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Minimal Models for $\mathcal{N}_\kappa^\infty$ -functions

Aad Dijksma, Annemarie Luger and Yuri Shondin

To Heinz Langer, wishing him a happy retirement

Abstract. We present explicit realizations in terms of self-adjoint operators and linear relations for a non-zero scalar generalized Nevanlinna function $N(z)$ and the function $\widehat{N}(z) = -1/N(z)$ under the assumption that $\widehat{N}(z)$ has exactly one generalized pole which is not of positive type namely at $z = \infty$. The key tool we use to obtain these models is reproducing kernel Pontryagin spaces.

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1. Introduction

An $n \times n$ matrix function N is called a generalized Nevanlinna function with κ negative squares if (i) it is defined and meromorphic on $\mathbb{C} \setminus \mathbb{R}$, (ii) it satisfies $N(z) = N(z^*)^*$ for all $z \in \mathcal{D}(N)$, the domain of holomorphy of N , and (iii) the kernel

$$K_N(\zeta, z) = \frac{N(\zeta) - N(z)^*}{\zeta - z^*}, \quad \zeta, z \in \mathcal{D}(N),$$

has κ negative squares. Here the expression on the right-hand side for $\zeta = z^*$ is to be understood as $N'(\zeta)$. If $\kappa = 0$, the function N is called a Nevanlinna function;

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in this case N is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, satisfies $N(z) = N(z^*)^*$ there and the kernel condition is equivalent to the condition

$$\frac{\operatorname{Im} N(z)}{\operatorname{Im} z} \geq 0, \quad \operatorname{Im} z \neq 0.$$

The class of $n \times n$ matrix functions with κ negative squares is denoted by $\mathcal{N}_\kappa^{n \times n}$ and by \mathcal{N}_κ when the functions are scalar.

A realization for a function $N \in \mathcal{N}_\kappa^{n \times n}$ in some Pontryagin space \mathcal{P} is a pair (A, Γ_z) consisting of a self-adjoint relation A in \mathcal{P} with a nonempty resolvent set $\rho(A)$ and a corresponding Γ -field Γ_z , that is, a family of mappings $\Gamma_z : \mathbb{C}^n \rightarrow \mathcal{P}$, $z \in \rho(A)$, which satisfy

$$\Gamma_z = (I_{\mathcal{P}} + (z - \zeta)(A - z)^{-1})\Gamma_\zeta, \quad \zeta, z \in \rho(A),$$

and

$$\frac{N(\zeta) - N(z)^*}{\zeta - z^*} = \Gamma_z^* \Gamma_\zeta, \quad \zeta, z \in \rho(A), \quad z \neq \zeta^*.$$

If a point $z_0 \in \rho(A)$ is fixed this implies the following representation of N :

$$N(z) = N(z_0)^* + (z - z_0^*)\Gamma_{z_0}^* (I_{\mathcal{P}} + (z - z_0)(A - z)^{-1})\Gamma_{z_0}, \quad z \in \mathcal{D}(N).$$

The function N is determined by the self-adjoint relation A in \mathcal{P} and the Γ -field Γ_z up to an additive constant hermitian $n \times n$ matrix. The space \mathcal{P} is called the state space of the realization (A, Γ_z) . The realization (A, Γ_z) can always be chosen minimal which means that

$$\overline{\operatorname{span}} \{ \Gamma_z \mathbf{c} \mid z \in \rho(A), \mathbf{c} \in \mathbb{C}^n \} = \mathcal{P}.$$

In that case the negative index of the state space \mathcal{P} is equal to the number of negative squares of the kernel $K_N(\zeta, z)$ and $\mathcal{D}(N) = \rho(A)$; see [16, Theorem 1.1]. Two minimal realizations of N are unitarily equivalent. With a minimal realization (A, Γ_z) often a symmetric restriction S of the relation A is associated and defined by

$$S = \{ \{f, g\} \in A \mid \Gamma_{z_0}^* (g - z_0^* f) = 0 \}.$$

This definition is independent of $z_0 \in \mathcal{D}(N)$, S is an operator, and Γ_z maps \mathbb{C}^n onto the defect subspace $\operatorname{ran} (S - z^*)^\perp$ of S at z . The triplet (A, Γ_z, S) is called a model in \mathcal{P} for the realization of N or, for short, a model for the function N in \mathcal{P} . The model will be called minimal if the realization is minimal.

If $n = 1$ the function

$$\varphi(z) = \Gamma_z 1 = (I_{\mathcal{P}} + (z - z_0)(A - z)^{-1})\varphi(z_0),$$

called a defect function for S and A , spans the defect subspace of S at z and the representation of N takes the form

$$N(z) = N(z_0)^* + (z - z_0^*) \langle \varphi(z), \varphi(z_0) \rangle_{\mathcal{P}}.$$

Every $N \in \mathcal{N}_\kappa$ admits a basic factorization of the form

$$N(z) = r^\#(z) N_1(z) r(z), \tag{1.1}$$

where $N_1 \in \mathcal{N}_0$ and r is a rational function whose zeros (poles) are the generalized zeros (poles) of N in $\mathbb{C}^+ \cup \mathbb{R}$ ($\mathbb{C}^- \cup \mathbb{R}$, respectively) which are not of positive type; for definitions and a proof of (1.1), see, for example, [10] and [9]. Here and in the sequel for a vector function f we denote by $f^\#$ the function $f^\#(z) = f(z^*)^*$. If κ_1 is the number of zeros of r and κ_2 is the number of poles of r (counted according to their multiplicities), then $\kappa = \max\{\kappa_1, \kappa_2\}$. If $\tau = \kappa_1 - \kappa_2$ is positive (negative) then $z = \infty$ is a generalized pole (zero) of N which is not of positive type and with degree of non-positivity $|\tau|$. In particular, if r is a polynomial (necessarily of degree κ), then $z = \infty$ is the only generalized pole of N and not of positive type; if on the other hand $\kappa_1 = 0$ (so that $\kappa_2 = \kappa$), then $z = \infty$ is the only generalized zero of N and not of positive type.

In this paper we are describing minimal models for functions $0 \neq N \in \mathcal{N}_\kappa$ and $\widehat{N} = -N^{-1}$ (which also belongs to \mathcal{N}_κ) under the assumption that the latter belongs to the class $\mathcal{N}_\kappa^\infty$ considered in [12]. By definition, a function \widehat{N} belongs to the class $\mathcal{N}_\kappa^\infty$ if and only if it belongs to \mathcal{N}_κ and has a representation of the form

$$\widehat{N}(z) = c^\#(z)N_0(z)c(z) + p(z), \quad (1.2)$$

where $N_0(z)$ is a Nevanlinna function with the properties

$$\lim_{y \rightarrow \infty} y \operatorname{Im} N_0(iy) = +\infty, \quad \lim_{y \rightarrow \infty} y^{-1} N_0(iy) = 0, \quad \operatorname{Re} N_0(i) = 0, \quad (1.3)$$

