M-theory and gauged supergravities

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Chapter 4

Dimensional Reduction

In the previous chapters we have learned that the most promising candidates for quantum gravity are M- and string theory. It is of interest to investigate which four-dimensional effective descriptions can be obtained from these ten- and eleven-dimensional theories. As a first step, in this chapter we will discuss the techniques of extracting different effective descriptions from a higher-dimensional field theory.

4.1 Introduction

Scalar Field and Kaluza-Klein States

Consider a complex scalar field[1] $\hat{\phi}$ in $\hat{D}$ dimensions, depending on the coordinates $x^{\hat{\mu}} = (x^{\mu}, z)$. One can expand the dependence on one of the coordinates via the Fourier decomposition:

$$\hat{\phi}(x, z) = \int dk e^{ikz} \phi_k(x) \,, \tag{4.1}$$

in terms of components ϕ_k with momentum k. If, in addition, the z direction is taken to be compact of length $2\pi R$ and we impose the boundary condition $\hat{\phi}(x, 0) = \hat{\phi}(x, 2\pi R)$, the integral becomes the sum

$$\hat{\phi}(x, z) = \sum_n e^{inz/R} \phi_n(x) \,, \tag{4.2}$$

over a discrete spectrum of fields ϕ_n with momentum $k = n/R$ in the compact direction.

[1]For our conventions concerning dimensional reduction, see appendix A.

Suppose the complex scalar $\hat{\phi}$ is subject to the Klein-Gordon equation $\hat{\Box}\hat{\phi} = 0$ where $\hat{\Box} = \partial_\mu \partial^\mu + \partial_z \partial^z$. Upon inserting the Fourier transform in this equation, one obtains separate equations for components with different momentum:

$$\Box \phi_k - k^2 \phi_k = \Box \phi_n - (n/R)^2 \phi_n = 0\,, \tag{4.3}$$

where $\Box = \partial_\mu \partial^\mu$. This is the equation for a scalar of (mass)^2 k^2 or $(n/R)^2$. Thus a massless scalar in $\hat{D}$ dimensions splits up in an infinite number of scalar fields in $D = \hat{D} - 1$ dimensions. In the context of dimensional reduction, these are called Kaluza-Klein states. Only one of these (the component ϕ_0) is massless, while the other ones are massive. The spectrum of Kaluza-Klein states is continuous for a non-compact internal direction and discrete for z compact. The latter spectrum therefore has a mass gap, which is an important ingredient when considering compactifications.

Consistency of Truncations

The fact that one obtains separate equations for the different Fourier components lies at the heart of dimensional reduction. First one expresses a higher-dimensional field in an infinite tower of lower-dimensional fields by expanding the dependence on the internal coordinates into harmonics on the internal manifold. Next, one observes that one can consistently truncate to a finite number of fields and set the rest of the spectrum equal to zero. Here, a consistent truncation refers to the origin in the higher-dimensional theory: every lower-dimensional solution should uplift to a higher-dimensional one.

Usually, one truncates to only the massless sector for the following reason. In dimensional reduction the masses are inversely proportional to the size of the internal manifold (as can be seen on dimensional grounds and in the example (4.3)). Since we live in an effectively four-dimensional world, any internal directions must be very small. But this means that the mass of states with non-zero momentum becomes very large. Therefore, these modes are too massive to be physically interesting and are usually discarded. In the above example, this would correspond to keeping only ϕ_0 and truncating the other components.

Note however that one does not need to take a very small size of the internal manifold for the massive modes to decouple; in many cases it is always a consistent truncation to retain only the massless modes, irrespective of whether the internal manifold is small or large or indeed, whether it is compact or non-compact. Again, the scalar field serves as an example: the Klein-Gordon equation for $\hat{\phi}$ splits up in many lower-dimensional equations, which are all solved by $\Box\phi_0 = 0$ and $\phi_k = 0$ (in the non-compact case) or $\phi_n = 0$ (in the compact case). Thus any solution to the equation for ϕ_0 will also solve the higher-dimensional Klein-Gordon equation for $\hat{\phi}$.

Another important point is that the lower-dimensional degrees of freedom are not always massless. In such cases, the Fourier expansion of a field over the internal manifold does not comprise any massless fields. A consistent truncation then only keeps the lightest modes of a field. The set of lower-dimensional fields then do not have the same mass: some may be massless (such as gravity and gauge vectors) while others are massive (such as scalars). In the above discussion of consistent truncation, this corresponds to replacing massless with lightest.

In the reduction procedures that we consider in this chapter, the number of degrees of freedom is unchanged by the dimensional reduction: every higher-dimensional degree of freedom corresponds, after the expansion and truncation, to one lower-dimensional degree of freedom. These lower-dimensional fields fall in multiplets of the isometry group of the internal manifold. In particular, when expanding a theory including gravity over a manifold with isometry group G, one expects non-Abelian gauge vectors of G to be among the massless lower-dimensional modes, see e.g. [120]. This will be an essential feature in sections 4.4 and 4.5.

Thus dimensional reduction consists of an expansion over an internal manifold and a subsequent truncation to the lightest subsector. However, this is usually not what is done in practice. Rather, a reduction Ansatz is constructed, relating higher-dimensional fields to a set of lower-dimensional fields. This is the result of the expansion and truncation: the lower-dimensional fields are the lightest modes of the expansion. Dimensional reduction then consists of substituting the reduction Ansatz in the field equations or Lagrangian. In many cases the reduction Ansatz contains a certain dependence on the internal coordinates. To be able to interpret the resulting equations as a lower-dimensional theory, this dependence should cancel at the end of the day. This requirement is equivalent to the consistency of truncations to the finite number of lower-dimensional fields.

In this chapter we will consider toroidal and twisted reductions and reductions over group and coset manifolds, all of which are consistent reductions. In the case of toroidal reduction, the reduction Ansatz is taken independent of the internal coordinates z^m. Toroidal reduction is therefore obviously consistent. The other three reductions require a certain z^m-dependence. For reductions with a twist and over a group manifold, the cancellation of the internal coordinate dependence is guaranteed on group-theoretical grounds, as will be explained in sections 4.3 and 4.4. In the remaining reduction over a coset manifold this cancellation is quite miraculous and poorly understood; it has been proven only in a small number of cases, see section 4.5. Examples of reductions whose consistency (in the above sense) has not been proven are Calabi-Yau compactifications[2], which we will not consider.

4.2 Toroidal Reduction

In this section we will consider the reduction Ansätze for toroidal reduction of gravity, gauge potentials and fermions. As indicated above, for reduction over a torus one does not include dependence on the internal coordinates and thus its consistency is guaranteed. More information can be found in e.g. [81].

[2]The metric of Calabi-Yau spaces are not known in full generality so explicit reduction Ansätze are not available.

Gravity on a Circle

We will now consider the reduction of gravity in $\hat{D}$ dimensions over a circle to $D = \hat{D} - 1$ dimensions. To this end, the coordinates are split up according to $x^{\hat{\mu}} = (x^\mu, z)$. We will use the following choice for the decomposition of $\hat{D}$-dimensional gravity into D-dimensional fields:

$$\hat{ds}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2 \,, \tag{4.4}$$

i.e. gravity decomposes into a lower-dimensional gravity plus a vector A_μ and a scalar ϕ. The constants α and β are in principle arbitrary. This Ansatz gives rise to a lower-dimensional theory with the Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2\,2!}e^{2(\beta-\alpha)\phi}F^2] \,, \tag{4.5}$$

with $F = dA$. Here we have chosen the constants to the values

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \,, \qquad \beta = -(D-2)\alpha \,, \tag{4.6}$$

to obtain the lower-dimensional Lagrangian in the conventional form (4.5), i.e. without dilaton coupling for the Ricci scalar and with the factor $\frac{1}{2}$ in the dilaton kinetic term. Note that this a system of the form that was considered in section 3.4 on brane solutions[3], with parameter Δ as defined in (3.34) equal to 4 for all dimensions D.

The appearance of the Maxwell kinetic term was the reason for Kaluza [12] and Klein [13] to consider such dimensional reductions: it seemed possible to unify gravity and electromagnetism in 4D by the introduction of a fifth coordinate. Note however that there is also an extra scalar, which can not be simply set equal to zero: this would be inconsistent with the higher-dimensional field equations. Often these extra fields are called the Kaluza-Klein scalar and vector. Also, the general procedure of obtaining a lower-dimensional description from a higher-dimensional theory is sometimes called Kaluza-Klein theory. We will not use this terminology, however, since we need to make a distinction between the different possibilities within Kaluza-Klein theory.

