M-theory and gauged supergravities

Roest, Diederik

Chapter 2

String and M-Theory

In this chapter we will consider string theory and M-theory, the main candidates for theories of quantum gravity. We will discuss the different string theories, their interrelations and the emergence of supergravities and M-theory. The literature on this introductory subject is vast. Good textbooks are the classics [40, 41] and the more recent reviews [42, 43], including the recent developments on D-branes and dualities.

2.1 Bosonic Strings

In this section we will discuss the classical dynamics and the quantisation of the bosonic string.

Free String Theory

For reasons of analogy, let us start with the more familiar notion of a particle. A particle moving in D-dimensional space-time is described by coordinates $X^\mu(\tau)$ with $\mu = 0, \ldots, D-1$. The parameter τ is the time-evolution parameter (which is not equal to the time coordinate $X^0(\tau)$). The motion of the particle is a function of τ and defines a trajectory in space-time, the world-line. A free particle, experiencing no force, moves in a straight line. Its dynamics are therefore dictated by the minimalisation of the length of its world-line:

$$S = -m \int d\tau \sqrt{-\det(\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu})}\,, \qquad \eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1) \tag{2.1}$$

where m is the mass and $\eta_{\mu\nu}$ the Minkowski space-time metric. To describe a particle in a more general background, e.g. in a curved space-time, it suffices to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}(\tau)$. The corresponding field equations for $X^\mu(\tau)$ will define a geodesic in this curved background.

A string in D-dimensional space-time is described by coordinates $X^\mu(\tau,\sigma)$. Apart from the time-evolution parameter the space-time coordinates depend on a spatial parameter $\sigma \in (0,l)$. The string motion sweeps out a two-dimensional surface in space-time, the world-sheet, which is parameterised by $\sigma^i = (\tau,\sigma)$. Again the dynamics of the free string are dictated by the minimalisation of the surface of the world-sheet:

$$S = -T\int d^2\sigma\,\sqrt{-\det(g_{ij})}\,, \qquad g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu}\,. \tag{2.2}$$

This is the Nambu-Goto action, where g_{ij} will be called the induced metric on the world-sheet and is the pullback of the space-time metric $\eta_{\mu\nu}$. The constant T is the tension of the string and has mass dimension 2. Often it is rewritten as $T = 1/\alpha' = 1/{l_s}^2$. The parameter $l_s = \sqrt{\alpha'}$ is called the string length, which introduces a fundamental scale in the theory. Agreement with general relativity, emerging later, requires $l_s \sim 10^{-33}$ cm. Generic strings are of the order of this size; only in this regime objects are no longer point-like.

The square root in the string action (2.2) troubles analysis. It is convenient to introduce an auxiliary field γ_{ij}, which we call the world-sheet metric. The alternative action then reads [44]

$$S = -\tfrac{1}{2}T\int d^2\sigma\,\sqrt{-\gamma}\gamma^{ij}g_{ij}\,, \tag{2.3}$$

giving rise to the equation of motion for the auxiliary field γ

$$\frac{\delta\mathcal{L}}{\delta\gamma^{ij}} = -\tfrac{1}{2}T\sqrt{-\gamma}[g_{ij} - \tfrac{1}{2}\gamma_{ij}\gamma^{mn}g_{mn}] = 0 \tag{2.4}$$

and is solved by $\gamma_{mn} = \Omega(\sigma^i)g_{mn}$. After substitution of this solution in the action (2.3) one easily obtains the original Nambu-Goto action (2.2). Instead of eliminating γ, consider the symmetries of (2.3). The action is invariant under [45]:

- global space-time Poincaré transformations: $X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu$,
- local world-sheet reparameterisations: $\sigma^i \to {\sigma'}^i = f^i(\sigma^j)$,
- local world-sheet Weyl invariance: $\gamma_{ij} \to {\gamma'}_{ij} = \Gamma(\sigma^i)\gamma_{ij}$.