$c(z) = (z - z_0)^m$ with $m \in \mathbb{N}_0$ and $z_0 \in \mathcal{D}(\widehat{N})$, and p is some real polynomial. As explained in [12], the representation (1.2) (with (1.3)) is irreducible and implies that $z = \infty$ is the only generalized pole of non-positive type of the function $\widehat{N}(z)$. The first two conditions in (1.3) are equivalent to the fact that in the minimal model for N_0 the symmetric operator is densely defined in the state space. The third condition is simply a normalization. In the definition of the class $\mathcal{N}_\kappa^\infty$ given in [12] it was required that the point z_0 belongs to the set $\mathbb{C} \setminus \mathbb{R}$, but in view of [12, Remark 1.3] z_0 may belong to the possibly larger set $\mathcal{D}(\widehat{N})$ and the definition is independent of the choice of $z_0 \in \mathcal{D}(\widehat{N})$. The minimal models, which we obtain for N and \widehat{N} and which are related to the irreducible representation (1.2) of \widehat{N} , have a state space of the form $\mathcal{K} = \mathcal{H}_0 \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$, $n = \max\{\deg p - 2m, 0\}$, equipped with the indefinite inner product $\langle G \cdot, \cdot \rangle_{\mathcal{K}}$, where \mathcal{H}_0 is the state space for a minimal model of the function N_0 and the Gram matrix G is the 4×4 block matrix given by (6.1) with blocks determined by the polynomials p and q from the realization (4.1) of $\widehat{N}(z)$. In [5], [6], [10], and [25] the minimal models for N related to the basic factorization (1.1) are studied. The model considered in [25] has a state space which is a subspace with finite co-dimension of $\mathcal{L} = \mathcal{H}_1 \oplus \mathbb{C}^\kappa \oplus \mathbb{C}^\kappa$ equipped with the indefinite inner product $\langle G_{\mathcal{L}} \cdot, \cdot \rangle_{\mathcal{L}}$, where \mathcal{H}_1 is the state space for a minimal model of the function N_1 and the Gram matrix is given by

$$G_{\mathcal{L}} = \begin{pmatrix} I_{\mathcal{H}_1} & 0 & 0 \\ 0 & 0 & I_{\mathbb{C}^\kappa} \\ 0 & I_{\mathbb{C}^\kappa} & 0 \end{pmatrix}.$$

The model in the present paper is more detailed than this model because we consider a more special class of generalized Nevanlinna functions.

To motivate our study of the model problem we list some applications where functions of the form (1.2) play a role. First we note that Nevanlinna functions $N_0(z)$ satisfying the asymptotic conditions in (1.3) (in the following we disregard the normalization condition) appear naturally (i) as a Q -function of the minimal operator associated with a self-adjoint boundary value problem for a formally symmetric ordinary differential expression (Titchmarsh-Weyl coefficient) and (ii) as the main ingredient in the formula of the resolvent for the singular perturbation

$$A_0 + \alpha^{-1}\chi\langle \cdot, \chi \rangle_0 \quad (1.4)$$

of an unbounded self-adjoint operator A_0 in a Hilbert space \mathcal{H}_0 with inner product $\langle \cdot, \cdot \rangle_0$, generated by a generalized element $\chi \in \mathcal{H}_{-1} \setminus \mathcal{H}_0$:

$$N_0(z) = (z - z_0^*)\langle \varphi_0(z), \varphi_0(z_0) \rangle_0 - i\text{Im } z_0 \langle \varphi_0(z_0), \varphi_0(z_0) \rangle_0 + c,$$

where $\varphi(z) = (A_0 - z)^{-1}\chi$ and c is a real number; see [1, 2]. Here, in the scale of Hilbert spaces associated with \mathcal{H}_0 and A_0 , the space \mathcal{H}_{-m} is the dual of the space $H_m = \text{dom } |A_0|^m$ equipped with the inner product $\langle (|A_0| + 1)^m \cdot, (|A_0| + 1)^m \cdot \rangle_0$, $m = 1, 2, \dots$. As explained in [13], generalized Nevanlinna functions of the form (1.2) with $\text{deg } p \leq 2m$ play a similar role as in (ii) but now for the strongly singular perturbation (1.4) with $\chi \in \mathcal{H}_{-m-1} \setminus \mathcal{H}_{-m}$. Furthermore, in [19] and [24] point-like perturbations of the Laplacian in \mathbb{R}^3 were constructed to describe the low energy asymptotic behavior

$$k \cot \delta_0(k) = \sum_{j=1}^n a_j k^{2j} + o(k^{2n})$$

of the quantum mechanical scattering data at zero orbital momenta, where $E = k^2$ is the energy of scattering particle and $\delta_0(k)$ is the scattering phase. This construction amounts to building a model of the generalized Nevanlinna function of the form (1.2) with $c(z) = 1$, $\text{deg } p > 0$, and $N_0(z) = -\sqrt{-z}$. In the two papers just mentioned two different models in Pontryagin spaces were given. To describe a given truncated series of low energy scattering with non-zero angular momentum models for generalized Nevanlinna functions (1.2) with arbitrary $\text{deg } c(z)$ and $\text{deg } p(z)$ are needed. Some models of this kind were considered in [8]. As a further motivation for the models in this paper, we discuss in Section 8 an approximation problem where generalized Nevanlinna functions of the form (1.2) with various values of $\text{deg } c(z)$ and $\text{deg } p(z)$ appear.

We summarize the contents of the seven sections which come after this introduction. In Section 2 we recall the main theorem from [12] which characterizes realizations of the functions N and $\widehat{N} = -1/N$ under the assumption that \widehat{N} belongs to the class $\mathcal{N}_\kappa^\infty$. The self-adjoint operator A and the self-adjoint relation in the models for N and \widehat{N} are related via infinite coupling. This notion from [20] is explained after Theorem 2.1. In the sequel we make it a point to indicate this

connection between various versions of the two models. To do this, we also consider minimal models for certain one-dimensional perturbations $A^{(\alpha)}$ of $A = A^{(0)}$, where α is a real number. The key tool in the further analysis of the realizations in Section 2 is the theory of reproducing kernel Pontryagin spaces and in Section 3 we collect some theorems from this theory which will be used in the sequel. In particular, we recall the so-called canonical models. The irreducible representation (1.2) induces a canonical model for the generalized Nevanlinna matrix functions

$$\tilde{N}(z) = \begin{pmatrix} N_0(z) & 0 & 0 & 0 \\ 0 & q(z) & 0 & 0 \\ 0 & 0 & p_0(z) & c^\#(z) \\ 0 & 0 & c(z) & 0 \end{pmatrix}, \quad M(z) = \begin{pmatrix} p_0(z) & c^\#(z) \\ c(z) & 0 \end{pmatrix},$$

where the real polynomials q and p_0 are uniquely determined by the polynomial p in (1.2) via the equality $p(z) = c^\#(z)q(z)c(z) + p_0(z)$ and the requirement that $\deg p_0 \leq 2m - 1$. In Section 4 we present models for N and \tilde{N} in which the reproducing kernel space $\mathcal{L}(\tilde{N})$ is the state space. See Theorem 4.1, where, as in all our theorems (unless stated otherwise), the case $n = \deg q > 0$ and $m > 0$ is considered. The resolvents of the corresponding self-adjoint operators/relation are given in Corollary 4.5. The cases where $n = 0$ or $m = 0$ are considered separately in Theorem 4.6 and Theorem 4.7; these cases are important in our examples. The space $\mathcal{L}(\tilde{N})$ admits the decomposition $\mathcal{L}(\tilde{N}) = \mathcal{L}(N_0) \oplus \mathcal{L}(q) \oplus \mathcal{L}(M)$, where the direct summands are the reproducing kernel spaces associated with the functions N_0 , q and M . In Section 5 we study special bases for the last two summands and the associated Gram matrices (see Lemma 5.1). These bases allow us to identify $\mathcal{L}(\tilde{N})$ with $\tilde{\mathcal{L}} = \mathcal{L}(N_0) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$. The corresponding matrix representations of the models in Theorems 4.1, 4.6, and 4.7 are given in Theorems 6.1, 6.3, and 6.4 in Section 6. In that section we also determine formulas for the compressions of the resolvents of the self-adjoint operators/relation in the models and the compressions of the operators/relation themselves to the subspaces $\mathcal{L}(N_0)$ and $\mathcal{L}(N_0) \oplus \mathbb{C}^n$; see Theorem 6.5 and Theorem 6.6. By changing the bases slightly, the self-adjoint operator in the model for N can be given in a block operator matrix form, and this result is shown in Section 7. In Section 8, the last section of this paper, we give some examples and discuss an approximation problem associated with the Bessel differential expression.

