identification $y \sim -y$ which superposes two strings of opposite orientation, see e.g. [215].

[15] The particle and strings source terms of [220] are smeared in the y-direction and directly relate to the charge distributions (6.81).

It should be noted that the linear behaviour of $\mathcal{Q}_1$ is an artifact of the coordinate frame for the transverse coordinate y. The important feature is that it is monotonically increasing when approaching the domain wall.

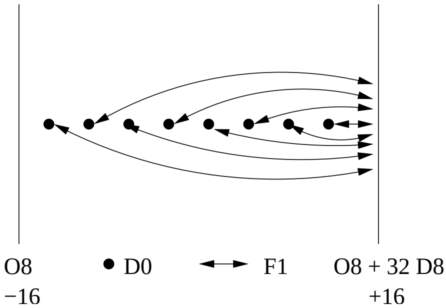

Figure 6.4: *The creation of strings in type I′: a (continuous) distribution of D-particles with a monotonically increasing distribution of unoriented F-strings ending on the D8-branes. The distribution of these F-strings has a maximum at the position of the D8-branes.*

Domain Walls with $SL(2,\mathbb{R})$ Strings in $D=9$

In [221] the D8-F1-D0 solution of massive IIA has been generalised to the 9D gauged supergravities with gauge group $CSO(p,q,r)$ with $p+q+r=2$.

We start from a general Ansatz, respecting $SO(7)$ symmetry. The fields are thus allowed to depend on $r=({x_1}^2+\ldots+{x_7}^2)^{1/2}$ and the transverse direction y. Our strategy will be to solve the BPS-equations obtained from the supersymmetry variations of the fermions. In analogy with the solution (6.78) for the Romans' mass parameter, we will assume that the dependence on r and y coordinates can be separated in a product, i.e. $f(y,r)=f(r)\,f(y)$. This assumption will simplify the equations drastically.

The BPS-equations are obtained by requiring the spinor ϵ to be annihilated by the projection operators for the relevant branes. We search for solutions, which include domain walls, strings and particles. Since we search for 1/4 BPS solutions the 3 projection operators corresponding to the domain walls, strings and particles should not be independent. In other words, once we have two of the operators, the third should follow as a combination of these. In contrast to the IIA solution (6.78), we have the possibility of $SL(2,\mathbb{R})$ doublets of both the particles and the strings in 9D. By analysing the supersymmetry variations in type IIB in $D=10$, it can be seen that the projectors for the F1- and the D1-strings are actually different, and this will therefore also be the case for the strings and particles in $D=9$. Choosing a specific string projector corresponds to choosing an $SL(2,\mathbb{R})$ frame. We take the following projectors

$$\Pi_{\mathrm{D0}}=\tfrac{1}{2}(1+\gamma^{\underline{0}}*)\,,\qquad \Pi_{\mathrm{F1}}=\tfrac{1}{2}(1+\gamma^{\underline{0y}}*)\,,\qquad \Pi_{\mathrm{DW}}=\tfrac{1}{2}(1+\gamma^{\underline{y}})\,,\tag{6.83}$$

where $*$ is seen as an operator, i.e. $*\epsilon = \epsilon^*$. Any third projector is implied by the other two:

$$\Pi_{\mathrm{DW}} = \Pi_{\mathrm{D0}} + \Pi_{\mathrm{F1}} - 4\,\Pi_{\mathrm{D0}}\Pi_{\mathrm{F1}}\,, \tag{6.84}$$

and cyclic. Since ϵ transforms under $SL(2,\mathbb{R})$, the choice of $SL(2,\mathbb{R})$ frame can be seen as a choice of ϵ. To get the most general solution, we should keep the mass parameters as general as possible. We can, however, still perform $SL(2,\mathbb{R})$ transformations, which are upper triangular, without changing ϵ. This can easily be seen by noting that ϵ transforms as

$$\epsilon \to \Big(\frac{c\tau^* + d}{c\tau + d}\Big)^{\frac{1}{4}}\epsilon \tag{6.85}$$

under the $SL(2,\mathbb{R})$ transformation

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.86}$$

We see that ϵ is invariant for $c = 0$. The mass matrix transforms under Λ as well. Even with $c = 0$ we can always use Λ to put m_1 to zero.