One can use the second symmetry to obtain the so-called orthonormal gauge where $\gamma_{ij} = \Delta(\tau,\sigma)\eta_{ij}$ with $\eta_{ij} = \mathrm{diag}(-1,1)$ and Δ arbitrary. In this gauge, the equation of motions for X^μ read

$$({\partial_\tau}^2 - {\partial_\sigma}^2)X^\mu = 0\,. \tag{2.5}$$

This is the wave equation and its solution splits up in a left- and a right-moving sector $X^\mu_\pm(\tau\pm\sigma)$. In addition one must satisfy the boundary condition

$$(\partial_\sigma X^\mu)\delta X_\mu\Big|_{\sigma=0}^{\sigma=l} = 0\,, \tag{2.6}$$

which can be done in three ways:

$$\begin{aligned}
&\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,l) = 0\,, && \text{- Neumann (open)}\,,\\
&\delta X_\mu(\tau,0) = \delta X_\mu(\tau,l) = 0\,, && \text{- Dirichlet (open)}\,,\\
&X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+l)\,, && \text{- Periodic (closed)}\,.
\end{aligned} \tag{2.7}$$

One can take Neumann boundary conditions for all coordinates, giving rise to a freely moving open string. Changing Neumann to Dirichlet boundary conditions for n of the spatial coordinates, one obtains an open string whose endpoints are stuck on a $(D-n-1)$-dimensional hypersurface, which is called a D-brane[1]. In contrast, one can also impose periodic boundary conditions for all coordinates but time, giving a closed string. The different possibilities are illustrated in figure 2.1. Note that one must also satisfy the equation of motion of γ (2.4), which now reads

$$g_{ij} - \tfrac{1}{2}\eta_{ij}\eta^{mn}g_{mn} = 0\,, \tag{2.8}$$

and requires the vanishing of the 2D energy-momentum tensor. This gives rise to the Virasoro constraints, which play an important role in the quantisation of the string.

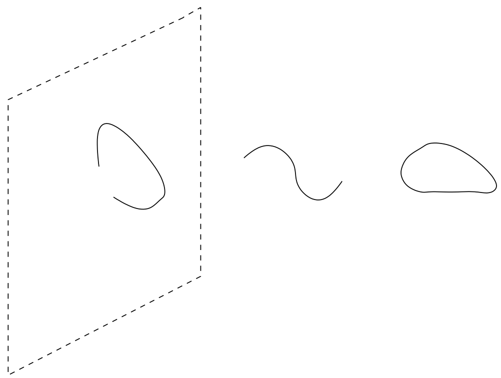

Figure 2.1: *From left to right: strings with Dirichlet, Neumann and periodic boundary conditions. The dashed lines indicate the D-brane on which open strings with Dirichlet boundary conditions can end.*

To obtain the spectrum of the string it is instructive to compare two ways of quantisation. Light-cone quantisation first solves for the constraints (2.8), after which quantisation is straight-forward. To regain Lorentz covariance in the final results this requires $D = 26$. To determine the correct number of states, one compares these results to those of covariant quantisation. This formalism is manifestly Lorentz covariant and imposes (2.8) by requirements

[1] For many years, this boundary condition was thought unphysical due to the breaking of Poincaré invariance; lately, such hypersurfaces are celebrated ingredients of the second superstring revolution.

on physical states (analogous to the Gupta-Bleuler method in electrodynamics). Physical null states arise, which are completely decoupled (having vanishing matrix element with any physical state). To determine the number of these null states, one combines the results of the two formalisms.

It turns out that the open string spectrum consists of the following modes of vibration:

- the vacuum $|0, k>$ with $M^2 = -\hbar/\alpha'$, corresponding to a tachyonic scalar,
- the first excited state $|1, k>$ with $M^2 = 0$, corresponding to a massless vector A_μ with 24 independent components,

plus an infinite tower of massive modes. The scalar particle, having imaginary mass, is a tachyon and moves faster than the speed of light. This unwanted feature of the theory will be projected out when considering the superstring. The mass gap between subsequent modes is $\sqrt{\hbar/\alpha'}$ and to make contact with gravity (which will emerge later) this must be of the order of 10^{19} GeV.