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2. Characterization of the class $\mathcal{N}_\kappa^\infty$

In [12] the following characterization of the class $\mathcal{N}_\kappa^\infty$ was established. We recall that if A is a self-adjoint operator or a self-adjoint relation in some Pontryagin space \mathcal{P} and w an element in \mathcal{P} , then w is called cyclic for A if

$$\overline{\text{span}}\{w, (A - z)^{-1}w \mid z \in \rho(A)\} = \mathcal{P},$$

or, equivalently, if for some (and then for every) $z_0 \in \rho(A)$, the function

$$\varphi(z) = w + (z - z_0)(A - z)^{-1}w$$

generates the space \mathcal{P} , that is,

$$\overline{\text{span}}\{\varphi(z) \mid z \in \rho(A)\} = \mathcal{P}.$$

If A is an operator then w is cyclic for A if and only if

$$\overline{\text{span}}\{(A - z)^{-1}w \mid z \in \rho(A)\} = \mathcal{P}.$$

Theorem 2.1. *For the functions $N(z)$ and $\widehat{N}(z) = -N(z)^{-1}$, the following four assertions are equivalent.*

(i) $N(z)$ has a representation :

$$N(z) = \langle (A - z)^{-1}w, w \rangle_{\mathcal{P}}, \quad z \in \mathcal{D}(N), \quad (2.1)$$

where A is a self-adjoint operator in a Pontryagin space \mathcal{P} with negative index κ , $w \in \mathcal{P}$ is a cyclic element for A with the property

$$w \in \text{dom } A^{m+n-1} \setminus \text{dom } A^{m+n}$$

for some integers $m, n \in \mathbb{N}_0$, $m+n > 0$, the subspace

$$\mathcal{L} = \text{span}\{w, Aw, \dots, A^{m-1}w, A^m w, \dots, A^{m+n-1}w\}$$

has index of non-positivity κ , and

$$\begin{aligned} \langle A^j w, A^k w \rangle &= 0, & 0 \leq j, k \leq m+n-1, \quad j+k \leq 2m+n-2, \\ \langle A^j w, A^k w \rangle &\neq 0, & 0 \leq j, k \leq m+n-1, \quad j+k = 2m+n-1. \end{aligned}$$

(ii) $N(z) \in \mathcal{N}_{\kappa}$, $z = \infty$ is the only generalized zero of non-positive type of $N(z)$, and $N(z)$ has a representation

$$N(z) = - \sum_{j=2m+n}^{2m+2n-1} \frac{s_{j-1}}{z^j} + \frac{1}{z^{2m+2n-1}} M(z) \quad (2.2)$$

with $m, n \in \mathbb{N}_0$, $m+n > 0$, real numbers s_j , $j = 2m+n-1, \dots, 2m+2n-2$, $s_{2m+n-1} \neq 0$ if $n > 0$, and a function $M(z)$ with the properties

$$\lim_{y \rightarrow \infty} M(iy) = 0, \quad \lim_{y \rightarrow \infty} y^2 \text{Re } M(iy) = +\infty.$$

(iii) $\widehat{N}(z)$ has a representation

$$\widehat{N}(z) = \widehat{N}(z_0^*) + (z - z_0^*) \langle (I_{\widehat{\mathcal{P}}} + (z - z_0)(\widehat{A} - z)^{-1})u, u \rangle_{\widehat{\mathcal{P}}}, \quad z \in \mathcal{D}(\widehat{N}),$$

where $z_0 \in \mathcal{D}(\widehat{N})$, \widehat{A} is a self-adjoint relation in a Pontryagin space $\widehat{\mathcal{P}}$ with negative index κ , $\rho(\widehat{A}) \neq \emptyset$, $u \in \widehat{\mathcal{P}}$ is a cyclic element for \widehat{A} , the root space $\widehat{\mathcal{L}}$ of \widehat{A} at $z = \infty$ is spanned by $m+n$ vectors w_1, w_2, \dots, w_{m+n} , which form a Jordan chain of \widehat{A} at ∞ , $\widehat{\mathcal{L}}$ has index of non-positivity κ and $\text{span}\{w_1, w_2, \dots, w_m\}$ is its isotropic subspace. If $m = 0$ and P_0 denotes the orthogonal projection onto $\mathcal{H}_0 = \widehat{\mathcal{P}} \ominus \widehat{\mathcal{L}}$, which is a uniformly positive subspace of $\widehat{\mathcal{P}}$, then $P_0 u \notin \text{dom } \widehat{A}$.

(iv) $\widehat{N}(z) \in \mathcal{N}_\kappa^\infty$, the irreducible representation of $\widehat{N}(z)$ being

$$\widehat{N}(z) = c(z) \# N_0(z) c(z) + p_\ell(z), \quad c(z) = (z - z_0)^m,$$

where $N_0 \in \mathcal{N}_0$ has the properties

$$\lim_{y \rightarrow \infty} y \operatorname{Im} N_0(iy) = \infty, \quad \lim_{y \rightarrow \infty} y^{-1} N_0(iy) = 0, \quad \operatorname{Re} N_0(i) = 0,$$

$m \in \mathbb{N}_0$, $z_0 \in \mathcal{D}(\widehat{N})$, $p_\ell(z) = \sum_{k=0}^{\ell} a_k z^k$ is a real polynomial of degree ℓ , and we set $n = \max\{\ell - 2m, 0\}$.

The Pontryagin spaces in (i) and (iii) can be chosen the same and then the element w_1 in (iii) can be chosen to coincide with w in (i) and to satisfy $\langle w_1, u \rangle = 1$; in this case $w_j = A^{j-1}w$, $j = 1, \dots, m+n$, and $\mathcal{L} = \widehat{\mathcal{L}}$. With A and w from (i) and the coefficients s_j , $2m+n-1 \leq j \leq 2m+2n-2$ in (2.2), $s_j = 0$ if $0 \leq j \leq 2m+n-2$, it holds

$$s_j = \langle A^r w, A^s w \rangle \quad \text{if } r+s=j, \quad 0 \leq r, s \leq m+n-1,$$

and, if $n > 0$, then $s_{2m+n-1} = 1/a_{2m+n} = 1/a_\ell$, where a_ℓ is the leading coefficient of the polynomial p in (iv). The relation between the negative index κ of the Pontryagin spaces in (i) and (iii) and the integers m , and n is given by

$$\kappa = \begin{cases} m & \text{if } n = 0, \\ m + \frac{n+1}{2} & \text{if } n > 0, \quad n \text{ odd}, \quad a_\ell < 0, \\ m + \left[\frac{n}{2} \right] & \text{otherwise.} \end{cases}$$

Note that in case $m = 0$, the root space $\widehat{\mathcal{L}}$ of \widehat{A} at ∞ in part (iii) is a regular subspace, whereas if $m > 0$ it is degenerate with an m -dimensional isotropic part. In the first case ∞ is called a critical singular point and in the second case it is called a singular critical point of \widehat{A} .