Analysing the BPS equations we find that, in order to make up the relevant projection operators, the following components must be put to zero:

$$F_{ty} = F_{tm} = F^1_{ty} = F^1_{tm} = H^i_{tmn} = H^2_{tym} + \chi H^1_{tym} = 0\,, \tag{6.87}$$

where m, n are indices of the spatial coordinates $x^m \neq y$. The Bianchi identity for F^1 reads $dF^1 = -q_1 H^2$. Since $F^1 = 0$, this will lead to further restrictions when q_1 is non-vanishing. We find that $H^2 = 0$ and, using (6.87), also $\chi H^1_{tyi} = 0$. We require $H^1 \neq 0$, since otherwise no F-strings would be present and we conclude that $\chi = 0$ if $q_1 \neq 0$. If $q_1 = 0$, one can draw the same conclusion but from a different argument. In this case, the BPS equations directly imply $\partial_\mu \chi = 0$ and therefore χ is a constant. The only non-zero mass parameter q_2 gauges the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$, which shifts the axion. Thus one can always use a global gauge transformation to set $\chi = 0$. Then (6.87) implies $H^2 = 0$. On top of this we take $F^2_{ty} = 0$ since a non-zero value requires D0-brane sources smeared on the domain-wall world-volume and we want to avoid such 'walls' of D0-branes. We thus find that, for all values of the mass parameters, we are left with just two non-vanishing tensor components, F^2_{tr} and H^1_{tyr}.

We now substitute our Ansatz in the supersymmetry variations of the fermions, which are given in (B.16) and (5.20). This leads to two undetermined functions, one depending on r and one depending on y. The latter can be fixed arbitrarily by using a general coordinate transformations in y. To determine the function of r, we need at least one field equation, e.g. the one for φ. We have computed this field equation, and the result is that the r-dependent function can be expressed in terms of a harmonic function. The resulting particle-string-domain wall solution can be expressed in a unified way, i.e. including all cases $\det(Q) = 0$,

$\det(Q) > 0$ and $\det(Q) < 0$, as follows

$$\begin{aligned}
ds^2 &= -(h_1h_2)^{1/14}\tilde{H}^{-11/7}dt^2 + (h_1h_2)^{-3/7}\tilde{H}^{-4/7}dy^2 + (h_1h_2)^{1/14}\tilde{H}^{3/7}dx_7^2\,,\\
e^{\phi} &= h_1^{1/2}h_2^{-1/2}\,, \quad e^{\sqrt{7}\varphi} = (h_1h_2)^{-1}\tilde{H}\,,\\
A_t^2 &= -h_2^{1/2}h^{-1}\,, \quad B_{ty}^1 = -h_2^{-1/2}h^{-1}\,,\\
\epsilon &= (h_1h_2)^{1/56}\tilde{H}^{-11/28}\,\epsilon_0\,.
\end{aligned} \tag{6.88}$$

The solution is given in terms of three harmonic functions

$$h_1 = 2q_1y + {l_1}^2\,, \qquad h_2 = 2q_2y + {l_2}^2\,, \qquad \tilde{H} = 1 + \frac{Q}{r^5}\,. \tag{6.89}$$

The q's are given in terms of the mass parameters in (5.85) and l_1 and l_2 are integration constants. Just as in $D = 10$, the solution is a harmonic superposition of D-particles, F-strings and domain walls with string and particle fluxes

$$\begin{aligned}
\mathcal{Q}_1 &= e^{-\phi-\varphi/\sqrt{7}} \star (dB^1) = -Q\,{h_2}^{1/2}\,d\Omega_6\,,\\
\mathcal{Q}_0 &= e^{\phi+3\varphi/\sqrt{7}} \star (dA^2 + q_2B^1) = -Q\,{h_2}^{-1/2}\,dy \wedge d\Omega_6\,,
\end{aligned} \tag{6.90}$$

with $d\Omega_6$ the volume form of the S^6. The charges are obtained by integrating the fluxes over S^6 and $S^6 \times \mathbb{R}$, respectively, where the S^6, together with the 7D radius r, spans $\mathbb{R}^7$ and $\mathbb{R}$ covers the y-range. The flux distributions are related by

$$d\mathcal{Q}_1 = q_2\,\mathcal{Q}_0\,, \tag{6.91}$$

as required by the B^2 equation of motion.

