M-theory and gauged supergravities

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Chapter 3

Supergravity

As we have mentioned in the previous chapter, supergravities in ten and eleven dimensions emerge as the effective low-energy description of string and M-theory. In this section we will first discuss supersymmetry and supergravity in various dimensions, some supersymmetric solutions and then discuss their interrelations.

3.1 Supersymmetry

Superalgebra and Supercharges

The symmetry of supergravity theories is the super-Poincaré symmetry, which is an extension of the usual Poincaré symmetry of gravity theories with the generators of supersymmetry. Thus, it contains the Lorentz generators, the generators of translations (a vector under the Lorentz symmetry) and the supersymmetry generators (spinors under the Lorentz symmetry). In addition, the super-Poincaré algebra, or superalgebra in short, can be extended with a number of gauge generators, which are bosonic generators whose parameter is a p-form.

Due to the intertwining of the fermionic generators of supersymmetry and the bosonic generators of translations and gauge symmetries in the superalgebra, the requirement of local supersymmetry [8] has profound implications. In particular, it leads to the inclusion of gravity, due to the presence of translations in the superalgebra. Thus any locally supersymmetric theory contains gravity and is usually called a supergravity.

For the discussion of supersymmetry in D dimensions we will now consider fermionic representations of the Lorentz group $SO(1, D-1)$. This is the Dirac representation and its generators are given by $[\Gamma_\mu, \Gamma_\nu]$, where the Γ-matrices Γ_μ satisfy the Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \, . \tag{3.1}$$

The minimal dimension[1] of the representation of the Clifford algebra is $2^{[D/2]+1}$, where the notation $[D/2]$ means the integer part of $D/2$.

Since spinors transform under the fermionic representation of the Lorentz group, their number of components in principle equals the dimension of the Dirac representation. These are called Dirac spinors. However, in certain dimensions Dirac spinors are reducible, allowing one to impose conditions that are preserved under Lorentz symmetry. For example, in even dimensions one can impose a chirality condition: spinors are required to have eigenvalue ± 1 under the chirality operator

$$\Gamma_c = i^{D/2-1}\Gamma_{\underline{01\cdots D-1}}\,, \tag{3.2}$$

giving rise to Weyl spinors. In other cases it is possible to impose a reality condition, leading to Majorana spinors. In addition it is possible that both these conditions can be imposed, leading to Majorana-Weyl spinors. In table 3.1 we give the minimal spinors in different dimensions and their number of components q, where minimal spinors have the smallest number of components, i.e. all possible and mutually consistent conditions are imposed. A more detailed account can be found in e.g. [65, 66].

Dimension	Spinors	Components (q)
2 mod 8	Maj.-Weyl	$2^{D/2-1}$
3,9 mod 8	Majorana	$2^{(D-1)/2}$
4,8 mod 8	Majorana	$2^{D/2}$
5,7 mod 8	Dirac	$2^{(D+1)/2}$
6 mod 8	Weyl	$2^{D/2}$

Table 3.1: *The different minimal spinors in different space-time dimensions and their number of components.*

The parameter of supersymmetry is a spinor and thus the number of supercharges Q, associated to supersymmetry generators, is always a multiple N of the dimension of the irreducible representation:

$$Q = Nq\,. \tag{3.3}$$

However, there is a bound on the number of supercharges [67]. For theories with global supersymmetry, thus not containing gravity, the bound is 16 supercharges. Theories with local supersymmetry, therefore including gravity, can have up to 32 supercharges. Superalgebras

[1] We always refer to the *real* dimension.

with more than 32 supercharges will only have representations that include states of helicity higher than two. When coupling these to other fields one breaks the associated gauge symmetry, thus rendering the interaction inconsistent. For this reason these higher-spin theories are usually discarded, although there are attempts to remedy the problems [68]. Theories with exactly 32 supercharges are called maximal supergravities.

Dimension	Supergravity (N)
11	1
10	1, IIA, IIB
7,8,9	1,2
6	1, iia, iib, 4

Table 3.2: *Supergravity in different space-time dimensions, labelled by their number of supersymmetry generators.*

Possible Supergravity Theories

When combining the bound on the number of supercharges with the dimension of the minimal spinor in the different dimensions, we can survey the different possibilities for N in different dimensions[2], as summarised in table 3.2. One dramatic conclusion is that in dimensions twelve or higher there are no supergravity theories[3] since the dimension of the minimum spinor is 64. Thus $D = 11$ is the tip of the pyramid of supergravities, where one can only have maximal supergravity with 32 supercharges. This agrees strikingly with the discussion of M-theory in section 2.3, which unifies all string theories in eleven dimensions and of which 11D supergravity is the low-energy description. We will discuss 11D supergravity in section 3.2.

In ten dimensions one can have either $N = 1$ or $N = 2$ supersymmetry, corresponding to 16 or 32 supercharges, respectively. Only the first of these cases does not necessarily contain gravity. The second case contains two Majorana-Weyl spinors of certain chiralities and thus allows for two different theories with spinors of either the opposite or the same chirality: type IIA and IIB supergravity with $(1, 1)$ or $(2, 0)$ supersymmetry, respectively (in this notation the first and second entries denote the number of supersymmetry generators with positive and negative chirality, respectively). In fact, $D = 10$ is the only dimension which has two inequivalent maximal supergravities; it is unique in all other dimensions. Again, this nicely

[2]We will always restrict ourselves to $D > 2$, since theories in two dimensions are special in many respects.
[3]At least with Lorentzian signature, as is our assumption here.

dovetails with our string theory findings, where we found maximally supersymmetric IIA and IIB string theory in $D = 10$. We will discuss IIA and IIB supergravity in subsection 3.2.

The next possibility for Weyl spinors occurs in six dimensions, where they have 8 components. Of the maximal superalgebras with $N = 4$, only the $(2, 2)$ case gives rise to a supergravity theory; other choices contain states with higher helicity. When considering 16 supercharges, there are two choices: one finds $(1, 1)$ and $(2, 0)$ supersymmetry as well, leading to two distinct $Q = 16$ supergravities in six dimensions, labelled iia and iib. In all other dimensions than six, the superalgebra with $Q = 16$ supercharges is unique.

We would like to make a few remarks about the explicit supergravity realisation of the superalgebras. The supergravity fields form massless multiplets under supersymmetry, called supermultiplets. These are usually christened after the field with the highest helicity. The best-known example is the graviton multiplet, which includes the graviton (spin 2), the gravitino (spin 3/2) and fields with lower spin. All supergravity theories contain this multiplet. Maximal supersymmetry only allows for this supermultiplet while a smaller amount of supersymmetry allows for other multiplets without gravity as well. Examples are the gravitino and the vector multiplet with highest spins $3/2$ and 1, respectively.

Name	Symbol	Spin	On-shell d.o.f.
Graviton	$g_{\mu\nu}$	2	$\frac{1}{2}(D-2)(D-1)-1$
Gravitino	ψ_μ	$3/2$	$\frac{1}{2}(D-3)\cdot q$
Rank-d potential	$C^{(d)}_{\mu_1\cdots\mu_d}$	1	$\begin{pmatrix} D-2 \\ d \end{pmatrix}$
Dilatino	λ	$1/2$	$\frac{1}{2}\cdot q$
Scalar	ϕ or χ	0	1

Table 3.3: *On-shell degrees of freedom of D-dimensional supergravity fields.*

For supersymmetry to be a consistent symmetry, all supermultiplets must have an on-shell matching of bosonic and fermionic degrees of freedom. The on-shell degrees of freedom are multiplets of the little group $SO(D-2)$ for massless fields and are given in table 3.3 for generic supergravity fields[4,5]. Note that a d-form potential $C^{(d)}$ carries the same amount of degrees of freedom as a $\tilde{d}$-form potential with $\tilde{d} = D - 2 - d$. What corresponds to an electric charge in one potential is a magnetic charge in its dual potential and vice versa.

[4]We distinguish between two types of scalars: dilatons ϕ and axions χ. Loosely speaking, the difference between these is that axions only appear with a derivative whereas the dilatons also occur without it. A stricter definition of this distinction will be discussed in section 3.3.