To the last part of the theorem can be added that $\widehat{A} = A^\infty$, where A^∞ is defined through infinite coupling of A and w . This means that it is obtained as the limit in the resolvent sense of the self-adjoint operator

$$A^{(\alpha)} = A + \alpha \langle \cdot, w \rangle w \tag{2.3}$$

by letting $\alpha \rightarrow \infty$: Since for $\alpha \in \mathbb{R} \setminus \{0\}$,

$$(A^{(\alpha)} - z)^{-1} = (A - z)^{-1} - \frac{\langle \cdot, \varphi(z^*) \rangle_{\mathcal{P}}}{N(z) + 1/\alpha} \varphi(z), \quad \varphi(z) = (A - z)^{-1}w, \tag{2.4}$$

we have

$$(A^\infty - z)^{-1} = (A - z)^{-1} - \frac{\langle \cdot, \varphi(z^*) \rangle_{\mathcal{P}}}{N(z)} \varphi(z). \tag{2.5}$$

For later reference we note that (2.4) implies

$$\langle (A^{(\alpha)} - z)^{-1}w, w \rangle = \frac{N(z)}{1 + \alpha N(z)}, \tag{2.6}$$

which for $\alpha = 0$ is consistent with (2.1) and for $\alpha = \infty$ with $(A^\infty - z)^{-1}w = 0$, which follows from (2.5).

Formula (2.2) is related to the moment problem for generalized Nevanlinna functions, the numbers s_j being the moments; see, for example, [23] and [3]. The purpose of this paper is to provide some explicit minimal models for the operator $A = A^{(0)}$ and the relation $\widehat{A} = A^\infty$. To derive these models we use the theory of reproducing kernels.

3. Reproducing kernel Pontryagin spaces and canonical models

A by now well-known model for $N \in \mathcal{N}_\kappa^{n \times n}$ is described in the following theorem. Here the state space is the reproducing kernel space $\mathcal{L}(N)$ associated with the kernel $K_N(\zeta, z)$. Recall that the elements of this space are n -vector functions defined and holomorphic on $\mathcal{D}(N)$, that the functions $K_N(\cdot, z)\mathbf{c}$, where z runs through $\mathcal{D}(N)$ and \mathbf{c} runs through \mathbb{C}^n , are dense in $\mathcal{L}(N)$, and that the kernel has the reproducing property:

$$\langle f, K_N(\cdot, z)\mathbf{c} \rangle_{\mathcal{L}(N)} = \mathbf{c}^* f(z), \quad f \in \mathcal{L}(N), \mathbf{c} \in \mathbb{C}^n.$$

Whenever defined we denote by R_z the difference-quotient operator and, for later use, by E_ζ the operator of evaluation at the point ζ , that is,

$$R_z f(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}, \quad E_\zeta f = f(\zeta), \tag{3.1}$$

where f is a vector function.

Theorem 3.1. *Let $N \in \mathcal{N}_\kappa^{n \times n}$ be given. Then:*

- (i) $A = \{ \{f, g\} \in \mathcal{L}(N)^2 \mid \exists \mathbf{c} \in \mathbb{C}^n : g(\zeta) - \zeta f(\zeta) \equiv \mathbf{c} \}$ is a self-adjoint relation in $\mathcal{L}(N)$ with $\rho(A) \neq \emptyset$, and

$$(\Gamma_z \mathbf{c})(\zeta) = K_N(\zeta, z^*)\mathbf{c} = \frac{N(\zeta) - N(z)}{\zeta - z}\mathbf{c}, \quad \mathbf{c} \in \mathbb{C}^n,$$

is a corresponding Γ -field. The pair (A, Γ_z) is a minimal realization of N .

- (ii) The resolvent of A is the difference-quotient operator in $\mathcal{L}(N)$:

$$(A - z)^{-1} = R_z, \quad z \in \rho(A).$$

- (iii) $S = \{ \{f, g\} \in \mathcal{L}(N)^2 \mid g(\zeta) - \zeta f(\zeta) \equiv 0 \}$ is a symmetric operator in the space $\mathcal{L}(N)$ with equal defect indices $n - d$, where $d = \dim \ker \Gamma_z$. Moreover, $\sigma_p(S) = \emptyset$ and the adjoint of S is given by

$$S^* = \overline{\text{span}} \{ \{ \Gamma_z \mathbf{h}, z \Gamma_z \mathbf{h} \} \mid \mathbf{h} \in \mathbb{C}^n, z \in \mathcal{D}(N) \}$$

$$= \{ \{f, g\} \in \mathcal{L}(N)^2 \mid \exists \mathbf{c}, \mathbf{d} \in \mathbb{C}^n : g(\zeta) - \zeta f(\zeta) \equiv \mathbf{c} - N(\zeta)\mathbf{d} \}.$$

For the proof of this theorem and remarks concerning its origin we refer to [10, Theorem 2.1]. The minimal realization of N described here is called the canonical realization of N and the triplet (A, Γ_z, S) is called the canonical model. For these canonical models only, to denote the dependence on N we often write A_N, Γ_{Nz}, S_N etc. instead of A, Γ_z, S , etc.

By \mathbf{N} we denote the operator of multiplication by the function N :

$$(\mathbf{N}f)(\zeta) = N(\zeta)f(\zeta). \quad (3.2)$$

Theorem 3.2. *Let $N \in \mathcal{N}_\kappa^{n \times n}$, assume that $N(z)$ is invertible for some point $z \in \mathcal{D}(N)$, and set $\widehat{N} = -N^{-1}$. Then $\widehat{N} \in \mathcal{N}_\kappa^{n \times n}$ and the following statements hold.*

- (i) \mathbf{N} as a mapping from $\mathcal{L}(\widehat{N})$ to $\mathcal{L}(N)$ is unitary; its inverse is the operator of multiplication by N^{-1} .
- (ii) We have

$$\widehat{A}_{\widehat{N}} = \mathbf{N}^{-1}A_N\mathbf{N} = \left\{ \{f, g\} \in \mathcal{L}(\widehat{N})^2 \mid \exists \mathbf{d} \in \mathbb{C}^n : g(\zeta) - \zeta f(\zeta) \equiv \widehat{N}(\zeta)\mathbf{d} \right\}$$

and hence $\rho(\widehat{A}_{\widehat{N}}) \neq \emptyset$.

- (iii) For $0 \neq \mathbf{c} \in \mathbb{C}^n$ and $j = 0, 1, \dots$, we have $\zeta^j N(\zeta)\mathbf{c} \in \mathcal{L}(N)$ if and only if $\zeta^j \mathbf{c} \in \mathcal{L}(\widehat{N})$.

The theorem coincides in part with [10, Corollary 2.3]. Part (i) follows from the kernel identity

$$N(\zeta)K_{\widehat{N}}(\zeta, z)N(z)^* = K_N(\zeta, z),$$

part (ii) from (i) and Theorem 3.1 (i), and part (iii) follows from (i). The inclusions in (iii) hold for $j = 0$ if and only if $z = \infty$ is a generalized zero of N or, equivalently, $z = \infty$ is a generalized pole of \widehat{N} . That we use the notation $\widehat{A}_{\widehat{N}}$ for the operator/relation $\mathbf{N}^{-1}A_N\mathbf{N}$ in part (ii) of Theorem 3.2 comes from applying our convention that if A is the self-adjoint operator/relation in a model for N then we write \widehat{A} for the corresponding operator/relation associated with \widehat{N} : If $N \in \mathcal{N}_\kappa^{n \times n}$ is invertible at some point in $\mathcal{D}(N)$, then the triplet $(A_{\widehat{N}}, K_{\widehat{N}}(\cdot, z^*), S_{\widehat{N}})$ is the canonical model for \widehat{N} and the triplet

$$(\widehat{A}_{\widehat{N}}, K_{\widehat{N}}(\cdot, z^*)N(z), S_{\widehat{N}}) \quad (3.3)$$

is a minimal model for the function $N = \widehat{\widehat{N}}$ in $\mathcal{L}(\widehat{N})$, because it is isomorphic under \mathbf{N} with the canonical model (A_N, Γ_{Nz}, S_N) for N in $\mathcal{L}(N)$.

