M-theory and gauged supergravities

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Chapter 5

Gauged Maximal Supergravities

5.1 Introduction

In this chapter we will consider a number of deformations of the massless supergravities with maximal supersymmetry, as discussed in chapter 3. These deformations are proportional to a parameter m of mass dimension one. Indeed, some fields will acquire masses proportional to the deformation parameter m. Often, another consequence of the parameter will be the gauging of a global symmetry of the massless theory. For this reason, such theories will be called gauged supergravities, which comprise the larger part of this chapter. In cases where the mass parameter does not induce a gauging, the theory is called a massive supergravity. The only known example of such a deformation of maximal supergravity is the massive IIA supergravity [74]. Examples with sixteen supercharges are the massive iia supergravities in six dimensions [138].

An important property of the massive deformations that we consider is that they do not break any supersymmetry. The gauged or massive supergravities therefore have the same number (i.e. 32) of supercharges as the corresponding ungauged or massless supergravity. This preservation of supersymmetry under the massive deformation is in many cases guaranteed due to a higher-dimensional origin: if a gauged supergravity can be obtained by any of the techniques of chapter 4, it necessarily has the same amount of supercharges as the higher-dimensional theory. Equivalently stated, reduction with a twist or over a group or coset manifold does not break supersymmetry[1]. Starting from a maximal higher-dimensional supergravity, one can apply the different reductions of chapter 4 to generate many gauged maximal supergravities. We will perform such reductions to construct gauged maximal supergravities in ten, nine and eight dimensions. In addition, we will include massive IIA su-

[1] This can be contrasted with e.g. Calabi-Yau compactifications, which break a fraction of the supersymmetry. Reduction of IIA supergravity over the four-dimensional Calabi-Yau manifold K3 with fluxes yields the massive iia supergravities with $N = 2$ in $D = 6$ [138], see also [139, 140].

pergravity, which is the only massive deformation of maximal supergravity without a known higher-dimensional origin.

Throughout this chapter we will reduce supersymmetry variations and field equations rather than Lagrangians, since some of the rigid symmetries we employ for reduction scale the Lagrangian. As was explained in detail in section 4.6, reduction with a symmetry that scales the Lagrangian can only be performed on the field equations and the supersymmetry variations, but not on the Lagrangian.

As a consequence of the non-trivial dimensional reduction, the supersymmetry variations and field equations receive two types of massive deformations. There are implicit mass terms that appear via the covariant field strengths, which generally acquire terms that are proportional to the mass parameter. In addition, the supersymmetry transformations and field equations have explicit mass terms. The explicit deformations of the massless supersymmetry variations δ_0 are denoted by δ_m, which are linear in the mass parameter m. The fermionic field equations, symbolically denoted by $X = 0$, also consist of a massless part X_0 plus linear deformations X_m. In contrast, the bosonic field equations receive quadratic massive deformations.

In the cases where it is possible to construct a Lagrangian, the quadratic deformations of the bosonic field equations can be derived from the explicit mass terms in the Lagrangian. These are also quadratic in m, only depend on the scalars and are therefore called the scalar potential V. In many cases, the scalar potential can be written in terms of a superpotential W, which is linear in the mass parameter:

$$V = \frac{1}{4}\left(g^{AB}\frac{\delta W}{\delta\Phi^A}\frac{\delta W}{\delta\Phi^B} - \frac{D-1}{D-2}W^2\right) . \tag{5.1}$$

Here g^{AB} is the inverse of the scalar metric g_{AB} which occurs as $-g_{AB}\partial\Phi^A\partial\Phi^B$ in the kinetic scalar terms, where Φ^A represents the different scalars of the theory (both dilatons and axions). This expression follows from the requirement of positive energy [141], as we will show in section 6.2. In supergravities with a scalar potential of this form, the explicit deformation δ_m of the gravitino ψ_μ and the dilatini λ are proportional to the superpotential W and its derivatives $\delta W/\delta\Phi^A$, respectively. We will encounter such deformations in all maximal supergravities except 11D and IIB.

A useful truncation of the full field content of the gauged or massive supergravities consists of the metric and the scalars only, for which we will derive the bosonic field equations. This subsector is interesting to us for two reasons. Firstly, it allows for an investigation of the feasibility of combinations of mass parameters. Suppose one has a massless theory with two different, separate deformations. One can wonder whether it is possible to combine these two while preserving all supersymmetry. As we will show, an investigation of the bosonic field equations for the metric and scalars suffices to answer this question. Secondly, the vacua of gauged or massive theories are often carried by the metric and scalars only, as we will see in chapter 6.

In the next section we will review the possible deformations in IIA supergravity. In the following two sections we will construct different gauged maximal supergravities in nine and eight dimensions, respectively. In the last section of this chapter we will consider a general structure of gauged maximal supergravities in various dimensions, which are obtainable via coset manifold and other reductions. Furthermore, we will discuss the relation to the gauged supergravities in ten, nine and eight dimensions of the preceding three chapters.

5.2 Massive and Gauged IIA Supergravity

In this section we will consider two deformations of IIA supergravity, one of which leads to a massive version of IIA while the other gives rise to the gauged IIA theory.

IIA Supergravity

As discussed in section 3.2, toroidal reduction of the eleven-dimensional theory over a circle yields the massless and ungauged IIA theory in ten dimensions. The appropriate reduction Ansätze given in (B.4) with $m_{11} = 0$. The field content of the $D = 10$ IIA supergravity theory is given by

$$\text{D=10 IIA:} \qquad \{e_\mu{}^a, B_{\mu\nu}, \phi, C^{(1)}_\mu, C^{(3)}_{\mu\nu\rho}; \psi_\mu, \lambda\}\,, \tag{5.2}$$

with corresponding Lagrangian (3.16) and supersymmetry transformations rules (B.5). As discussed in section 3.2 and indicated in table 5.1, the IIA theory has two scaling symmetries[2]. One is called α, which scales the Lagrangian and is the reduction of the 11D trombone symmetry. The other is β, which leaves the Lagrangian invariant and stems from the internal coordinate transformations of 11D supergravity.

$\mathbb{R}^+$	$e_\mu{}^a$	B	e^ϕ	$C^{(1)}$	$C^{(3)}$	ψ_μ	λ	ϵ	$\mathcal{L}$	Origin
α	$\frac{9}{8}$	3	$\frac{3}{2}$	0	3	$\frac{9}{16}$	$-\frac{9}{16}$	$\frac{9}{16}$	9	11D
β	0	$\frac{1}{2}$	1	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0	0	0	

Table 5.1: *The scaling weights of the $D = 10$ IIA supergravity fields and action under the scaling symmetries α and β and their origin as higher-dimensional scaling symmetries.*

Note that the Ramond-Ramond vector A is invariant under α while it scales under β. This has important consequences when considering the possible gaugings of IIA supergravity. Since gauge vectors transform in the adjoint of the gauge group and the adjoint of $\mathbb{R}^+$ is trivial, only the symmetry α can be gauged while this is impossible for the symmetry β [75]. Indeed, we will encounter the gauging of α below. In addition, the IIA theory allows for another deformation, which we will first discuss.

[2]We use a different basis of these symmetries in this section than in section 3.2.

Massive IIA Supergravity

The first massive deformation, with mass parameter $m_{\rm R}$, was already encountered in section 3.2 and was constructed by Romans [74]. The explicit deformations of the supersymmetry transformations are denoted by $\delta_{m_{\rm R}}$ and are given in terms of a superpotential W and its derivative with respect to the dilaton:

$$\delta_{m_{\rm R}}\psi_\mu = -\tfrac{1}{32}W\Gamma_\mu\epsilon\,, \qquad \delta_{m_{\rm R}}\lambda = \delta_\phi W\epsilon\,, \qquad W = e^{5\phi/4}m_{\rm R}\,, \tag{5.3}$$

where $\delta_\phi W = \delta W/\delta\phi$. Furthermore, there are implicit massive deformations to the original supersymmetry rules δ_0, given in (B.5), due to the fact that one must replace all massless field strengths by the following massive counterparts:

$$G^{(2)} = {\rm d}C^{(1)} + m_{\rm R}B\,, \quad H = {\rm d}B\,, \quad G^{(4)} = {\rm d}C^{(3)} + C^{(1)}\wedge H + \tfrac{1}{2}m_{\rm R}B\wedge B\,. \tag{5.4}$$

The Lagrangian contains terms linear and quadratic in $m_{\rm R}$. Again there are implicit deformations, via the massive field strengths, and explicit deformations. The explicit deformations of the bosonic sector are quadratic in the mass parameter and define the scalar potential, which can be written in terms of the superpotential W and its derivative via the general expression (5.1):

$$V_{m_{\rm R}} = \tfrac{1}{2}(\delta_\phi W)^2 - \tfrac{9}{32}W^2 = \tfrac{1}{2}e^{5\phi/2}m_{\rm R}^2\,. \tag{5.5}$$

Note that this scalar potential can be naturally included in the massless IIA Lagrangian (3.16) by including the case of $d = -1$ in the summation (and identifying $G^{(0)} = m_{\rm R}$).

In the fermionic sector, one finds the following linear deformations of the gravitino and dilatino field equations in the massive IIA theory:

$$\begin{aligned} X_{m_{\rm R}}(\psi^\mu) &= m_{\rm R}e^{5\phi/4}\Gamma^{\mu\nu}(\tfrac{1}{4}\psi_\nu + \tfrac{5}{288}\Gamma_\nu\lambda)\,,\\ X_{m_{\rm R}}(\lambda) &= m_{\rm R}e^{5\phi/4}\Gamma^{\nu}(-\tfrac{5}{4}\psi_\nu - \tfrac{21}{160}\Gamma_\nu\lambda)\,. \end{aligned} \tag{5.6}$$

The undeformed equations, $X_0(\psi^\mu) = 0$ and $X_0(\lambda) = 0$, are given in (B.7).

Supersymmetry transforms the fermionic field equations, $X_0 + X_{m_{\rm R}} = 0$, into the bosonic equations of motion. For later purposes it is convenient to truncate away all bosonic fields except the metric and the dilaton. After this truncation we find that the fermionic field equations transform into

$$\begin{aligned} (\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\psi^\mu) &= \tfrac{1}{2}\Gamma^\nu\epsilon\,[R^\mu{}_\nu - \tfrac{1}{2}Rg^\mu{}_\nu - \tfrac{1}{2}(\partial^\mu\phi)(\partial_\nu\phi) + \tfrac{1}{4}(\partial\phi)^2 g^\mu{}_\nu +\\ &\qquad + \tfrac{1}{4}m_{\rm R}^2 e^{5\phi/2}g^\mu{}_\nu] = 0\,,\\ (\delta_0 + \delta_{m_{\rm R}})(X_0 + X_{m_{\rm R}})(\lambda) &= \epsilon\,[\Box\phi - \tfrac{5}{4}m_{\rm R}^2 e^{5\phi/2}] = 0\,. \end{aligned} \tag{5.7}$$

At the right hand side, we thus find the massive IIA bosonic field equations for the metric and the dilaton. Indeed, these field equations can be derived from the massless Lagrangian plus the scalar potential (5.5).

The parameter $m_{\rm R}$ breaks both symmetries α and β of the IIA theory. This can easily be seen from the scalar potential (5.5): the former symmetry is broken since the dilaton scales while the Lagrangian is invariant, while the trombone symmetry is broken since the scalar potential is not a two-derivative term like the other bosonic terms. However, there is a linear combination that is not broken by the massive terms: it is given by the linear combination $12\beta - 5\alpha$.

As argued in section 3.2, the mass parameter $m_{\rm R}$ should be seen as a zero-form Ramond-Ramond field strength: it appears naturally in the democratic formulation, including all Ramond-Ramond potentials and field strengths. The scalar potential (5.5) then appears as the kinetic term for the zero-form field strength. The corresponding D-brane is the D8-brane of section 6.1, which is magnetically charged with respect to $m_{\rm R}$ [24].

The massive IIA theory is different from the other massive deformations that we will consider in this chapter. Firstly, it is not known to have a higher-dimensional supergravity origin[3]. Secondly, it is not a gauged supergravity: no global symmetry of the massless theory has been promoted to a local one. Therefore, this deformation of IIA gives rise to a massive rather than gauged supergravity.

Gauged IIA Supergravity

The second massive deformation, with mass parameter m_{11}, does give rise to a gauged IIA supergravity, where the symmetry α has been gauged. It was first obtained in [132], on whose procedure we will comment below. Afterwards, it was shown in [133] that the same theory can also be obtained by a twisted reduction of $D = 11$ supergravity using the trombone symmetry (3.7). The corresponding twisted reduction Ansätze are given in (B.4) with $m_{11} \neq 0$.

This leads to the following explicit massive deformations of the $D = 10$ IIA supersymmetry rules:

$$\delta_{m_{11}} \psi_\mu = \tfrac{9}{16} m_{11} e^{-3\phi/4} \Gamma_\mu \Gamma_{11} \epsilon \,, \qquad \delta_{m_{11}} \lambda = \tfrac{3}{2} m_{11} e^{-3\phi/4} \Gamma_{11} \epsilon \,. \tag{5.8}$$

The implicit massive deformations of the original supersymmetry rules δ_0 arise from the massive bosonic field strengths

$$\begin{aligned} \mathrm{D}\phi &= \mathrm{d}\phi + \tfrac{3}{2} m_{11} C^{(1)} \,, \qquad & G^{(2)} &= \mathrm{d}C^{(1)} \,, \\ H &= \mathrm{d}B + 3 m_{11} C^{(3)} \,, \qquad & G^{(4)} &= \mathrm{d}C^{(3)} + C^{(1)} \wedge H \,, \end{aligned} \tag{5.9}$$

while the covariant derivative of the supersymmetry parameter is given by

$$D_\mu \epsilon = (\partial_\mu + \omega_\mu + \tfrac{9}{16} m_{11} \Gamma_\mu \not{C}^{(1)}) \epsilon \,. \tag{5.10}$$

[3] For different approaches to the M-theory origin of massive IIA supergravity, see [142–144].