[5]In the case $d = (D-2)/2$ one can impose a self-duality constraint on the $(d+1)$-form field strength. The potential would then give rise to half the degrees of freedom as listed in table 3.3.

This equivalence between two potentials is called Hodge duality and is a generalisation of the well-known electric-magnetic duality in 4D to higher ranks d and $\tilde{d}$ and dimension D.

3.2 Maximal Supergravities in 11D and 10D

Supergravity in 11D

In eleven dimensions one has maximal supersymmetry. The superalgebra allows for the inclusion of a rank-2 and a rank-5 gauge symmetry. As is always the case with maximal supersymmetry, there is only one massless supermultiplet: the graviton multiplet, which we already encountered in section 2.3. It consists of the on-shell degrees of freedom

$$\text{D=11:} \qquad (\mathbf{44} + \mathbf{84})_{\text{B}} + (\mathbf{128})_{\text{F}} \,, \tag{3.4}$$

which are multiplets of $SO(9)$. In 11D supergravity theory the graviton multiplet is usually represented by the fields

$$\text{D=11:} \qquad \{e_\mu{}^a, C_{\mu\nu\rho}; \psi_\mu\} \,. \tag{3.5}$$

These are the Vielbein, a three-form gauge potential and a Majorana gravitino, respectively. The bosonic part of the corresponding Lagrangian [69] reads

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2} G \cdot G - \tfrac{1}{6} \star (G \wedge G \wedge C)] \,, \tag{3.6}$$

where $G = dC$. Note that it consists of the Einstein-Hilbert term, a kinetic term for the rank-three potential and a Chern-Simons term. The latter only depends on the rank-three potential and is independent of the metric; for this reason it is also called a topological term.

The 11D supergravity theory has an $\mathbb{R}^+$ symmetry which acts as

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \,, \qquad C_{\mu\nu\rho} \to \lambda^3 C_{\mu\nu\rho} \,, \qquad \psi_\mu \to \lambda^{1/2} \psi_\mu \,, \tag{3.7}$$

with $\lambda \in \mathbb{R}^+$. Two remarks are in order here. The above symmetry acts covariantly on the field equations (as all symmetries) but does not leave the Lagrangian invariant: it transforms as $\mathcal{L} \to \lambda^9 \mathcal{L}$. All terms in $\mathcal{L}$ scale with the same weight: for this reason it is called a trombone symmetry [70]. Secondly, the covariant scaling of $\mathcal{L}$ only holds at lowest order. Higher-derivative corrections will scale with different weights and thus break the symmetry (3.7) of the field equations.

The occurrence of trombone symmetries will be a generic feature in ungauged or massless supergravities. The weights of the fields are always determined by a simple rule: for the bosonic fields the weights equal the number of Lorentz indices while for the fermions it is one-half less. The Lagrangian will scale as $\mathcal{L} \to \lambda^{D-2} \mathcal{L}$ under such symmetries. The scaling of bosonic terms is easily understood from the two derivatives they contain. Thus this symmetry is broken by terms with less (as in scalar potentials, to be encountered in chapter 5) or more (as in higher-order corrections) than two derivatives.

Minimal Supergravity in 10D

In 10D the minimal spinor is a 16-component Majorana-Weyl spinor. Minimal $N = 1$ supersymmetry in 10D therefore has 16 supercharges. Its superalgebra allows for the inclusion of a rank-one and a self-dual rank-five gauge symmetry. Being non-maximal supersymmetry, one finds different supermultiplets [67], some of which we already encountered in section 2.2:

$$\text{N=1:} \qquad \begin{cases} \text{vector:} & (\mathbf{8}_{\rm v})_{\rm B} + (\mathbf{8}_{\rm c})_{\rm F}\,, \\ \text{graviton:} & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times \mathbf{8}_{\rm v} = (\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\rm B} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\rm F}\,, \\ \text{gravitino A:} & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times \mathbf{8}_{\rm s} = (\mathbf{56}_{\rm v} + \mathbf{8}_{\rm v})_{\rm B} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\rm F}\,, \\ \text{gravitino B:} & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times \mathbf{8}_{\rm c} = (\mathbf{35}_{\rm c} + \mathbf{28} + \mathbf{1})_{\rm B} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\rm F}\,, \end{cases} \tag{3.8}$$

Note that the little group $SO(8)$ has three 8-dimensional representations: one bosonic, the vector $\mathbf{8}_{\rm v}$, and two fermionic, spinors of opposite chirality $\mathbf{8}_{\rm c}$ and $\mathbf{8}_{\rm s}$. This special property of $SO(8)$ is known as triality.

Due to the appearance of several supermultiplets, non-maximal supergravity is not unique. It always contains the graviton multiplet, which can be coupled in various ways to vector multiplets, leading to e.g. a Yang-Mills sector. An example was encountered in section 2.2, where the low-energy limit of the three $N = 1$ string theories consisted of the graviton multiplet plus 496 vector multiplets to obtain the $SO(32)$ or $E_8 \times E_8$ gauge groups [15].

IIA and IIB Supergravity

Turning to maximal $N = 2$ supersymmetry in 10D, one has two possibilities: one can choose Majorana-Weyl spinors of either opposite or equal chirality, leading to the non-chiral IIA or the chiral IIB supergravity theories with $(1, 1)$ and $(2, 0)$ supersymmetry, respectively. The IIA superalgebra can be extended with gauge symmetries of rank 0,1,2,4 and 5, while IIB allows for 1,1,3,5^+,5^+ and 5^+, where all five-form gauge parameters 5^+ are self-dual. In fact, the IIB superalgebra has an additional $SO(2)$ R-symmetry, rotating the two supersymmetry spinors of equal chirality. Under this R-symmetry, the central charges form doublets (for rank 1 and 5^+) and singlets (for rank 3 and 5^+). We will discuss R-symmetries of lower-dimensional superalgebras in section 3.3.

As always, maximal supersymmetry allows for only one massless multiplet, whose on-shell degrees of freedom are given by

$$\begin{aligned} \text{IIA:} \quad & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm s}] = [(\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\text{NS-NS}} + (\mathbf{56}_{\rm v} + \mathbf{8}_{\rm v})_{\text{R-R}}]_{\rm B} \\ & \qquad + [(\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{NS-R}} + (\mathbf{56}_{\rm c} + \mathbf{8}_{\rm c})_{\text{R-NS}}]_{\rm F}\,, \\ \text{IIB:} \quad & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] = [(\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\text{NS-NS}} + (\mathbf{35}_{\rm c} + \mathbf{28} + \mathbf{1})_{\text{R-R}}]_{\rm B} \\ & \qquad + [(\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{NS-R}} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{R-NS}}]_{\rm F}\,, \end{aligned} \tag{3.9}$$

Note that these $N = 2$ supermultiplets are constructed from the $N = 1$ supermultiplets: both $N = 2$ graviton multiplets consist of the $N = 1$ graviton and a gravitino multiplet. This is possible in 10D due to triality, which yields $N=1$ graviton and gravitino multiplets of equal size.

We will now consider the field-theoretic realisation of the graviton multiplet. The common bosonic subsector, which is called the NS-NS subsector, contains gravity, a rank-two potential and a dilaton. The remaining bosonic part is called the Ramond-Ramond subsector and will only contain R-R rank-d potentials where d is odd in IIA and even in IIB. The standard forms of the theories have $d=1,3$ for IIA and $d=0,2,4$ for IIB:

$$\begin{aligned} &\text{IIA:} \qquad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(1)}_{\mu}, C^{(3)}_{\mu\nu\rho}; \psi_\mu, \lambda\}\,, \\ &\text{IIB:} \qquad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)+}_{\mu\nu\rho\sigma}; \psi_\mu, \lambda\}\,. \end{aligned} \tag{3.10}$$

In the IIA case the fermions are real and contain two minimal spinors of both chiralities, while in the IIB case they are complex and contain two minimal spinors of the same chirality. The field strength of the IIB rank-four potential $C^{(4)+}$ satisfies a self-duality constraint, halving the number of degrees of freedom.