One can understand the lower-dimensional symmetries of the Lagrangian (4.5) by considering its higher-dimensional origin. In particular, the $\hat{D}$-dimensional Einstein-Hilbert action is invariant under general coordinate transformations

$$\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}} \,, \qquad \Rightarrow \delta\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} + \hat{g}_{\hat{\rho}\hat{\nu}}\partial_{\hat{\mu}}\hat{\xi}^{\hat{\rho}} + \hat{g}_{\hat{\mu}\hat{\rho}}\partial_{\hat{\nu}}\hat{\xi}^{\hat{\rho}} \,. \tag{4.7}$$

In general, such a transformation will not preserve the form of the reduction Ansatz (4.4), i.e. the resulting metric will not be expressible as (4.4) with transformed fields. The Ansatz

[3] The corresponding electric and magnetic brane solutions will uplift to the gravitational wave and Kaluza-Klein monopole in $\hat{D}$ dimensions, respectively, as also seen in figure 3.2.

will only transform covariantly under transformations with specific parameters. Such Ansatz-preserving transformations and their effect on the lower-dimensional fields are the following:

$$\begin{aligned}
\delta x^\mu &= -\xi^\mu(x)\,, &&\Rightarrow \begin{cases} \delta g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho\,, \\ \delta A_\mu = \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho\,, \\ \delta\phi = \xi^\rho \partial_\rho \phi\,, \end{cases} \\
\delta z &= -\lambda(x)\,, &&\Rightarrow \begin{cases} \delta A_\mu = \partial_\mu \lambda\,, \end{cases} \\
\delta z &= -cz\,, &&\Rightarrow \begin{cases} \delta g_{\mu\nu} = -2\alpha c g_{\mu\nu}/\beta\,, \\ \delta A_\mu = -cA_\mu\,, \\ \delta\phi = c/\beta\,. \end{cases}
\end{aligned} \tag{4.8}$$

These can respectively be understood as D-dimensional general coordinate transformations, $U(1)$ gauge transformations and a global scale symmetry.

The latter can be integrated to give a finite, rather than infinitesimal, transformation. In addition one has the higher-dimensional trombone symmetry $\hat{g}_{\hat{\mu}\hat{\nu}} \to \lambda^2 \hat{g}_{\hat{\mu}\hat{\nu}}$, which also reduces to a finite scale symmetry of the lower-dimensional theory. One can construct linear combinations of these symmetries to obtain the following transformations

$$g_{\mu\nu} \to {\lambda_1}^2 g_{\mu\nu}\,, \qquad A_\mu \to \lambda_1 A_\mu\,, \tag{4.9}$$

where $\lambda_1 \in \mathbb{R}^+$. This is the lower-dimensional trombone symmetry (with coefficients as explained in section 3.2), which scales all terms in the Lagrangian with the same factor, and is only a symmetry of the field equations. The other combination reads

$$A_\mu \to {\lambda_2}^{\alpha-\beta} A_\mu\,, \qquad e^\phi \to \lambda_2 e^\phi\,, \tag{4.10}$$

also with $\lambda_2 \in \mathbb{R}^+$. This corresponds to the only scale symmetry of the Lagrangian. Indeed, this explains the two $\mathbb{R}^+$ symmetries of IIA supergravity: they stem from combinations of the 11D trombone symmetry and internal coordinate transformations.

Gravity on a Torus

The reduction of gravity over a torus T^n can be seen as successive reductions over n circles. The reduction Ansatz of $\hat{D}$-dimensional gravity over an n-torus to $D = \hat{D} - n$ dimensions reads (with a coordinate split $x^{\hat{\mu}} = (x^\mu, z^m)$ where $m = 1, \ldots, n$)

$$\hat{ds}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} M_{mn}(dz^m + A^m_\mu dx^\mu)(dz^n + A^n_\mu dx^\mu)\,. \tag{4.11}$$

The lower-dimensional field strength is a generalisation of the result of a torus reduction: in addition to gravity one finds n vectors A^m_μ, a dilaton ϕ and a scalar matrix M_{mn} which parameterises a coset $SL(n,\mathbb{R})/SO(n)$ (see section 3.3 for scalar cosets). The latter corresponds to $n-1$ dilatons and $\frac{1}{2}n(n-1)$ axions. Again, one can obtain the lower-dimensional Lagrangian by a reduction of the Einstein Hilbert term:

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{4}\mathrm{Tr}(\partial M \partial M^{-1}) - \tfrac{1}{2\,2!}e^{2(\beta-\alpha)\phi}M_{mn}F^mF^n]\,, \tag{4.12}$$

with $F^m = dA^m$. The convenient values for α and β now read

$$\alpha^2 = \frac{n}{2(D+n-2)(D-2)}\,, \qquad \beta = -\frac{(D-2)\alpha}{n}\,, \tag{4.13}$$

yielding the Lagrangian in the conventional form (4.12).

As in the reduction over the circle, one can wonder which general coordinate transformations (4.7) preserve the form of the reduction Ansatz and induce a lower-dimensional transformation. For the torus reduction (4.11) these turn out to be

$$\hat{\xi}^\mu = \xi^\mu(x)\,, \qquad \hat{\xi}^m = \lambda^m(x) + \Lambda^m{}_n z^n\,. \tag{4.14}$$

These can respectively be understood as D-dimensional general coordinate transformations, $U(1)^n$ gauge transformations and a global $GL(n,\mathbb{R})$ symmetry. As in the torus case, the global transformations can be integrated to finite transformations, where it is convenient to use a split into $SL(n,\mathbb{R})$ and $\mathbb{R}^+$. The former acts in the obvious way on the $SL(n,\mathbb{R})$ indices while the latter can again be combined with the reduced trombone symmetry to yield the lower-dimensional trombone symmetry and the dilaton scale symmetry (formulae (4.9) and (4.10) with an extra m index for A_μ). Thus, in comparison with the circle case, the new features of the n-torus reduction are the n Abelian gauge symmetries and the global $SL(n,\mathbb{R})$ symmetry.

Inclusion of Gauge Potentials

We will now consider the reduction of a gauge potential of rank d over a circle. The dynamics of the higher-dimensional potential $\hat{C}^{(d)}$, coupled to gravity and possibly a dilaton $\hat{\varphi}$, is determined by

$$\mathcal{L} = \sqrt{-\hat{g}}[-\tfrac{1}{2}(\partial\hat{\varphi})^2 - \tfrac{1}{2}e^{a\hat{\varphi}}\hat{G}^{(d+1)}\cdot\hat{G}^{(d+1)}]\,. \tag{4.15}$$

with $\hat{G}^{(d+1)} = d\hat{C}^{(d)}$, where we have included the dilaton kinetic term. The parameter a characterises the dilaton coupling. For gravity we will take the reduction Ansatz (4.4) while the rest of the reduction Ansatz reads

$$\hat{C}^{(d)} = C^{(d)} + (dz + A)\wedge C^{(d-1)}\,, \qquad \hat{\varphi} = \varphi\,. \tag{4.16}$$

where A is the Kaluza-Klein vector field of the gravity Ansatz (4.4). The resulting Lagrangian is described by

$$\mathcal{L} = \sqrt{-g}[-\tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}e^{a\varphi-2d\alpha\phi}G^{(d+1)}\cdot G^{(d+1)} - \tfrac{1}{2}e^{a\varphi+2(D-d-1)\alpha\phi}G^{(d)}\cdot G^{(d)}]\,, \tag{4.17}$$

with field strengths $G^{(d+1)} = dC^{(d)} + F{\wedge}C^{(d-1)}$ and $G^{(d)} = dC^{(d-1)}$. Note that Δ, defined in (3.34), is preserved under the operation of dimensional reduction; the value associated to $\hat{G}^{(d+1)}$ is also found for both $G^{(d+1)}$ and $G^{(d)}$:

$$\begin{aligned}\Delta &= a^2 + \frac{2d(\hat{D}-d-2)}{\hat{D}-2}\,,\\ &= a^2 + (2d\alpha)^2 + \frac{2d(D-d-2)}{D-2}\,,\\ &= a^2 + (2(D-d-1)\alpha)^2 + \frac{2(d-1)(D-d-1)}{D-2}\,.\end{aligned} \tag{4.18}$$

Indeed, this corresponds to the statement from section 3.4 that the parameter Δ is invariant under toroidal reduction.

The reduction of a d-form gauge potential over a circle can be performed a number of times. This corresponds to the reduction over a torus. We will not discuss the explicit Ansatz here since it follows from (4.16) but clearly there are general formulae for the reduction of a gauge potential over a torus, similar to (4.11). However, it is useful to know the resulting field content. From subsequent applications of (4.16) it can be seen that the reduction of a d-form over an n-torus gives rise to an amount of

$$\left(\begin{array}{c} n \\ d-\tilde{d} \end{array}\right)\,, \qquad \text{where } \; d-n \leq \tilde{d} \leq d\,, \tag{4.19}$$

forms of rank $\tilde{d}$. For example, reduction of a 2-form over a 2-torus gives rise to a 2-form, two vectors and a scalar.

Upon reduction over a torus, the gauge symmetry $\delta\hat{C}^{(d)} = d\hat{\lambda}^{(d-1)}$ splits up in different lower-dimensional gauge transformations, corresponding to the different $\tilde{d}$-form potentials. In the case of a circle, for example, the gauge transformations that act covariantly on the lower-dimensional potentials are

$$\hat{\lambda}^{(d-1)} = \lambda^{(d-1)} + (dz + A){\wedge}\lambda^{(d-2)}\,, \tag{4.20}$$

where A is the Kaluza-Klein vector. The gauge parameters $\lambda^{(d-1)}$ and $\lambda^{(d-2)}$ correspond to the potentials $C^{(d)}$ and $C^{(d-1)}$, respectively.