There is no Lagrangian for the IIA gauged supergravity, but there are field equations. The linear deformations of the fermionic field equations read in this case

$$\begin{aligned} X_{m_{11}}(\psi^\mu) &= m_{11} e^{-3\phi/4} \Gamma_{11} \Gamma^{\mu\nu} (-\tfrac{9}{2}\psi_\nu + \tfrac{17}{48}\Gamma_\nu \lambda)\,, \\ X_{m_{11}}(\lambda) &= m_{11} e^{-3\phi/4} \Gamma_{11} \Gamma^{\nu} (\tfrac{3}{2}\psi_\nu - \tfrac{9}{16}\Gamma_\nu \lambda)\,. \end{aligned} \tag{5.11}$$

We first consider the truncation with all bosonic fields equal to zero except the metric and the dilaton. Under supersymmetry the fermionic field equations transform into

$$\begin{aligned} (\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\psi^\mu) &= \tfrac{1}{2}\Gamma^\nu \epsilon \left[R^\mu{}_\nu - \tfrac{1}{2} R g^\mu{}_\nu - \tfrac{1}{2}(\partial^\mu \phi)(\partial_\nu \phi) + \tfrac{1}{4}(\partial\phi)^2 g^\mu{}_\nu + \right. \\ &\quad \left. + 36 m_{11}^2 e^{-3\phi/2} g^\mu{}_\nu\right] + \\ &\quad + \Gamma_{11}\epsilon[3 m_{11} e^{-3\phi/4}\partial^\mu\phi] = 0\,, \\ (\delta_0 + \delta_{m_{11}})(X_0 + X_{m_{11}})(\lambda) &= \epsilon\,[\Box\phi] + \Gamma^\nu\Gamma_{11}\epsilon[9 m_{11} e^{-3\phi/4}\partial_\nu\phi] = 0\,. \end{aligned} \tag{5.12}$$

The terms involving Γ_{11} are part of the vector field equation. Therefore, to obtain a consistent truncation, we must further truncate the dilaton to zero. One is then left with only the metric satisfying the Einstein equation with a positive cosmological constant.

The reduced theory is a gauged supergravity, where the scaling symmetry α of table 5.1 has been gauged. In particular, the gauge parameter and transformation of the Ramond-Ramond potentials read as follows[4]:

$$\Lambda = e^{w_\alpha m_{11}\lambda}\,, \qquad C^{(1)} \to C^{(1)} - \mathrm{d}\lambda\,, \qquad C^{(3)} \to e^{3m_{11}\lambda}(C^{(3)} - \mathrm{d}\lambda\, B)\,, \tag{5.13}$$

where w_α are the weights under α. One can take two different limits of the α gauge transformations. Firstly, the limit $m_{11} \to 0$ leads to the massless gauge transformations of the Ramond-Ramond potential. Secondly, one can take the limit where α is constant. This leads to the ungauged scaling symmetry α of table 5.1.

A noteworthy feature of the $D = 10$ gauged supergravity is that no Lagrangian can be defined for it, since the symmetry that is gauged is not a symmetry of the Lagrangian but only of the equations of motion. This is clear from its higher-dimensional origin, which involves a twisting with a symmetry of the field equations only. As discussed in section 4.6, this generally gives rise to field equations that can not be interpreted as Euler-Lagrange equations.

As mentioned above, there exists an alternative way to construct this theory. In [132] it was constructed by allowing for a more general solution of the Bianchi identities of $D = 11$ superspace involving a conformal spin connection. This generalised connection is equivalent to standard $D = 11$ supergravity for a topologically trivial space-time but leads to a new possibility for a non-trivial space-time of the form $M_{10} \times S^1$. The reduction over the circle leads to the $D = 10$ gauged supergravity theory. It is not properly understood why these two procedures give rise to the same lower-dimensional description.

[4] It is understood that each field with $w_\alpha \neq 0$ is multiplied by Λ. Also, the gauge parameter should not be confused with the dilatino, which is also denoted by λ.

Combinations of Mass Parameters and α' Corrections

In the previous subsections we have considered two deformations of IIA supergravity. We would like to examine the possibility to combine these massive deformations [145]. If possible, the resulting theory will have two mass parameters characterising the different deformations. However, not all combinations are necessary consistent with supersymmetry. This complication only appears when investigating the bosonic field equations: the supersymmetry algebra with a combination of massive deformations always closes, as can be seen from the following argument.

Suppose one has a supergravity with one massive deformation m and supersymmetry transformations $\delta_0 + \delta_m$. In all cases discussed in this chapter, only the supersymmetry variations of the fermions receive explicit massive corrections: $\delta_m(\text{boson}) = 0$. This implies that the issue of the closure of the supersymmetry algebra is a calculation with m-independent parts and parts linear in m, but no parts of higher order[5] in m. On the one hand $[\delta(\epsilon_1), \delta(\epsilon_2)]$ has no terms quadratic in m, since one of the two δ's acts on a boson. On the other hand the supersymmetry algebra closes modulo fermionic field equations, which also only have terms independent of and linear in m. Therefore, given the closure of the massless algebra, the closure of the massive supersymmetry algebra only requires the cancellation of terms linear in m.

The closure of the supersymmetry algebras with a single massive deformation is guaranteed by their higher-dimensional origin. The argument of linearity then allows one to combine different massive deformations. Suppose one has two massive supersymmetry algebras with transformations $\delta_0 + \delta_{m_a}$ and $\delta_0 + \delta_{m_b}$. Both supersymmetry algebras close modulo fermionic field equations with (different) massive deformations. Then the combined massive algebra with transformation $\delta_0 + \delta_{m_a} + \delta_{m_b}$ also closes modulo fermionic field equations whose massive deformations are given by the sum of the separate massive deformations linear in m_a and m_b. The closure of the combined algebra is guaranteed by the closure of the two massive algebras, since it requires a cancellation at the linear level.

Under supersymmetry variation of the fermionic field equations, one in general finds linear and quadratic deformations of the bosonic equations of motion. In addition to these corrections, we find that there are also algebraic equations posing constraints on the mass parameters. Solving these equations generically excludes the possibility of combining massive deformations by requiring mass parameters to vanish. At first sight, it might seem surprising that the supersymmetry variation of the fermionic equations of motion leads to constraints other than the bosonic field equations. However, one should keep in mind that the multiplets involved cannot be linearised around a Minkowski vacuum solution. Therefore, the usual rules for linearised Minkowski multiplets do not apply here.

As a first application of this rationale, let us try to combine the two massive deformations m_{R} and m_{11} of IIA supergravity theory. Based on the linearity argument presented Above,

[5] That is, up to cubic order in fermions. We have not checked the higher-order fermionic terms, but we do not expect these to affect any of our findings.

one would expect a closed supersymmetry algebra. The bosonic field equations (with up to quadratic deformations) can be derived by applying the supersymmetry transformations (with only linear deformations) to the fermionic field equations (containing only linear deformations). For simplicity, we truncate all bosonic fields to zero except the metric and the dilaton, since inclusion of the full field content will not change the conclusions. We thus find

$$\begin{aligned}
&(\delta_0 + \delta_{m_{\rm R}} + \delta_{m_{11}})(X_0 + X_{m_{\rm R}} + X_{m_{11}})(\psi^\mu) = \\
&= \tfrac{1}{2}\Gamma^\nu \epsilon \,[R^\mu{}_\nu - \tfrac{1}{2}Rg^\mu{}_\nu - \tfrac{1}{2}(\partial^\mu\phi)(\partial_\nu\phi) + (\tfrac{1}{4}(\partial\phi)^2 + \tfrac{1}{4}m_{\rm R}^2 e^{5\phi/2} + 36 m_{11}^2 e^{-3\phi/2})g^\mu{}_\nu] \\
&\quad + \Gamma_{11}\epsilon[3m_{11}e^{-3\phi/4}\partial^\mu\phi] + \Gamma_{11}\Gamma^\mu\epsilon\,[\tfrac{15}{4}m_{\rm R}m_{11}e^{\phi/2}] = 0\,, \\
&(\delta_0 + \delta_{m_{\rm R}} + \delta_{m_{11}})(X_0 + X_{m_{\rm R}} + X_{m_{11}})(\lambda) = \\
&= \epsilon\,[\Box\phi - \tfrac{5}{4}m_{\rm R}^2 e^{5\phi/2}] + \Gamma^\nu\Gamma_{11}\epsilon[9m_{11}e^{-3\phi/4}\partial_\nu\phi] + \Gamma_{11}\epsilon\,[\tfrac{33}{2}m_{\rm R}m_{11}e^{\phi/2}] = 0\,. \qquad (5.14)
\end{aligned}$$

At the right hand sides we find four different structures. Three of them correspond to the field equations of the metric, dilaton and Ramond-Ramond vector. The vector field equation corresponds to the term containing $m_{11}\partial_\mu\phi$, which implies that truncating away the Ramond-Ramond vector forces one to set $\phi = c$, provided $m_{11} \neq 0$. More interesting is the fourth structure which is bilinear in the mass parameters, leading to the requirement $m_{\rm R}m_{11} = 0$. This constraint cannot be a remnant of a higher-rank form field equation due to its lack of Lorentz indices. It could only fit in the scalar field equation, but the Γ_{11} factor prevents this. It is an extra constraint which does not restrict degrees of freedom but rather restricts mass parameters.

Independent of this constraint from supersymmetry, one can question whether the mass parameters $m_{\rm R}$ and m_{11} are consistent with higher-order corrections of IIA string theory to supergravity. Starting with the former, it is believed that the massive IIA deformation is allowed at all orders in α', due to the connection with the D8-brane. As for the second mass parameter, it arises from the trombone symmetry of 11D supergravity. However, the higher-order derivative terms which arise as corrections in M-theory break this symmetry. The twisted reduction of [133] will therefore be prohibited by M-theory corrections to 11D supergravity. Presumably this also means that the method of [132] involving the generalised spin connection does not work in the presence of higher-order corrections.

Concluding, IIA supergravity allows for two massive deformations with parameters $m_{\rm R}$ and m_{11}. While the closure of the algebra is a linear calculation and therefore always works for combinations, the bosonic field equations rule out the possibility of including both mass parameters [145]. Moreover, string theory corrections to IIA supergravity exclude the m_{11} massive deformations. We therefore conclude that only Romans' massive IIA supergravity is consistent with supersymmetry and string theory.

5.3 Gauged Maximal Supergravities in $D = 9$

In this section we will consider a number of massive deformations of maximal supergravity in $D = 9$, which all give rise to gauged supergravities and have a higher-dimensional origin. In addition, we will find relations between these parameters and investigate to which extent one can combine the different deformations. To end with, we will discuss the quantisation of a certain class of mass parameters. Many of the results of this section were first obtained in [145].

Maximal Supergravity in $D = 9$

Toroidal reduction of both massless IIA and IIB supergravity over a circle yields the unique $D = 9$, $N = 2$ massless supergravity theory, as explained in section 3.2. Its field content is given by

$$\text{D=9:} \qquad \{e_\mu{}^a, \phi, \varphi, \chi, A_\mu, A^i_\mu, B^i_{\mu\nu}, C_{\mu\nu\rho}; \psi_\mu, \lambda, \tilde{\lambda}\}\,, \tag{5.15}$$

with $SL(2,\mathbb{R})$ indices $i = 1, 2$. These indices are raised and lowered with $\varepsilon_{ij} = -\varepsilon^{ij}$ with $\varepsilon_{12} = -\varepsilon_{21} = 1$.

The supersymmetry rules δ_0 of the massless or ungauged 9D supergravity are given in (B.16). The theory inherits several global symmetries from its higher-dimensional parents. Among these is the $SL(2,\mathbb{R})$ symmetry[6] from IIB supergravity. The latter comprises an elliptic $SO(2)$ symmetry θ, a hyperbolic $SO(1,1)^+ \sim \mathbb{R}^+$ symmetry γ and a parabolic $\mathbb{R}$ symmetry ζ. With a fixed gauge of the local $SO(2)$ symmetry (see section 3.3), the $SL(2,\mathbb{R})$ transformations in 9D read

$$\begin{gathered}
\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad A_i \to \Omega_i{}^j A_j\,, \quad B_i \to \Omega_i{}^j B_j\,, \quad \Omega_i{}^j = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,. \\
\psi_\mu \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \psi_\mu\,, \qquad \lambda \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{3/4} \lambda\,, \\
\tilde{\lambda} \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{-1/4} \tilde{\lambda}\,, \qquad \epsilon \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \epsilon\,,
\end{gathered} \tag{5.16}$$

while φ and C are invariant.

In addition to $SL(2,\mathbb{R})$, including the scaling symmetry γ, the 9D theory inherits two other scaling symmetries α and β from IIA and one trombone symmetry δ from IIB. The weights of the different scaling symmetries are given in table 5.2. It turns out that only three of the four scaling symmetries are linearly independent:

$$8\alpha - 48\beta = 18\gamma + 9\delta\,. \tag{5.17}$$

[6] As can be seen in (5.16), the symmetry transformations of both the scalars and the fermions do not change if we replace Ω by $-\Omega$; therefore these fields transform under $PSL(2,\mathbb{R})$. In this section, we will usually only consider group elements Ω that are continuously connected to the identity.

	$e_\mu{}^a$	e^ϕ	e^φ	χ	A	A^1	A^2	B^1	B^2	C	ψ_μ, ϵ	$\lambda, \tilde\lambda$	$\mathcal{L}$	Orig.
α	$\frac{9}{7}$	0	$\frac{6}{\sqrt{7}}$	0	3	0	0	3	3	3	$\frac{9}{14}$	$-\frac{9}{14}$	9	11D
β	0	$\frac{3}{4}$	$\frac{\sqrt{7}}{4}$	-$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	IIA
γ	0	-2	0	2	0	1	-1	1	-1	0	0	0	0	IIB
δ	$\frac{8}{7}$	0	$-\frac{4}{\sqrt{7}}$	0	0	2	2	2	2	4	$\frac{4}{7}$	$-\frac{4}{7}$	8	IIB

Table 5.2: *The scaling weights of the 9D supergravity fields and action under the scaling symmetries α, β, γ and δ, subject to the relation* (5.17)*, and their origin as higher-dimensional scaling symmetries.*

This relation gives rise to the following pattern. Using (5.17) to eliminate one of the scaling symmetries, we are left with three independent scaling symmetries. Each of the three gauge fields A_μ, A^1_μ, A^2_μ has weight zero under two linear combinations of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only. As we found in 10D, the symmetries that leave a vector invariant can be gauged. We will now construct the corresponding massive deformations by performing twisted reductions of IIA and IIB supergravity.