For use in the next section we recall the following theorem (see [10, Theorem 2.4]). A function $N \in \mathcal{N}_\kappa^{n \times n}$ is called strict if for some non-real point $z_0 \in \mathcal{D}(N)$ it holds

$$\bigcap_{\zeta \in \mathcal{D}(N)} \ker K_N(\zeta, z_0) = \{0\}.$$

Theorem 3.3. *Suppose that $N \in \mathcal{N}_\kappa^{n \times n}$ is strict and let (A, Γ_z, S) be the canonical model for N . Then:*

- (i) *A relation is a canonical self-adjoint extension of S if and only if it is of the form*

$$A_{\mathcal{A}, \mathcal{B}} = \left\{ \{f, g\} \in \mathcal{L}(N)^2 \mid \exists \mathbf{h} \in \mathbb{C}^n : g(\zeta) - \zeta f(\zeta) \equiv (\mathcal{A} + N(\zeta)\mathcal{B})\mathbf{h} \right\}$$

with $n \times n$ matrices \mathcal{A} and \mathcal{B} satisfying the relations

$$\text{rank} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = n, \quad \mathcal{A}^* \mathcal{B} - \mathcal{B}^* \mathcal{A} = 0.$$

If $A_{\mathcal{A},\mathcal{B}}$ and $A_{\mathcal{A}',\mathcal{B}'}$ are two such canonical self-adjoint extensions of S then $A_{\mathcal{A}',\mathcal{B}'} = A_{\mathcal{A},\mathcal{B}}$ if and only if $\mathcal{A}' = \mathcal{A}C$ and $\mathcal{B}' = \mathcal{B}C$ for some invertible $n \times n$ matrix C .

- (ii) $\rho(A_{\mathcal{A},\mathcal{B}}) \neq \emptyset$ if and only if for some non-real point $z_0 \in \mathcal{D}(N)$ the matrices $\mathcal{A} + N(z_0)\mathcal{B}$ and $\mathcal{A} + N(z_0)^*\mathcal{B}$ are invertible. In this case for $z \in \rho(A_{\mathcal{A},\mathcal{B}}) \cap \rho(A)$:

$$(A_{\mathcal{A},\mathcal{B}} - z)^{-1} = (A - z)^{-1} - \Gamma_z \mathcal{B} (\mathcal{A} + N(z)\mathcal{B})^{-1} \Gamma_{z^*}^*. \tag{3.4}$$

We specialize to case $n = 1$ and assume $0 \neq N \in \mathcal{L}(N)$. Then on account of Theorem 3.1 (i) and (ii), $w = N$ is a cyclic element of $\mathcal{L}(N)$ for A_N and

$$N(z) = \langle (A_N - z)^{-1} w, w \rangle_{\mathcal{L}(N)}.$$

The operator

$$A_N^{(\alpha)} = A_N + \alpha \langle \cdot, w \rangle_{\mathcal{L}(N)} w, \quad \alpha \in \mathbb{R},$$

can also be written as

$$A_N^{(\alpha)} = \{ \{f, g\} \in \mathcal{L}(N)^2 \mid \exists c \in \mathbb{C} : g(\zeta) - \zeta f(\zeta) = (1 + \alpha N(\zeta))c \}. \tag{3.5}$$

This can be seen by comparing the resolvents (2.4) and (3.4) applied to this situation. If we let $\alpha \rightarrow \infty$, then in the resolvent sense $A_N^{(\alpha)}$ converges to

$$\begin{aligned} A_N^\infty &= \{ \{f, g\} \in \mathcal{L}(N)^2 \mid \exists c \in \mathbb{C} : g(\zeta) - \zeta f(\zeta) = N(\zeta)c \} \\ &= S_N + \{ \{0, cw\} \mid c \in \mathbb{C} \} = S_N + \{0\} \times (\text{dom } S_N)^\perp. \end{aligned}$$

The set on the right-hand side after the first equality can at least formally be obtained from the set on the right-hand side of (3.5) by replacing $\{f, g\}$ by $\{\alpha f, \alpha g\}$ and letting $\alpha \rightarrow \infty$. Now we apply the unitary map \mathbf{N} and find that (the constant function) $1 \in \mathcal{L}(\widehat{N})$, 1 is a cyclic element for $\widehat{A}_{\widehat{N}}$, and

$$\begin{aligned} (\widehat{A}_{\widehat{N}})^{(\alpha)} &= \mathbf{N}^{-1} A_N^{(\alpha)} \mathbf{N} = \widehat{A}_{\widehat{N}} + \alpha \langle \cdot, 1 \rangle_{\mathcal{L}(\widehat{N})} \\ &= \{ \{f, g\} \in \mathcal{L}(\widehat{N})^2 \mid \exists c \in \mathbb{C} : g(\zeta) - \zeta f(\zeta) = (\alpha - \widehat{N}(\zeta))c \}. \end{aligned} \tag{3.6}$$

Hence in the infinite coupling of $\widehat{A}_{\widehat{N}}$ and 1 we have

$$(\widehat{A}_{\widehat{N}})^\infty = A_{\widehat{N}} = \mathbf{N}^{-1} A_N^\infty \mathbf{N}. \tag{3.7}$$

4. Minimal models in the space $\mathcal{L}(\widetilde{N})$

From now on we assume that

- (1) N is a non-zero scalar generalized Nevanlinna function in \mathcal{N}_κ ,
- (2) $\widehat{N} = -1/N \in \mathcal{N}_\kappa^\infty$,

- (3) \widehat{N} has representation (1.2), and
- (4) $z_0 \in \mathcal{D}(\widehat{N})$ belongs to the possibly smaller set $\mathcal{D}(N_0)$.

We rewrite the irreducible representation (1.2) of \widehat{N} in the form

$$\widehat{N}(z) = c^\#(z)(N_0(z) + q(z))c(z) + p_0(z), \quad c(z) = (z - z_0)^m, \quad (4.1)$$

where q and p_0 are real polynomials such that $c^\#(z)q(z)c(z) + p_0(z) = p(z)$, $\ell_0 = \deg p_0 < 2m$, and $n = \deg q$ is the number appearing in Theorem 2.1. With the decomposition (4.1) we associate the generalized Nevanlinna matrix functions

$$\widetilde{N}(z) = \begin{pmatrix} N_0(z) & 0 & 0 & 0 \\ 0 & q(z) & 0 & 0 \\ 0 & 0 & p_0(z) & c^\#(z) \\ 0 & 0 & c(z) & 0 \end{pmatrix}, \quad M(z) = \begin{pmatrix} p_0(z) & c^\#(z) \\ c(z) & 0 \end{pmatrix}.$$

It follows that the reproducing kernel space $\mathcal{L}(\widetilde{N})$ with kernel $K_{\widetilde{N}}(\cdot, \cdot)$ can be decomposed as the orthogonal sum $\mathcal{L}(\widetilde{N}) = \mathcal{L}(N_0) \oplus \mathcal{L}(q) \oplus \mathcal{L}(M)$. If $n > 0$ and $m > 0$, then the elements of $\mathcal{L}(q)$ are the polynomials of degree $< n$ and the elements of $\mathcal{L}(M)$ are 2-vector functions with polynomial entries. Unless stated otherwise we assume $n > 0$ and $m > 0$. If $n = 0$ or $m = 0$, then $\mathcal{L}(q) = \{0\}$ or $\mathcal{L}(M) = \{0\}$ and the formulas simplify; we consider these cases separately.