We would also like to present a special formulation of IIA and IIB supergravity which emphasises the equivalence of dual R-R potentials, based on [71], and introduces an extra feature of IIA supergravity. To this end we will enlarge the field content by including all odd or even R-R potentials, thus allowing for the ranges $d=1,3,5,7,9$ and $d=0,2,4,6,8$. The field contents of IIA and IIB supergravity read in the double formulation

$$\begin{aligned} &\text{IIA :} \qquad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(1)}_{\mu}, C^{(3)}_{\mu\nu\rho}, C^{(5)}_{\mu\cdots\rho}, C^{(7)}_{\mu\cdots\rho}; \psi_\mu, \lambda\}\,, \\ &\text{IIB :} \qquad \{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\cdots\rho}, C^{(6)}_{\mu\cdots\rho}, C^{(8)}_{\mu\cdots\rho}; \psi_\mu, \lambda\}\,. \end{aligned} \tag{3.11}$$

To get the correct number of degrees of freedom, one must by hand impose duality relations between the field strengths of rank-d and rank-$(8-d)$ potentials, which read [71]

$$G^{(d+1)} = (-)^{[(d+1)/2]}\, e^{(d-4)\phi/2} \star G^{(9-d)}\,, \qquad G^{(d+1)} = dC^{(d)} - dB{\wedge}C^{(d-2)}\,. \tag{3.12}$$

for vanishing fermions. The (bosonic part of the) field equations for $C^{(d)}$ can be derived from the action [71]

$$L = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-\phi} H\cdot H - \sum_d \tfrac{1}{4} e^{(4-d)\phi/2} G^{(d+1)}\cdot G^{(d+1)}]\,, \tag{3.13}$$

subject to the duality relations (3.12). Due to these constraints, the above is called a pseudo-action [72]. Note that the doubling of Ramond-Ramond potentials has two effects: the kinetic terms have coefficients $1/4$ instead of the canonical $1/2$ and there are no explicit Chern-Simons terms in the action.

We would like to make the following two remarks. Note that the duality constraint on the five-form field strength of IIB can not be eliminated, in contrast to the other duality relations; it is a constraint on one field strength $G^{(5)}$ while the others relate two different field strengths $G^{(d+1)}$ and $G^{(9-d)}$ for $d \neq 4$.

Secondly, one can include a nine-form potential $C^{(9)}$ in (3.11), which carries no degrees of freedom (and thus is consistent with (3.9)) but is very natural from the point of view of R-R equivalence [24]. The corresponding field strength trivially satisfies the Bianchi identity. Its Hodge dual is a rank-zero field strength, which has no corresponding potential nor a field equation. Its Bianchi identity implies it to be constant. Thus we have effectively introduced a mass parameter in the theory, given by

$$G^{(0)} = e^{-5\phi/2} \star G^{(10)} \,. \tag{3.14}$$

The corresponding action is given by (3.13) with $d = -1, 1, \ldots, 9$ [71] and the field strengths [73]

$$G^{(d+1)} = dC^{(d)} - dB \wedge C^{(d-2)} + \frac{1}{(d+1)/2\,!}\, G^{(0)} B \wedge \ldots \wedge B \,. \tag{3.15}$$

Due to the equivalence of the different formulations, one should expect this mass parameter to appear in the normal formulation as well. Indeed this deformation of massless to massive IIA supergravity has been found [74], shortly after the inception of its massless counterpart [75, 76]. In this chapter, we concentrate on the massless part and we will come back to the massive deformations in sections 5.2 and 6.1. Also, we leave the formulation with R-R equivalence here and return to the standard formulation (3.10).

The (bosonic part of the massless) IIA Lagrangian is given by

$$\begin{aligned} L_{IIA} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-\phi} H \cdot H - \sum_{d=1,3} \tfrac{1}{2} e^{(4-d)\phi/2} G^{(d+1)} \cdot G^{(d+1)} + \\ - \tfrac{1}{2} \star (dC^{(3)} \wedge dC^{(3)} \wedge B)] \,, \end{aligned} \tag{3.16}$$

The IIA theory has two $\mathbb{R}^+$ symmetries. The first is a symmetry of the Lagrangian (3.16) and is given by

$$e^{\phi} \to \lambda e^{\phi} \,, \quad B_{\mu\nu} \to \lambda^{1/2} B_{\mu\nu} \,, \quad C^{(1)}_{\mu} \to \lambda^{-3/4} C^{(1)}_{\mu} \,, \quad C^{(3)}_{\mu\nu\rho} \to \lambda^{-1/4} C^{(3)}_{\mu\nu\rho} \,, \tag{3.17}$$

with $\lambda \in \mathbb{R}^+$ and other fields invariant. The other is the 10D analog of the 11D trombone symmetry (3.7) with weights as explained below the 11D weights.

The (bosonic part of the) field equations for IIB supergravity [77, 78] can be derived from the Lagrangian

$$\begin{aligned} L_{IIB} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{-\phi} H \cdot H - \sum_{d=0,2,4} \tfrac{1}{2} e^{(4-d)\phi/2} G^{(d+1)} \cdot G^{(d+1)} + \\ + \tfrac{1}{2} \star (C^{(4)} \wedge dC^{(2)} \wedge dB)] \,, \end{aligned} \tag{3.18}$$

which has to be supplemented[6] with the self-duality relation (3.12) for $d = 4$ (for this reason it is called a pseudo-action [72]). The IIB supergravity theory has a global $SL(2, \mathbb{R})$ symmetry

[6] An action without extra constraints can only be constructed when including auxiliary fields [79].

[80]

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad B^i \to (\Omega^{-1})_j{}^i B^j\,, \quad C^{(4)} \to C^{(4)}\,, \quad \Omega_i{}^j = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,,$$
$$\psi_\mu \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \psi_\mu\,, \quad \lambda \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{3/4} \lambda\,, \quad \epsilon \to \left(\frac{c\,\tau^* + d}{c\,\tau + d}\right)^{1/4} \epsilon\,, \qquad (3.19)$$

where we have defined the doublet $B^i = (-B, C^{(2)})$ and the complex scalar $\tau = \chi + ie^{-\phi}$ with the axion $\chi = C^{(0)}$. In terms of the real and imaginary parts of τ the action of $SL(2,\mathbb{R})$ reads

$$e^\phi \to (c\chi + d)^2 e^\phi + c^2 e^{-\phi}\,, \qquad \chi \to \frac{ac + e^{2\phi}(a\chi + b)(c\chi + d)}{c^2 + e^{2\phi}(c\chi + d)^2}\,, \qquad (3.20)$$

Note that the scalars transform non-linearly. We will discuss a more covariant way to view this $SL(2,\mathbb{R})$ symmetry in section 3.3. The $SL(2,\mathbb{R})$ symmetry of IIB supergravity is broken to $SL(2,\mathbb{Z})$ in IIB string theory, as we found in section 2.3. The element $(a,b;c,d) = (0,1;-1,0)$ corresponds to the transformation $\phi \to -\phi$ (for vanishing axion background), which relates the strong and weak string coupling. For this reason this transformation is called S-duality. In addition the IIB symmetry also has a trombone symmetry.

T-duality in Supergravity

In (2.19) we showed the decomposition of the on-shell 11D $SO(9)$ into IIA $SO(8)$ representations. This can also be done in terms of supergravity fields and amounts to dimensionally reducing the 11D supergravity, a procedure which is being elaborated upon in section 4.1, while only retaining the massless modes. These relations are given in (B.4). Indeed, the full 11D Lagrangian (3.6) and supersymmetry transformations (B.1) in this way give rise to the IIA counterparts (3.16) and (B.5).

Similarly, we have seen in (2.18) that the decomposition of the $SO(8)$ representations of IIA and IIB under $SO(7)$ coincides. At the supergravity level, the corresponding reduction Ansätze for IIA and IIB supergravity are given in (B.9) and (B.14), respectively. These reduce the IIA and IIB supersymmetry transformations and field equations to their 9D counterparts. Also the IIA Lagrangian (3.16) can be reduced to the correct 9D action. The IIB case requires a bit more discussion due to the self-duality constraint on the 5-form field strength. Upon reduction it gives rise to a 4-form and a 5-form field strength and a duality relation between the two. The latter can be used to eliminate either of the field strengths, which is usually the 5-form. If properly treated the IIB pseudo-Lagrangian (3.18) can also be reduced to the 9D Lagrangian. In accordance with their accompanying string theories, the map between the (dimensionally reduced) IIA and IIB supergravities is usually called T-duality.