Twisted Reduction of IIB using $SL(2,\mathbb{R})$

We will start with the case that has received most attention in the literature: twisted reductions of $D = 10$ IIB supergravity using the $SL(2,\mathbb{R})$ symmetry. It has been treated in increasing generality by [133, 146, 147].

The reduction Ansätze are given in (B.14) with $m_{\text{IIB}} = 0$. This yields three mass parameters $\vec{m} = (m_1, m_2, m_3)$ in 9D, parameterising the algebra element

$$C_i{}^j = \tfrac{1}{2} \begin{pmatrix} m_1 & m_2 + m_3 \\ m_2 - m_3 & -m_1 \end{pmatrix} . \tag{5.18}$$

The massive deformations will always occur via the superpotential, containing the scalars via the $SL(2,\mathbb{R})/SO(2)$ coset M:

$$W = e^{2\varphi/\sqrt{7}}\, \text{Tr}(MQ)\,, \qquad Q^{ij} = \varepsilon^{ik} C_k{}^j = \tfrac{1}{2} \begin{pmatrix} -m_2 + m_3 & m_1 \\ m_1 & m_2 + m_3 \end{pmatrix} , \tag{5.19}$$

where $\varepsilon^{12} = -\varepsilon^{21} = -1$, giving rise to the symmetric matrix Q.

The twisted reduction results in explicit deformations of the supersymmetry transformations, given in [148]

$$\delta_{\vec m}\psi_\mu = \tfrac{1}{28}\gamma_\mu W\epsilon\,, \qquad \delta_{\vec m}\lambda = i(\delta_\phi W + i e^{-\phi}\delta_\chi W)\epsilon^*\,, \qquad \delta_{\vec m}\tilde\lambda = i\delta_\varphi W \epsilon^*\,, \tag{5.20}$$

while the implicit massive deformations read

$$\begin{aligned} \mathrm{D}\tau &= \mathrm{d}\tau + e^{-2\varphi/\sqrt{7}-\phi}(\delta_\phi W + i e^{-\phi}\delta_\chi W)A\,, \\ F &= \mathrm{d}A\,, \qquad F_i = \mathrm{d}A_i - C_i{}^j B_j\,, \qquad H^i = \mathrm{d}B^i - AF^j\,, \\ G &= \mathrm{d}C + B_i F^i + \tfrac{1}{2} C_i{}^j B^i B_j\,, \end{aligned} \tag{5.21}$$

for the bosons and

$$D_\mu \epsilon = (\partial_\mu + \omega_\mu + \tfrac{i}{4} e^{\phi}\partial_\mu \chi - \tfrac{1}{4} i e^{-2\varphi/\sqrt{7}} W A_\mu)\epsilon \tag{5.22}$$

for the supersymmetry parameter.

The bosonic sector of the field equations is deformed via a scalar potential, that has the generic form for twisted reductions (4.38):

$$\begin{aligned} V_{\vec{m}} &= \tfrac{1}{2} e^{4\varphi/\sqrt{7}} \mathrm{Tr}[C^2 + C^T M^{-1} C M] \\ &= \tfrac{1}{2} e^{4\varphi/\sqrt{7}} \left[2\mathrm{Tr}(MQMQ) - (\mathrm{Tr}(MQ))^2\right] \\ &= \tfrac{1}{2}(\delta_\phi W)^2 + \tfrac{1}{2} e^{-2\phi}(\delta_\chi W)^2 + \tfrac{1}{2}(\delta_\varphi W)^2 - \tfrac{2}{7} W^2\,, \end{aligned} \tag{5.23}$$

which we have also written in terms of the mass matrix Q and the form (5.1) with the superpotential W and its derivatives. The field equations of the 9D fermions receive the following explicit massive corrections:

$$\begin{aligned} X_{\vec{m}}(\psi^\mu) &= -\tfrac{1}{4}\gamma^{\mu\nu}[W\psi_\nu - \tfrac{1}{16} i(\delta_\phi W + i e^{-\phi}\delta_\chi W)\gamma_\nu \lambda^* - \tfrac{1}{16} i \delta_\varphi W \gamma_\nu \tilde{\lambda}^*]\,, \\ X_{\vec{m}}(\lambda) &= -i\gamma^\mu[(\delta_\phi W + i e^{-\phi}\delta_\chi W)\psi^*_\mu - \tfrac{1}{12} i W \gamma_\mu \lambda - \tfrac{2}{9\sqrt{7}} i(\delta_\phi W + i e^{-\phi}\delta_\chi W)\gamma_\mu \tilde{\lambda}]\,, \\ X_{\vec{m}}(\tilde{\lambda}) &= -i\gamma^\nu[\delta_\varphi W \psi^*_\nu - \tfrac{2}{9\sqrt{7}} i(\delta_\phi W - i e^{-\phi}\delta_\chi W)\gamma_\nu \lambda - \tfrac{1}{28} i W \gamma_\nu \tilde{\lambda}]\,. \end{aligned} \tag{5.24}$$

The inclusion of the three mass parameters breaks the $SL(2,\mathbb{R})$ invariance. Rather than being a symmetry, the transformations now relate theories with different mass parameters:

$$C \to \Omega^{-1} C \Omega\,. \tag{5.25}$$

This can always be used to set $m_1 = 0$, yielding an off-diagonal matrix C and a diagonal matrix Q. Due to (5.25), one says that the massive theories are covariant under $SL(2,\mathbb{R})$ transformations rather than invariant. Note that the combination $\det(C) = \det(Q) = \frac{1}{4}(-{m_1}^2 - {m_2}^2 + {m_3}^2)$ is always invariant under these transformations, which can therefore be used to label the different massive deformations.

As discussed in section 4.3, the mass matrix is only invariant under (5.25) if

$$\Omega = \exp(C\lambda)\,, \tag{5.26}$$

The transformations of this one-dimensional subgroup have special properties; for example, the superpotential W is invariant under it. In fact, this subgroup of the global $SL(2,\mathbb{R})$ symmetry has been gauged by the massive deformations $\vec{m}$:

$$\Omega = e^{C\lambda}\,, \qquad A \to A - \mathrm{d}\lambda\,, \qquad B_i \to \Omega_i{}^j(B_j - A_j\,\mathrm{d}\lambda)\,, \tag{5.27}$$

with gauge vector A and parameter λ. We distinguish three distinct cases depending on the value of $\det(Q)$ [143, 149, 150]:

- $\det(Q) = 0$: we gauge the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$ with parameter ζ,
- $\det(Q) < 0$: we gauge the $SO(1,1)^+$ subgroup of $SL(2,\mathbb{R})$ with parameter γ,
- $\det(Q) > 0$: we gauge the $SO(2)$ subgroup of $SL(2,\mathbb{R})$ with parameter θ.

All these three cases are one-parameter massive deformations. At the end of this section we will discuss the quantisation of the mass parameters m_1, m_2 and m_3 in the context of string theory.

Toroidal Reduction of Massive IIA

In addition to the twisted reductions, one can also generate mass terms in nine dimensions by reducing higher-dimensional deformations, i.e. the massive and gauged IIA supergravity theories of section 5.3. We will start with reducing the first possibility.

Toroidal reduction of the massive IIA supergravity, with reduction Ansätze (B.9) with $m_4 = m_{\mathrm{IIA}} = 0$, leads to a gauged nine-dimensional supergravity. Its deformations coincide with those parameterised by the mass parameters $\vec{m}$ with the identifications [146]

$$\vec{m} = (0, m_{\mathrm{R}}, m_{\mathrm{R}})\,. \tag{5.28}$$

Thus the reduction of massive IIA supergravity corresponds to a twisted reduction of IIB supergravity, employing the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$. This nine-dimensional equivalence is called massive T-duality and can be seen as a deformation of the massless T-duality.

An interesting feature of massive T-duality is that massive IIA becomes a gauged theory upon reduction. The emergence of this gauging can be seen as a generalisation of the enhanced gaugings discussed in section 4.3, in which the extra gauge vector comes from a higher-dimensional vector. In the massive IIA case, however, the gauge vector is A, which comes from the Neveu-Schwarz two-form B in IIA.

Overview of Massive Deformations in 9D

In addition to the $SL(2,\mathbb{R})$ twisted reduction of IIB, we can also perform twisted reductions of both IIA and IIB using the scaling symmetries α, β and δ; the corresponding mass parameters are denoted by m_{IIA}, m_4 and m_{IIB}, respectively. The reduction Ansätze are given in

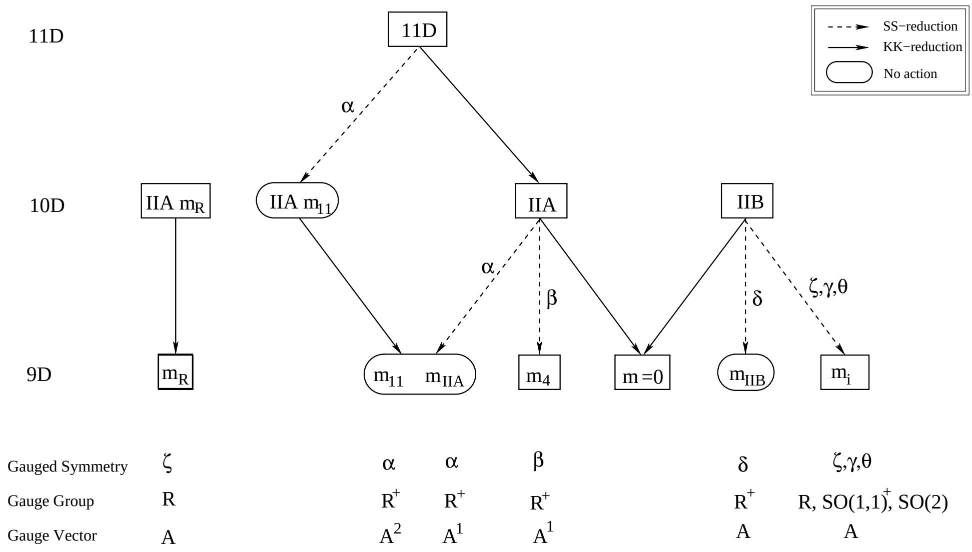

Figure 5.1: *Overview of all twisted reductions performed in this section with the employed symmetries and resulting mass parameters. Mass parameters in the same box form a multiplet under $SL(2,\mathbb{R})$ (see table 5.3). We also give the gauged symmetry and gauge vector in 9D.*

(B.9) and (B.14). Also, like the massive IIA theory, the gauged version of IIA supergravity can be toroidally reduced to nine dimensions. The different possibilities are illustrated in figure 5.1, while the resulting implicit and explicit deformations of the 9D theory are given in appendix B.4. In total, this amounts to seven deformations of the unique $D = 9$ supergravity, with parameters $m_1, m_2, m_3, m_4, m_{\text{IIA}}, m_{\text{IIB}}$ and m_{11}. As noted before, the parameter m_{R} is not independent but yields a subset of the parameters $\vec{m}$.

However, various massive deformations are related. Symmetries of the massless theory become field redefinitions in the gauged theory, that only act on the massive deformations (exactly like in (5.25)). This means that the mass parameters transform under such transformations: they have a scaling weight under the different scaling symmetries and fall in multiplets of $SL(2,\mathbb{R})$. In table 5.3, the multiplet structure of the massive deformations under $SL(2,\mathbb{R})$ is given. The mass parameter $\tilde{m}_4$ is defined as the S-dual partner of m_4 and can not be obtained by a twisted reduction of IIA supergravity.

As an example, consider the two mass parameters $(m_{11}$ and $m_{\text{IIA}})$, which form a doublet under $SL(2,\mathbb{R})$ field redefinitions. This can be understood from their higher-dimensional origin. For m_{IIA} one first performs an ordinary toroidal reduction and next a twisted reduction with α, while for m_{11} the order of these reductions is reversed: one first performs a twisted reduction with α and next a toroidal reduction. Since $SL(2,\mathbb{R})$ in 9D comes from the reparameterisations of the two-torus, it also relates the two mass parameters m_{11} and m_{IIA}.

Mass parameters	$SL(2,\mathbb{R})$
(m_1, m_2, m_3)	adjoint
$(m_4, \tilde{m}_4)$	doublet
(m_{11}, m_{IIA})	doublet
m_{IIB}	singlet

Table 5.3: *The $D = 9$ mass parameters of the different reduction schemes (see figure 5.1) form different multiplets under $SL(2,\mathbb{R})$.*

All the 9D deformations correspond to a gauging of a global symmetry. As shown in section 4.3, it is always the symmetry that is employed in the twisted reduction Ansatz that becomes gauged upon reduction. The corresponding gauge vector is provided by the metric, i.e. it is the Kaluza-Klein vector of the dimensional reduction (being A^1 for IIA and A for IIB). In all cases but one, this is the complete story and one finds an Abelian gauged supergravity. The exception is the mass parameter m_4, which leads to a non-Abelian symmetry. Indeed, the 10D vector of IIA has a non-trivial scaling under
$beta$; as discussed in section 4.3, this leads to symmetry enhancement. In the other cases such enhancement is impossible, due to the absence of gauge vectors with a non-trivial scaling weight.

Combining Massive Deformations in 9D and α' Corrections

We would like to consider the feasibility of combinations of massive deformations in nine dimensions. One might hope that, due to the large amount of mass parameters, the bosonic field equations do not exclude all possible combinations, as we found in $D = 10$.

For the present purposes, we will focus on specific terms in the supersymmetry variations of the fermionic field equations. In the following, δ_m and X_m are understood to mean the supersymmetry variation and fermionic field equation at linear order containing the sum of all seven possible massive deformations derived in the previous subsections. Variation of the fermionic field equations gives, amongst other γ-structures, the terms

$$
\begin{aligned}
(\delta_0 + \delta_m)(X_0 + X_m)(\psi^\mu) &\sim i\,\gamma^\mu \epsilon[\ldots] + \gamma^\mu \epsilon^*[\ldots] + i\,\gamma^\mu \epsilon^*[\ldots]\,,\\
(\delta_0 + \delta_m)(X_0 + X_m)(\lambda) &\sim \epsilon[\ldots] + i\,\epsilon[\ldots]\,,\\
(\delta_0 + \delta_m)(X_0 + X_m)(\tilde{\lambda}) &\sim \epsilon[\ldots] + i\,\epsilon[\ldots] + \epsilon^*[\ldots]\,,
\end{aligned}
\tag{5.29}
$$

where the $[\ldots]$ denote different bosonic real expressions of bilinear mass terms and scalar factors. These are the analogue of the ten-dimensional expression $[m_{\text{R}} m_{11} e^{\phi/2}]$ (see (5.14)) and give rise to constraints on the mass parameters. Requiring all expressions $[\ldots]$ to vanish, one is led to the following possible combinations (with the other mass parameters vanishing):

- **Case 1** with $\{m_{\rm IIA}, m_4\}$: this combination can also be obtained by twisted reduction of IIA employing a linear combination of the symmetries α and β, which guarantees its consistency. It is also a gauging of both this symmetry and (for $m_4 \neq 0$) the parabolic subgroup of $SL(2,\mathbb{R})$ in 9D, giving a non-Abelian gauge group.