In addition, the higher-dimensional Lagrangian is of course invariant under the general coordinate transformations (4.7). As in the case of gravity, the reduction Ansatz for gauge potentials over a torus is only covariant for the restricted transformations (4.14). The lower-dimensional potentials transform in the usual way under the lower-dimensional coordinate transformations and they can also be assigned a weight under the global scale symmetries. Moreover, the $\tilde{d}$-form potentials, the number of which is given by (4.19), form linear representations of the global $SL(n, \mathbb{R})$ symmetry.

Global Symmetry Enhancement

However, this is not the full story of gravity and gauge potentials on tori. It turns out that, in special cases, one obtains a larger symmetry group than the $SL(n, \mathbb{R})$ whose appearance was guaranteed by the higher-dimensional coordinate transformations. An obvious example is provided by the Lagrangian (4.5), which is the reduction of the Einstein-Hilbert action over a circle. Reduction of (4.5) over an n-torus will lead to the global symmetry $SL(n{+}1, \mathbb{R})$ rather than $SL(n, \mathbb{R})$. In this case one can understand the symmetry enhancement by the higher-dimensional origin of (4.5). However, there are also examples where such an explanation is not available.

As an example, consider the bosonic string, whose low-energy limit was given in (2.11). It consists of gravity, a dilaton and a rank-two gauge potential. After appropriate field redefinitions, the action takes the canonical form (3.33) of the gravity-dilaton-potential system of section 3.4, with a dilaton coupling corresponding to $\Delta = 4$. Upon reduction over an n-torus, it turns out that the global symmetry group is enhanced from $SL(n, \mathbb{R})$ to $SO(n, n)$, see e.g. [66]. In addition, the scalar coset is enhanced as well:

$$\frac{SL(n, \mathbb{R})}{SO(n)} \qquad \Rightarrow \qquad \frac{SO(n, n)}{SO(n) \times SO(n)} \,. \tag{4.21}$$

For this to be possible, there is a conspiracy between the scalars coming from the metric (giving rise to the smaller coset) and those coming from the two-form, together giving rise to the larger coset.

Another example is provided by the reduction of (the bosonic sector of) eleven-dimensional supergravity, whose symmetry groups are given in table 3.4. Again, the symmetry groups and scalar cosets are larger than the naive $SL(n, \mathbb{R})$. In this case this requires a collaboration between the scalars coming from the metric and those coming from the three-from gauge potential. Although often appearing in the low-energy limits of string or M-theory, it should be stressed that such symmetry enhancement is a miraculous phenomenon and strongly dependent on the details of interactions.

Fermionic Sector

If one wants to dimensionally reduce a supergravity theory, clearly a recipe is required for the fermionic sector. Since this is rather strongly dependent on the dimensions of the higher- and lower-dimensional theories, we will not present explicit formulae but only discuss the conceptual aspects. In the explicit reduction of supergravities that we will perform later, such explicit formulae are given while in this chapter however, we will mainly consider the bosonic part. For more detail see e.g. [81, 120].

The essential idea in fermionic dimensional reduction is to split up the spinors as a tensor product of spinors in the lower-dimensional space and the internal space. For toroidal reduction, the internal spinors are taken constant. Thus, the reduction Ansatz for a dilatino sketchily reads

$$\hat{\lambda} = \sum_i \lambda^i \otimes \eta^i \,, \tag{4.22}$$

where λ^i are the lower-dimensional spinors and η^i the internal spinors. The range of i is equal to the number of independent spin-$1/2$ components on the internal manifold and therefore strongly depends on $\hat{D} - D$. This range corresponds to the quotient of the degrees of freedom of the minimal spinors in the higher- and lower-dimensional theory. For example, reducing over a seven-torus, the 32-component minimal spinor $\hat{\lambda}$ splits up in 4-component minimal spinors λ^i and therefore i ranges from 1 to 8. This corresponds for example to the reduction of $N = 1$ supergravity in 11D to $N = 8$ supergravity in 4D over the seven-torus, which indeed allows for eight constant internal spinors.

In the case of spin-$3/2$ fermions, i.e. if the fermions are carrying a space-time index as well, the procedure is a combination of the bosonic and fermionic Ansätze. Both spinorial and space-time indices are split up into the lower-dimensional ranges:

$$\hat{\psi}_\mu = \sum_i \psi^i_\mu \otimes \eta^i \,, \qquad \hat{\psi}_m = \sum_j \lambda^j \otimes \eta^j_m \,, \tag{4.23}$$

where η^i and η^j_m are constant fermions on the internal space of spin 1/2 and 3/2, respectively. Thus the resulting fermions are the gravitini ψ^i_μ and the dilatini λ^j.

We will indicate the changes in the fermionic Ansätze in the upcoming cases of twisted reduction and reductions over group manifolds.

4.3 Reduction with a Twist

We will now discuss a generalisation of toroidal reduction, leading to a different lower-dimensional description including e.g. a scalar potential. This generalisation is possible whenever the higher-dimensional theory contains a global symmetry [38].

Boundary Conditions and Twisted Expansions

In section 4.1 we considered the expansion of a complex scalar field over an internal dimension under the assumption $\hat{\phi}(x, 2\pi R) = \hat{\phi}(x, 0)$, i.e. a periodic boundary condition. One can also impose the generalised boundary condition

$$\hat{\phi}(x, 2\pi R) = e^{2\pi i m R}\hat{\phi}(x, 0)\,, \tag{4.24}$$

for some constant m. We will call this the twisted boundary condition, giving rise to reduction with a twist. It leads to the expansion

$$\hat{\phi}(x, z) = \sum_n e^{i(m+n/R)z}\phi_n(x)\,, \tag{4.25}$$

with a discrete spectrum of fields ϕ_n. Note that this twisted expansion is invariant under the transformation

$$m \to m + 1/R\,, \qquad \phi_n \to \phi_{n+1}\,. \tag{4.26}$$

For this reason one can always take $|m| \leq \frac{1}{2}/R$ without loss of generality. Substitution into the Klein-Gordon equation yields

$$\Box\phi_n - (m + n/R)^2\phi_n = 0\,. \tag{4.27}$$

Again, the higher-dimensional equation decouples into separate equations for all components ϕ_n of (mass)^2 $(m + n/R)^2$.

Again, we would like to truncate to the sector with the lowest mass; to which component ϕ_n this corresponds to is determined by the parameter m. Adhering to the above convention of taking $|m| \leq \frac{1}{2}/R$, the lowest sector corresponds to the component ϕ_0, as in the massless case. Note however that the lower-dimensional description is different; the periodic boundary condition gave rise to a massless scalar while the twisted boundary condition leads to a scalar of (mass)^2 m^2. However, both reductions are consistent: the field equations for ϕ_n with $n \neq 0$ are satisfied and, equivalently, the dependence on the internal coordinate z has dropped out.

Note that one can take $m = n/R$, leaving the above convention, and truncate consistently to the component ϕ_0. However, this does not correspond to the lightest mode. Indeed, due to the above symmetry (4.26), this corresponds to a toroidal reduction with expansion (4.2) and subsequent truncation to a heavier mode. The ambiguity in the lower-dimensional description (i.e. a massless or massive scalar) stems from the possibility to consistently truncate the Kaluza-Klein tower (4.2) in infinitely many ways.

Global Symmetries and Monodromy

One can extend the generalised boundary condition (4.24) for $U(1)$ to other groups if the theory is invariant under a global symmetry group G. Consider a set of fields $\hat{\phi}$, which we take to be scalars for concreteness but the discussion can easily be extended to other fields. The fields $\hat{\phi}$ are taken to transform linearly under a global transformation: $\hat{\phi} \to g\hat{\phi}$ with

$g \in G$, where we suppress group indices. This allows us to impose a more general twisted boundary condition:

$$\hat{\phi}(x, 2\pi R) = \mathcal{M}(g)\hat{\phi}(x, 0)\,. \tag{4.28}$$

Upon traversing the circle, the fields come back to themselves up to a symmetry transformation: this transformation is called the monodromy. This boundary condition leads to the twisted reduction Ansatz (i.e. expansion and truncation to the lightest modes)

$$\hat{\phi}(x, z) = g(z)\phi(x)\,, \qquad \Rightarrow \qquad \mathcal{M}(g) = g(z = 2\pi R)g(z = 0)^{-1}\,, \tag{4.29}$$

with an element $g(z) \in G$ which depends on z. This is the generalisation to arbitrary groups G of the $U(1)$ twisted Ansatz (4.25) with $\phi_{n\neq 0} = 0$. For general groups G, the element $g(z)$ has to satisfy a consistency criterium: the combination

$$C = g(z)^{-1}\partial_z g(z)\,. \tag{4.30}$$

must be a constant, which is required by the cancellation of the z-coordinate in the lower-dimensional field equations and thus ensures consistency of the truncation to the lightest modes ϕ. Clearly, it can be solved by the z-dependence

$$g(z) = \exp(Cz)\,, \qquad \text{with } \mathcal{M} = \exp(2\pi RC)\,. \tag{4.31}$$

Thus the constants C constitute an element of the Lie algebra of G. It determines which linear combination of the generators of G is employed in the twisted reduction.