3.3 Scalar Cosets and Global Symmetries in $D \leq 9$

We now turn to the remaining maximal supergravities in $D \leq 9$. Being unique these can all be obtained by dimensional reduction of any of the higher-dimensional theories, in the same way that IIA supergravity can be obtained from 11 dimensions. Their construction is rather straightforward and we will not consider it in great detail. One aspects deserves proper discussion however: the scalar section and its transformation under the global symmetries of the theory. See [81] for a clear discussion.

Scalar Cosets

The field content of any $D \leq 9$-dimensional maximal supergravity is easily obtained by dimensional reduction; its bosonic subsector consisting of gauge potentials is given in table 3.4. The same holds for the Lagrangians and general formulae for maximal supergravity in any dimension have been obtained [82]. The bosonic part generically reads

$$\mathcal{L}_D = \sqrt{-g}[R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \sum_{d,i} \tfrac{1}{2} e^{\vec{\alpha}^i_d \cdot \vec{\phi}} G^{(d+1)}_i \cdot G^{(d+1)}_i] + \mathcal{L}_{CS} \,, \tag{3.21}$$

where the $G^{(d+1)}_i$ are rank-$(d+1)$ field strengths of gauge potentials $C^{(d)}_i$ with $d = 0, \ldots, 3$. The index i denotes the different d-form potentials; its range can be inferred from table 3.4. The number of dilatons $\vec{\phi}$ always equals $11 - D$ since all reduced dimensions will give rise to one dilaton. The length of the vectors $\vec{\alpha}^i_d$ will always be given by

$$\vec{\alpha}^i_d \cdot \vec{\alpha}^i_d = 4 - \frac{2d(D-d-2)}{D-2} \,, \tag{3.22}$$

in maximal supergravity.

Of special interest in this Lagrangian is the scalar sector, which we rewrite as

$$\mathcal{L}_{\text{scalars}} = \sqrt{-g}[-\tfrac{1}{2}(\partial\vec{\phi})^2 - \sum_{i} \tfrac{1}{2} e^{\vec{\alpha}^i \cdot \vec{\phi}} G^{(1)}_i \cdot G^{(1)}_i] \,. \tag{3.23}$$

where $G^{(1)}_i$ are the one-form field strengths of the axions χ^i and where we have dropped the subscript 0 on the vectors $\vec{\alpha}^i$. The vectors $\vec{\alpha}^i$ can be interpreted as positive root vectors of a simple Lie algebra. In the Cartan-Weyl basis, the generators of this algebra are the Cartan generators $\vec{H}$, the positive root generators $E_{\vec{\alpha}^i}$ and the negative root generators $E_{-\vec{\alpha}^i}$ with commutation relations

$$[\vec{H}, \vec{H}] = 0 \,, \qquad [\vec{H}, E_{\vec{\alpha}^i}] = \vec{\alpha}^i E_{\vec{\alpha}^i} \,, \qquad [E_{\vec{\alpha}^i}, E_{\vec{\alpha}^j}] = N(\vec{\alpha}^i, \vec{\alpha}^j) E_{\vec{\alpha}^i + \vec{\alpha}^j} \,, \tag{3.24}$$

and similarly for the negative root generators (replacing $\vec{\alpha}^i \rightarrow -\vec{\alpha}^i$). The coefficients $N(\vec{\alpha}^i, \vec{\alpha}^j)$ are constants (possibly zero) and characterise the algebra. We will now show that the scalar sector (3.23) is invariant under the action of the corresponding semi-simple group G.

To this end we construct a particular representative of G, defined by[7]

$$L = \exp(\sum_i \chi^i E_{\vec{\alpha}^i})\exp(-\vec{\phi}\vec{H}/2)\,, \tag{3.25}$$

with parameters $\vec{\phi}$ corresponding to the Cartan generators and χ^i to the positive root generators. This parameterises the coset G/H with H the maximal compact subgroup of G. The group H will turn out to be the R-symmetry group of the superalgebra. Upon acting with a group element $g \in G$ from the left, the element gL will generically no longer have the form of the G/H representative (3.25), i.e. this can in general not be expressed as a transformation $\vec{\phi} \to \vec{\phi}'$ and $\chi^i \to \chi^{i\prime}$. However, one can employ the Isawa decomposition, which states that

$$L \to gL = L'h\,, \tag{3.26}$$

i.e. the resulting matrix can be decomposed as L' of the form (3.25) and a remainder $h \in H$. The latter will be dependent on $\vec{\phi}$ and χ^i in general. Due to the Isawa decomposition we have defined a transformation

$$L(\vec{\phi}, \chi^i) \to L' = gL(\vec{\phi}, \chi^i)h^{-1} = L(\vec{\phi}', \chi^{i\prime})\,, \tag{3.27}$$

consisting of a left-acting G element and a compensating right-acting H element. Note that for global G transformations, the action of H will be local due to the field dependence via $\vec{\phi}$ and χ^i. The H transformation is called compensating since it compensates for the G transformation that does not preserve the G/H representative (3.25).

The relevance of the transformation properties of L stems from the fact that the scalar kinetic terms (3.23) can be written as

$$\mathcal{L}_{\text{scalars}} = \sqrt{-g}[\tfrac{1}{4}\text{Tr}(\partial M \partial M^{-1})]\,, \tag{3.28}$$

where we have defined $M = LL^T$. Note that M does not see the compensating H transformation: it transforms as $M \to gMg^T$ under (3.27). Thus the scalar section is by construction invariant under global G transformations. It turns out that this group is a symmetry not only of the scalar subsector but of the entire theory[8], i.e. when also including the potentials of higher rank and the fermions.

Let us take a step back and consider the significance of the compensating transformation H. We have shown that the scalar kinetic terms (3.23) are invariant under the global symmetry G by constructing a particular G/H representative L. Every G transformation is accompanied by a compensating H transformation to keep L of the same form. This can be

[7] Other choices for this representative are related by field redefinitions.

[8] In many cases, however, the group G is a symmetry of the equations of motion rather than the Lagrangian, since it requires e.g. the dualisation of some gauge potentials.

seen as the gauge fixed version (with gauge choice (3.25)) of a more covariant system with global G and local H symmetry. The covariant system has kinetic term (3.28) for arbitrary $L \in G$. The extra degrees of freedom that are introduces in L are cancelled by the extra gauge degrees of freedom $L \to Lh$ with $h \in H$ local. This is a completely equivalent formulation of the scalar sector with advantages due to its covariance.

Example: $SL(2,\mathbb{R})$ **Symmetry of IIB**

To make matters more concrete let us discuss the scalar sector of IIB supergravity as an example. From its Lagrangian one reads off that it has one dilaton and one axion with positive root vector $\alpha = 2$. This corresponds to the simple Lie algebra $sl(2)$ with generators (in the fundamental representation)

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad E_{+2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad E_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,, \tag{3.29}$$

satisfying the algebra (3.24). Next we define the $SL(2,\mathbb{R})/SO(2)$ representative

$$L = e^{\chi E_{+2}} e^{-\phi H/2} = \begin{pmatrix} e^{-\phi/2} & e^{\phi/2}\chi \\ 0 & e^{\phi/2} \end{pmatrix}\,. \tag{3.30}$$

Any left-acting $SL(2,\mathbb{R})$ transformation on L can be compensated by a right-acting field-dependent $SO(2)$ transformation. Indeed one can easily identify these in the explicit $SL(2,\mathbb{R})$ transformations (3.19) of IIB supergravity. The two-form potentials transform linearly under G while the fermions only transform under the compensating $SO(2)$ transformations. Without gauge fixing the G transformations would read (omitting $SL(2,\mathbb{R})$ indices)

$$\begin{aligned} &L \to g\,L\,h^{-1}_{SO(2)}\,, \qquad B \to (g^{-1})^T B\,, \qquad C^{(4)} \to C^{(4)}\,, \\ &\psi_\mu \to h^{1/2}_{U(1)}\psi_\mu\,, \qquad \lambda \to h^{3/2}_{U(1)}\lambda\,, \qquad \epsilon \to h^{1/2}_{U(1)}\epsilon\,. \end{aligned} \tag{3.31}$$

where g and h are given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,, \quad h_{SO(2)} = \exp \begin{pmatrix} 0 & \theta(x) \\ -\theta(x) & 0 \end{pmatrix}\,, \quad h_{U(1)} = \exp(i\theta(x))\,.$$

This clearly shows the two different symmetries that act independently in the covariant formulation. The gauge fixing condition translates in the role of H as compensating transformation with

$$\theta = -\arccos\left(\frac{e^{\phi}(c\chi + d)}{\sqrt{c^2 + e^{2\phi}(c\chi+d)^2}}\right)\,. \tag{3.32}$$

Indeed, the transformations (3.31) with constraint (3.32) reduce to the non-linear transformations (3.19).