Of course one can perform an $SL(2,\mathbb{R})$ transformation on the solution (6.88) and obtain intersections with more general strings and particles. The $SL(2,\mathbb{R})$ generalised flux distributions are given by

$$\begin{aligned}
\mathcal{Q}_{i\,1} &= e^{-\varphi/\sqrt{7}}\mathcal{M}_{ij} \star (dB^j) = q_{i\,1}\mathcal{Q}_1\,,\\
\mathcal{Q}_{i\,0} &= e^{3\varphi/\sqrt{7}}\mathcal{M}_{ij} \star (dA^j - Q^{jk}B_k) = q_{i\,0}\mathcal{Q}_0\,,
\end{aligned} \tag{6.92}$$

where the massive field strengths are taken from (5.21). In this notation the F-strings and D-particles (6.90) have charges $q_{i\,1} = (1,0)$ and $q_{i\,0} = (0,1)$. A transformation with parameter

$$\Lambda = \begin{pmatrix} r & p \\ s & q \end{pmatrix} \in SL(2,\mathbb{R})\,, \tag{6.93}$$

would take the distributions of F-strings and D-particles (6.90) to $q_{i\,1} = (r,s)$ and $q_{j\,0} = (p,q)$. This corresponds to (p,q)-particles and (r,s)-strings subject to the condition $qr - ps =$

1. Furthermore, the $SL(2,\mathbb{R})$ transformation (6.93) rotates the diagonal background (5.85) into

$$Q^{ij} = \tfrac{1}{2}\begin{pmatrix} -m_2' + m_3' & m_1' \\ m_1' & m_2' + m_3' \end{pmatrix} = \begin{pmatrix} q^2 q_1 + s^2 q_2 & -pqq_1 - rsq_2 \\ -pqq_1 - rsq_2 & p^2 q_1 + r^2 q_2 \end{pmatrix} . \qquad (6.94)$$

From now on we will omit the primes on the mass parameters. Thus we find that the most general intersection of (p,q)-particles, (r,s)-strings and an (m_1,m_2,m_3)-domain wall is subject to two conditions:

- The $SL(2,\mathbb{R})$ condition $qr - ps = 1$ should be satisfied. This condition requires orthogonality of the strings and particle charges. It can be expressed as $\varepsilon^{ij} q_{i\,0} q_{i\,1} = 1$.
- The form of the mass matrix is given in (6.94). This mass matrix contains only two independent parameters q_1 and q_2 rather than three for an arbitrary but symmetric mass matrix. This restriction corresponds to $Q^{ij} q_{i\,0} q_{i\,1} = 0$.

The two orthogonality conditions are manifestly $SL(2,\mathbb{R})$ invariant and the parameters q_1 and q_2 specify the only $SL(2,\mathbb{R})$ orbits that solves the BPS equations.

The physical picture consists of a distribution of particles from which strings are emanating towards the domain wall, like in the IIA case. However, we now have an $SL(2,\mathbb{R})$ generalisation of (r,s)-strings stretching between (p,q)-particles in an (m_1,m_2,m_3) background with two orthogonality conditions. The two conditions reduce the seven parameters to five, three of which correspond to the $SL(2,\mathbb{R})$ freedom while the two remaining parameters are q_1 and q_2. In addition the charge Q is the unit string charge. The general solution is illustrated in figure 6.5. The interval in this case is $\mathrm{Max}(-q_1/{l_1}^2, -q_2/{l_2}^2) < y < 0$ with all q_i positive. Note that the charge distribution of the strings is not linear, as opposed to the massive IIA solution in 10D. This is due to the freedom of reparameterisation of the y-coordinate; the important feature is that $\mathcal{Q}_0$ is continuous and positive, implying $\mathcal{Q}_1$ to be monotonically increasing on this interval.

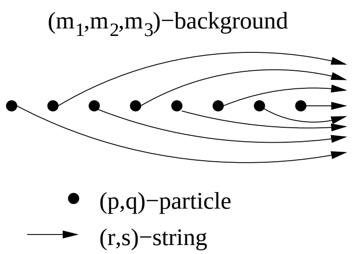

Figure 6.5: *The creation of strings in 9D: a (continuous) distribution of* (p,q)*-particles with a monotonically increasing distribution of emanating oriented* (r,s)*-strings in an* (m_1,m_2,m_3)*-background. There are two orthogonality conditions on the charges.*