This reduction Ansatz brings one from the higher-dimensional massless Klein-Gordon equations to lower-dimensional massive Klein-Gordon equations:

$$\hat{\Box}\hat{\phi} = 0 \qquad \Rightarrow \qquad \Box\phi + C^2\phi = 0\,. \tag{4.32}$$

For this reason, the matrix C is usually called the mass matrix. The eigenvalues of C^2 are related to the (masses)^2 of the fields ϕ: negative eigenvalues correspond to positive (masses)^2 and vice versa. This depends on the compactness of the subgroups of G generated by C.

Note that the symmetry G is generically broken upon twisted reduction: elements of G do not preserve the field equations but rather transform the mass matrix by

$$C \to g^{-1}Cg\,. \tag{4.33}$$

Only transformations for which the two mass matrices C and $g^{-1}Cg$ are equal preserve the lower-dimensional field equations. This is in general only met by group elements of the form as employed in the twisted reduction, i.e. of the form $\exp(\lambda C)$. Note that G is always preserved for $C = 0$, i.e. under toroidal reduction.

A special case consists of a mass matrix $C \neq 0$ that gives rise to a trivial monodromy $\mathcal{M} = \mathbb{I}$. This is possible when G contains a compact subgroup and is the equivalent of the choice $m = 1/R$ considered in the previous subsection: it corresponds to an expansion without twist (yielding trivial monodromy) which is truncated to a massive mode (giving rise to the mass matrix), rather than the massless mode [121]. This situation will be encountered in section 5.3.

In this toy example, the group G plays a central role. If G is not a symmetry of the theory, the reduction will not be consistent: one will (generally) not find cancellation of all z-dependence in the lower-dimensional field equations. Thus, the existence of G allows for the twisted reduction Ansatz, as was first recognised by Scherk and Schwarz[4] [38].

Another important point is the fact that G is only a global symmetry in the higher-dimensional theory. It is impossible to perform twisted reductions of this kind with local symmetries, as can easily be seen from our toy example. Suppose that the higher-dimensional theory had a local symmetry G. Then the group element $g(z)$ in the reduction Ansatz $\hat{\phi}(x,z) = g(z)\phi(x)$ can be brought to the left-hand side, where it acts on $\hat{\phi}$. But this is just a symmetry transformation, which leaves the higher-dimensional theory invariant. Thus the reduction Ansätze $\hat{\phi}(x,z) = \phi(x)$ and $\hat{\phi}(x,z) = g(z)\phi(x)$ will give the same result, a massless lower-dimensional theory, if $g \in G$ is a local symmetry acting on $\hat{\phi}$.

Gravity and Gaugings

We would like to apply our twisted reductions to supergravity in chapter 5. For this reason it is imperative to include gravity, which will bring in a number of new features.

A useful subsector of supergravities to consider consists of only gravity and the scalars. As in all maximal supergravities, the scalars parameterise a coset G/H, denoted by M. Examples of G and H are given in table 3.4. The Lagrangian reads

$$\hat{\mathcal{L}} = \sqrt{-\hat{g}}[\hat{R} + \tfrac{1}{4}\mathrm{Tr}(\partial\hat{M}\partial\hat{M}^{-1})] \,. \tag{4.34}$$

Toroidal reduction of this theory would correspond to the reduction Ansatz (in the case of a circle)

$$\hat{ds}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2 \,, \qquad \hat{M} = M \,, \tag{4.35}$$

with the constants α and β given in 4.6. However, this theory has a global symmetry[5], which acts as $M \to \Omega M \Omega^T$ with $\Omega \in G$. Therefore it also allows for a twisted reduction, parameterised by a mass matrix C of the Lie algebra of G. The corresponding reduction Ansatz reads

$$\hat{ds}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2 \,, \qquad \hat{M} = U(z)MU(z)^T \,, \tag{4.36}$$

[4] Their motivation was the spontaneous breaking of supersymmetry.

[5] We restrict to symmetries of the action here. In the case of symmetries that scale the action, e.g. trombone symmetries, there is a subtlety that is addressed in section 4.6.

for an element $U(z) = \exp(Cz) \in G$. The resulting lower-dimensional Lagrangian is given by

$$\mathcal{L} = \sqrt{-g}\,[R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{4}\mathrm{Tr}(DMDM^{-1}) - \tfrac{1}{2\,2!}e^{2(\beta-\alpha)\phi}F^2 - V]\,, \tag{4.37}$$

where we have defined

$$DM = dM + (CM + MC^T)A\,, \qquad V = \tfrac{1}{2}e^{2(\alpha-\beta)\phi}\mathrm{Tr}[C^2 + C^T M^{-1} CM]\,, \tag{4.38}$$

where DM and V are the scalar field strength and the scalar potential, respectively. These contain the deformations in terms of the mass matrix C.

In the previous discussion we have found the fate of the symmetry G under twisted reduction. Only a one-dimensional subgroup (with generator C) was preserved while the remaining transformations were broken. When including gravity, it is interesting to consider the action of the general coordinate transformations on the fields in the twisted reduction. Recall the decomposition (4.14) of the higher-dimensional coordinate transformations $\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}}$ into lower-dimensional coordinate transformations, a $U(1)$ gauge symmetry and a global symmetry. The first of these transformations is unchanged, i.e. also the lower-dimensional theory has diffeomorphism invariance. The latter two are modified due to the twist, however.

The $U(1)$ factor corresponds to the parameter $\hat{\xi}^z = \lambda(x)$. Note that the scalar reduction Ansatz (4.36) is not invariant under this coordinate transformation:

$$\hat{M} = U(z)MU(z)^T \qquad \rightarrow \qquad \hat{M} = U(z-\lambda)MU(z-\lambda)^T\,. \tag{4.39}$$

Using $U(z) = \exp(Cz) \in G$, an internal coordinate transformation corresponds to the lower-dimensional transformation

$$M \rightarrow \exp(-C\lambda)M\exp(-C^T\lambda)\,, \qquad A_\mu \rightarrow A_\mu + \partial_\mu\lambda\,. \tag{4.40}$$

Indeed, the scalar field strength transforms covariantly under this local transformation. Thus it turns out that the one-dimensional subgroup of G generated by C is in fact gauged. This means that the global parameter of this transformation is elevated to a local one. For this reason we say that twisted reduction leads to a non-trivial gauging in the lower-dimensional theory.

The remaining parameter of the higher-dimensional diffeomorphisms, the constant c in the decomposition (4.14), acts as

$$\hat{M} = U(z)MU(z)^T \qquad \rightarrow \qquad \hat{M} = U(z-cz)MU(z-cz)^T\,. \tag{4.41}$$

However, unlike the local $U(1)$ action (4.39), this can not be interpreted as a lower-dimensional (i.e. z-independent) transformation on M. For this reason the extra scale symmetry is broken by the mass parameters C. Another way to see this stems from the scale weight of the scalar potential under the global symmetry with parameter c given in (4.14). It is easily seen that the kinetic terms scale differently than the scalar potential, which therefore breaks this symmetry.

In addition to the Ansätze for gravity and scalars presented in this section, one can construct similar formulae for the twisted reduction of e.g. gauge potentials and fermions. The guiding principle is the global symmetry: one modifies the toroidal Ansätze by inserting the transformation $U(z)$ in the appropriate places, while the consistency of such reductions is guaranteed by the global symmetry G. We will perform twisted reductions of supergravities in section 5.3.

Enhanced Gaugings

However, one feature of enlarged field contents is noteworthy. In special cases, the existence of extra gauge potentials in twisted reduction gives rise to an enhancement of the gauging. This means that, in addition to the gauging of the twisted symmetry, one finds other symmetries that have been elevated to local ones in the gauged theory. Clearly, for this to be possible, one needs the global part of these symmetries to be present in the ungauged theory. An additional requirement is the presence of the corresponding gauge vectors, which are necessary to gauge the extra symmetries.

Rather than the most general possibility we will consider a specific example, which will be important in section 5.3. In addition to gravity and the scalar coset $\hat{M}_{mn}$ of the previous subsection, we include a gauge vector $\hat{V}$. The twist symmetry that we employ scales this gauge vector with a certain weight α, i.e. $\hat{V} \to \Omega^\alpha \hat{V}$ with $\Omega \in \mathbb{R}^+$. In addition, we have the gauge transformation $\delta\hat{V} = d\hat{\lambda}$.

As explained in the previous subsection, the transformation under the twist symmetry determines the internal dependence of the reduction Ansatz, which therefore reads

$$\hat{V} = U^\alpha (V + \chi(dz + A)) \,, \qquad \hat{\lambda} = U^\alpha \lambda \tag{4.42}$$

where $\hat{V}$ splits up in a vector V and an axion χ, while the vector A comes from the metric. We have also included the reduction Ansatz for the gauge parameter $\hat{\lambda}$. The internal dependence is inserted via the $\mathbb{R}^+$ group element $U = \exp(mz)$.