Global Symmetries of Maximal Supergravities

Having dealt with the simplest example in $D = 10$, we now turn to lower-dimensional scalar cosets. In table 3.4 we give the groups G and H that one encounters. The dimension of the scalar coset G/H equals the number of scalars; the number of axions is given by the number of positive roots of the algebra corresponding to G while the number of dilatons equals $(11 - D)$ (one for every reduced dimension). In table 3.4 we also give the bosonic potentials of higher rank and their transformation under the G groups. The potentials form linear representations of G while they are invariant under H. We do not give the fermionic field content; see e.g. [66]. In contrast to the bosons, the fermions are invariant under G but transform under H. One can check these statements in the example of $SL(2,\mathbb{R})$ symmetry in IIB supergravity, see (3.19) and (3.31).

D	G	H	Dim$[G/H]$	$d=1$	$d=2$	$d=3$	$d=4$
11	1	1	–	–	–	$\mathbf{1}$	–
IIA	$\mathbb{R}^+$	1	1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	–
IIB	$SL(2,\mathbb{R})$	$SO(2)$	2	–	$\mathbf{2}$	–	$\mathbf{1}^+$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	3	$\mathbf{2}+\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	–
8	$SL(3,\mathbb{R})\times$ $\times SL(2,\mathbb{R})$	$SO(3)\times$ $\times SO(2)$	7	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})^+$	–
7	$SL(5,\mathbb{R})$	$SO(5)$	14	$\mathbf{10}$	$\mathbf{5}$	–	–
6	$SO(5,5)$	$SO(5)\times SO(5)$	25	$\mathbf{16}$	$\mathbf{10}^+$	–	–
5	$E_{6(+6)}$	$USp(8)$	42	$\mathbf{27}$	–	–	–
4	$E_{7(+7)}$	$SU(8)$	70	$\mathbf{56}^+$	–	–	–

Table 3.4: *The groups G and H and the d-form gauge potentials as representations of G of maximal supergravity in $6 \leq D \leq 11$. The $+$ denotes self-dual representations: there are duality constraints that halve the number of degrees of freedom. In all but the IIB case the constraints can be eliminated at the cost of breaking manifest G covariance.*

Note that the global symmetry group G in D dimensions is often larger than the $SL(11-D,\mathbb{R})$ that is expected from the connection with eleven-dimensional supergravity (as will be explained in section 4.2). For this reason, the group G is known as a hidden symmetry [83–85]. Another important feature in even dimensions is that they are only symmetries of the equations of motion and not of the Lagrangian. For example, this can come about when the symmetry transformation involves a Hodge dualisation of a gauge potential, which can only be performed straightforwardly on the field equations and not on the Lagrangian. This is the origin of the self-dual representations in table 3.4.

A number of complications turn up in $D \leq 5$, as can be inferred from table 3.4. First of all, the exceptional groups of the A-D-E-classification (of simply-laced simple Lie algebras) appear. Secondly, one needs to dualise potentials of higher rank to axions to realise the symmetry group G. Also the H groups are no longer orthogonal and one needs a generalised notion of orthogonality. Some details can be found in [86].

The appearing symmetry groups G can be represented by Dynkin diagrams. Here each node represents a simple root (spanning the space of positive roots) and the number of lines (zero, one, two or three) between two nodes corresponds to an angle of 90, 120, 135 or 150 degrees between the associated simple roots. In the algebras that we encounter all simple root vectors have the same length (simply laced algebras) and angles of 120 degrees with respect to each other. The Dynkin diagrams of maximal supergravity are distilled into figure 3.1. Indeed, continuation to $D < 6$ brings one to the exceptional Lie algebras.

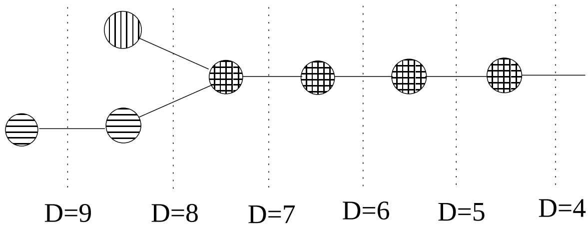

Figure 3.1: *The Dynkin diagrams of the symmetry groups G of maximal supergravity in different dimensions summarised in one picture. Given D, the part to the left of the corresponding split is relevant. The horizonal and vertical fillings correspond to the 11D and IIB origin, respectively.*

Note that the Dynkin diagram is very reminiscent of the possible maximal supergravities; with a highest node in 11D, two possibilities in 10D and unique possibilities in $D \leq 9$. Indeed, one can view the symmetry group G as coming from the higher-dimensional origin: reduction over a d-torus gives rise to an $SL(d, \mathbb{R})$ symmetry (as explained in section 4.2). Thus, one can understand the horizontally filled nodes as coming from 11D while the vertical fillings come from IIB. Together, these two subgroups generate the full duality group G in any dimension $D \leq 9$ [87].

As an amusing note we would like to mention that the same phenomenon occurs in $Q = 16$ supergravity. As discussed in section 3.1, these are unique in all dimensions but six, where one encounters non-chiral iia and chiral iib, similar to IIA and IIB in $D = 10$. Again, the existence of this extra supergravity in six dimensions gives rise to an extra $SL(2, \mathbb{R})$ symmetry in four dimensions.

However, despite many similarities, the above discussion does not directly carry over to theories with less supersymmetry. For example, the global symmetry group G is always maximally non-compact for the case of maximal supersymmetry. In less supersymmetric cases this is not necessarily true, in which case one should not exponentiate all Cartan generators but only the non-compact ones. Some of these issues are discussed in [81].

3.4 Supergravity Solutions

Generic Brane Solutions

In the previous subsections we have seen that supergravity theories generically contain bosonic fields of spin 0,1 and 2, corresponding to a scalar, a rank-d potential and the graviton. In this subsection we will take a step back and discuss generic solutions to this system called p-brane solutions. These are generalisations of the extremal Reissner-Nordström charged black hole to $d \neq 1$ (the rank of the gauge potential) and $D \neq 4$ (the dimension of space-time). These will occur frequently as supersymmetric solutions of supergravities, as we will find below. For reviews see e.g. [88, 89].

The starting point is the D-dimensional toy model Lagrangian

$$\mathcal{L} = \sqrt{-g}[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{a\phi}G^{(d+1)} \cdot G^{(d+1)}] , \tag{3.33}$$

with the rank-(d+1) field strength $G^{(d+1)} = dC^{(d)}$. It consists of an Einstein-Hilbert term, a dilaton kinetic term and a kinetic term for a rank-d potential with arbitrary dilaton coupling, parameterised by a. For future use we define the constants [90]

$$\Delta = a^2 + \frac{2d\tilde{d}}{D-2} , \qquad \tilde{d} = D - d - 2 . \tag{3.34}$$

The constant Δ will play an important role in the characterisation of solutions. In particular, in many supergravities it will be given by $4/n$ with n a positive integer and the corresponding p-brane solutions will preserve a fraction $1/2^n$ of the supersymmetry.