- **Case 2,3,4** with $\{\vec{m}, m_{\rm IIB}\}$: as in the case with $m_{\rm IIB} = 0$ and only $\vec{m}$ this combination contains three different, inequivalent cases depending on $\det(Q)$ (depending crucially on the fact that $m_{\rm IIB}$ is a singlet under $SL(2,\mathbb{R})$):

  - **Case 2** with $\{\vec{m}, m_{\rm IIB}\}$ and $\det(Q) = 0$.
  - **Case 3** with $\{\vec{m}, m_{\rm IIB}\}$ and $\det(Q) > 0$.
  - **Case 4** with $\{\vec{m}, m_{\rm IIB}\}$ and $\det(Q) < 0$.

  All these combinations can also be obtained by twisted reduction of IIB employing a linear combination of the symmetries δ and (one of the subgroups of) $SL(2,\mathbb{R})$, implying consistency of the combinations. All cases (assuming that $m_{\rm IIB} \neq 0$) correspond to the gauging of an Abelian scaling symmetry in 9D.

- **Case 5** with $\{5m_4 = -12m_{\rm IIA}, m_2 = m_3\}$: this case can be understood as the twisted reduction of Romans' massive IIA theory, employing the scaling symmetry that is not broken by the $m_{\rm R}$ deformations: it is given by the combination $12\beta - 5\alpha$ of table 5.1. This deformation gauges both the linear combination of scaling symmetries and the parabolic subgroup of $SL(2,\mathbb{R})$ in 9D, which form a non-Abelian gauge group.

Another solution to the quadratic constraints has parameters $\{m_{\rm IIA}, m_{11}\}$, but this combination does not represent a new case: it can be obtained from only $m_{\rm IIA}$ (and thus a truncation of case 1) via an $SL(2,\mathbb{R})$ field redefinition (since they form a doublet). Thus the most general deformations are the five cases given above, all containing two mass parameters. All of these are gauged theories and have a higher-dimensional origin. Both case 1 and case 5 have a non-Abelian gauge group provided $m_4 \neq 0$.

We will now consider the viability of the different mass parameters in string theory rather than supergravity. The massive deformations that are based on a symmetry that is broken by α' corrections do not correspond to a sector of compactified string theory. Only the symmetries that are preserved by the higher-order string corrections to supergravity give rise to gauged supergravities that are embeddable in string theory. We have two such symmetries:

- The $SL(2,\mathbb{R})$ (or rather its $SL(2,\mathbb{Z})$ subgroup) symmetry of IIB. Thus the $\vec{m} = (m_1, m_2, m_3)$ deformations correspond to the low-energy limits of three different sectors of compactified IIB string theory (depending on $\det(Q) = \frac{1}{4}(-{m_1}^2 - {m_2}^2 + {m_3}^2)$).

- The linear combination $\alpha + 12\beta$ of scaling symmetries of IIA. Thus one can define a massive deformation m_s within case 1 with $\{m_{\rm IIA} = m_s, m_4 = 12m_s\}$ which corresponds to the low-energy limit of a sector of compactified IIA string theory.

One gains a better understanding of the m_s massive deformation and the $\alpha + 12\beta$ symmetry of IIA from the following point of view. This combination of scaling symmetries of IIA can be understood from its 11D origin as the general coordinate transformation $x^{11} \to \lambda\, x^{11}$. This explains why all α' corrections transform covariantly under this specific scaling symmetry: the higher-order corrections in 11D are invariant under general coordinate transformations and upon reduction they must transform covariantly under the reduced coordinate transformations, among which is the $\alpha + 12\beta$ scaling symmetry.

In fact, the twisted reduction from IIA to 9D using the transformation $x^{11} \to \lambda\, x^{11}$ is equivalent to the unique group manifold reduction from 11D to 9D: upon relating the components of ${f_{ab}}^c$ (of which only one is independent for 2D groups) to m_s, the deformations from the twisted and group reductions coincide. Indeed, this explains why the m_s deformations correspond to a gauging of the 2D non-Abelian group rather than only the scaling symmetry $\alpha + 12\beta$. This is an example of the relation between the different methods of dimensional reduction, as indicated in section 4.4.

Quantisation Conditions on $SL(2,\mathbb{R})$ Mass Parameters

The classical $SL(2,\mathbb{R})$ symmetry of IIB supergravity is broken to $SL(2,\mathbb{Z})$ by string theory, as discussed in section 2.3. We would like to consider the effect of this on the twisted reductions of IIB with the $SL(2,\mathbb{R})$ symmetry of section 5.3. In particular, it implies that the monodromy matrix must be an element of $SL(2,\mathbb{Z})$, the arithmetic subgroup of $SL(2,\mathbb{R})$:

$$M(x + 2\pi R) = \Lambda\, M(x)\Lambda^T \qquad \text{with } \Lambda = e^{2\pi R C} \in SL(2,\mathbb{Z})\,, \tag{5.30}$$

where C is given by (5.18). This will imply a quantisation of the mass parameters $\vec{m}$.

We will apply the following procedure. The mass parameters will be parameterised by $\vec{m} = \tilde{m}\,(p,q,r)$. Then, given the radius of compactification R and the relative coefficients (p,q,r) of the mass parameters, one should choose the overall coefficient $\tilde{m}$ such that the monodromy lies in $SL(2,\mathbb{Z})$. This is not always possible; a necessary requirement in all cases but one will be that (p,q,r) are integers and satisfy a so-called diophantic equation, i.e. an equation for integer numbers. Furthermore we must require q and r to be either both even or both odd. Thus we get all $SL(2,\mathbb{Z})$ monodromies that can be expressed as products of the elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{5.31}$$

and their inverses. The conjugacy classes of $SL(2,\mathbb{Z})$ have been classified in [151, 152]. We will discuss the results for the different possibilities of $\det(Q)$ [149, 150].

The case $\det(Q) < 0$ gives rise to a monodromy $\Lambda \in SL(2,\mathbb{Z})$ provided we have

$$\tilde{m} = \frac{\text{arccosh}(n/2)}{\pi R\sqrt{n^2-4}} \;\text{ and }\; p^2 + q^2 - r^2 = n^2 - 4\,, \tag{5.32}$$

for some integer $n \geq 3$. One set of solutions to this diophantic equation is $(p, q, r) = (\pm n, 0, \pm 2)$ with monodromy $\Lambda = (S\,T^{-n})^{\pm 1}$. There are other conjugacy classes, however: not all other solutions are related to it by $SL(2, \mathbb{Z})$.

For $\det(Q) = 0$, we find that Λ is an element of $SL(2, \mathbb{Z})$ provided we have

$$\tilde{m} = \frac{1}{2\pi R} \text{ and } p^2 + q^2 - r^2 = 0\,. \tag{5.33}$$

All the solutions of the diophantic equation are related via $SL(2, \mathbb{Z})$ to the solution $(p, q, r) = (0, n, n)$ with n an arbitrary integer. This gives rise to the monodromy $\Lambda = T^n$. The quantisation on $\tilde{m}$ is the same charge quantisation condition as found in [146].

For the remaining case, $\det(Q) > 0$, we find that there are three distinct possibilities for Λ to be an element of $SL(2, \mathbb{Z})$. For the first possibility we must have

$$\tilde{m} = \frac{1}{4R} \text{ and } p^2 + q^2 - r^2 = -4\,. \tag{5.34}$$

One solution to this diophantic equation is $(p, q, r) = (0, 0, \pm 2)$, yielding $\Lambda = S^{\pm 1}$. All other solutions to the diophantic equation are related by $SL(2, \mathbb{Z})$. For the second possibility one must require

$$\tilde{m} = \frac{1}{3\sqrt{3}R} \text{ and } p^2 + q^2 - r^2 = -3\,.,\tag{5.35}$$

which is solved by $(p, q, r) = (\pm 1, 0, \pm 2)$ with monodromy $\Lambda = (T^{-1}\,S)^{\pm 1}$. Again all other solutions are related by $SL(2, \mathbb{Z})$. The third possibility is of a different sort: it requires

$$\tilde{m} = \frac{1}{R} \text{ and } p^2 + q^2 - r^2 = -4\,,\tag{5.36}$$

but (p, q, r) are not necessarily integer-valued. This gives rise to trivial monodromy $\Lambda = \mathbb{I}$ and thus corresponds to a truncation of the untwisted Kaluza-Klein tower to a set of massive rather than massless modes, see section 4.3 and [121].

5.4 Gauged Maximal Supergravities in $D = 8$

In this section we will perform all possible 3D group manifold reductions of 11D supergravity, resulting in different 8D gauged maximal supergravities. These results were first obtained in [136, 153].

The Bianchi Classification of 3D Groups

We will first review the Bianchi classification[7] [159] of three-dimensional Lie groups. The generators of the group satisfy the commutation relations $(m, n, p = (1, 2, 3))$

$$[T_m, T_n] = f_{mn}{}^p T_p \,, \tag{5.37}$$

with constant structure coefficients $f_{mn}{}^p$ subject to the Jacobi identity $f_{[mn}{}^q f_{p]q}{}^r = 0$. For three-dimensional Lie groups, the structure constants have nine components, which can be conveniently parameterised by

$$f_{mn}{}^p = \varepsilon_{mnq} Q^{pq} + 2\delta_{[m}{}^p a_{n]} \,, \qquad Q^{pq} a_q = 0 \,. \tag{5.38}$$

Here Q^{pq} is a symmetric matrix with six components, and a_m is a vector with three components. The constraint on their product follows from the Jacobi identity. Having $a_q = 0$ corresponds to an algebra with traceless structure constants: $f_{mn}{}^n = 0$. The Bianchi classification distinguishes between class A and B algebras which have vanishing and non-vanishing trace, respectively.

Of course Lie algebras are only defined up to changes of basis: $T_m \to R_m{}^n T_n$ with $R_m{}^n \in GL(3, \mathbb{R})$. The corresponding transformation of the structure constants and its components reads

$$f_{mn}{}^p \to f'_{mn}{}^p = R_m{}^q R_n{}^r (R^{-1})_s{}^p f_{qr}{}^s \; : \quad \begin{cases} Q^{mn} \to \det(R)((R^{-1})^T Q R^{-1}))^{mn} \,, \\ a_m \to R_m{}^n a_n \,. \end{cases} \tag{5.39}$$

These transformations are naturally divided into two complementary sets. First there is the group of automorphism transformations with $f_{mn}{}^p = f'_{mn}{}^p$, whose dimension is given in table 5.4 for the different algebras [160]. Then there are the transformations that change the structure constants, and these can always be used [160, 161] to transform Q^{pq} into a diagonal form and a_q to have only one component. We will explicitly go through the argument.

Consider an arbitrary symmetric matrix Q^{mn} with eigenvalues λ_m and orthogonal eigenvectors $\vec{u}_m$. Taking

$$R^T = (\sqrt{d_2 d_3}\, \vec{u}_1, \sqrt{d_1 d_3}\, \vec{u}_2, \sqrt{d_1 d_2}\, \vec{u}_3) \,, \tag{5.40}$$

with $d_m \neq 0$ and $\mathrm{sgn}(d_1) = \mathrm{sgn}(d_2) = \mathrm{sgn}(d_3)$ we find that

$$Q^{mn} \to \mathrm{diag}(d_1\lambda_1, d_2\lambda_2, d_3\lambda_3) \,. \tag{5.41}$$

We now distinguish between four cases, depending on the rank of Q^{mn}:

[7] Actually, the classification method used nowadays and presented here is not Bianchi's original one, but it is due to Schücking and Behr (see Kundt's paper based on the notes taken in a seminar given by Schücking [154] and the editorial notes [155]), and the earliest publications in which this method is followed are [156, 157]. The history of the classification of three- and four-dimensional real Lie algebras is also reviewed in [158]. We will adhere to the common use of Bianchi classification, however.