In this section we give minimal models for N and \widehat{N} in the space $\mathcal{L}(\widetilde{N})$. For this we introduce the vector function

$$v(z) = (c(z) \quad c(z) \quad 1 \quad c(z)(N_0(z) + q(z)))^\top$$

and the following 4×4 matrices

$$\mathcal{A}_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{B} = - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad \widehat{\mathcal{B}} = - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1. *Assume the conditions (1)–(4) hold.*

- (i) *The minimal models of N and \widehat{N} in $\mathcal{L}(\widetilde{N})$ are given by the triplets*

$$(B, K_{\widetilde{N}}(\cdot, z^*)v(z)N(z), \widetilde{S}) \quad \text{and} \quad (\widehat{B}, K_{\widetilde{N}}(\cdot, z^*)v(z), \widetilde{S}),$$

where

$$B = \{ \{ \widetilde{f}, \widetilde{g} \} \in \mathcal{L}(\widetilde{N}) \mid \exists \mathbf{h} \in \mathbb{C}^4 : \widetilde{g}(\zeta) - \zeta \widetilde{f}(\zeta) = (\mathcal{A}_0 + \widetilde{N}(\zeta)\mathcal{B})\mathbf{h} \}, \quad (4.2)$$

$$\widehat{B} = \{ \{ \widetilde{f}, \widetilde{g} \} \in \mathcal{L}(\widetilde{N}) \mid \exists \mathbf{h} \in \mathbb{C}^4 : \widetilde{g}(\zeta) - \zeta \widetilde{f}(\zeta) = (I_{\mathbb{C}^4} + \widetilde{N}(\zeta)\widehat{\mathcal{B}})\mathbf{h} \}, \quad (4.3)$$

and

$$\widetilde{S} = \{ \{ \widetilde{f}, \widetilde{g} \} \in \mathcal{L}(\widetilde{N}) \mid \exists \mathbf{h} \in \mathbb{C}^4 \text{ with } h_3 = 0 : \widetilde{g}(\zeta) - \zeta \widetilde{f}(\zeta) = (\mathcal{A}_0 + \widetilde{N}(\zeta)\widehat{\mathcal{B}})\mathbf{h} \}.$$

(ii) \tilde{S} has defect $(1, 1)$ and the family of all its self-adjoint extensions in $\mathcal{L}(\tilde{N})$ is given by \widehat{B} and $B^{(\alpha)}$, $\alpha \in \mathbb{R}$, where

$$\begin{aligned} B^{(\alpha)} &= B + \alpha \langle \cdot, \tilde{w} \rangle_{\mathcal{L}(\tilde{N})} \tilde{w} \\ &= \{ \{ \tilde{f}, \tilde{g} \} \in \mathcal{L}(\tilde{N}) \mid \exists \mathbf{h} \in \mathbb{C}^4 : \tilde{g}(\zeta) - \zeta \tilde{f}(\zeta) = (\mathcal{A}_\alpha + \tilde{N}(\zeta)\mathcal{B})\mathbf{h} \} \end{aligned} \quad (4.4)$$

with

$$\tilde{w} = (0 \quad 0 \quad 1 \quad 0)^\top \in \mathcal{L}(\tilde{N}).$$

Moreover, $B = B^{(0)}$ and $\widehat{B} = B^\infty$, the limit of $B^{(\alpha)}$ in the resolvent sense by letting $\alpha \rightarrow \infty$.

Note that \tilde{S} has defect $(1, 1)$ shows that it does not coincide with $S_{\tilde{N}}$ in the canonical representation of \tilde{N} in $\mathcal{L}(\tilde{N})$, which has defect $(4, 4)$ if $n \neq 0$ and $(3, 3)$ if $n = 0$.

On account of (1)–(4) the four equivalent statements in Theorem 2.1 hold for N and \widehat{N} . For the Pontryagin space $\widehat{\mathcal{P}}$ we take the reproducing kernel Pontryagin space $\mathcal{L}(\widehat{N})$. For \mathcal{P} we take $\mathcal{L}(N)$ but we identify it with $\mathcal{L}(\widehat{N})$ via the unitary map \mathbf{N} defined by (3.2). Since ∞ is a generalized zero of N , we have $N \in \mathcal{L}(N)$ (see [10, Corollary 2.3(iii)]). It follows that (2.1) holds with $w = N$ and $A = A_N$ in \mathcal{P} and in the identification of \mathcal{P} with $\mathcal{L}(\widehat{N})$ we have $w = w_1 = 1$ and $A = \widehat{A}_{\widehat{N}}$ and so

$$N(z) = \langle (\widehat{A}_{\widehat{N}} - z)^{-1} 1, 1 \rangle_{\mathcal{L}(\widehat{N})}.$$

The expansion (2.2) for N implies that the functions w_1, w_2, \dots, w_{m+n} (or, what amounts to the same, $w, Aw, \dots, A^{m+n-1}w$) are given by $1, \zeta, \dots, \zeta^{m+n-1}$ (see [3, Lemma 5.2]). Here we use that the moments s_j are zero for $0 \leq j \leq 2m + n - 2$. The representation for \widehat{N} in statement (iii) of Theorem 2.1 holds with $\widehat{A} = A_{\widehat{N}}$ and

$$u = K_{\widehat{N}}(\cdot, z_0^*). \quad (4.5)$$

Notice that $\langle u, w_1 \rangle_{\mathcal{L}(\widehat{N})} = 1$ by the reproducing property of the kernel. Since $(A_{\widehat{N}} - z)^{-1} 1 = R_z 1 = 0$, we see directly that $A_{\widehat{N}}$ is a relation with a nontrivial multi-valued part: $1 \in A_{\widehat{N}}(0)$. In Section 3 we showed that the triplets

$$(\widehat{A}_{\widehat{N}}, K_{\widehat{N}}(\cdot, z^*)N(z), S_{\widehat{N}}) \quad \text{and} \quad (A_{\widehat{N}}, K_{\widehat{N}}(\cdot, z^*), S_{\widehat{N}})$$

are minimal models of N and \widehat{N} in $\mathcal{L}(\widehat{N})$ and that $A_{\widehat{N}}$ is the limit in the resolvent sense of $\widehat{A}_{\widehat{N}}^{(\alpha)}$; see (3.3), Theorem 3.1 applied to \widehat{N} , and (3.6) and (3.7). The main idea of the proof of the theorem is that $B^{(\alpha)}$ and \widehat{B} are isomorphic copies of $\widehat{A}_{\widehat{N}}^{(\alpha)}$ and $A_{\widehat{N}}$. The isomorphism is given in Lemma 4.4 below. We begin with two technical lemmas.