One can take different limits of the general solution (6.88). First of all, one can set the parameter $q_1 = 0$. This case corresponds to the reduction of the massive IIA solution of [219] and indeed the Kaluza-Klein reduction (B.9) of (6.78) along one of the world-volume directions of the D8-brane gives (changing y to $\tilde{y}$ for reasons that will become clear shortly)

$$\begin{aligned} ds^2 &= -H^{1/7}\tilde{H}^{-11/7}dt^2 + H^{8/7}\tilde{H}^{-4/7}d\tilde{y}^2 + H^{1/7}\tilde{H}^{3/7}dx_7^2\,, \\ e^{\phi} &= H^{-1}\,, \quad e^{\sqrt{7}\varphi} = H^{-2}\tilde{H}\,, \\ B^1_{t\tilde{y}} &= -\tilde{H}^{-1}\,, \quad A^2_t = -H\tilde{H}^{-1}\,, \end{aligned} \tag{6.95}$$

where the harmonic functions are defined as

$$H = c + 2m_{\mathrm{R}}\tilde{y}\,, \quad \tilde{H} = 1 + \frac{Q}{r^5}\,. \tag{6.96}$$

The above is a special solution to the nine-dimensional gauged supergravity where the mass parameters obey $q_1 = 0$ and $q_2 = m_{\mathrm{R}}$. Exactly the same identifications were found in the case of the reduced massive IIA supergravity, see (5.28). It is related to the generic solution (6.88) by a coordinate transformation $y = y(\tilde{y})$ defined by $h_2(y) = H(y)^2$, which is a special case of the coordinate transformation of footnote 6 of this chapter.

Another possible truncation of the general solution (6.88) is obtained by setting both mass parameters q_1 and q_2 equal to zero. This yields a harmonic superposition of the F-string solution with a distribution of D-particles on it. The two charge distributions are related (both are linear in Q) and therefore it is impossible to obtain either one separately.

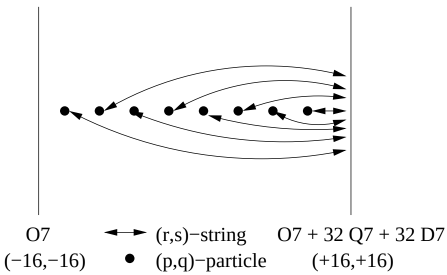

Figure 6.6: *The creation of strings in 9D on $S^1/\mathbb{Z}_2$: a (continuous) distribution of* (p,q)-*particles with a monotonically increasing distribution of unoriented* (r,s)-*strings ending on the D7- and Q7-branes.*

The 9D particle-string-domain wall solution (6.88) corresponds to a region between two domain walls on the $S^1/\mathbb{Z}_2$, as illustrated in figure 6.5. One might wonder about the possibility of extending this to a globally well-defined solution by including source terms for the domain walls and the particle-string intersection. Note that all tensor components of (6.88) are even under the relevant 9D $\mathbb{Z}_2$-symmetry. In fact, the reason to discard the possibility of non-zero F^1_{ty} was its odd transformation under this $\mathbb{Z}_2$-symmetry. Thus one is led to think that it is possible to embed the solution (6.88) in a globally well-defined solution on $S^1/\mathbb{Z}_2$, as is illustrated in figure 6.6. It would be interesting to investigate the boundary conditions in a manner analogous to the IIA analysis of [220].

Recapitulating, the 9D solution (6.88) consists of a smeared distribution of (p,q)-particles, from which (r,s)-strings are emanating and ending on the (m_1,m_2,m_3)-domain wall. There are two orthogonality conditions on the seven parameters, $\varepsilon^{ij}q_{1\,0}q_{i\,1}=1$ and $Q^{ij}q_{1\,0}q_{i\,1}=0$, which are manifestly $SL(2,\mathbb{R})$ invariant. This is the natural generalisation of the 10D IIA solution (6.78). These solutions suggest new possibilities of string creation in nine dimensions that are not the result of the reduced type I$'$ mechanism.

In the light of the general domain wall solution (6.19) of section 6.2, it would be very interesting to extend our 10D and 9D analysis here and consider intersections of domain walls and strings in all dimensions. It is not obvious to us, however, what the general structure in $D\leq 7$ will be. Consider as example the $SO(5)$ gauged theory in $D=7$. Due to the lack of one- and two-forms in the fundamental representation of $SO(5)$, our construction does not trivially carry over. We do expect solutions like (6.88) in the $ISO(4)$ gauged theory in $D=7$, however. It would be very interesting to investigate such solutions in lower-dimensional gauged supergravities and their uplift to the higher-dimensional theories.