- Rank$(Q^{mn}) = 3$: in this case all components of a_m necessarily vanish (due to the Jacobi identity), and we can take $d_m = \pm 1/|\lambda_m|$ to obtain

$$Q^{mn} = \pm \mathrm{diag}(\mathrm{sgn}(\lambda_1), \mathrm{sgn}(\lambda_2), \mathrm{sgn}(\lambda_3)) \,, \qquad a_m = (0,0,0) \,. \tag{5.42}$$

- Rank$(Q^{mn}) = 2$: in this case one eigenvalue vanishes which we take to be λ_1. Then we set $d_i = \pm 1/|\lambda_i|$, with $i = 2, 3$, to obtain $Q^{mn} = \pm \mathrm{diag}(0, \mathrm{sgn}(\lambda_2), \mathrm{sgn}(\lambda_3))$. From the Jacobi identity, it then follows that $a_m = (a, 0, 0)$. We distinguish between vanishing and non-vanishing vector. In the case $a \neq 0$, one might think that one can use d_1 to set $a = 1$, but from the transformation rule of a_m (5.39) and the form of R (5.40) it can be seen that $a \sim \sqrt{d_2 d_3}$, and therefore a can not be fixed by d_1. In this case we thus have a one-parameter family of Lie algebras:

$$Q^{mn} = \pm \mathrm{diag}(0, \mathrm{sgn}(\lambda_2), \mathrm{sgn}(\lambda_3)) \,, \qquad \begin{cases} a_m = (0,0,0) \,, \\ a_m = (a,0,0) \,. \end{cases} \tag{5.43}$$

- Rank$(Q^{mn}) = 1$: in this case two eigenvalues vanish, e.g. $\lambda_1 = \lambda_2 = 0$. We set $d_3 = \pm 1/|\lambda_3|$ to obtain $Q^{mn} = \pm \mathrm{diag}(0, 0, \mathrm{sgn}(\lambda_3))$. Again one distinguishes between $a_m = 0$ and $a_m \neq 0$. In the latter case one is left with a vector $a_m = (a_1, a_2, 0)$, of which $a_1 \sim \sqrt{d_2 d_3}$ and $a_2 \sim \sqrt{d_1 d_3}$. Thus, one can use d_1 and d_2 to adjust the length of $\vec{a}$ to 1, after which an $O(3)$ transformation in the $(1, 2)$-subspace gives the final result:

$$Q^{mn} = \pm \mathrm{diag}(0, 0, \mathrm{sgn}(\lambda_3)) \,, \qquad \begin{cases} a_m = (0,0,0) \,, \\ a_m = (1,0,0) \,. \end{cases} \tag{5.44}$$

- Rank$(Q^{mn}) = 0$: in this case all three eigenvalues vanish and therefore $Q^{mn} = 0$. Thus, the transformation with matrix (5.40) is irrelevant. For $a_m \neq 0$, it follows from (5.39) that one can first do a scaling to get $|\vec{a}| = 1$ and then an $O(3)$ transformation to obtain:

$$Q^{mn} = \mathrm{diag}(0, 0, 0) \,, \qquad \begin{cases} a_m = (0,0,0) \,, \\ a_m = (1,0,0) \,. \end{cases} \tag{5.45}$$

Thus, we find that the most general three-dimensional Lie algebra can be described by

$$Q^{mn} = \mathrm{diag}(q_1, q_2, q_3) \,, \qquad a_m = (a, 0, 0) \,. \tag{5.46}$$

In this basis the commutation relations take the form

$$[T_1, T_2] = q_3 T_3 - a T_2 \,, \qquad [T_2, T_3] = q_1 T_1 \,, \qquad [T_3, T_1] = q_2 T_2 + a T_3 \,. \tag{5.47}$$

The different three-dimensional Lie algebras are obtained by taking different signatures of Q^{mn} and are given in table 5.4. Naïvely one might conclude that the classification as given above leads to ten different algebras. However, it turns out that one has to treat the subcase $a = 1/2$ of (5.43) as a separate case[8]. Thus, the total number of inequivalent three-dimensional Lie algebras is eleven, two of which are one-parameter families.

Bianchi	a	(q_1, q_2, q_3)	Class	Algebra	Dim(Aut)
I	0	$(0,0,0)$	A	$u(1)^3$	9
II	0	$(0,0,1)$	A	$heis_3$	6
III	1	$(0,-1,1)$	B		4
IV	1	$(0,0,1)$	B		4
V	1	$(0,0,0)$	B		6
VI_0	0	$(0,-1,1)$	A	$iso(1,1)$	4
VI_a	a	$(0,-1,1)$	B		4
VII_0	0	$(0,1,1)$	A	$iso(2)$	4
VII_a	a	$(0,1,1)$	B		4
VIII	0	$(1,-1,1)$	A	$so(2,1)$	3
IX	0	$(1,1,1)$	A	$so(3)$	3

Table 5.4: *The Bianchi classification of three-dimensional Lie algebras in terms of the components a and q_1, q_2, q_3 of their structure constants. Note that there are two one-parameter families VI_a and VII_a with special cases VI_0, VII_0 and $\text{VI}_{a=1/2}$=III. The algebra $heis_3$ denotes the three-dimensional Heisenberg algebra. The table also gives the dimensions of the automorphism groups.*

Of the eleven Lie algebras, only $SO(3)$ and $SO(2,1)$ are simple while the rest are all non-semi-simple [160, 162]. In the non-semi-simple cases, we can always choose $q_1 = 0$. In

[8]The distinction between $a = 1/2$ and $a \neq 1/2$ arises when considering the isometries on the group manifold, see also [153].

this case, the Abelian invariant subgroup consists of T_2 and T_3, since T_1 does not appear on the right-hand side in (5.47). The algebras of class B with non-vanishing trace $f_{mn}{}^n$ always give rise to non-compact groups [163]. In contrast, the algebras of class A correspond to both compact and non-compact groups; an example is the algebra of type IX, which always gives rise to the compact $SO(3)$ group. All algebras of class A can be seen as group contractions and analytic continuations of $so(3)$, see section 5.5.

Reduction over a 3D Group Manifold

In this subsection we perform the reduction of $D = 11$ supergravity over a three-dimensional group manifold to $D = 8$ dimensions. The prime example is the reduction over the three-sphere S^3, which gives rise to the $SO(3)$ gauged supergravity of Salam and Sezgin [164]. By choosing other structure constants, corresponding to other three-dimensional Lie algebras, one employs other group manifolds, some of which give rise to non-compact gaugings. Since these algebras are ordered via the Bianchi classification, the different group manifold reductions give rise to a Bianchi classification of 8D gauged maximal supergravities [153].

To perform the dimensional reduction, it is convenient to make an $8 + 3$ split of the eleven-dimensional space-time: $x^{\hat{\mu}} = (x^\mu, z^m)$ with $\mu = (0, 1, \ldots 7)$ and $m = (1, 2, 3)$. Eleven-dimensional fields will be hatted while unhatted quantities are 8D. Using a particular Lorentz frame the reduction Ansatz for the eleven-dimensional fields is

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{-\varphi/6} e_\mu{}^a & e^{\varphi/3} L_m{}^i A^m{}_\mu \\ 0 & e^{\varphi/3} L_n{}^i U^n{}_m \end{pmatrix}, \tag{5.48}$$

and

$$\begin{aligned} \hat{C}_{abc} &= e^{\varphi/2} C_{abc}\,, & \hat{C}_{abi} &= L_i{}^m B_{m\,ab}\,, \\ \hat{C}_{aij} &= e^{-\varphi/2} \varepsilon_{mnp} L_i{}^m L_j{}^n V_a{}^p\,, & \hat{C}_{ijk} &= e^{-\varphi} \varepsilon_{ijk} \ell\,, \end{aligned} \tag{5.49}$$

for the bosonic fields and

$$\hat{\psi}_a = e^{\varphi/12}(\psi_a - \tfrac{1}{6}\Gamma_a \Gamma^i \lambda_i)\,, \qquad \hat{\psi}_i = e^{\varphi/12}\lambda_i\,, \qquad \hat{\epsilon} = e^{-\varphi/12}\epsilon\,, \tag{5.50}$$

for the fermions. Thus the full eight-dimensional field content consists of the following $128 + 128$ field components (omitting space-time indices on the potentials):

$$\text{8D}: \quad \{e_\mu{}^a, L_m{}^i, \varphi, \ell, A^m, V^m, B_m, C; \psi_\mu, \lambda_i\}\,. \tag{5.51}$$

We will now describe the quantities appearing in this reduction Ansatz.

The matrix $L_m{}^i$ describes the five-dimensional $SL(3,\mathbb{R})/SO(3)$ scalar coset of the internal space. It transforms under a global $SL(3,\mathbb{R})$ acting from the left and a local $SO(3)$ symmetry acting from the right. We take the following explicit representative (3.25), thus fixing the gauge of the local $SO(3)$ symmetry:

$$L_m{}^i = \begin{pmatrix} e^{-\sigma/\sqrt{3}} & e^{-\phi/2+\sigma/2\sqrt{3}}\chi_1 & e^{\phi/2+\sigma/2\sqrt{3}}\chi_2 \\ 0 & e^{-\phi/2+\sigma/2\sqrt{3}} & e^{\phi/2+\sigma/2\sqrt{3}}\chi_3 \\ 0 & 0 & e^{\phi/2+\sigma/2\sqrt{3}} \end{pmatrix} , \qquad (5.52)$$

which contains two dilatons ϕ, σ and three axions χ_1, χ_2, χ_3. It is useful to define the $SO(3)$ invariant scalar matrix

$$M_{mn} = L_m{}^i L_n{}^j \eta_{ij} , \qquad (5.53)$$

where $\eta_{ij} = \mathbb{I}_3$ is the internal flat metric. Similarly, the two-dimensional $SL(2,\mathbb{R})/SO(2)$ scalar coset is parameterised by the dilaton φ and the axion ℓ via the $SO(2)$ invariant scalar matrix

$$W_{IJ} = e^{\varphi} \begin{pmatrix} \ell^2 + e^{-2\varphi} & \ell \\ \ell & 1 \end{pmatrix} . \qquad (5.54)$$

The only dependence on the internal coordinates z^m comes in via the $GL(3,\mathbb{R})$ matrices $U^m{}_n$. These can be interpreted as the components of the three Maurer-Cartan one-forms $\sigma^m = U^m{}_n dz^n$ of some three-dimensional Lie group. By definition they satisfy the Maurer-Cartan equations (4.46), giving rise to the structure constants $f_{mn}{}^p$ of the group, which are independent of z^m. Using a particular frame in the internal directions, the explicit coordinate dependence of the Maurer-Cartan one-forms is given by

$$U^m{}_n = \begin{pmatrix} 1 & 0 & -s_{1,3,2} \\ 0 & e^{az^1}\, c_{2,3,1} & e^{az^1}\, c_{1,3,2}\, s_{2,3,1} \\ 0 & -e^{az^1}\, s_{3,2,1} & e^{az^1}\, c_{1,3,2}\, c_{2,3,1} \end{pmatrix} , \qquad (5.55)$$

where we have used the following abbreviations

$$c_{m,n,p} = \cos(\sqrt{q_m}\sqrt{q_n}\, z^p) , \qquad s_{m,n,p} = \sqrt{q_m}\sin(\sqrt{q_m}\sqrt{q_n}\, z^p)/\sqrt{q_n} , \qquad (5.56)$$

This gives rise to structure constants (5.38) with (5.46). It is understood that the structure constants satisfy the Jacobi identity, amounting to $q_1 a = 0$.

A subtlety which is not obvious from the analysis by Scherk and Schwarz [39] is that one only can reduce the action for traceless structure constants ($f_{mn}{}^n = 0$). These cases lead to the class A gauged supergravities. For structure constants with non-vanishing trace ($f_{mn}{}^m \neq 0$), one has to resort to a reduction of the field equations, see section 4.6. These cases lead to the class B gauged supergravities. Note that the adjoint of the gauge group $\mathcal{G}$ in embedded in the fundamental of $GL(3,\mathbb{R})$:

$$g_n{}^m = e^{\lambda^k f_{kn}{}^m} , \qquad (5.57)$$

where λ^k are the parameters of the gauge transformations. Therefore, in the case of a non-vanishing trace, the gauge group $\mathcal{G}$ is a subgroup of $GL(3,\mathbb{R})$ and not of $SL(3,\mathbb{R})$.

The relation between the Maurer-Cartan one-forms σ^m and the three-dimensional isometry groups is as follows. The metric on the group manifold reads

$$ds_G^2 = e^{2\varphi/3} M_{mn} \sigma^m \sigma^n \,, \tag{5.58}$$

where the scalars φ and M are constants from the three-dimensional point of view. A vector field L defines an isometry if it leaves the metric invariant

$$L_L g_{mn} = 0 \,. \tag{5.59}$$

For all values of the scalars, the group manifold has three isometries generated by the left invariant Killing vector fields, as explained in section 4.4. These fulfill the stronger requirement

$$L_{L_m} \sigma^n = 0 \tag{5.60}$$

for all three Maurer-Cartan forms on the group manifold and generate the algebra as given in (4.46). In the class A case, i.e. $a = 0$, the left-invariant Killing vectors generating the three isometries are given by

$$\begin{aligned} L_1 &= \frac{c_{1,2,3}}{c_{1,3,2}} \frac{\partial}{\partial z^1} - s_{2,1,3} \frac{\partial}{\partial z^2} + \frac{c_{1,2,3}\, s_{3,1,2}}{c_{1,3,2}} \frac{\partial}{\partial z^3} \,, \\ L_2 &= \frac{s_{1,2,3}}{c_{1,3,2}} \frac{\partial}{\partial z^1} + c_{1,2,3} \frac{\partial}{\partial z^2} - \frac{s_{1,2,3}\, s_{1,3,2}}{c_{1,3,2}} \frac{\partial}{\partial z^3} \,, \\ L_3 &= \frac{\partial}{\partial z^3} \,, \end{aligned} \tag{5.61}$$

whereas in the class B case, i.e. $q_1 = 0$ and $a \neq 0$, they are given by

$$\begin{aligned} L_1 &= \frac{\partial}{\partial z^1} - (a z^2 + q_2 z^3) \frac{\partial}{\partial z^2} + (q_3 z^2 - a z^3) \frac{\partial}{\partial z^3} \,, \\ L_2 &= \frac{\partial}{\partial z^2} \,, \qquad\qquad L_3 = \frac{\partial}{\partial z^3} \,. \end{aligned} \tag{5.62}$$

Here, $\partial/\partial z^2$ and $\partial/\partial z^3$ are manifest isometries. This follows from the fact that the matrix $U^n{}_m$ is independent of z^2 and z^3.

In this section, we have not heeded any global issues concerning the group manifold reductions. This amounts to taking the universal cover of the group manifold. For this reason, the manifolds of types I-VIII are non-compact and have the topology of $\mathbb{R}^3$, while the type IX manifold has the topology of S^3. The latter case therefore does not raise any issues when compactifying. In the case of non-compact groups, there are two approaches:

- One reduces over a non-compact group manifold. Supersymmetry is preserved, but the non-compact internal manifold leads to a continuous spectrum in the lower-dimensional theory; this spectrum can be consistently truncated to an 8D gauged maximal supergravity, however. This is the so-called non-compactification scheme.

- The group manifold is compactified by dividing out by discrete symmetries [165]. For all Bianchi types except types IV and VI_a, it is possible to construct compact manifolds in this way [166]. Sometimes, supersymmetry is preserved under this operation, like for the three-torus. In other cases, in particular for class B group manifolds, we do not know whether any supersymmetry is preserved under such an identification.

In this thesis, we will concentrate on local aspects, and therefore not take sides regarding this issue.