Due to the presence of the gauge potential, solutions to this system can carry electric and magnetic charge, defined by

$$Q_e = \int_{S^{\tilde{d}+1}} e^{a\phi} \star G^{(d+1)} , \qquad Q_m = \int_{S^{d+1}} G^{(d+1)} . \tag{3.35}$$

These are conserved due to the field equation of $C^{(d)}$ and the Bianchi identity of $G^{(d+1)}$, respectively, and can be seen as generalisations of the Maxwell charges in 4D. Hodge dualisation interchanges the electric and magnetic charges since the dual field strengths are related by (in analogy to (3.12))

$$e^{a\phi}G^{(d+1)} = \star G^{(\tilde{d}+1)} , \tag{3.36}$$

where $G^{(\tilde{d}+1)} = dC^{(\tilde{d})}$. Under this dualisation the field equations for $C^{(d)}$ is transformed to the Bianchi identity for the dual field strength $G^{(\tilde{d}+1)}$ while the Bianchi identity for $G^{(d+1)}$

corresponds to the field equation for the dual potential $C^{(\tilde{d})}$. Also Δ is invariant under Hodge dualisation since this interchanges d and $\tilde{d}$ and flips the sign of a.

The system (3.33) allows for two p-brane solutions, where p refers to the dimensionality of the spatial extension of the brane, that carry one of the charges (3.35):

$$\begin{aligned} \text{electric } p\text{-brane:} \qquad & p = d - 1\,, \\ \text{magnetic } p\text{-brane:} \qquad & p = \tilde{d} - 1\,. \end{aligned}$$

The dimension of the world-volume equals $p + 1$ while the remainder $D - p - 1$ is the dimension of the transverse space and is called the codimension.

We will discuss the electric and magnetic p-brane solutions at the same time. To this end, we split up the coordinates in the world-volume t, x^i with $i = 1, \ldots, p$ and the transverse space x^m with $m = p + 1, \ldots, D - 1$. The metric and dilaton are given by

$$ds^2 = H^{-4\tilde{d}/(\Delta(D-2))}(-dt^2 + dx_i{}^2) + H^{4d/(\Delta(D-2))}dx_m{}^2\,, \quad e^{\phi} = H^{\pm 2a/\Delta}\,, \tag{3.37}$$

where the electric and magnetic solutions have a $+$ and a $-$ sign, respectively. The corresponding field strengths are given by[9,10]

$$G_e^{(d+1)} = \frac{2}{\sqrt{\Delta}}\, dt\wedge dx^1\wedge\cdots\wedge dx^p\wedge dH^{-1}\,, \quad G_m^{(d+1)} = \frac{2}{\sqrt{\Delta}}\star(dt\wedge dx^1\wedge\cdots\wedge dx^p\wedge dH)\,. \tag{3.38}$$

The p-branes are characterised by the function $H(x^m)$, which is given by (for the moment we assume $p < D - 3$; we will discuss the other cases later)

$$H = c + \frac{Q}{r^{D-p-3}}\,, \tag{3.39}$$

with $r = \|x^m\|$. The integration constants c and Q are taken both positive to avoid naked singularities at finite r. All such p-brane solutions have $ISO(1,p)\times SO(D-p-1)$ isometry. For branes with $a \neq 0$ the constant c can be related to the asymptotic value of ϕ via $g_s = \exp(\phi)_\infty = c^{\pm 2a/\Delta}$.

The p-brane solutions have a horizon at $r = 0$. Depending on D, p and Δ the horizon may coincide with a singularity or it may be possible to find a geodetically complete extension of the solution. We will not pursue the solution behind the horizon and will content ourselves with the description of the $0 < r < \infty$ part of space-time, thus avoiding the possible necessity for a source term. This part interpolates between two different vacua of the theory [91]: one

[9] We give only the so-called brane solutions with positive charge; anti-brane solutions carry negative charge and have an extra $-$ sign in (3.38).

[10] An additional possibility for $d = D/2 - 1$ is the dyonic brane carrying both electric and magnetic charge. In such cases, both lines of (3.38) are valid, with an extra factor of $1/2$ on the right-hand sides.

finds D-dimensional Minkowski space for $r \to \infty$ and a metric which is conformal to a product of Anti-de Sitter space[11] and a higher-dimensional sphere:

$$ds^2 = H^{2a^2/(\Delta(D-p-3))}(ds^2(AdS_{p+2}) + ds^2(S^{D-p-2}))\,, \tag{3.40}$$

for $r \to 0$, which is called the near-horizon limit.

The p-brane solutions carry mass and charge density. The ADM mass per unit p-brane volume is given by

$$M = \frac{4Q(D-p-3)g_s^{-a/2}\Omega_{D-p-2}}{\sqrt{\Delta}}\,, \tag{3.41}$$

where Ω_{D-p-2} is the volume of the unit $(D-p-2)$-sphere that surrounds the $p+1$-dimensional world-volume. Computing the charge densities from (3.35), one finds that there is an equality between the mass and (the absolute value of) the charge density: $M = |Q_e|$ for the electric solution and $M = |Q_m|$ for the magnetic solution. In supergravity theories this will generically lead to an amount of preserved supersymmetry.

There are several generalisations of the prime examples (3.37), (3.38) of p-brane solutions. For instance, one can replace the function $H = H(x^m)$ by any solution to the Laplace equation

$$\Box H(x^m) = \partial_n \partial^n H(x^m) = 0\,, \tag{3.42}$$

in $(D-p-1)$-dimensional flat transverse space. Examples are

- the multi-center p-brane solution with

$$H = c + \sum_i \frac{Q_i}{||x^m - x_i^m||^{D-p-3}}\,, \tag{3.43}$$

  with all Q_i positive to avoid naked singularities at finite x^m. Its interpretation consists of a number of p-branes located at x_i^m. Physically, this solution is possible since all separate p-branes have equal mass and charge; for this reason their attractive force (due to gravity and the scalar) cancels their repulsive force (due to the rank-d potential).

- the smeared p-brane solution with $H = H(x^m)$ a harmonic function in a subspace of the full transverse space. An example is the following function for $p < D-4$:

$$H = c + \frac{Q}{||x^{\tilde{m}}||^{D-p-4}}\,, \tag{3.44}$$

  where $\tilde{m} = p+1, \ldots, D-2$, i.e. the harmonic function does not depend on x^{D-1}. This can be interpreted as the configuration of a smooth distribution of p-brane in the x^{D-1}-direction. The smeared solutions will be very relevant later for the relation between the different solutions.

[11] An exception is the case $a^2 = 2\tilde{d}^2/(D-2)$: in this case the radius of the Anti-de Sitter space-time becomes infinite and the AdS-part reduces to (p+2)-dimensional Minkowski space-time [91, 92].

These generalisations break part of the isometry group. However, since the mass and charge of these solutions are still equal, they will preserve supersymmetry in a supergravity theory.

It is also possible to add mass to the p-brane solution without affecting its charge: $Q_{e,m} < M$. This generically breaks the supersymmetry and (part of the) isometry of the solutions. For example, one can construct non-supersymmetric solutions with isometry group $\mathbb{R} \times ISO(p) \times SO(D-p-1)$ [93, 94]. Such deformations are only possible for the single-center solution (3.39) and not for its multi-center generalisation (3.43), as can physically be understood from the inequality of mass and charge: the attractive and repulsive forces between different constituents no longer cancel.

Branes with Little Transverse Space

Let us now discuss branes with $p \geq D-3$, starting with the case that saturates this bound. Such branes are sometimes called vortex branes and have a two-dimensional transverse space. The most symmetric harmonic function reads (with $r = \|x^m\|$)

$$H = c + Q\log(r)\,, \tag{3.45}$$

giving rise to $ISO(1, D-3) \times SO(2)$ isometry. The limit $r \to \infty$ in this case does not yield D-dimensional Minkowski but an asymptotically locally flat space-time; locally this is Minkowski but a global difference occurs in the form of a deficit angle in the 2D transverse space, stemming from the mass density of the $(D-3)$-brane solution. The other limit, $r \to 0$, is not well-defined since the harmonic function becomes negative at finite r, thus rendering this solution valid only for r large enough. However, there are modifications of this solution with the same large-r behaviour and a well-defined interior [95].