Note that in the lower-dimensional theories there are two vectors: the Kaluza-Klein vector A coming from the metric and the vector V coming from the higher-dimensional vector. We will call the gauge parameters λ_A and λ_V, respectively. Their action on the axion χ reads

$$\delta_A \chi = m\lambda_A \chi \,, \qquad \delta_V \chi = m\lambda_V \,. \tag{4.43}$$

Thus one mass parameter yields two independent local transformations: we find gauge symmetry enhancement. In fact, in this case the two gaugings are non-Abelian, since

$$[\delta_A, \delta_V] = m^2 \lambda_A \lambda_V \,. \tag{4.44}$$

These form the unique two-dimensional non-Abelian group, which we will denote by $A(1)$.

Though a general proof on the appearance of enhanced gaugings is lacking, the above example seems to be typical for this phenomenon. The generic rule, applicable throughout this thesis, is that any higher-dimensional gauge vector that transforms under the twist symmetry will give rise to an extra gauging upon reduction. We will encounter different examples of enhanced gaugings in section 5.3 and 5.4, including the two-dimensional group $A(1)$.

4.4 Reduction over a Group Manifold

In the previous section we have seen how twisted reduction employs the global symmetries of the higher-dimensional theory. In this section we will focus on the global symmetries of the internal space instead, leading to group manifolds as internal spaces. For this reason, the corresponding reduction procedure is only possible for theories which include gravity.

Group Manifolds

A group manifold G with coordinates z^m consists of group elements $g = g(z^m) \in G$ (omitting group indices): points on the manifold correspond to elements of the group and the dimension n of the manifold equals dim(G). Group multiplication, e.g. $g \to \Lambda_L g$ or $g \to g\Lambda_R$, corresponds to a coordinate transformation. Both left and right multiplications correspond to transitively acting coordinate transformations[6] due to the group structure.

However, these coordinate transformations are not necessarily isometries of the metric. To ensure that left multiplication gives rise to an isometry of the metric, we make the choice

$$ds_G^2 = g_{mn}\sigma^m\sigma^n \,, \qquad T_m\sigma^m = g^{-1}dg \,, \tag{4.45}$$

with g_{mn} arbitrary, T_m generators and $g = g(z^m)$ elements of the group G. The combinations σ^m are called the Maurer-Cartan one-forms and can be written as $\sigma^m = U^m{}_n dz^n$ with $U^m{}_n = U^m{}_n(z^p)$. Since left multiplication $g \to \Lambda_L g$ leaves σ^m invariant it is an isometry of the metric, which is therefore called the left-invariant metric. Note that the group manifold with metric (4.45) is homogeneous[7] for all values of g_{mn} due to the transitively acting isometries of left multiplication. These isometries are generated by the Killing vectors L_n, which by definition satisfy the Maurer-Cartan equations

$$[L_m, L_n] = f_{mn}{}^p L_p \,, \tag{4.46}$$

where the $f_{mn}{}^p$ are given by

$$f_{mn}{}^p = -2(U^{-1})^r{}_m(U^{-1})^s{}_n\,\partial_{[r}U^p{}_{s]} \,. \tag{4.47}$$

Due to Lie's second theorem, these are always independent of z^m and indeed are the structure constants of the group G. Thus a group manifold with metric (4.45) has n transitively acting

[6] Coordinate transformations are said to act transitively if they relate all points on the manifold.

[7] We call a manifold homogeneous if its metric allows for transitively acting isometries.

isometries that span the group G. Explicit examples of such Killing vectors are given in section 5.4.

With the choice of metric (4.45), right multiplication does not give rise to an isometry for general g_{mn}: the transformation $g \to g\Lambda_R$ is an isometry of the metric (4.45) if and only if g_{mn} is given by the Cartan-Killing metric of the group G. Such a particular metric is referred to as the bi-invariant metric since its isometry group is $G_L \times G_R$.

Gravity on a Group Manifold

To see how such group manifolds arise in reductions, we start out with the Ansatz for toroidal reduction

$$\hat{ds}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} M_{mn}(dz^m + A^m_\mu dx^\mu)(dz^n + A^n_\mu dx^\mu)\,, \tag{4.48}$$

with α and β given in (4.13). As noted before, this reduction Ansatz transforms covariantly under general coordinate transformations of the special form

$$\hat{\xi}^\mu = \xi^\mu(x)\,, \qquad \hat{\xi}^m = \lambda^m(x) + \Lambda^m{}_n z^n\,. \tag{4.49}$$

The latter term corresponds to $GL(n,\mathbb{R})$ transformations on the internal coordinates z^m. These will reduce to global symmetries of the lower-dimensional theory.

As is the case of global symmetries of the higher-dimensional theories, these internal transformations can also be used for a generalised reduction procedure [39]. In complete analogy to the twisted reduction, one can take the toroidal reduction Ansatz and perform a $GL(n,\mathbb{R})$ transformation on the lower-dimensional fields, whose parameter we call U. If this is a constant parameter, the lower-dimensional theory is clearly unchanged due to its global symmetry. However, we allow for a certain internal coordinate dependence of the transformation parameter: $U^m{}_n = U^m{}_n(z^p)$. Thus, for reduction of gravity, the Ansatz can be obtained by applying $U^m{}_n$ transformations on all fields in the toroidal Ansatz (4.48) and reads

$$\begin{aligned}\hat{ds}^2 &= e^{2\alpha\phi} ds^2 + e^{2\beta\phi} U^m{}_p U^q{}_n M_{pq}(dz^m + (U^{-1})^m{}_r A^r_\mu dx^\mu)(dz^n + (U^{-1})^n{}_s A^s_\mu dx^\mu)\,,\\ &= e^{2\alpha\phi} ds^2 + e^{2\beta\phi} M_{mn}(\sigma^m + A^m_\mu dx^\mu)(\sigma^n + A^n_\mu dx^\mu)\,,\end{aligned} \tag{4.50}$$

with $\sigma^m = U^m{}_n dz^n$. Thus the internal part of this metric, given by

$$ds^2_G = e^{2\beta\phi} M_{mn}\sigma^m\sigma^n\,, \tag{4.51}$$

corresponds to the left-invariant metric of a group manifold. Therefore this reduction procedure corresponds to the reduction over a group manifold G, where one uses the left-invariant metric on the group manifold [39, 122, 123].

Upon reduction of the Einstein-Hilbert term, the $GL(n,\mathbb{R})$ transformation will cancel in many places, due to the fact that it is a global symmetry of the lower-dimensional theory.

Only when the parameters $U^m{}_n$ run into internal derivatives, the cancellation of such terms is no longer guaranteed. However, it turns out that the only combination of $U^m{}_n$'s that survives upon reduction is exactly the combination $f_{mn}{}^p$ of (4.47). Therefore, to obtain a lower-dimensional theory without z^m dependence, one has to require that the combinations $f_{mn}{}^p$ are z^m independent. As we have seen in the previous subsection, this is guaranteed if one takes the internal dependence of $U^m{}_n$ such that

$$T_m U^m{}_n dz^n = g^{-1} dg \,, \tag{4.52}$$

for group elements $g = g(z^m)$.

Explicitly, the lower-dimensional result of the reduction of the higher-dimensional action with Einstein-Hilbert term is given by[8]

$$\mathcal{L} = \sqrt{-g}\left[R + \tfrac{1}{4}\mathrm{Tr}(D\mathcal{M}D\mathcal{M}^{-1}) - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{2(\alpha-\beta)\phi}F^m\mathcal{M}_{mn}F^n - V\right], \tag{4.53}$$

where the field strengths are given by

$$F^m = 2\partial A^m - f_{np}{}^m A^n A^p \,, \qquad D\mathcal{M}_{mn} = \partial\mathcal{M}_{mn} + 2 f_{q(m}{}^p A^q \mathcal{M}_{n)p} \,. \tag{4.54}$$

In addition, one has a scalar potential

$$V = \tfrac{1}{4}e^{2(\beta-\alpha)\phi}\left[2\mathcal{M}^{nq} f_{mn}{}^p f_{pq}{}^m + \mathcal{M}^{mq}\mathcal{M}^{nr}\mathcal{M}_{ps} f_{mn}{}^p f_{qr}{}^s\right]. \tag{4.55}$$

Thus we find two differences when compared with toroidal reduction: the modification of field strengths and the appearance of a scalar potential. These deformations of the massless theory are linear and quadratic in the structure constants, respectively.

Gaugings from Group Manifolds

Again, it is natural to wonder about the lower-dimensional symmetries. The higher-dimensional coordinate transformations that act covariantly on the reduction Ansatz are

$$\hat{\xi}^\mu = \xi^\mu(x) \,, \qquad \hat{\xi}^m = U^m{}_n \lambda^n(x) \,, \tag{4.56}$$

consisting of lower-dimensional coordinate transformations with parameter $\xi^\mu(x)$ and gauge transformations with parameter $\lambda^n(x)$. The effect of the latter on the lower-dimensional fields is given by

$$\delta A^m_\mu = \partial_\mu \lambda^m + f_{np}{}^m \lambda^n A^p_\mu \,, \qquad \delta M_{mn} = f_{mp}{}^q \lambda^p M_{qn} + f_{np}{}^q \lambda^p M_{mq} \,, \tag{4.57}$$

while the metric is invariant. These are non-Abelian gauge transformations with gauge vectors A^m_μ and structure constants $f_{mn}{}^p$.