Supersymmetry Transformations and Global Symmetries

With the Ansatz above, all class A and B gauged supergravities can be obtained. We will first consider the supersymmetry transformations of these theories. Reduction of the 11D supersymmetry rules (B.1) yields

$$
\begin{aligned}
\delta e_\mu{}^a =& -\frac{i}{2}\bar\epsilon\Gamma^a\psi_\mu \\
\delta\psi_\mu =& 2\partial_\mu\epsilon - \frac{1}{2}\not\psi_\mu\epsilon + \frac{1}{2}L_{[i|}{}^m\mathcal{D}_\mu L_{m|j]}\Gamma^{ij}\epsilon + \frac{1}{24}e^{-\varphi/2}f_{ijk}\Gamma^{ijk}\Gamma_\mu\epsilon - \frac{1}{6}\,e^{-\varphi/2}f_{ij}{}^j\Gamma_\mu\Gamma^i\epsilon \\
&+ \frac{1}{24}e^{\varphi/2}\Gamma^i L_i{}^m(\Gamma_\mu{}^{\nu\rho} - 10\delta_\mu{}^\nu\Gamma^\rho)F_{m\nu\rho}\epsilon - \frac{i}{12}e^{-\varphi}\Gamma^{ijk}L_i{}^m L_j{}^n L_k{}^p G^{(1)}_{\mu mnp}\epsilon \\
&+ \frac{i}{96}e^{\varphi/2}(\Gamma_\mu{}^{\nu\rho\delta\epsilon} - 4\delta_\mu{}^\nu\Gamma^{\rho\delta\epsilon})G_{\nu\rho\delta\epsilon}\epsilon + \frac{i}{36}\Gamma^i L_i{}^m(\Gamma_\mu{}^{\nu\rho\delta} - 6\delta_\mu{}^\nu\Gamma^{\rho\delta})H_{\nu\rho\delta m}\epsilon \\
&+ \frac{i}{48}e^{-\varphi/2}\Gamma^i\Gamma^j L_i{}^m L_j{}^n(\Gamma_\mu{}^{\nu\rho} - 10\delta_\mu{}^\nu\Gamma^\rho)F_{\nu\rho mn}\epsilon\,, \\
\delta\lambda_i =& \frac{1}{2}L_i{}^m L^{jn}\not{\mathcal{D}}M_{mn}\Gamma_j\epsilon - \frac{1}{3}\not\partial\varphi\Gamma_i\epsilon - \frac{1}{4}e^{-\varphi/2}(2f_{ijk} - f_{jki})\Gamma^{jk}\epsilon \\
&+ \frac{1}{4}e^{\varphi/2}L_i{}^m M_{mn}\not{F}^n\epsilon + \frac{i}{144}e^{\varphi/2}\Gamma_i\not{G}\epsilon + \frac{i}{36}(2\delta_i{}^j - \Gamma_i{}^j)L_j{}^m\not{H}_m\epsilon \\
&+ \frac{i}{24}e^{-\varphi/2}\Gamma^j L_j{}^m L_k{}^n(3\delta_i{}^k - \Gamma_i{}^k)\not{F}_{mn}\epsilon + \frac{i}{6}e^{-\varphi}\Gamma^{jk}L_i{}^m L_j{}^n L_k{}^p\not{G}^{(1)}_{mnp}\epsilon\,, \\
\delta A^m{}_\mu =& -\frac{i}{2}e^{-\varphi/2}L_i{}^m\bar\epsilon(\Gamma^i\psi_\mu - \Gamma_\mu(\eta^{ij} - \frac{1}{6}\Gamma^i\Gamma^j)\lambda_j)\,, \\
\delta V_{\mu\,mn} =& \varepsilon_{mnp}[-\frac{i}{2}e^{\varphi/2}L_i{}^p\bar\epsilon(\Gamma^i\psi_\mu + \Gamma_\mu(\eta^{ij} - \frac{5}{6}\Gamma^i\Gamma^j)\lambda_j) - \ell\,\delta A^p{}_\mu]\,, \\
\delta B_{\mu\nu\,m} =& L_m{}^i\bar\epsilon(\Gamma_{i[\mu}\psi_{\nu]} + \frac{1}{6}\Gamma_{\mu\nu}(3\delta_i^j - \Gamma_i\Gamma^j)\lambda_j) - 2\,\delta A^n{}_{[\mu}V_{\nu]\,mn}\,, \\
\delta C_{\mu\nu\rho} =& \frac{3}{2}e^{-\varphi/2}\bar\epsilon\Gamma_{[\mu\nu}(\psi_{\rho]} - \frac{1}{6}\Gamma_{\rho]}\Gamma^i\lambda_i) - 3\delta A^m{}_{[\mu}B_{\nu\rho]\,m}\,, \\
L_i{}^n\delta L_{nj} =& \frac{i}{4}e^{\varphi/2}\bar\epsilon(\Gamma_i\delta_j{}^k + \Gamma_j\delta_i{}^k - \frac{2}{3}\eta_{ij}\Gamma^k)\lambda_k\,,
\end{aligned}
$$

$$\begin{aligned} \delta\varphi &= -\frac{i}{2}\bar{\epsilon}\Gamma^i\lambda_i\,, \\ \delta\ell &= -\frac{i}{2}e^{\varphi}\bar{\epsilon}\Gamma^i\lambda_i\,. \end{aligned} \tag{5.63}$$

where reduction of the 11D field strength $\hat{G}$ gives rise to the 8D field strengths

$$\begin{aligned} G &= dC + F^m\wedge B_m\,, & F_{mn} &= \mathcal{D}V_{mn} - f_{mn}{}^p B_p + \ell\varepsilon_{mnp}F^p\,, \\ H_m &= \mathcal{D}B_m + F^n\wedge V_{mn}\,, & G^{(1)}_{mnp} &= \varepsilon_{mnp}d\ell + 3\left(V_{r[m} + \ell A^q\varepsilon_{qr[m]}\right) f_{np]}{}^r\,, \end{aligned} \tag{5.64}$$

and where the field strengths of the Kaluza-Klein vectors are given by

$$F^m = dA^m - \tfrac{1}{2}f_{np}{}^m A^n\wedge A^p\,, \tag{5.65}$$

which are the non-Abelian gauge field strengths.

The ungauged theory has a global symmetry group (see table 3.4)

$$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})\,. \tag{5.66}$$

The first group acts on the indices m, n, p of the bosonic sector in the obvious way. For $SL(2,\mathbb{R})$ covariance, one needs to construct the $SL(2,\mathbb{R})/SO(2)$ scalar coset W_{IJ} given in (5.54) and the doublet of vector field strengths $F^{I\,m} = (\epsilon^{mnp}F_{np}, F^m)$, with $I = 1, 2$. The $SO(1,1)^+ \sim \mathbb{R}^+$ subgroup of $SL(2,\mathbb{R})$ can be combined with the $SL(3,\mathbb{R})$ group to yield the full $GL(3,\mathbb{R})$, that one would expect from the 11D origin.

In the gauged theory, this $GL(3,\mathbb{R})$ is in general no longer a symmetry, since it does not preserve the structure constants. The unbroken part is exactly given by the automorphism group of the structure constants as given in table 5.4. Of course, this always includes the gauge group, which is embedded in $GL(3,\mathbb{R})$ via (5.57). However, the full automorphism group can be bigger. For instance, it is nine-dimensional in the $U(1)^3$ case; this amounts to the fact that the ungauged $D = 8$ theory has a $GL(3,\mathbb{R})$ symmetry. Note that all other cases have Dim(Aut) < 9 and thus break the $GL(3,\mathbb{R})$ symmetry to some extent. The scaling symmetry that corresponds to the determinant of the $GL(3,\mathbb{R})$ element (or, equivalently, to the $SO(1,1)^+$ subgroup of $SL(2,\mathbb{R})$), is broken by all non-vanishing structure constants. To understand the fate of the other subgroups of $SL(2,\mathbb{R})$, one needs to define the doublet $f^I_{mn}{}^p = (f_{mn}{}^p, 0)$. Under a global $SL(2,\mathbb{R})$ transformation the full theory is invariant up to a transformation of the structure constants:

$$f^I_{mn}{}^p \to \Omega^I{}_J f^I_{mn}{}^p\,, \qquad \Omega^I{}_J \in SL(2,\mathbb{R})\,. \tag{5.67}$$

From this transformation, one can see that the $SO(2)$ and $\mathbb{R}^+$ subgroups of $SL(2,\mathbb{R})$ are broken by any non-zero structure constants and thus in all theories except the Bianchi type I. In contrast, the doublet of structure constants (5.67) is invariant under an $\mathbb{R}$ subgroup of the $SL(2,\mathbb{R})$ symmetry.

Lagrangian for Class A Theories

The bosonic part of the eight-dimensional action for class A theories reads

$$\begin{aligned}\mathcal{L} = \sqrt{-g}\ \big[& R + \tfrac{1}{4}\mathrm{Tr}(\mathcal{D}M\mathcal{D}M^{-1}) + \tfrac{1}{4}\mathrm{Tr}(\partial W \partial W^{-1}) - \tfrac{1}{4}F^{I\,m}M_{mn}W_{IJ}F^{J\,n} + \\ & - \tfrac{1}{2\cdot 3!}H_m M^{mn} H_n - \tfrac{1}{2\cdot 4!}e^{\varphi}G^2 - V - \tfrac{1}{6}\star(CS)\big]\ , \end{aligned} \tag{5.68}$$

with Chern-Simons term

$$\begin{aligned} CS = & \ell\, G\wedge G + 2\epsilon^{mnp}G\wedge H_m \wedge V_{np} - 2G\wedge(\tilde{F}^m + \ell F^m)\wedge B_m + 2G\wedge\partial\ell\wedge C + \\ & + \epsilon^{mnp}H_m\wedge H_n\wedge B_p + 2H_m\wedge(\tilde{F}^m + \ell F^m)\wedge C\,, \end{aligned} \tag{5.69}$$

where we have defined $\tilde{F}^m = \epsilon^{mnp}G_{np}$. The scalar potential V reads

$$\begin{aligned} V &= \tfrac{1}{4}e^{-\varphi}\left[2M^{nq}f_{mn}{}^{p}f_{pq}{}^{m} + M^{mq}M^{nr}M_{ps}f_{mn}{}^{p}f_{qr}{}^{s}\right] \\ &= -\tfrac{1}{2}\,e^{-\varphi}\left[(\mathrm{Tr}(MQ))^2 - 2\mathrm{Tr}(MQMQ)\right], \end{aligned} \tag{5.70}$$

where we have used the relation (5.38) between the structure constants and the mass matrix.

The massive deformations of class A can be written in terms of a superpotential W, which is given by

$$W = e^{-\varphi/2}\mathrm{Tr}(MQ)\,. \tag{5.71}$$

The deformations of the supersymmetry transformation of the gravitino can be written in terms of W, while the dilatino variations contain terms with $\delta_\Phi W$, where Φ denotes a generic scalar. The scalar potential (5.70) can also be written in terms of the superpotential and its derivatives via the general formula (5.1). We will come back to this in section 5.5.

Lagrangians for Truncations of Class B Theories

The class B gaugings and group manifolds are parameterised by three parameters $a \neq 0$ and (q_2, q_3) while $q_1 = 0$. The full set of field equations for class B gaugings cannot be derived from an action. However, for specific truncations this is possible, as discussed in section 4.6. We know of three such cases, leading to a Lagrangian with a single exponential potential [167]:

- Type III with the truncation[9]

$$M = \begin{pmatrix} e^{-\sigma/\sqrt{3}} & 0 & 0 \\ 0 & e^{\sigma/2\sqrt{3}}\cosh(\tfrac{1}{2}\sqrt{3}\sigma) & -e^{\sigma/2\sqrt{3}}\sinh(\tfrac{1}{2}\sqrt{3}\sigma) \\ 0 & -e^{\sigma/2\sqrt{3}}\sinh(\tfrac{1}{2}\sqrt{3}\sigma) & e^{\sigma/2\sqrt{3}}\cosh(\tfrac{1}{2}\sqrt{3}\sigma) \end{pmatrix} \tag{5.72}$$

[9] The off-diagonal components of M (corresponding to non-zero axions) are consequences of our basis choice for the structure constants. An $SO(2)$ rotation renders M diagonal but introduces off-diagonal components in Q.

which corresponds to the manifold $S^1 \times \mathbb{H}^2$. It leads to the Lagrangian

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\sigma)^2 - \tfrac{3}{2}e^{-\varphi-\sigma/\sqrt{3}}] \,, \tag{5.73}$$

which has $\Delta = -1$.

- Type V with $M = \mathbb{I}_3$, corresponding to the manifold $\mathbb{H}^3$:

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{3}{2}e^{-\varphi}] \,, \tag{5.74}$$

with a dilaton coupling giving rise to $\Delta = -4/3$.

- Type VII$_a$ with $M = \mathbb{I}_3$, also corresponding to the manifold $\mathbb{H}^3$ and leading to the same Lagrangian (5.74).

Note that in all three cases the group manifold (partly) reduces to a hyperbolic manifold, i.e. the maximally symmetric space of constant negative curvature with enhanced isometry and isotropy groups.

Nine-dimensional Origin

In this subsection, we will discuss how all $D = 8$ gauged supergravities, except those whose gauge group is simple (i.e. $SO(3)$ or $SO(2,1)$), can be obtained by a twisted reduction of maximal $D = 9$ ungauged supergravity using its global symmetry group $\mathbb{R}^+ \times SL(2,\mathbb{R})$. This is possible since all these theories follow from the reduction over a non-semi-simple group manifold, which has two commuting isometries. These can always be arranged to be manifest, as in (5.55) with $q_1 = 0$. In these cases, one first can perform a toroidal reduction over T^2 to nine dimensions, followed by a twisted reduction to eight dimensions.

Restricting ourselves to symmetries that are not broken by α'-corrections, the $D = 9$ global symmetry group is given by

$$SL(2,\mathbb{R}) \times \mathbb{R}^+ \,. \tag{5.75}$$

Here the duality group $SL(2,\mathbb{R})$ is a symmetry of the action and is not broken by α'-corrections, since it descends from the duality group $SL(2,\mathbb{R})$ of type IIB string theory. We denote its elements by Ω. The explicit $\mathbb{R}^+$ symmetry with elements Λ is given by[10] the combination $4\alpha - 3\delta$ of table 5.2 and is valid on the equations of motion only. Since it has an M-theory origin as the scaling symmetry $x^\mu \to \Lambda\, x^\mu$ for $\mu = 10, 11$, this symmetry is not broken by α'-corrections either. This scaling symmetry is precisely the transformation with parameter $\Lambda = \exp(az^1)$, generated by the matrix $U^m{}_n$, see (5.55), for $q_1 = q_2 = q_3 = 0$. Note that this scaling symmetry scales the volume-element of the two-torus, which explains why it is only a symmetry of the $D = 9$ equations of motion.