The next case concerns $(D-2)$-branes which are usually referred to as domain walls. Their transverse space is one-dimensional, on which the most general harmonic function reads (where $y = x^{D-1}$)

$$H = c + Qy\,, \tag{3.46}$$

where we take Q positive. Note that a potential of rank $D-1$, corresponding to an electric domain wall, carries no degrees of freedom (see table 3.3). Its Hodge dual

$$G^{(0)} = e^{a\phi} \star G^{(D)} = 2Q/\sqrt{\Delta}\,, \tag{3.47}$$

is a constant zero-form field strength and can be interpreted as a mass parameter. We thus find that mass parameters can support domain walls. A necessary condition for this is the quadratic term in (3.33) with $d = 0$. Rather than a kinetic term it is called a scalar potential (due to the coupling to the dilaton) and its form determines the possible properties of domain wall solutions. We will encounter many examples of scalar potentials in gauged supergravities, see section 5.

Again, one might wonder if the domain wall solution interpolates between different vacua. Due to the one-dimensional transverse space, the domain walls differ in this respect from the other p-branes. One can always do a reparameterisation of the transverse coordinate [96] to obtain the metric of either conformal Anti-de Sitter space-time or of conformal Minkowski space-time. However, the domain wall as it stands is certainly not a globally well-defined solution[12]: one finds that the harmonic function vanishes for finite y. To remedy the resulting singularity, one has to patch solutions with different values for the mass parameters. This requires the presence of source terms, whose charge is related to the difference between the values of the mass parameters on both sides of the domain wall. We will discuss an example of such a source term in section 6.1.

Domain walls of the above type are usually called thin domain walls: the source term corresponds to a object of infinitesimal thickness in the transverse direction. Such source terms are always necessary with potentials of the form (3.33) with $p = D - 2$, which have only one asymptotic minimum (with $\phi \to \pm\infty$). In contrast, potentials with more than one (local) minima allow for solutions interpolating between two minima. Such smooth configurations are called thick domain walls. We will mostly encounter the thin version in this thesis, however.

Taking the p-brane classification one step further by considering $p = D - 1$ brings us to space-time-filling branes. All of space-time is world-volume and there is no transverse space. Though not very interesting from a supergravity point of view there is an appreciation of space-time filling branes in string theory [24, 97].

Maximally Supersymmetric Solutions

In section 3.2 we have encountered different supergravity theories in eleven and ten dimensions. In this subsection we will discuss solutions of these theories that preserve a fraction of supersymmetry.

From the supersymmetry transformations one can deduce which solutions can preserve supersymmetry. We will only consider bosonic solutions. For these to preserve supersymmetry, the right-hand side of the supersymmetry transformations of the fermions must vanish. These conditions are the Killing spinor equations. Here one distinguishes two possibilities: either all terms in the variation of the fermions vanish separately, leading to maximally supersymmetric solutions, or there is a cancellation between non-zero terms. The latter case will involve a condition on the supersymmetry parameter ϵ due to the different Γ-structures. The supersymmetry parameter subject to this condition is called the Killing spinor. Since it is constrained this will lead to solutions preserving only fractions of supersymmetry.

All maximally supersymmetric solutions to maximal supergravity in eleven and ten dimensions have been classified [98]. Minkowski space-time without field strengths is a maximally supersymmetric solution to 11D, IIA and IIB supergravity. In addition to this trivial vacuum, there are so-called AdS $\times$ S and plane wave solutions that preserve all supersymmetry. The AdS $\times$ S metric consists of a product of a $(d + 1)$-dimensional Anti-

[12] Except for the case $a = 0$, in which the domain wall solution yields Anti-de Sitter space-time (without conformal factor). Indeed, the scalar potential becomes a pure cosmological constant in this limit.

de Sitter space-time and an $(D-d-1)$-dimensional sphere, whose isometry group is $SO(1,d+1)\times SO(D-d)$ (which is considerably larger than that of the brane solutions with rank-$(d+1)$ field strengths). In addition, there is a flux of the rank-$(d+1)$ field strength though the sphere. In eleven dimensions one has such solutions with $d=3$ and $d=6$ [99,100] while IIB allows for the $d=4$ case. The plane wave solution, found in 11D [101] and in IIB [102], has the metric of a gravitational plane wave and a constant null flux of the rank-four and self-dual rank-five field strength, respectively. Only recently has it been appreciated [103] that the maximally supersymmetric plane wave is the Penrose limit [104, 105] of the AdS $\times$ S solutions.

Half-supersymmetric Solutions

The solutions preserving half supersymmetry have also received a lot of attention. Here the Killing spinor is subject to a projection:

$$\tfrac{1}{2}(1\pm O)\epsilon=\epsilon\,,\qquad O^2=1\,. \tag{3.48}$$

The possible projectors of 11D, IIA and IIB supergravity are given in table 3.5. Each theory has a number of p-brane solutions while they have the plane wave and Kaluza-Klein monopole in common.

O	11D, IIA, IIB solution	O	11D solution
$\Gamma_{\underline{01}}$	pp-wave	$\Gamma_{\underline{012}}$	M2-brane
$\Gamma_{\underline{1234}}$	Kaluza-Klein monopole	$\Gamma_{\underline{12345}}$	M5-brane
O	IIA solution	O	IIB solution
$\Gamma_{\underline{0}}\Gamma_{11}$	D0-brane	$\mathrm{i}\Gamma_{\underline{01}}\star$	D1-brane
$\Gamma_{\underline{01}}\Gamma_{11}$	F1-brane	$\Gamma_{\underline{01}}\star$	F1-brane
$\Gamma_{\underline{012}}$	D2-brane	$\Gamma_{\underline{01234}}\star$	D3-brane
$\Gamma_{\underline{12345}}$	D4-brane	$\mathrm{i}\Gamma_{\underline{1234}}\star$	D5-brane
$\Gamma_{\underline{1234}}\Gamma_{11}$	NS5-brane	$\Gamma_{\underline{1234}}\star$	NS5-brane
$\Gamma_{\underline{123}}\Gamma_{11}$	D6-brane	$\mathrm{i}\Gamma_{\underline{12}}$	D7-brane

Table 3.5: *Possible projection operators of the supersymmetry transformations of 11D, IIA and IIB supergravity and the corresponding half-supersymmetric solutions.*

The branes of table 3.5 are labelled by their value of p, which equals $d-1$ for the electric solution and $\tilde{d} - 1 = D - d - 3$ for the magnetic solution. Their metric, dilaton and field strength are given in (3.37) and (3.38). In addition, their values of a (the dilaton coupling to the field strength kinetic term in the electric formulation) can be read off from (3.6), (3.16) and (3.18):

- $a = 0$ for the M-branes [106, 107],
- $a = -1$ for the F1-brane [108],
- $a = \frac{1}{2}(3 - p)$ for the D-branes [93],
- $a = +1$ for the NS5-brane [109].

From (3.34) it follows that these branes all have $\Delta = 4$. Such branes preserve half of supersymmetry. Note that a vanishes for the M2-, M5- and D3-brane[13]. This has an important consequence: their near-horizon limits (3.40) are of the form $AdS^4 \times S^7$, $AdS^7 \times S^4$ and $AdS_5 \times S^5$ (without conformal factor), respectively, which are maximally supersymmetric vacua of 11D and IIB supergravity. Thus one finds isometry and supersymmetry enhancement in the near-horizon limit for these branes.

We can now interpret the brane solutions of IIA and IIB supergravity in the context of string theory. An important tool will be the dependence of the mass on the coupling constant g_s, which is given by[14]

$$M \sim g_s^{-(2a+p+1)/4} \,. \tag{3.49}$$

The F1-solution corresponds to the fundamental string, which is charged with respect to the NS-NS 2-form B. Its mass scales like ${g_s}^0$. The Dp-brane solutions are interpreted as the $p{+}1$-dimensional hyperplanes on which open strings can end [62], due to the Dirichlet boundary condition (2.7) (see figure 2.1). These carry charge of the corresponding R-R potential $C^{(p+1)}$ and their masses scale as $1/g_s$, which is in between fundamental and solitonic behaviour. The microscopic understanding of D-branes in terms of open strings with Dirichlet boundary conditions was one of the key insights that led to the second superstring revolution. Note that the remaining brane solution, the NS5-brane, has a mass that scales like $1/{g_s}^2$ and can thus be considered truly solitonic.