The closed string spectrum is a bit richer; its lowest modes are

- the vacuum $|0, k>$ with $M^2 = -4\hbar/\alpha'$, which corresponds to a tachyonic scalar,
- the first excited state $|1, k>$ with $M^2 = 0$. This state is the product of two vector representations and has 24^2 independent components. These split up into massless representations of the little group in 26 dimensions, $SO(24)$:

$$\mathbf{24} \times \mathbf{24} = \mathbf{299} + \mathbf{276} + \mathbf{1}\,, \tag{2.9}$$

  corresponding to the space-time fields $g_{\mu\nu}, B_{\mu\nu}, \phi$ of spin $2, 1$ and 0, respectively.

The tachyonic states present in both spectra will be lost when going to the superstring and therefore we will not worry about them. Rather we will focus on the massless states of the closed string spectrum.

Interactions and Background

Interactions are introduced by coupling the string to the background fields. The most general covariant action for the closed string with two world-sheet derivatives reads

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\sqrt{-\gamma}\gamma^{ij} g_{ij} - \epsilon^{ij} B_{ij} - \alpha' \sqrt{-\gamma}\, \phi\, \mathcal{R}\right] \tag{2.10}$$

where $g_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}$ and $B_{ij} = \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}$ are the pullbacks of the space-time fields $g_{\mu\nu}$ and $B_{\mu\nu}$ and $\mathcal{R}$ is the Ricci scalar of the world-sheet metric γ_{ij}. This action is called the non-linear σ-model. Note that, due to the space-time dependence of the background fields, this action is no longer bilinear in X^μ.

One can now calculate scattering amplitudes between different string modes via the string path integral. Here one sums over the different topologies of the world-sheet; perturbative string theory can be seen as a sum of conformal field theories on different Riemann surfaces of genus g. This is the string analogue of Feynman diagrams. There is of course a number of differences. While the number of Feynman diagrams grows rapidly with the number of loops, there is a (topologically) unique Riemann surface of genus g. In addition, all such scattering amplitudes are finite, in contrast to QFT and GR.

The third term of the action (2.10) has a special feature. Assuming a constant mode of the dilaton ϕ_0, this part is proportional to the Euler characteristic χ:

$$\chi = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{-\gamma}\,\mathcal{R} = 2-2g$$

where the genus g is the number of holes of the world-sheet. Therefore, we find that the amplitude of a string diagram of genus g is multiplied by $(e^{\phi_0})^{2g-2}$. This implies that we can associate the expectation value of e^{ϕ} with the string coupling constant g_s. String theory can therefore generate its own coupling constant by restricting the possible dilaton backgrounds.

The scattering amplitudes for the massless modes can be summarised by the effective action. For the closed bosonic string, it reads

$$S = \frac{1}{2\kappa^2}\int d^{26}X\,\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \tfrac{1}{2}H\cdot H\right]\,. \tag{2.11}$$

where R is the Ricci scalar for the background metric $g_{\mu\nu}$ and $H = dB$ is the field strength tensor of $B_{\mu\nu}$ (where we use form notation, see appendix A). This action is a low energy approximation since we have included only the massless modes of the spectrum and the zeroth order in α'. Note that these terms all comes from tree-level contributions ($g = 0$), as can be seen from the factor of $e^{-2\phi}$. Another way to obtain this effective action is by requiring the classical Weyl symmetry to hold at the quantum level as well [46], in which case the vanishing of the β-functions are interpreted as field equations of (2.11). After a suitable rescaling of the metric $g_{\mu\nu} \to e^{4\phi/(D-4)}g_{\mu\nu}$, the above action contains the known Einstein-Hilbert term in 26 dimensions. This rescaling takes one from string frame (which is the geometry seen by strings) to Einstein frame (which is more convenient for gravity). Including terms of higher order in α' leads to corrections to general relativity.