[10]The symmetry $\alpha + 12\beta$ considered in section 5.3 is a linear combination of the explicit $\mathbb{R}^+$ and the $SO(1,1)^+ \sim \mathbb{R}^+$ symmetry of $SL(2,\mathbb{R})$.

$D=9 \Rightarrow D=8$ Reduction Ansatz	$\Lambda = 1$ ($\Rightarrow$ class A)	$\Lambda \neq 1$ ($\Rightarrow$ class B)
$\Omega = \mathbb{I}_2$	I = $U(1)^3$	V
$\Omega \in \mathbb{R}$	II = Heis_3	VI
$\Omega \in \mathbb{R}^+$	$\mathrm{VI}_0 = ISO(1,1)$	III = $\mathrm{VI}_{a=1/2}$, VI_a
$\Omega \in SO(2)$	$\mathrm{VII}_0 = ISO(2)$	VII_a

Table 5.5: *The $D=8$ non-semi-simple gauged maximal supergravities, resulting from reduction of $D=9$ ungauged maximal supergravity by using the different global symmetries in $D=9$. Here Ω and Λ denote elements of $SL(2,\mathbb{R})$ and $\mathbb{R}^+$, respectively.*

When performing the $D=9$ to $D=8$ twisted reduction [38], we distinguish between the cases where $\Lambda = 1$ $(a=0)$ and where $\Lambda \neq 1$ $(a \neq 0)$. Furthermore, we allow Ω to be either the identity or an element of the three subgroups of $SL(2,\mathbb{R})$. Reduction to $D=8$ thus gives rise to eight different possibilities, one of which has to be split in two. These correspond to the nine $D=8$ gauged maximal supergravities with non-semi-simple gauge groups, i.e. all Bianchi types except type VIII with gauge group $SO(2,1)$ and type IX with gauge group $SO(3)$. The result is given in table 5.5.

It can be seen that class A gauged supergravities are obtained by using only a subgroup of $SL(2,\mathbb{R})$, which is a reduction that can be performed on the $D=9$ ungauged action. Class B gauged supergravities, however, require the use of the extra scaling symmetry which indeed can only be performed at the level of the field equations.

An alternative to the twisted reduction of 9D ungauged theories is the trivial reduction of the gauged theories of section 5.3. When restricting to gauge groups that are embeddable in string theory, we have four possibilities in nine dimensions: the three subgroups ζ, γ and θ of $SL(2,\mathbb{R})$ and the scaling symmetry $\alpha + 12\beta$. Upon reduction, we find that these theories are related to Bianchi types up to $SO(2) \subset SL(2,\mathbb{R})$ rotation of 90 degrees. The specific types are II, VI_0 and VII_0 (of class A) and III (of class B), respectively.

5.5 CSO Gaugings of Maximal Supergravities

In this section we will discuss CSO gauged maximal supergravities, appearing in diverse dimensions, and describe the relation to the previously constructed theories. We will conclude by mentioning some other possibilities of gauged maximal supergravities.

CSO Algebras and Groups

An important role in gauged maximal supergravity is played by the so-called CSO groups, see e.g. [168–170]. These groups can be seen as analytic continuations and group contractions of SO groups, as is demonstrated below.

We start with the algebra $so(n)$ with generators in the fundamental representation (with $i, j, \ldots = 1, \ldots, n$)

$$(g_{ij})^k{}_l = \delta^k_{[i} Q_{j]l} \,, \tag{5.76}$$

with Q equal to the identity matrix for $so(n)$. The generators are labelled by an anti-symmetric pair of indices, giving rise to $\frac{1}{2}n(n-1)$ different generators. These satisfy the commutation relations

$$[g_{ij}, g_{kl}] = f_{ij,kl}{}^{mn} g_{mn} \,, \qquad f_{ij,kl}{}^{mn} = 2\delta^{[m}_{[i} Q_{j][k} \delta^{n]}_{l]} \,. \tag{5.77}$$

The corresponding group elements leave the matrix Q invariant:

$$\text{exp}(\lambda^{ij} g_{ij}) Q \text{exp}(\lambda^{ij} g^T_{ij}) = Q \,, \tag{5.78}$$

where λ^{ij} are the (real) parameters of the group elements. The above properties hold for an arbitrary matrix Q, which equals $\mathbb{I}_n$ for the $SO(n)$ group.

Consider the following scaling of the $so(n)$ algebra, where $i, j = 1, \ldots, n-1$:

$$g_{ij} \to g_{ij} \,, \qquad g_{in} \to \lambda g_{in} \tag{5.79}$$

A straightforward calculation shows that the only effect on the above algebra is a scaling of the matrix Q:

$$Q = \mathbb{I}_n \to \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & \lambda^{-2} \end{pmatrix} . \tag{5.80}$$

Therefore, different choices for λ result in different algebras:

- $\lambda \to 1$ is the trivial case, retaining the $so(n)$ algebra,
- $\lambda \to i$ is an analytic continuation, yielding the $so(n-1,1)$ algebra and
- $\lambda \to \infty$ corresponds to a group contraction, giving the $iso(n-1)$ algebra,

as can be seen from the defining equation (5.78). Thus, the (imaginary or infinite) rescaling of the generators (5.79) takes one from the $so(n)$ algebra with $Q = \mathbb{I}_n$ to the algebras $so(n-1,1)$ or $iso(n-1)$.

One can perform the operation (5.79) a number of times with different generators, leading to the algebra (5.76) with the matrix

$$Q = \begin{pmatrix} \mathbb{I}_p & 0 & 0 \\ 0 & -\mathbb{I}_q & 0 \\ 0 & 0 & 0_r \end{pmatrix} , \tag{5.81}$$

with $p+q+r = n$. The corresponding algebra is called the $cso(p,q,r)$ algebra, satisfying the equations (5.76)-(5.78). Therefore, the $cso(p,q,r)$ algebras with $p+q+r = n$ are analytic continuations and group contractions of the prime example $so(n)$. This generalises the $so(n)$ algebra to $[n^2/4+n]$ different possible algebras.

Note that a generator g_{ij} vanishes if and only if $Q_{ii} = Q_{jj} = 0$. For this reason, the matrix (5.81) gives rise to $\frac{1}{2}r(r-1)$ vanishing generators. The number of non-trivial generators of a $cso(p,q,r)$ algebra therefore equals

$$\tfrac{1}{2}(p+q+r)(p+q+r-1) - \tfrac{1}{2}r(r-1) = \tfrac{1}{2}(p+q)(p+q+2r-1) \,. \tag{5.82}$$

Also note that $cso(p,q,r)$ and $cso(q,p,r)$ are isomorphic, while $cso(p,q,0) = so(p,q)$ and $cso(p,q,1) = iso(p,q)$.

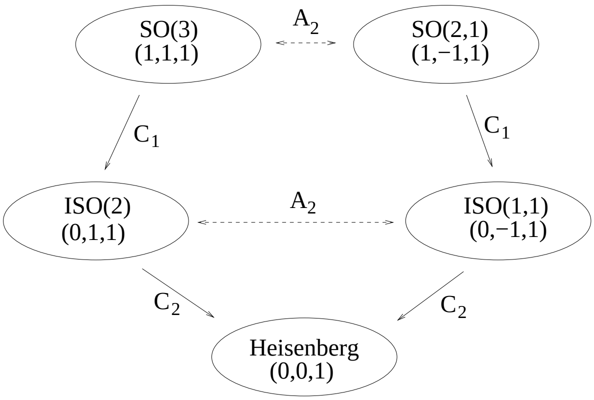

Figure 5.2: *Relations between the different CSO groups with $n = 3$ under analytic continuations* A *and group contractions* C. *The boxes give the groups and the diagonal components of Q.*

The corresponding CSO group elements satisfy (5.78). The simplest examples are

- $n = 2$: $SO(2)$, $SO(1,1)$, $ISO(1) \sim \mathbb{R}$,
- $n = 3$: $SO(3)$, $SO(2,1)$, $ISO(2)$, $ISO(1,1)$, $CSO(1,0,2) \sim Heis_3$.

The $n = 2$ case are the one-dimensional subgroups of $SL(2,\mathbb{R})$, while the $n = 3$ case exactly comprises the class A groups of the Bianchi classification (see table 5.4). The relations under analytic continuations and group contractions are illustrated in figure 5.2 for $n = 3$.

Gauged Maximal Supergravity

One might have noticed a certain familiarity with the $CSO(p, q, r)$ groups with $p+q+r = n$ for $n = 2$ and $n = 3$. Indeed, these are exactly the gauge groups for a subset of the gaugings considered in sections 5.3 and 5.4. Such CSO groups also emerge in lower-dimensional gauged maximal supergravities[11], as we will now discuss.

It has been known for long that certain gauged maximal supergravities with global symmetry groups $SL(n, \mathbb{R})$ allow for the gauging of the $SO(n)$ subgroup of the global symmetry. An example is the $SO(8)$ gauging in four dimensions [173]. Subsequently, it was realised that such gauged supergravities could be obtained by the reduction of a higher-dimensional supergravity over a sphere, with a flux of some field strength through the sphere. An example is the reduction of 11D supergravity over S^7, with magnetic flux of the four-form field strength through the seven-sphere, yielding the $SO(8)$ theory [127]. Other examples are given in table[12,13] 5.6.

D	n	ϕ	Origin
10	1	$\surd$	Massive IIA [74]
9	2	$\surd$	IIB with $SO(2)$ twist [147]
8	3	$\surd$	IIA on S^2 [164]
7	5	–	11D on S^4 [125, 126]
6	5	$\surd$	IIA on S^4 [175]
5	6	–	IIB on S^5 [128, 174]
4	8	–	11D on S^7 [127]

Table 5.6: *The different gauged maximal supergravities in D dimensions with n mass parameters. The relevant scalar subsector consists of the coset $SL(n, \mathbb{R})/SO(n)$ plus, for the cases with a $\surd$ in the third column, an extra dilaton ϕ. We also give the higher-dimensional origin of the $SO(n)$ prime examples.*

In addition to $SO(n)$, the global symmetry group $SL(n, \mathbb{R})$ has more subgroups that can be gauged. It was found that many more gaugings could be obtained from the $SO(n)$ prime

[11] For the purposes of uniformity, we will restrict ourselves to $D \geq 4$. Gauged maximal supergravities in $D = 3$ have a number of remarkable properties, see e.g. [171, 172].

[12] We have included massive IIA supergravity in table 5.6, even though it is not a gauged theory and its higher-dimensional origin is unknown, for reasons that will be discussed in the next subsection.

[13] The S^5 reduction of IIB has not (yet) been proven in full generality. The linearised result was obtained by [174] while the full reduction of the $SL(2, \mathbb{R})$ invariant part of IIB supergravity was performed by [128].

examples by analytic continuation or group contraction of the gauge group [176, 177]. This leads one from $SO(n)$ to the group $CSO(p,q,r)$ with $p+q+r=n$, as we have seen in the previous subsection.

At first the generalisation of $SO(p,q)$ to $CSO(p,q,r)$ was thought to be possible only for even-dimensional gauged supergravities, due to problems with the number of degrees of freedom of gauge potentials in odd dimensions. The resolution lies in the role played by the massive self-dual gauge potentials in odd dimensions [178]. For example, the resulting field content in $D=5$ contains $15+r$ gauge vectors and $12-r$ massive self-dual two-form potentials [169]. In $D=7$ one would expect r massless two-forms and $5-r$ massive self-dual three-forms, of which the case $r=1$ is confirmed in [175]. Surprisingly, this phenomenon does not occur in $D=9$, where one has one massless three-form potential for all values of r [147]. This is related to the fact that the 9D potential is a singlet, while the lower-dimensional potentials transform non-trivially under the gauge group, see table 3.4. In this section, we will be concerned with the scalar subsector of these theories and therefore not mind the subtleties associated with the gauge potentials.

The question of the higher-dimensional origin[14] of the $CSO(p,q,r)$ gaugings was clarified in [131], where the same operations of analytic continuations and group contractions were applied to the internal manifold. The resulting manifolds are hypersurfaces defined by

$$\sum_{i=1}^{n} q_i {\mu_i}^2 = 1\,, \tag{5.83}$$

with n parameters[15] q_i of which p are positive, q are negative and r are vanishing; hence $p+q+r=n$. The manifold corresponding to (5.83) is denoted by $H^{p,q}\times T^r$ [131]. The hyperbolic manifold $H^{p,q}$ can be endowed with a positive-definite metric, which generically is inhomogeneous [181]; the exceptions are the (maximally symmetric) coset spaces

$$S^n = H^{n+1,0} \simeq \frac{SO(n+1)}{SO(n)}\,, \qquad H^n = H^{1,n} \simeq \frac{SO(1,n)}{SO(n)}\,, \tag{5.84}$$

i.e. the sphere and the hyperboloid. Generically the spaces $H^{p,q}$ are non-compact; the only exception is the sphere with $q=0$.

Thus non-compact gauge groups $CSO(p,q,r)$ with $q\neq 0$ are obtained from reduction over non-compact manifolds, as first suggested in [182]. It can be argued that the corresponding reduction is consistent provided the compact case, with reduction over S^{n-1}, has been proven consistent [131].

A special case of this reduction is provided by $p+q=1$ or 2. In such cases, $H^{p,q}$ corresponds to a one-dimensional manifold, over which one performs a twisted reduction (see section 4.3). The difference between $(p,q,r)=(2,0,0)$, $(1,1,0)$ and $(1,0,1)$ is the

[14] For discussions of the higher-dimensional origin of self-duality relations, see [125, 137, 178].

[15] Another approach to the introduction of these parameters in the lower dimension is the inclusion of n Killing vectors in 11D supergravity [142, 147, 179, 180].

flux of the scalars: the different values correspond to twisting with the subgroups $SO(2)$, $SO(1,1)$ and $\mathbb{R}$ of a global symmetry group $SL(2,\mathbb{R})$, respectively.