Lemma 4.2. *If $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{L}(M)$ and $h_2 = 0$, then $\deg h_1 < m$.*

Proof. Since $\deg p_0 < 2m$, the space $\mathcal{L}(M)$ is spanned by the $2m$ linearly independent 2-vector functions

$$R_0^j \begin{pmatrix} p_0 \\ c \end{pmatrix}, \quad R_0^j \begin{pmatrix} c^\# \\ 0 \end{pmatrix}, \quad j = 1, 2, \dots, m,$$

where R_0 is the difference-quotient operator given by (3.1) with $z = 0$. If $h_2 = 0$, then

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \text{span} \left\{ R_0^j \begin{pmatrix} c^\# \\ 0 \end{pmatrix} \mid j = 1, 2, \dots, m \right\},$$

which implies $\deg h_1 < m$. □

Lemma 4.3. *Assume $N_0 \in \mathcal{N}_0$ satisfies the conditions in (1.3). If h_1 and h_2 are polynomials and $N_0 h_1 + h_2 \in \mathcal{L}(N_0)$, then $h_1 = h_2 = 0$.*

Proof. Whenever defined we have

$$R_{w_0}(fg)(\zeta) = R_{w_0}(f)(\zeta)g(w_0) + f(\zeta)R_{w_0}(g)(\zeta).$$

Let h_1 and h_2 be polynomials such that $N_0 h_1 + h_2 \in \mathcal{L}(N_0)$. Assume that h_1 and h_2 are not both identically equal to 0. If we apply R_{w_0} with $w_0 \in \mathcal{D}(N_0)$ a number of times to the function $N_0 h_1 + h_2 \in \mathcal{L}(N_0)$ and use the above formula and that $R_{w_0}(N_0)(\zeta)c \in \mathcal{L}(N_0)$, $c \in \mathbb{C}$, we find that there is pair of complex numbers $(c_1, c_2) \neq (0, 0)$ such that $N_0 c_1 + c_2 \in \mathcal{L}(N_0)$. If $N_0 c_1 + c_2 = 0$, then N_0 is a real constant and therefore, on account of the last equality in (1.3) equal to zero. But this is in contradiction with the first equality in (1.3). If $N_0 c_1 + c_2 \neq 0$, then, by Theorem 3.1(iii) applied to N_0 ,

$$\{0, 0\} \neq \{0, N_0 c_1 + c_2\} \in S_{N_0}^*,$$

which implies that the minimal operator S_{N_0} in $\mathcal{L}(N_0)$ is not densely defined. This is in contradiction with the first two equalities in (1.3). These contradictions imply that h_1 and h_2 are identically equal to 0. □

Lemma 4.4. *The mapping $\mathbf{V} : \mathcal{L}(\tilde{N}) \rightarrow \mathcal{L}(\hat{N})$ defined by $(\mathbf{V}\tilde{f})(\zeta) = v^\#(\zeta)\tilde{f}(\zeta)$ is unitary.*

The corresponding mapping in [10, Lemma 3.1], where realizations of N related to its basic factorization (1.1) are considered, is a partial isometry but not necessarily injective.

Proof of Lemma 4.4. A straightforward calculation shows

$$v^\#(\zeta)K_{\tilde{N}}(\zeta, z)v(z^*) = K_{\hat{N}}(\zeta, z). \tag{4.6}$$

We claim that the number of negative squares of the kernels $K_{\tilde{N}}$ and $K_{\hat{N}}$ coincide. Indeed, the first number is the sum of the number of negative squares of the scalar function q and the matrix function M . The first of which is equal to $(n + 1)/2$ if n is odd and the leading coefficient of the polynomial q is negative, and otherwise it is $\lceil n/2 \rceil$. The kernel of M has m negative squares since z_0 is the only zero of M in the closed upper half-plane and its multiplicity is m . By Theorem 2.1, the sum of these numbers equals κ , which is the number of negative squares of $K_{\hat{N}}$. Hence

[4, Theorem 1.5.7.] implies that \mathbf{V} is a surjective partial isometry. We show that it is in fact a unitary mapping, that is, $\ker \mathbf{V}$ is trivial. Assume there is an element $\tilde{f}(\zeta) = (f(\zeta) \ a(\zeta) \ b(\zeta) \ d(\zeta))^\top \in \ker \mathbf{V}$. Then

$$c^\#(\zeta)f(\zeta) + c^\#(\zeta)a(\zeta) + b(\zeta) + c^\#(\zeta)(N_0(\zeta) + q(\zeta))d(\zeta) \equiv 0. \quad (4.7)$$

Since N_0 is holomorphic at the point z_0^* , we have that $f \in \mathcal{L}(N_0)$ is holomorphic at z_0^* also and the equality (4.7) implies that the polynomial b has a zero of order at least m at z_0^* . So we have $b(\zeta) = (\zeta - z_0^*)^m b_1(\zeta)$ for some polynomial b_1 . Equality (4.7) implies $N_0 h_1 + h_2 = -f \in \mathcal{L}(N_0)$ with polynomials $h_1 = d$ and $h_2 = a + b_1 + qd$. By Lemma 4.3, $h_1 = h_2 = 0$, that is, $d = 0$ and $a + b_1 = 0$. Now we use Lemma 4.2: Since $(b \ d)^\top \in \mathcal{L}(M)$ and $d = 0$, the lemma yields that $\deg b < m$, which implies $b_1 = 0$ and hence $b = 0$ and $a = 0$. Finally, on account of (4.7), we have $f = 0$. We conclude that $\tilde{f} = 0$, that is, $\ker \mathbf{V} = \{0\}$. \square

Proof of Theorem 4.1. We first define B , \widehat{B} , and \widetilde{S} by the formulas

$$B = \mathbf{V}^{-1} \widehat{A}_{\widehat{N}} \mathbf{V}, \quad \widehat{B} = \mathbf{V}^{-1} A_{\widehat{N}} \mathbf{V}, \quad \widetilde{S} = B \cap \widehat{B} = \mathbf{V}^{-1} S_{\widehat{N}} \mathbf{V}$$

and claim that they coincide with the relations in part (i) of the theorem. Assuming the claim is true, we have, according to (4.6), that under the unitary mapping \mathbf{V} the element $K_{\widehat{N}}(\cdot, z^*)v(z)$ in $\mathcal{L}(\widehat{N})$ is the isomorphic copy of the element $K_{\widetilde{N}}(\cdot, z^*)$ in $\mathcal{L}(\widetilde{N})$. Hence the triplets in (i) are isomorphic copies of the minimal models (3.3) and $(A_{\widehat{N}}, K_{\widehat{N}}(\cdot, z^*), S_{\widehat{N}})$ for N and \widehat{N} in $\mathcal{L}(\widetilde{N})$. It remains to prove the claim. It is easy to see that

$$\begin{aligned} B &= \{ \{ \tilde{f}, \tilde{g} \} \in \mathcal{L}(\widetilde{N})^2 \mid \exists d \in \mathbb{C} : v^\#(\zeta)(\tilde{g}(\zeta) - \zeta \tilde{f}(\zeta)) = \widehat{N}(\zeta)d \}, \\ \widehat{B} &= \{ \{ \tilde{f}, \tilde{g} \} \in \mathcal{L}(\widetilde{N})^2 \mid \exists c \in \mathbb{C} : v^\#(\zeta)(\tilde{g}(\zeta) - \zeta \tilde{f}(\zeta)) = c \}. \end{aligned}$$

First we prove formula (4.2). Denote by $B_{\mathcal{A}_0, \mathcal{B}}$ the relation defined by the right-hand side of (4.2). If $\{ \tilde{f}, \tilde{g} \} \in B_{\mathcal{A}_0, \mathcal{B}}$ and $\mathbf{h} = (h_1 \ h_2 \ h_3 \ h_4)^\top$ then

$$v^\#(\zeta)(\tilde{g}(\zeta) - \zeta \tilde{f}(\zeta)) = v^\#(\zeta)(\mathcal{A}_0 + \widetilde{N}(\zeta)\mathcal{B})\mathbf{h} = -\widehat{N}(\zeta)h_3, \quad (4.8)$$

hence $B_{\mathcal{A}_0, \mathcal{B}} \subset B$. Since $B_{\mathcal{A}_0, \mathcal{B}}$ and B are self-adjoint operators, see Theorem 3.3, equality holds. In the same way, if $\{ \tilde{f}, \tilde{g} \} \in B_{I_{\mathbb{C}^4}, \widehat{\mathcal{B}}}$, the operator defined by the right-hand side of (4.3), then

$$v^\#(\zeta)(\tilde{g}(\zeta) - \zeta \tilde{f}(\zeta)) = v^\#(\zeta)(I_{\mathbb{C}^4} + \widetilde{N}(\zeta)\widehat{\mathcal{B}})\mathbf{h} = h_3, \quad (4.9)$$

hence $B_{I_{\mathbb{C}^4}, \widehat{\mathcal{B}}} \subset \widehat{B}$ and equality prevails because both self-adjoint relations have nonempty resolvent sets. The formula for \widetilde{S} follows from (4.8) and (4.9). This completes the proof of part (i).