Note that one can distinguish two different perturbative expansions in string theory. The first one is in the string coupling constant g_s, which is the string analogue of $\hbar$. This parameter controls the contribution of higher-loop diagrams to the effective action. The limit $g_s \to 0$ corresponds to the inclusion of only zero-genus diagrams. The second expansion is in α', which is the square of the string length. This parameter has no counterpart in field theory and arises from the intrinsic scale of strings. The limit $\alpha' \to 0$ corresponds to the field theory limit.

2.2 Superstrings

In the previous section we have obtained the spectra of the open and closed string. Associating space-time particles with every mode, only bosons are present in these spectra. To obtain space-time fermions one needs to introduce Grassmann-odd variables on the world-sheet[2]. This will be done in such a way as to obtain world-sheet and space-time supersymmetry. Supersymmetry relates bosons and fermions since its parameter is a fermion; for a more extensive discussion see section 3.1.

To the bosonic string action (2.3) in orthonormal gauge we add a term including D fermions $\psi^\mu = (\psi_+^\mu, \psi_-^\mu)$ with real components of opposite chirality:

$$S = -\tfrac{1}{2}T \int d^2\sigma \, [\partial_i X^\mu \partial^i X_\mu + i\bar{\psi}^\mu \gamma^i \partial_i \psi_\mu] \, , \tag{2.12}$$

where γ^i are Γ-matrices in two dimensions, satisfying $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$. This action is invariant under the global world-sheet supersymmetry

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu \, , \qquad \delta\psi^\mu = \gamma^i \partial_i X^\mu \epsilon \, , \tag{2.13}$$

with fermionic parameter $\epsilon = (\epsilon_-, \epsilon_+)$. Note that the bosonic and fermionic sectors decouple in this free theory. We therefore take the same bosonic solutions with left- and right-moving sectors $X_\pm^\mu(\tau \pm \sigma)$. In addition, the fermionic solutions also split up in $\psi_\pm^\mu(\tau \pm \sigma)$.

There is also a fermionic boundary condition, which can be satisfied in two ways for open strings. Choosing $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ without loss of generality, we have

- $\psi_+^\mu(\tau, l) = +\psi_-^\mu(\tau, l)$ Ramond
- $\psi_+^\mu(\tau, l) = -\psi_-^\mu(\tau, l)$ Neveu-Schwarz

So, as for the bosonic string, we can impose different boundary conditions. Upon quantisation these yield different spectra: the Neveu-Schwarz sector contains a tachyonic scalar and a massless vector whereas the Ramond sector comprises two space-time spinors of opposite chirality. Another feature of the superstring is a different critical dimension: its quantisation is only consistent if $D = 10$. A second goal is space-time supersymmetry, a fermionic symmetry between the bosonic and fermionic background fields. A necessary requirement for this is an equal number of degrees of freedom in the bosonic and fermionic sector (see section 3.1). To achieve this, we apply the Gliozzi-Scherk-Olive (GSO) projection [50], which boils down to the truncation of states that do not have a counterpart in the other sector. After this projection, the entire spectrum is invariant under supersymmetry transformations with one fermionic parameter: we have obtained the so-called $N = 1$ space-time supersymmetry. In

[2]We will discuss the Neveu-Schwarz-Ramond superstring [47,48] here, which has world-sheet fermions. Another approach is due to Green and Schwarz [49], with world-volume bosons and manifest space-time supersymmetry from the onset.

the massless sector we are left with the Neveu-Schwarz vector and a Ramond spinor:

$$N = 1 \text{ vector:} \qquad (\mathbf{8}_{\rm v})_{\rm B} + (\mathbf{8}_{\rm c})_{\rm F}\,, \tag{2.14}$$

corresponding to the space-time fields A_μ and λ. These form a multiplet under the supersymmetry, which is denoted by the vector multiplet of $N = 1$ supersymmetry in $D = 10$.