[8]Here we restrict to unimodular groups, having structure constants with vanishing trace: $f_{mn}{}^n = 0$. For non-unimodular groups there is a number of complications which will be addressed in section 4.6.

As in the twisted reduction, the global symmetry employed in the reduction is generically broken for the larger part. In the group manifold case, this symmetry is $GL(n,\mathbb{R})$ and comes from the internal coordinate transformations with $\hat{\xi}^m = \Lambda^m{}_n z^n$. In the gauged theory, the $GL(n,\mathbb{R})$ is in general no longer a symmetry since it does not preserve the structure constants. The unbroken part is exactly given by the automorphism group of the structure constants, i.e. the transformations satisfying

$$f_{mn}{}^p = \Lambda_m{}^q \Lambda_n{}^r (\Lambda^{-1})_s{}^p f_{qr}{}^s \, . \tag{4.58}$$

Of course it always includes the gauge group, which is embedded in the global symmetry group $GL(n,\mathbb{R})$ via

$$\Lambda_n{}^m = e^{\lambda^k f_{kn}{}^m} \, , \tag{4.59}$$

where λ^k are the local parameters of the gauge transformations. However, the full automorphism group can be bigger; for instance, its dimension is n^2 in case of $f_{mn}{}^p = 0$. Of course this amounts to the fact that the ungauged theory has a $GL(n,\mathbb{R})$ symmetry. All other cases have Dim(Aut) $< n^2$ and thus break the $GL(n,\mathbb{R})$ symmetry to some extent.

Thus reduction over a group manifold leads to a gauging, where the adjoint representation of the gauge group is embedded in the fundamental representation of the global symmetry group (4.59). Therefore, reduction over a torus T^n leads to a theory without gauging, since the adjoint of $U(1)^n$ is trivial; we call this an ungauged theory. In contrast, gauge groups with non-trivial adjoints lead to the gauging of a number of global symmetries; these are called gauged theories.

Although we have only discussed gravity on a group manifold in this section, the same reasoning can be applied to other fields, as was already done in [39]. The behaviour under the internal transformations (4.49) determines the reduction Ansatz and guarantees consistency of the reduction. In section 5.4 we will apply group manifold reductions to $D = 11$ maximal supergravity.

Consistency of Reduction over Group Manifolds

The consistency of this procedures is guaranteed by group-theoretical arguments: there is always an internal dependence such that only the structure constants appear in the lower-dimensional theory. An equivalent statement is that the Kaluza-Klein tower of fields, stemming from the expansion over the group manifold, is truncated to fields that are singlets under G_L. Since singlets can never generate non-singlets, this guarantees that the field equations for the non-singlets are automatically satisfied. In other words, the consistency of this reduction can be understood from the presence of the transitively acting isometries of G_L, over which one can reduce.

The metric (4.50) includes deformations from the bi-invariant metric, which are parameterised by the lower-dimensional fields. Since the metric always retains a set of transitively acting isometries, these are called homogeneous deformations and reduce the isometry group from $G_L \times G_R$ to $G_L \times H_R$ where $H_R \subset G_R$. In the literature, such deformations are referred to as squashings of the maximally symmetric metric [120].

Another result from group theory is that the matrix $U^m{}_n$, parameterising the dependence on the internal coordinates, can be taken independent of a set of coordinates that correspond to commuting isometries. Clearly, an extreme case is the torus, having all isometries commuting and indeed allowing for a constant $U^m{}_n$. The opposite extreme has no two isometries that commute, in which case $U^m{}_n$ depends on all but one internal coordinates.

Twisted vs. Group Manifold Reductions

Having treated both twisted and group manifold reductions, we would like to comment on some similarities and differences.

An important common feature of the two reduction schemes is the reliance on global symmetries in the reduction Ansatz. The twisted reduction employs a global symmetry of the higher-dimensional theory while group manifold reduction makes use of the global symmetries of the internal manifold. Due to these global symmetries, one can introduce a certain dependence on the internal coordinate via $U(z^p)$, which will either cancel or appear in the specific combinations $C_m{}^n$ or $f_{mn}{}^p$ defined in (4.30) and (4.47). Thus, to interpret the emerging equations as lower-dimensional, one has to require these combinations to be z-independent. For $C_m{}^n$ this implies that it is the Lie algebra element corresponding to the twisted reduction while the $f_{mn}{}^p$'s are interpreted as the structure constants of the isometry group of the internal manifold.

This brings us to an equally important difference: due to the different dependences on the internal coordinates, the resulting deformations will be different as well. In the twisted case, the mass matrix $C_m{}^n$ induces a gauging which is always one-dimensional and therefore Abelian (in the generic cases without enhanced gaugings). On the contrary, the structure constants $f_{mn}{}^p$ necessarily involve non-Abelian gaugings. Both gaugings induce a scalar potential.

However, in certain cases there is a relation between the two reduction schemes. Consider reduction over group manifolds with $n-1$ commuting isometries: these can be split up in a toroidal reduction over $n-1$ dimensions followed by a twisted reduction over the remaining dimension. In this scenario, the twist symmetry is a subgroup of the $GL(n-1,\mathbb{R})$ global symmetry obtained from the toroidal reduction of gravity. Thus, a twisted reduction with a symmetry that has a higher-dimensional origin can also be interpreted as a group manifold reduction. Indeed, in such cases one must encounter the phenomenon of gauging enhancement, as discussed in section 4.3: the twisted reduction must lead to a non-Abelian gauging. In this example, the extra gauge vectors, transforming under the twist symmetry, are provided by the reduction of gravity over the T^{n-1}. Explicit cases will be discussed in sections 5.3 and 5.4.

4.5 Reduction over a Coset Manifold

We turn to the most complicated reduction procedure that we will discuss, reduction over a coset manifold. Unlike the preceding reductions, its consistency is not secured by group-theoretical arguments and has been proven only in very special cases.

Coset Manifolds

A coset manifold is defined as follows. Consider a group manifold G with group elements $g \in G$. A subgroup of G, denoted by H, can be used to construct a coset manifold by identifying group elements that are related by a right-acting transformation of an element of H:

$$g \cong gh \,, \qquad \forall g \in G \,, \qquad \forall h \in H \subset G \,. \tag{4.60}$$

The corresponding coset manifold is denoted by G/H. Its dimension n is equal to dim$[G]$ − dim$[H]$.

Remember that a group manifold has coordinate transformations corresponding to left- and right-acting group multiplication. Indeed, the bi-invariant metric has isometry group $G_L \times G_R$. For coset manifolds, only the left-acting group multiplication corresponds to coordinate transformations, while right-acting multiplication takes one outside of the coset manifold:

$$g \to g\Lambda_R \ncong gh\Lambda_R \leftarrow gh \,, \tag{4.61}$$

since $\Lambda_R^{-1} h \Lambda_R$ is not an element of H in general. Therefore, the most symmetric metric on a coset manifold G/H will have isometry group G (omitting the subscript) rather than $G_L \times G_R$. This metric is usually called the round metric. The subgroup H is known as the isotropy group.

An important example of a coset manifold is the sphere S^n, which has isometry group $G = SO(n+1)$ and isotropy group $H = SO(n)$. Indeed, for every point on the sphere, one can perform $SO(n)$ rotations that leave this point invariant. This corresponds to the identification (4.60).

Coset Reductions

The maximal isometry group of a coset manifold G/H is G. However, for generic metrics, the coset manifold has no isometries at all. Therefore, the deformations from the maximally symmetric metric are called inhomogeneous: they break all isometries and thus also homogeneïty. The lower-dimensional fields parameterise these deformations and fall in multiplets of the maximal isometry group G. In particular, one expects massless gauge vectors corresponding to G.

The lack of isometries is an important issue in reductions over coset manifolds. Due to this feature, reduction over a coset is a highly non-trivial procedure whose consistency is not guaranteed by group-theoretical arguments. Only in very special cases the consistency has been proven, though not at all understood. Most of these cases are concerned with spheres S^n, resulting in massless $SO(n+1)$ gauge vectors upon reduction. A necessary requirement for this to be possible is the presence of $\frac{1}{2}n(n+1)$ gauge vectors in the lower dimensions. In addition, which is the condition we will focus on, the ungauged theory must have a global symmetry that contains $SO(n+1)$. This rules out coset reductions of pure gravity: we have seen in section 4.2 that reduction of gravity over T^n leads to an $SL(n,\mathbb{R})$ global symmetry group, which does not contain $SO(n+1)$ and therefore does not allow for such a gauge group.