Examples of these cases are provided by the reduction of IIB with an $SL(2,\mathbb{R})$ twist, giving rise to CSO gauged supergravity in 9D with $n=2$ (see section 5.3). This requires the identification

$$Q = \tfrac{1}{2}\left(\begin{array}{cc} -m_2+m_3 & m_1 \\ m_1 & m_2+m_3 \end{array} \right) = \left(\begin{array}{cc} q_1 & 0 \\ 0 & q_2 \end{array} \right) , \qquad (5.85)$$

between the parameters $\vec{m}=(m_1,m_2,m_3)$ of the $SL(2,\mathbb{R})$ twisted reduction (B.14) and the parameters (q_1,q_2) of the reduction over the hypersurface (5.83). The choice of diagonal Q corresponds to vanishing m_1, which can always be obtained by $SL(2,\mathbb{R})$ field redefinitions (as explained in section 5.3). Note that generic twisted reductions (4.36) give rise to a traceless matrix C, which only for $n=2$ can be related to a symmetric matrix Q, see (5.19). The explicit relation between the twisted reduction coordinate y and the Cartesian coordinates μ_i reads

$$\mu_1 = \sin(\sqrt{q_1q_2}y)/\sqrt{q_1} \,, \qquad \mu_2 = \cos(\sqrt{q_1q_2}y)/\sqrt{q_2} \,. \qquad (5.86)$$

This explains the relation between twisted reduction and the case $p+q \leq 2$ of (5.83).

Another noteworthy remark concerns the next case, $p+q=3$. This defines two-dimensional spaces, e.g. S^2 and H^2, over which one can perform coset reductions. Alternatively, these cases can be viewed as group manifold reductions over three-dimensional group manifolds, e.g. $SO(3)$ and $SO(2,1)$. For example, one can either perform a two-dimensional coset reduction of IIA or a three-dimensional group manifold reduction of 11D to obtain the class A gauged supergravities in 8D [136]. The structure constants of these class A group manifolds are given by

$$f_{mn}{}^p = \varepsilon_{mnq} Q^{pq} \,, \qquad Q^{mn} = \text{diag}(q_1,q_2,q_3) \,, \qquad (5.87)$$

which relates the parameters of the group manifold reduction and the reduction over the hypersurface. Note that the structure constants only contain a symmetric matrix Q for the case $n=3$, confirming the relation between 3D group manifolds and (5.83) with $n=3$. Explicitly, the relations between the three-dimensional group manifold reductions and the reductions over the two-dimensional hypersurface (5.83) are

$$\begin{aligned} \mu_1 &= \sin(\sqrt{q_2q_3}\, y^2)/\sqrt{q_1} \,, \\ \mu_2 &= \sin(\sqrt{q_1q_3}\, y^1)\cos(\sqrt{q_2q_3}\, y^2)/\sqrt{q_2} \,, \\ \mu_3 &= \cos(\sqrt{q_1q_3}\, y^1)\cos(\sqrt{q_2q_3}\, y^2)/\sqrt{q_3} \,, \end{aligned} \qquad (5.88)$$

where $y^{1,2}$ are the two coordinates of the 3D group manifold that remain after reduction over the manifest isometry direction y^3.

We expect the following relations between the different maximal supergravities with CSO gauge groups upon toroidal reduction. Consider the mass parameters in dimensions D and $d < D$, denoted by n_D and $n_d \geq n_D$, respectively. Then the n_D mass parameters in D dimensions reduce to the n_d mass parameters in d dimensions with $n_d - n_D$ vanishing entries:

$$Q_D \qquad \overset{T^{D-d}}{\Longrightarrow} \qquad Q_d = \begin{pmatrix} Q_D & 0 \\ 0 & 0_{n_d - n_D} \end{pmatrix} . \tag{5.89}$$

Therefore, the set of all CSO gaugings in D dimensions reduces to (generically) a subset of all CSO gaugings in d dimensions. In the reduction Ansatz from 11D or 10D to d dimensions, the $n_d - n_D$ vanishing mass parameters correspond to a torus T^{D-d} over which one can reduce first, as can be seen from (5.83). This conjecture relating the different CSO gauged supergravities will be proven below for the scalar subsector of the theories.

Scalar Potential

In addition to the gauging of the group $CSO(p, q, r)$, the non-trivial reduction over the spaces $H^{p,q} \times T^r$ gives rise to a scalar potential. To this end, we consider the scalar subsector of these theories.

In all cases, it contains a scalar coset $SL(n, \mathbb{R})/SO(n)$, which is parameterised by a symmetric matrix M. We will restrict ourselves to a diagonal matrix, for reasons that will be explained in section 6.2. The diagonal part of the scalar is given by

$$M = \text{diag}(e^{\vec{\alpha}_1 \cdot \vec{\phi}}, \ldots, e^{\vec{\alpha}_n \cdot \vec{\phi}}) , \tag{5.90}$$

where the n vectors $\vec{\alpha}_i = \{\alpha_{iI}\}$ are weights of $SL(n, \mathbb{R})$ fulfilling the following relations

$$\sum_i \alpha_{iI} = 0 , \qquad \sum_i \alpha_{iI} \, \alpha_{iJ} = 2 \, \delta_{IJ} , \qquad \vec{\alpha}_i \cdot \vec{\alpha}_j = 2 \, \delta_{ij} - \frac{2}{n} . \tag{5.91}$$

In addition, the scalar coset can contain an extra scalar ϕ, as indicated in table 5.6. Note that M and ϕ generically do not correspond to the full scalar coset, as can be inferred from table 3.4; however, they do constitute the part that is relevant to the CSO gauging and scalar potential. Similarly, the full global symmetry will often be larger than $SL(n, \mathbb{R})$; it is for example given by $SO(5, 5)$ in 6D. Its $SL(n, \mathbb{R})$ subgroup will generically be the largest symmetry of the Lagrangian, however, and is the only part of the symmetry group that is relevant for the present discussion.

The scalar potential of all CSO gaugings has the universal form

$$V = -\tfrac{1}{2} e^{a\phi} ((\text{Tr}[QM])^2 - 2\text{Tr}[QMQM]) , \qquad Q = \text{diag}(q_1, \ldots, q_n) , \tag{5.92}$$

in terms of the mass parameters q_i of the hypersurface (5.83). The dilaton coupling a is given by

$$a^2 = \frac{8}{n} - 2\,\frac{D-3}{D-2}\,, \tag{5.93}$$

for the different cases. This scalar potential can be written in terms of the superpotential

$$W = e^{a\phi/2}\mathrm{Tr}[QM]\,, \tag{5.94}$$

via the general formula (5.1) for the scalar potential:

$$V = \tfrac{1}{2}(\delta_\phi W)^2 + \tfrac{1}{2}(\delta_{\vec{\phi}} W)^2 - \frac{D-1}{4(D-2)} W^2\,. \tag{5.95}$$

This superpotential also parameterises the explicit deformations of the supersymmetry transformations: the gravitino variation will be proportional to W while the dilatini variations will be proportional to $\delta W/\delta\vec{\phi}$ and $\delta W/\delta\phi$.

In accordance with table 5.6, a vanishes for $(D,n) = (7,5)$, $(5,6)$ and $(4,8)$, for which the extra dilaton ϕ is absent. The $SL(2,\mathbb{R})$ twisted reduction of IIB and class A group manifold reduction of 11D yield scalar potentials (5.23) and (5.70) that coincide with (5.92) for $(D,n) = (9,2)$ and $(8,3)$, respectively. In addition, the scalar potential (5.5) of massive IIA also is of exactly this form with $(D,n) = (10,1)$ and is therefore included in table 5.6.

For the $SO(n)$ cases, i.e. all $q_i = 1$, the scalar subsector can be truncated by setting $M = \mathbb{I}$. In this truncation, the scalar potential reduces to a single exponential potential

$$V = -\tfrac{1}{2}n(n-2)e^{a\phi}\,, \tag{5.96}$$

Note the dependence of the sign of the potential on n: it is positive for $n = 1$, vanishing for $n = 2$ and negative for $n \geq 3$. If $a = 0$ (which necessarily implies $n \geq 3$ in $D \geq 4$), the scalar potential becomes a cosmological constant and allows for a fully supersymmetric AdS solution; for this reason, such theories are called AdS supergravities. Theories with $a \neq 0$ are called DW supergravities since the natural vacuum is a domain wall solution, see section 6.2.

Group Contraction and Dimensional Reduction

We would like to consider two operations on the scalar sector of the CSO gauged supergravity. The first operation corresponds to a contraction of the CSO gauge group and corresponds to setting one mass parameter equal to zero, as explained above. For concreteness, it is taken to be the last one: $q_i = (q_p, 0)$, where we have split up $i = (p, n)$ and $p = 1, \ldots, n-1$. The superpotential now reads

$$W = e^{a\phi/2}\sum_p q_p e^{\vec{\alpha}_p\cdot\vec{\phi}} = e^{a\phi/2+\vec{\beta}\cdot\vec{\phi}}\sum_p q_p e^{\vec{\beta}_p\cdot\vec{\phi}}\,, \tag{5.97}$$

where we have chosen to extract an overall part $\vec{\beta}\cdot\vec{\phi}$ according to $\vec{\alpha}_p = \vec{\beta}+\vec{\beta}_p$. A convenient choice for $\vec{\beta}$ is

$$\vec{\beta} = -\frac{1}{n-1}\vec{\alpha}_n = (0,\ldots,0,\frac{1}{\sqrt{n(n-1)/2}})\,. \tag{5.98}$$

This corresponds to the scalar coset split

$$M = \begin{pmatrix} e^{\vec{\beta}\cdot\vec{\phi}}\tilde{M} & 0 \\ 0 & e^{-(n-1)\vec{\beta}\cdot\vec{\phi}} \end{pmatrix}\,, \qquad \tilde{M} = \text{diag}(e^{\vec{\beta}_1\cdot\vec{\phi}},\ldots,e^{\vec{\beta}_{n-1}\cdot\vec{\phi}})\,, \tag{5.99}$$

where the weight vectors $\vec{\beta}_p$ are subject to the reduction of (5.91):

$$\sum_p \beta_{pI} = 0\,, \qquad \sum_p \beta_{pI}\,\beta_{pJ} = 2\,\delta_{IJ}\,, \qquad \vec{\beta}_p\cdot\vec{\beta}_q = 2\,\delta_{pq} - \frac{2}{n-1}\,, \tag{5.100}$$

while the last component of all vectors $\vec{\beta}_p$ vanishes: $\beta_{pn} = 0.$. Therefore, the contracted superpotential (5.97) only depends on the smaller coset $SL(n-1,\mathbb{R})/SO(n-1)$. Also note that the overall dilaton coupling has changed due to the contraction. For the scalar potential, this will amount to $a\phi + 2\vec{\beta}\cdot\vec{\phi}$ instead of $a\phi$. After a change of basis, corresponding to an $SO(n+1)$ rotation in $(\phi,\vec{\phi})$-space, this takes the form $\tilde{a}\tilde{\phi}$ with

$$\tilde{a}^2 = a^2 + 4\vec{\beta}\cdot\vec{\beta} = \frac{8}{n-1} - 2\,\frac{D-3}{D-2} > a\,, \tag{5.101}$$

which is exactly the original relation (5.93) with n decreased by one. It should be clear that this contraction can be employed several times, each time reducing n by one.

The second operation we wish to perform corresponds to dimensionally reducing the scalar sector. We take trivial Ansätze for the scalars, $\hat{M} = M$ and $\hat{\phi} = \phi$, and the usual Ansatz (4.4) for the metric (obtaining Einstein frame with a canonically normalised Kaluza-Klein scalar φ in the lower dimension):

$$d\hat{s}_D{}^2 = e^{2\gamma\varphi}ds_{D-1}{}^2 + e^{-2(D-3)\gamma\varphi}dz^2\,, \qquad \gamma^2 = \frac{1}{2(D-2)(D-3)}\,, \tag{5.102}$$

where we have truncated the Kaluza-Klein vector away. The resulting scalar potential is of the same form (5.92), but again the dilaton coupling has changed: the factor $a\phi$ is replaced by $a\phi + 2\gamma\varphi$. After a field redefinition, this corresponds to $\tilde{a}\tilde{\phi}$ with

$$\tilde{a}^2 = a^2 + 4\gamma^2 = \frac{8}{n} - 2\,\frac{D-4}{D-3} > a\,, \tag{5.103}$$

which is exactly the original relation (5.93) with D decreased by one. Again, dimensional reduction can be performed any number of times, reducing D by one at each step.

Concluding, after any number of group contractions or dimensional reductions, the scalar subsector will always have a scalar potential (5.92) with dilaton coupling (5.93). The only effect of these operations is to decrease D or n by one, respectively: the resulting system still satisfies all equations with the new values of the parameters D and n. This proves that the scalar subsectors of different gauged supergravities reduce onto each other upon matching D and n by dimensional reductions and/or group contractions. We expect this to hold for the full theories as well.

Other Gauged Maximal Supergravities

The CSO gaugings generalise the gaugings of subgroups of $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$ in nine and eight dimensions, respectively. These are not the only possibilities in lower dimensions, however. Other examples were constructed in e.g. [170, 183].

An interesting approach was taken in [171, 184], where possible gaugings were classified by a purely group-theoretical analysis. For example, different gaugings were found in 4D, depending on the global symmetry group of the Lagrangian[16] [184]. The Lagrangian with $SL(8,\mathbb{R})$ invariance allows for the $CSO(p,q,r)$ gauging with $p+q+r=n$, as found above, but other gaugings in $D=4$ and $D=5$ were also found. For example, after a number of Hodge duality transformations can bring one to an equivalent Lagrangian with $SL(6,\mathbb{R})\times SL(2,\mathbb{R})\times SO(1,1)$ invariance, which allows for other gaugings. These gauged theories are obtainable from dimensional reduction of IIB supergravity [185]. Indeed, the global symmetry group has a natural origin from the IIB point of view: the $SL(6,\mathbb{R})$ stems from the six internal coordinates, while the $SL(2,\mathbb{R})$ is already present in ten dimensions.

In addition to theories with a Lagrangian, it was found in sections 5.3 and 5.4 that M-theory allows for other gauged supergravities, that do not have an action but only field equations. In nine dimensions, there was one such theory with parameter m_s. In eight dimensions, there were five theories, with parameters q_2, q_3 and a. Clearly, one can expect such theories also in the lower dimensions. It is not clear to us what the general pattern will be, however.

[16] Hodge duality relates electric and magnetic vectors in 4D. While this does not affect the symmetry group of the field equations, the different choices give rise to different global symmetries of the Lagrangian.