For closed strings one can take periodic or anti-periodic boundary conditions for the fermions:

- $\psi^\mu_\pm(\tau,\sigma) = +\psi^\mu_\pm(\tau,\sigma+l)$ Ramond
- $\psi^\mu_\pm(\tau,\sigma) = -\psi^\mu_\pm(\tau,\sigma+l)$ Neveu-Schwarz

Since we can impose (anti-)periodicity for the left- and right-moving components $\psi^\mu_\pm$ separately, we get four different sectors, denoted by e.g. (NS,R) with a Neveu-Schwarz boundary condition on ψ^μ_+ and Ramond on ψ^μ_-. Upon quantisation one obtains a rich spectrum, with a tachyon in the (NS,NS) sector and several massless modes. As for the fermionic open string, we apply a GSO projection. This can be done separately on the left- and right-moving sectors, giving rise to two possibilities, leading to IIA and IIB string theory (depending on whether the GSO projections of the two sectors have opposite or equal signs, respectively). In truncating the spectrum by the GSO projection we have obtained supersymmetry with two fermionic parameters: this is called $N = 2$ space-time supersymmetry. Again, the massless modes form multiplets under this symmetry, which are denoted by the graviton multiplets of IIA and IIB supersymmetry in $D = 10$, respectively:

$$\begin{aligned}
\text{IIA:} \qquad & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm s}] = [(\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\text{NS-NS}} + (\mathbf{56}_{\rm v} + \mathbf{8}_{\rm v})_{\text{R-R}}]_{\rm B} \\
& \qquad\qquad + ([(\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{NS-R}} + (\mathbf{56}_{\rm c} + \mathbf{8}_{\rm c})_{\text{R-NS}}]_{\rm F}\,, \\
\text{IIB:} \qquad & [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] = [(\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\text{NS-NS}} + (\mathbf{35}_{\rm c} + \mathbf{28} + \mathbf{1})_{\text{R-R}}]_{\rm B} \\
& \qquad\qquad + [(\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{NS-R}} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\text{R-NS}}]_{\rm F}\,,
\end{aligned} \tag{2.15}$$

where $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$ are the $SO(8)$ vector representation and spinor representations of opposite chirality, respectively. The corresponding space-time fields will be discussed in section 3.2.

Due to the decoupling of the left- and right-moving sectors of closed strings one can take different choices for either sector, as we have seen in the GSO-projection to IIA and IIB string theory. A more radical difference would be to take a bosonic left-moving and a supersymmetric right-moving sector, leading to the so-called heterotic string [16]. Due to the different critical dimensions ($D = 26$ and $D = 10$, respectively), one has to compactify 16 internal dimensions of the left-moving sector. This can be done consistently in two different ways, yielding gauge groups $SO(32)$ or $E_8 \times E_8$ in $D = 10$. Both heterotic string theories have $N = 1$ space-time supersymmetry.

The remaining possibility consists of both open and closed strings. Note that it is not consistent to only include open strings, since interactions will then lead to closed strings as well. This leads to type I string theory, which is required to have an $SO(32)$ gauge group and also has $N = 1$ space-time supersymmetry. Alternatively, this theory can be obtained by modding out IIB string theory with world-sheet parity Ω, flipping the sign of σ.

The spectrum of the $N = 1$ string theories is free of tachyons, while its massless modes consist of the $N = 1$ graviton multiplet, given by

$$N = 1: \qquad [\mathbf{8}_{\rm v} + \mathbf{8}_{\rm c}] \times \mathbf{8}_{\rm v} = (\mathbf{35}_{\rm v} + \mathbf{28} + \mathbf{1})_{\rm B} + (\mathbf{56}_{\rm s} + \mathbf{8}_{\rm s})_{\rm F}\,, \tag{2.16}$$

coupled to 496 vector multiplets, giving rise to the $SO(32)$ or $E_8 \times E_8$ gauge groups [15]. In addition there is an infinite tower of massive modes, with mass gap proportional to $\sqrt{\hbar/\alpha'}$.