Extending gravity with gauge fields and scalars, the situation looks more promising. We already encountered examples of such global symmetry enhancement in section 4.2. Indeed, as we will discuss, these all allow for coset reductions. We will consider gravity, a dilaton ϕ and an n-form field strength $G^{(n)} = dC^{(n-1)}$ whose coupling to the dilaton is parameterised by the constant a. Its Lagrangian reads

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{a\phi}G^{(n)} \cdot G^{(n)}]\,, \tag{4.62}$$

which is identical to the system giving rise to the brane solutions of section 3.4 with $n = d+1$. Note that the dilaton decouples for $a = 0$ and can be consistently truncated away. Due to Hodge duality, the field strengths $G^{(n)}$ and $G^{(D-n)}$ are equivalent and therefore we restrict to $n \leq \frac{1}{2}D$. It turns out [124] that reduction of this system over T^n gives rise to an $SL(n+1,\mathbb{R})$ global symmetry (rather than just the $SL(n,\mathbb{R})$ that follows from gravity) if the dilaton coupling is given by

$$a^2 = \frac{8 - 2(n-3)(D-n-3)}{D-2}\,, \tag{4.63}$$

corresponding to the value $\Delta = 4$. This is only a necessary and (in general) not a sufficient condition. The following cases do allow for coset reductions:

- $n = 1$: This is clearly not the most interesting of all cases since the manifold $S^1 \sim SO(2)/SO(1)$ is not a coset since $SO(1)$ is trivial. A related point is that the necessary $SL(2,\mathbb{R})$ symmetry is already present in the higher-dimensional system (4.62). Therefore, in the discussion of coset reductions, we will not consider this case.

- $n = 2$: In this case the system (4.62) can be exactly obtained from the reduction of pure gravity over a circle (4.4). This higher-dimensional origin clearly explains the appearance of the $SL(3,\mathbb{R})$ symmetry rather than $SL(2,\mathbb{R})$, as noted in section 4.2. The consistent reduction of this system over S^2 has been proven in [124]. An equivalent way to view this coset reduction of the Einstein-Maxwell-dilaton system is to perform an $SO(3)$ group manifold reduction on the higher-dimensional gravity [34]. We will encounter an example of this in section 5.4.

- $n = 3$: This is exactly the effective action of the bosonic string in D dimensions, encountered in section 2.1 (after field redefinitions). Indeed, the reduction of this effective action on an n-torus gives a global symmetry group $SO(n, n)$, as discussed in section 4.2. The case $n = 3$ then corresponds to $SO(3, 3) \sim SL(4, \mathbb{R})$, which allows for a gauging of $SO(4)$. The consistency of the corresponding S^3 coset reduction was proven in [124].

- $n = 4$: Reality of a implies $D \leq 11$. Let us first consider $D = 11$, in which case a vanishes. It has been proven that the reduction of the system (4.62) is inconsistent: one needs an extra interaction term, which is called a Chern-Simons term and which is exactly present in (the bosonic sector of) 11D supergravity, see (3.6). The corresponding reduction Ansätze on S^4 [125, 126] and S^7 [127] have been proven to be consistent. Indeed, maximal supergravity in $D = 7$ and $D = 4$ include global symmetry groups $SL(5, \mathbb{R})$ and $SL(8, \mathbb{R})$. Other cases with $D < 11$ and $a \neq 0$ correspond to toroidal reduction of 11D supergravity to D dimensions, followed by the coset reductions.

- $n = 5$: Reality of a implies $D \leq 10$. Again, in the limiting case $D = 10$ one finds a vanishing dilaton coupling a. Reduction of the system (4.62) is not consistent, however: one needs to impose a self-duality constraint on the five-form field strength. The corresponding reduction Ansatz has been constructed in [128]. Note that one again encounters a (bosonic subsector of) supergravity, in this case IIB supergravity[9]. Indeed, 5D maximal supergravity includes a global symmetry group $SL(6, \mathbb{R})$. Lower-dimensional cases with $a \neq 0$ are obtainable by toroidal reduction of the prime example in $D = 10$.

This concludes all possible sphere reductions. Cases with $n > 5$ and real a are all related to any of the above cases by Hodge duality.

Of these coset reductions, the first case with $n = 2$ is readily understood from its higher-dimensional origin. Indeed, one can always split up a reduction over the group G into a reduction over the subgroup $H \subset G$ followed by a coset reduction G/H [129]. Clearly, the consistency of such a coset reduction is implied by its higher-dimensional origin. The above example corresponds to $G = SO(3)$ and $H = SO(2)$.

The next case, which has $n = 3$, allows for a reduction over the coset manifold $S^3 = SO(4)/SO(3)$, leading to an $SO(4)$ gauge group, of which three corresponding vectors are provided by the metric while the remaining three are provided by the three-form field strength. This can be contrasted to the reduction of the same theory over the group manifold $SO(3)$. As discussed in section 4.4, this gives rise to the gauge group $SO(3)$ of which the vectors are provided by the metric only. The peculiar feature in this case is that the group and coset manifolds coincide for the maximally symmetric case, having isometries $SO(4) \sim SO(3) \times SO(3)$. The two reduction schemes differ in the deformations that are included in the reduction Ansätze. In the group manifold these only parameterise homogeneous

[9]However, the consistency of the S^5 reduction of the full IIB supergravity has not been proven so far.

deformations, keeping a transitively acting $SO(3)$ group of isometries. In contrast, the coset manifold reduction includes also inhomogeneous deformations, breaking all isometries.

Another noteworthy feature of the bosonic string effective action is the global symmetry group $SO(n,n)$ that appears upon reduction over an n-torus. This has led to the conjecture [130] that it allows for a consistent truncation over the coset manifold $(G \times G)/G$, where G has dimension n and is a compact subgroup of $SO(n)$. Though not proven in generality, such a truncation is believed to be consistent, of which the above case $n = 3$ (whose consistency has been proven [124]) provides an example.

It is remarkable that for the remaining cases $n = 4$ and $n = 5$, purely bosonic considerations lead to subsectors of the highest-dimensional supergravities, while consistency of the reduction requires exactly the interactions provided by supergravity. These spherical reductions have been employed to generate lower-dimensional gauged supergravities. We will discuss these in more detail in section 5.5.

Method	Requirement	Manifold	Gauging	Min.	Max.
Toroidal	–	$U(1)^n$	–	$U(1)^n$	$U(1)^n$
Twisted	Global symmetry	$U(1)$	$U(1)$	$U(1)$	$U(1)$
Group manifold	Gravity	G	G_R	G_L	$G_L \times G_R$
Coset manifold	Gravity and flux	G/H	G	–	G

Table 4.1: *The different reduction schemes with the requirements, the internal manifolds and the resulting gaugings of the lower-dimensional theories. We also give the minimum and maximum possible isometry groups of the internal manifold. Adapted from [129].*

In addition to the aforementioned spherical reductions, one can also consider reductions over hyperboloid spaces defined by a hypersurface

$$\sum_{i=1}^{n} q_i {\mu_i}^2 = 1\,, \tag{4.64}$$

with parameters $q_i = \pm 1$. The case with all $q_i = +1$ is the only compact manifold, corresponding to the sphere, while the other cases are non-compact. Despite its non-compactness, one can still perform consistent reductions over such spaces, giving rise to non-compact $SO(p,q)$ gaugings with $p + q = n$. The consistency of reductions over such hyperboloids can be deduced from analytical continuation of the corresponding spherical reduction [131]. We will encounter examples of such hyperboloids in section 5.5.

4.6 Lagrangian vs. Field Equations

In the preceding sections on toroidal, twisted and group manifold reductions, we have often substituted the reduction Ansatz in the Lagrangian to obtain the lower-dimensional Lagrangian. However, the reduction Ansatz comprises a truncation to the lightest modes, and the consistency of truncations is determined by the field equations. In general, there is no reason to assume that substitution in the Lagrangian yields the same result as substitution in the field equations, as illustrated in figure 4.1. In this section we will first discuss an explicit example in which this issue arises and then discuss general conditions in which the two schemes yield the same result, i.e. in which the operations in figure 4.1 do commute.

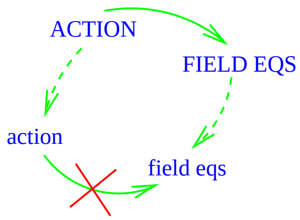

Figure 4.1: *Reductions of the action or the field equations do not necessarily yield equivalent lower-dimensional field equations, i.e. the operations of minimalisation (denoted by the solid arrows) and reduction (denoted by the dashed arrows) of the action do not necessarily commute.*

Toy Example

As a toy model in 10D, we start with the truncation of IIA and IIB supergravity to the metric and the dilaton. The Lagrangian reads

$$\hat{\mathcal{L}} = \sqrt{-\hat{g}}[\hat{R} - \tfrac{1}{2}(\partial\hat{\phi})^2]\,, \tag{4.65}$$

while the corresponding Euler-Lagrange equations are given by

$$\begin{aligned} [\hat{g}^{\hat{\mu}\hat{\nu}}]: &\qquad \hat{R}_{\hat{\mu}\hat{\nu}} - \tfrac{1}{2}\hat{R}\hat{g}_{\hat{\mu}\hat{\nu}} - \tfrac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} + \tfrac{1}{4}(\partial\hat{\phi})^2\hat{g}_{\hat{\mu}\hat{\nu}} = 0\,, \\ [\hat{\phi}]: &\qquad \Box\hat{\phi} = 0\,. \end{aligned} \tag{4.66}$$

This system has two global symmetries, as discussed in section 3.2: one can either scale the metric or one can shift the dilaton, parameterised by m_g and m_ϕ, respectively:

$$\hat{g}_{\hat{\mu}\hat{\nu}} \to e^{2m_g}\hat{g}_{\hat{\mu}\hat{\nu}}\,, \qquad \hat{\phi} \to \hat{\phi} + m_\phi\,. \tag{4.67}$$

The shift of the dilaton is a symmetry of the Lagrangian. The trombone symmetry, scaling the metric, is a symmetry of the field equations only; it scales the Lagrangian. This will prove an important difference when performing twisted reductions. We will show that one has to reduce field equations, rather than the Lagrangian, when performing reductions with twist symmetries of the field equations only.