In addition to the brane solutions one has so-called pp-wave solutions[15]. Its metric and field strength read (in light-cone coordinates $x^{\pm} = t \pm x^1$ and x^m with $m = 2, \ldots, D-1$):

$$ds^2 = 2dx^+dx^- + H(x^m, x^-)(dx^-)^2 + (dx^m)^2 \,, \qquad G^{(d+1)} = dx^- {\wedge} \xi^{(d)} \,, \tag{3.50}$$

[13]In fact, the D3-brane carries both electric and magnetic charge (it is dyonic), due to the self-duality condition on its five-form field strength. For this reason, in contrast to all other branes, both lines of (3.38) are valid, but with an extra factor of $1/2$ on the right-hand sides.

[14]The difference with (3.41) is due to the field redefinition $g_{\mu\nu} \to e^{\phi/2} g_{\mu\nu}$ between Einstein and string frame.

[15]Here *pp* stands for *plane fronted with parallel rays*. The former refers to the planar nature of the wave fronts while the latter denotes the existence of a covariantly constant null vector.

where H and $\xi^{(d)}$ satisfy the requirements

$$\Box H = -\tfrac{1}{4}\|\xi^{(d)}\|^2\,, \qquad d\xi^{(d)} = d\star\xi^{(d)} = 0\,, \tag{3.51}$$

which are all defined on the transverse Euclidean space with coordinates x^m (for all x^-). The field strength $G^{(d+1)}$ can be the four-form field strength of 11D or several field strengths of IIA and IIB. This pp-wave solution generically preserves half supersymmetry (with the projector as given in table 3.5) but special choices of H and $\xi^{(d)}$ give rise to more supersymmetry [110–113]. For 11D and IIB one obtains maximal supersymmetry for the truncation to the plane wave

$$\begin{aligned}
&\text{11D:} \qquad \begin{cases} \xi^{(3)} = \mu dx^2{\wedge}dx^3{\wedge}dx^4\,, \\ H(x^m,x^-) = -\tfrac{1}{9}\mu^2((x^2)^2+(x^3)^2+(x^4)^2) - \tfrac{1}{36}\mu^2((x^5)^2+\cdots+(x^{10})^2)\,, \end{cases}\\
&\text{IIB:} \qquad \begin{cases} \xi^{(4)} = \mu dx^2{\wedge}dx^3{\wedge}dx^4{\wedge}dx^5 + \mu dx^6{\wedge}dx^7{\wedge}dx^8{\wedge}dx^9\,, \\ H(x^m,x^-) = -4\mu^2((x^2)^2+\cdots+(x^9)^2)\,. \end{cases}
\end{aligned} \tag{3.52}$$

Another special case is the Brinkmann wave [114], a purely gravitational solution with $\xi^{(d)} = 0$. It is described in terms of one harmonic function, i.e. a function satisfying $\Box H = 0$. There is in general no supersymmetry enhancement for this case.

Another purely gravitational solution of 11D, IIA and IIB is provided by the Kaluza-Klein monopole [115, 116] ($m = 1,2,3$ and $i = 5,\dots,D-1$):

$$ds^2 = -dt^2 + dx_i^2 + H^{-1}(dx^4 + A_m dx_m)^2 + H dx_m^2\,, \tag{3.53}$$

where the functions $H = H(x^m)$ and $A_m = A_m(x^n)$ are subject to the condition

$$F_{mn} = \tfrac{1}{2}(\partial_m A_n - \partial_n A_m) = \varepsilon_{mnp}\partial_p H\,. \tag{3.54}$$

This metric is the product of a Minkowski space-time and the 4D Euclidean Taub-NUT space with isometry direction x^4. The $SO(3)$ isometric case is given by (where $r = \|x^m\|$)

$$H = c + \frac{Q}{r}\,. \tag{3.55}$$

This gives rise to a regular geometry if the isometry direction x^4 is compact with period $4\pi Q$ [117]. Its near-horizon limit $r \to 0$ gives rise to flat space-time and thus indeed gives rise to both isometry and supersymmetry enhancement. In addition to the $SO(3)$ isometric case, one can take also take multi-centered solutions or smeared versions, as discussed in 3.4. The Kaluza-Klein monopole also preserves half of supersymmetry for generic choices of the harmonic function.

Relations between Half-Supersymmetric Solutions

The above solutions constitute all known maximally and half-supersymmetric solutions of eleven- and ten-dimensional maximal supergravity. Since the theories in 10D and 11D are related to each other upon dimensional reduction, as we found in section 3.2, one can also relate their solutions. One provision is that the solution must have the correct isometry to allow for this reduction. Reduction in a transverse direction is therefore only possible for smeared solutions with harmonic functions that have an extra isometry. Reduction in a world-volume direction is always possible. Thus, reduction of the two M-branes gives rise to four different brane solutions of IIA supergravity. Similar remarks hold for the relations between IIA and IIB solutions.

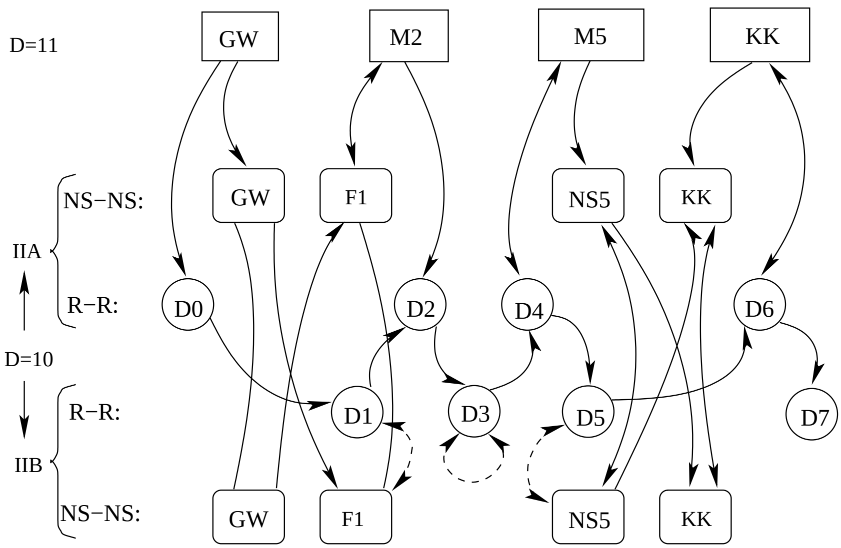

Figure 3.2: *The web of half-supersymmetric solutions and their relations in D=10 and D=11 maximal supergravities. Solid lines correspond to dimensional reduction or T-duality, the dashed lines correspond to S-duality. If an arrow ends with a head, the operation leads to the maximally isometric solution; if not, one obtains a smeared version. Adapted from [97].*

In figure 3.2 we show the relations between the different solutions that preserve half of supersymmetry. Note that the solutions in the NS-NS sectors of IIA and IIB transform into each other; the same holds for the D-branes[16] of the R-R sectors. As for the pp-wave solutions, we have only considered their purely gravitational limit (the gravitational wave) since the solution is then expressible in terms of a harmonic function, which greatly simplifies

[16]Indeed, T-duality interchanges the Neumann and Dirichlet boundary conditions of section 2.1; for this reason it also relates the different D-branes in string theory.

the T-duality discussion.

Furthermore, solutions with less than 1/2 supersymmetry have been studied extensively. For example, it has been known for long that 11D supergravity allows for solutions preserving 1/4 or 1/8 supersymmetry [107]. Only later these were understood as intersections of different solutions preserving 1/2 supersymmetry [118]. A lot of intersections have been studied since, see [119] for a review.