Recapitulating, we find that there are five consistent ten-dimensional string theories. These have $N = 1$ or $N = 2$ space-time supersymmetry and the massless modes consist of the graviton multiplet (plus vector multiplets for $N = 1$). Analogous to the discussion of section 2.1, one can investigate their effective actions. These turn out to be so-called supergravity actions (to lowest order and in string frame). Thus, supergravity in ten dimensions appears as the low-energy effective action of the corresponding string theories; for this reason we will discuss these extensively in section 3.

Note however that there are two important modifications. Firstly, the supergravity actions will have a tower of α'-corrections. These introduce higher-derivative terms, as can be anticipated on dimensional grounds. Secondly, there is a tower of massive modes of mass $\sqrt{\hbar/\alpha'}$, which will become important when considering higher energies. These modifications cure the (alleged) non-renormalisable divergences of supergravity.

2.3 M-theory and Dualities

In the previous section we have considered perturbative superstring theory. It turned out that there were five different candidates for the alleged Theory of Everything; this is often called an embarrassment of riches. In this section we will relate the different $D = 10$ theories and sketch the appearance of an 11D theory, called M-theory. Excellent introductions are [51,52].

In choosing the periodic boundary conditions for the closed string in (2.7) in the previous section we have assumed that the space-time manifold is non-compact. If one assumes one compact direction one can impose a generalised periodic boundary condition in the compact direction X^9:

$$X^9(\tau,\sigma) = X^9(\tau,\sigma + l) + 2\pi n\,R\,, \tag{2.17}$$

with R the radius of the circle. This corresponds to a closed string, winding n times around the circle. It turns out that one can no longer distinguish between IIA and IIB string theory on such manifolds. To be precise, IIA on a circle with radius R is equivalent to IIB on a circle with radius $\tilde{R}$ with the relation $\tilde{R} = \alpha'/R$ [53, 54]. Generically, such a relation between theories on different compactification manifold is called T-duality [55].

To understand how this comes about in the effective actions, we expand the 10D supergravities over the circle. As explained in section 4.1, this yields 9D massless and massive modes. The massless modes can be seen to coincide: the decomposition of $SO(8)$ representations under $SO(7)$ reads

$$\begin{aligned}
&\text{IIA:}\begin{cases}
\text{NS-NS:} & \mathbf{35}_\text{v} \to \mathbf{27}+\mathbf{7}+\mathbf{1}\,, \quad \mathbf{28} \to \mathbf{21}+\mathbf{7}\,, \quad \mathbf{1} \to \mathbf{1}\,,\\
\text{R-R :} & \mathbf{56}_\text{v} \to \mathbf{35}+\mathbf{21}\,, \quad \mathbf{8}_\text{v} \to \mathbf{7}+\mathbf{1}\,,\\
\text{NS-R:} & \mathbf{56}_\text{s} \to \mathbf{48}+\mathbf{8}\,, \quad \mathbf{8}_\text{s} \to \mathbf{8}\,,\\
\text{R-NS:} & \mathbf{56}_\text{c} \to \mathbf{48}+\mathbf{8}\,, \quad \mathbf{8}_\text{c} \to \mathbf{8}\,,
\end{cases}\\
&\text{IIB:}\begin{cases}
\text{NS-NS:} & \mathbf{35}_\text{v} \to \mathbf{27}+\mathbf{7}+\mathbf{1}\,, \quad \mathbf{28} \to \mathbf{21}+\mathbf{7}\,, \quad \mathbf{1} \to \mathbf{1}\,,\\
\text{R-R :} & \mathbf{35}_\text{c} \to \mathbf{35}\,, \quad \mathbf{28} \to \mathbf{21}+\mathbf{7}\,, \quad \mathbf{1} \to \mathbf{1}\,,\\
\text{NS-R:} & \mathbf{56}_\text{s} \to \mathbf{48}+\mathbf{8}\,, \quad \mathbf{8}_\text{s} \to \mathbf{8}\,,\\
\text{R-NS:} & \mathbf{56}_\text{s} \to \mathbf{48}+\mathbf{8}\,, \quad \mathbf{8}_\text{s} \to \mathbf{8}\,,
\end{cases}
\end{aligned} \tag{2.18}$$