Using an arbitrary linear combination of the two global symmetries we make the following Ansatz for twisted reduction over z to nine dimensions:

$$\hat{g}_{\hat{\mu}\hat{\nu}} = e^{2m_g z} \begin{pmatrix} e^{\sqrt{7}\varphi/14} g_{\mu\nu} & 0 \\ 0 & e^{-\sqrt{7}\varphi/2} \end{pmatrix} , \qquad \hat{\phi} = \phi + m_\phi z \,, \tag{4.68}$$

where we have omitted the Kaluza-Klein vector A_μ for simplicity. Using this Ansatz the 10D field equations yield the following 9D equations:

$$\begin{aligned} [\hat{g}^{\mu\nu}] : \qquad & R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{4}(\partial\phi)^2 g_{\mu\nu} - \tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \tfrac{1}{4}(\partial\varphi)^2 g_{\mu\nu} + \\ & + e^{4\varphi/\sqrt{7}}(\tfrac{1}{4}{m_\phi}^2 + 28{m_g}^2) g_{\mu\nu} = 0 \,, \\ [\hat{\phi}] : \qquad & \Box\phi + 8 m_g m_\phi e^{4\varphi/\sqrt{7}} = 0 \,, \\ [\hat{g}^{zz}] : \qquad & \Box\varphi - \tfrac{2}{\sqrt{7}} {m_\phi}^2 e^{4\varphi/\sqrt{7}} = 0 \,. \end{aligned} \tag{4.69}$$

Note that the field equations of the metric and both scalars get bilinear massive deformations. In addition one has the reduction of the $\hat{g}^{z\mu}$ field equation

$$[\hat{g}^{z\mu}] : \qquad 2\sqrt{7} m_g \partial_\mu\varphi + \tfrac{1}{2} m_\phi \partial_\mu \phi = 0 \,, \tag{4.70}$$

which is the equation of motion for the Kaluza-Klein vector A_μ.

We would like to discuss whether the field equations can be reproduced by a Lagrangian. We will not consider the field equation for the vector (4.70) since it is not important for our argument, and restrict to (4.69). If one performs the twisted reduction on the 10D Lagrangian, instead of on the field equations, the result reads $\hat{\mathcal{L}} = e^{8m_g z}\mathcal{L}$ with the 9D Lagrangian given by

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 - V(\phi,\varphi)] \quad \text{with} \quad V(\phi,\varphi) = e^{4\varphi/7}(\tfrac{1}{2}{m_\phi}^2 + 72{m_g}^2) \,. \tag{4.71}$$

The corresponding Euler-Lagrange equations read

$$\begin{aligned} [g^{\mu\nu}] : \qquad & R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{4}(\partial\phi)^2 g_{\mu\nu} - \tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \tfrac{1}{4}(\partial\varphi)^2 g_{\mu\nu} + \\ & + e^{4\varphi/\sqrt{7}}(\tfrac{1}{4}{m_\phi}^2 + 36{m_g}^2) g_{\mu\nu} = 0 \,, \\ [\phi] : \qquad & \Box\phi = 0 \,, \\ [\varphi] : \qquad & \Box\varphi - \tfrac{4}{\sqrt{7}} e^{4\varphi/\sqrt{7}}(\tfrac{1}{2}{m_\phi}^2 + 72{m_g}^2) = 0 \,. \end{aligned} \tag{4.72}$$

These Euler-Lagrange equations only coincide with the reduction of the 10D Euler-Lagrange equations (4.69) provided $m_g = 0$. Thus the twisted reduction of the Lagrangian does not give the correct answer if the Lagrangian scales: the Euler-Lagrange equations (4.72) are not equal to the field equations (4.69) for $m_g \neq 0$. In fact, the situation is worse [132, 133]: for $m_g \neq 0$ there is no Lagrangian $\mathcal{L}$ with potential $V(\phi, \varphi)$ whose Euler-Lagrange equations are the correct field equations (4.69). The metric field equation would require

$$V(\phi, \varphi) = e^{4\varphi/\sqrt{7}}(\tfrac{1}{2}{m_\phi}^2 + 56{m_g}^2)\,, \tag{4.73}$$

but this is inconsistent with the ϕ and φ field equations for $m_g \neq 0$.

Thus we conclude that twisted reduction of the Lagrangian is only legitimate when the exploited symmetry leaves the Lagrangian invariant rather than covariant. For symmetries that scale the Lagrangian one has to reduce the field equations. Including the full field content, such as the Kaluza-Klein vector A_μ, does not change this conclusion.

However, it is possible that certain truncations do lead to the possibility of an action. In our toy model, an example hereof is provided by the identification

$$2\sqrt{7} m_g \varphi = -\tfrac{1}{2} m_\phi \phi\,. \tag{4.74}$$

It can be seen that this truncation is fully consistent with the field equations (4.69) and (4.70). The resulting field equations can be derived from the Lagrangian

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}c^2(\partial\varphi)^2 - \tfrac{1}{2}c^2{m_\phi}^2 e^{4\varphi/7}]\,, \tag{4.75}$$

with $c^2 = 1 + 112{m_g}^2/{m_\phi}^2$. However, note that this is not the same result as the insertion of this truncation in the reduced Lagrangian (4.71).

General Requirements

In the above example we have found that in the case of twisted reduction with a trombone symmetry, one should reduce the field equations and not the action. The lower-dimensional field equations do not even allow for a corresponding Lagrangian, i.e. the field equations can not be interpreted as Euler-Lagrange equations stemming from the minimalisation of an action. The general rule for twisted reduction seems to be that the Lagrangian should be invariant under the twist symmetry to allow for a lower-dimensional Lagrangian. Reduction of the Lagrangian and the field equations are equivalent in such cases. An example is provided by the twisted reduction with the global symmetry of a scalar coset, as considered in section 4.3. Though we know of no general proof of this statement, it is generally believed to be consistent and no counterexamples are known.

As for group manifold reductions, one finds a rather similar condition. It turns out [134, 135] that only group manifolds with traceless structure constants, i.e. ${f_{mn}}^n = 0$, allow for reduction of the Lagrangian[10]. Indeed, such manifolds employ a symmetry (stemming from the higher-dimensional diffeomorphisms) that leaves the higher-dimensional Lagrangian invariant. In terms of ${U^m}_n$, this corresponds to the $SL(n, \mathbb{R})$ subgroup of the full

[10]Note that this also proves the correctness of the reduction of the Lagrangian for toroidal reduction, having ${f_{mn}}^p = 0$.

$GL(n,\mathbb{R})$. Reduction over such group manifolds give rise to gauge groups whose adjoint is embedded in the $SL(n,\mathbb{R})$ global symmetry group.

In contrast, group manifolds with traceful structure constants allow for reduction of the field equations. Indeed, these employ a symmetry that scales the higher-dimensional Lagrangian. Such symmetries are only embeddable in $GL(n,\mathbb{R})$ and not in $SL(n,\mathbb{R})$. The corresponding group manifold reduction gauges a subgroup of the $GL(n,\mathbb{R})$ global symmetry group of the theory, of which only $SL(n,\mathbb{R})$ is a symmetry of the Lagrangian. Examples of such reduction spaces are hyperbolic group manifolds, which we will encounter in sections 5.3 and 5.4.

This distinction between traceless and traceful structure constants, corresponding to unimodular and non-unimodular groups respectively, has been the cause of some confusion in the literature on group manifold reduction. It has been claimed [39, 135] that it is inconsistent to reduce over group manifolds with $f_{mn}{}^{n} \neq 0$. Another point of view, however, puts emphasis on the consistency of reduction of the field equations [129, 136, 137], where lower-dimensional theories are consistent if every solution uplifts to a higher-dimensional solution as well. In this thesis, we will adhere to the latter, yielding lower-dimensional theories without actions that uplift consistently to the higher dimension. The same distinction directly carries over to twisted reductions with symmetries of the Lagrangian and the field equations, respectively.

Indeed, the same situation is encountered in coset reductions, in which one reduces field equations rather than Lagrangians as well. However, in contrast to the twisted reduction with a trombone symmetry or over a non-unimodular group manifold, the lower-dimensional field equations can be obtained from an action. This action can not be derived by substitution the reduction Ansatz in the higher-dimensional action, though. This is very much like the truncation (4.74) in our toy model, leading to field equations that allow for an action but that do not follow from the reduced action. We will encounter such situations after reduction over non-unimodular group manifolds in section 5.4.