while the corresponding relations between supergravity fields is given in (B.9) and (B.14). Thus the massless modes of IIA and IIB supergravity on S^1 are equivalent and indeed are described by the same effective theory, the unique $D = 9$ maximal supergravity. However, the IIA and IIB massive modes, sometimes called momentum modes, are distinct. For this reason IIA and IIB supergravity are only equivalent for $R \to 0$ and $\tilde{R} \to 0$, where such modes become infinitely massive (for more detail, see section 4.1). String theory modifies this situation in the following way. Due to the fact closed strings can wind around the internal direction (2.17), there is an entire tower of massive winding multiplets. Note that this phenomenon is intrinsic to string theory and does not have a counterpart in field theory. It turns out that the combination of massive momentum states and massive winding states yields the same result for IIA and IIB string theory; indeed, these towers of massive states are interchanged under the T-duality transformation[3] on S^1 [53, 54].

Yet more surprises are in store when considering the strong coupling limit of IIA string theory. It is believed that one obtains symmetry enhancement in the strong coupling limit: rather than ten-dimensional one finds eleven-dimensional Lorentz covariance [18]! This means that such a theory lives in 11D; an extra dimension opens up in the strong coupling limit of IIA string theory. The eleven-dimensional theories goes under the name of M-theory, where the meaning of M can range from Membrane to Mother.

A first piece of evidence can be found in the low-energy limit of IIA string theory. Indeed the IIA graviton multiplet can be grouped to representations of $SO(9)$, the little group in 11D:

$$\begin{aligned}
\text{B :} \quad & \mathbf{35}_\text{v}+\mathbf{8}_\text{v}+\mathbf{1} \to \mathbf{44}\,, \quad \mathbf{56}_\text{v}+\mathbf{28} \to \mathbf{84}\,,\\
\text{F :} \quad & \mathbf{56}_\text{s}+\mathbf{8}_\text{s}+\mathbf{56}_\text{c}+\mathbf{8}_\text{c} \to \mathbf{128}\,,
\end{aligned} \tag{2.19}$$

[3]A first confirmation can be found in the gauge vectors of 9D supergravity that couple to these momentum and winding states. These are A^1 and A, respectively, for the IIA theory and interchanged for the IIB theory, see (B.9) and (B.14). For a more extensive discussion of the role of these massive states in 9D supergravity, see [56].

with the corresponding relations between supergravity fields given in (B.4). Indeed, IIA supergravity can be obtained as a reduction of the unique supergravity theory in $D = 11$. Due to the relations with IIA string theory and supergravity, this 11D supergravity is the low-energy effective theory of M-theory. In particular, from the relation (B.4) between the supergravity fields one can read off the following relations between the parameters of IIA and 11D on a circle:

$$l_s{}^2 = \frac{l_p{}^3}{R}\,, \qquad g_s = \left(\frac{R}{l_p}\right)^{3/2}, \tag{2.20}$$

where l_p is the 11D Planck length and R the radius of the internal circle. Indeed, strong coupling in IIA corresponds to a large radius. Though the appearance of eleven-dimensional Lorentz covariance can not be proven in perturbative IIA string theory (since its size is proportional to e^{ϕ}), a lot of evidence for the existence of M-theory has been put forward [18, 57, 58]. For example, the massive Kaluza-Klein states of 11D supergravity are interpreted as the D0-brane states of IIA string theory [58, 59].

The strong coupling limit of IIB string theory can be understood from its conjectured $SL(2,\mathbb{Z})$ symmetry [19]. Indeed, this symmetry is shared by its low-energy approximation and one of its generators acts on the IIB supergravity fields as $\phi \to -\phi$ (for vanishing axion background). This corresponds to a strong-weak coupling transformation due to the interpretation of the dilaton and is called S-duality. For this reason, IIB string theory is understood to be self-dual[4]. At weak coupling, strings are the fundamental, perturbative degrees of freedom while at strong coupling, this role is played by the Dp-branes with $p = 1$.

The $N = 1$ string theories are also related via dualities. The two heterotic theories, with gauge groups $SO(32)$ and $E_8 \times E_8$, are T-dual to each other: upon compactification on a circle these yield the same theory. One can understand this T-duality from the construction of the heterotic string. Remember that one combines a bosonic left-moving sector with a fermionic right-moving sector. The former is then reduced over 16 dimensions, which can be done in two different ways. However, these two compactification procedures turn out to be equivalent upon reducing over an extra circle [61].

As for the strong coupling of $N = 1$ theories, the two $N = 1$ string theories with $SO(32)$ gauge groups have been found to be S-dual [18, 62]. The strong coupling of the remaining theory, heterotic string theory with $E_8 \times E_8$, turns out to be M-theory on an interval $S^1/\mathbb{Z}_2$, breaking half of the supersymmetry [63, 64].

Thus we obtain a web of dualities, as sketched in figure 2.2. M-theory is the fundamental theory for all values of the different parameters, corresponding to the bulk of the octagon; not too much is known about this theory for generic parameters, however. Only around the corners of the octagon does one have a well-defined theory, corresponding to perturbative string theory or 11D supergravity. The different points are related by the web of dualities. Here Ω maps IIB onto type I, as mentioned in section 2.2. This point of view of M-theory is reminiscent of the definition of a manifold, consisting of charts related by transition functions.

[4] This is very similar to the conjectured $SL(2,\mathbb{Z})$ duality of $N = 4$ super-Yang Mills theory in 4D [60].

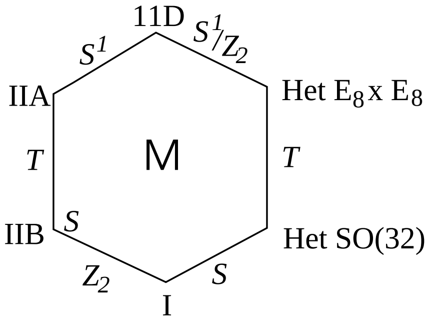

Figure 2.2: *M-theory and the web of relations between the different theories at the corners in the parameter space of M-theory.* S *and* T *stand for S- and T-duality,* S^1 *and* S^1/Z_2 *denote uplift or reduction over a circle and an interval, respectively, while* Z_2 *represents modding out with the* $\mathbb{Z}_2$ *symmetry* Ω.

The web of dualities in figure 2.2 deals with the dualities between different descriptions of M-theory on S^1. One expects a generalisation when considering M-theory on T^n for $n \geq 2$. Indeed, it has been conjectured [19] that M-theory on a torus is invariant under the duality groups given in table 2.1. This duality group of M-theory consists of a combination of T- and S-dualities of string theory and is called the U-duality group. For example, M-theory on T^2 would have an $SL(2,\mathbb{Z})$ U-duality group; this implies that M-theory on different backgrounds that are related by $SL(2,\mathbb{Z})$ are equivalent. These U-duality groups will reappear as global symmetry groups of the corresponding supergravity theories, see section 3.3.

D	U-duality	T-duality
9	$SL(2,\mathbb{Z})$	–
8	$SL(3,\mathbb{Z})\times$ $\times SL(2,\mathbb{Z})$	$SL(2,\mathbb{Z})\times$ $\times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{Z})$	$SO(3,3;\mathbb{Z})$
6	$SO(5,5;\mathbb{Z})$	$SO(4,4;\mathbb{Z})$

Table 2.1: *The U-duality groups of M-theory on* T^{11-D} *and T-duality groups of IIA and IIB string theory on* T^{10-D} *for* $6 \leq D \leq 9$.