As to (ii) we first define the operators $B^{(\alpha)}$ by

$$B^{(\alpha)} = \mathbf{V}^{-1} (\widehat{A}_{\widehat{N}})^{(\alpha)} \mathbf{V}, \quad \alpha \in \mathbb{R}.$$

On account of (3.6), we have

$$B^{(\alpha)} = B + \alpha \langle \cdot, \tilde{w} \rangle_{\mathcal{L}(\tilde{N})} \tilde{w},$$

where

$$\tilde{w} = \mathbf{V}^{-1} \mathbf{1} = (0 \quad 0 \quad 1 \quad 0)^\top. \quad (4.10)$$

We now show (4.4). From (2.4), applied to this situation, we have

$$(B^{(\alpha)} - z)^{-1} = (B - z)^{-1} - \frac{\langle \cdot, (B - z^*)^{-1} \tilde{w} \rangle_{\mathcal{L}(\tilde{N})}}{N(z) + 1/\alpha} (B - z)^{-1} \tilde{w}$$

and, by Theorem 3.3,

$$(B_{\mathcal{A}_\alpha, \mathcal{B}} - z)^{-1} = (A_{\tilde{N}} - z)^{-1} - \Gamma_{\tilde{N}z} \mathcal{B} (\mathcal{A}_\alpha + \tilde{N}(\zeta) \mathcal{B})^{-1} E_z,$$

where $E_z = \Gamma_{\tilde{N}z^*}^*$ is the operator of evaluation at the point z . For $\alpha = 0$, the last equality yields

$$(B - z)^{-1} = (A_{\tilde{N}} - z)^{-1} - \Gamma_{\tilde{N}z} \mathcal{B} (\mathcal{A}_0 + \tilde{N}(\zeta) \mathcal{B})^{-1} E_z$$

and, on account of (4.10), $(B - z)^{-1} \tilde{w} = N(z) \Gamma_{\tilde{N}z} v(z)$. Combining these relations we find

$$\begin{aligned} & (B^{(\alpha)} - z)^{-1} - (B_{\mathcal{A}_\alpha, \mathcal{B}} - z)^{-1} \\ &= \Gamma_{\tilde{N}z} \left[-\mathcal{B} (\mathcal{A}_0 + \tilde{N}(\zeta) \mathcal{B})^{-1} - \frac{N(z)^2}{N(z) + \frac{1}{\alpha}} v(z) v^\#(z) + \mathcal{B} (\mathcal{A}_\alpha + \tilde{N}(\zeta) \mathcal{B})^{-1} \right] E_z. \end{aligned}$$

A straightforward calculation shows that the expression in square brackets vanishes, which implies (4.4) for all $\alpha \in \mathbb{R}$.

Clearly, $B = B^{(0)}$ and, because of (3.7), $\widehat{B} = B^\infty$, where the relation on the right-hand side is obtained via infinite coupling of B and \tilde{w} , that is, by taking the limit of $B^{(\alpha)}$ in the resolvent sense by letting $\alpha \rightarrow \infty$. Finally, the statement concerning \tilde{S} and its extensions follow from the corresponding results for $S_{\tilde{N}}$ and the unitarity of \mathbf{V} . \square

The following corollary is an immediate consequence of Theorem 3.3. We set

$$K_\alpha(z) = \begin{pmatrix} c(z)c^\#(z) & c(z)c^\#(z) & c(z) & \alpha - p_0(z) \\ c(z)c^\#(z) & c(z)c^\#(z) & c(z) & \alpha - p_0(z) \\ c^\#(z) & c^\#(z) & 1 & (N_0(z) + q(z))c^\#(z) \\ \alpha - p_0(z) & \alpha - p_0(z) & (N_0(z) + q(z))c(z) & (N_0(z) + q(z))(\alpha - p_0(z)) \end{pmatrix}$$

and

$$\widehat{K}(z) = \lim_{\alpha \rightarrow \infty} \frac{1}{\alpha} K_\alpha(z) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & N_0(z) + q(z) \end{pmatrix}.$$

Corollary 4.5. *The resolvents of $B^{(\alpha)}$ and \widehat{B} are given by*

$$(B^{(\alpha)} - z)^{-1} = (A_{\widetilde{N}} - z)^{-1} + \frac{N(z)}{1 + \alpha N(z)} \Gamma_{\widetilde{N}z} K_{\alpha}(z) E_z, \quad (4.11)$$

$$(\widehat{B} - z)^{-1} = (A_{\widetilde{N}} - z)^{-1} + \Gamma_{\widetilde{N}z} \widehat{K}(z) E_z, \quad (4.12)$$

where $(A_{\widetilde{N}} - z)^{-1}$ is the difference-quotient operator in the space $\mathcal{L}(\widetilde{N})$,

$$\Gamma_{\widetilde{N}z} = \text{diag} \{K_{N_0}(\cdot, z^*), K_q(\cdot, z^*), K_M(\cdot, z^*)\},$$

and $E_z = (\Gamma_{\widetilde{N}z^*})^*$ is the evaluation operator at the point z .

From (4.11) it readily follows that

$$\langle (B^{(\alpha)} - z)^{-1} \widetilde{w}, \widetilde{w} \rangle = \frac{N(z)}{1 + \alpha N(z)}, \quad \alpha \in \mathbb{R} \cup \{\infty\},$$

which is consistent with (2.6).

It remains to discuss the simplifications if $n = 0$ or $m = 0$.

The case $n = 0$ and $m > 0$: Then $q(z) = q_0$ with $q_0 \in \mathbb{R}$, hence $K_q(\zeta, z) = 0$ and $\mathcal{L}(q) = \{0\}$; as to the 2×2 matrix function $M(z)$: $\deg p_0(z) < 2m$ and the space $\mathcal{L}(M)$ is nontrivial. Now $\widetilde{N}(z)$ becomes the 3×3 matrix function

$$\widetilde{N}(z) = \text{diag} \{N_0(z) + q_0, M(z)\} \quad \text{and} \quad \mathcal{L}(\widetilde{N}) = \mathcal{L}(N_0) \oplus \mathcal{L}(M).$$

With $v(z) = (c(z) \ 1 \ c(z)(N_0(z) + q_0))^\top$ the operator $\mathbf{V} : \mathcal{L}(\widetilde{N}) \rightarrow \mathcal{L}(\widehat{N})$ of multiplication by $v^\#(z) = (c^\#(z) \ 1 \ c^\#(z)(N_0(z) + q_0))$ is unitary and \widetilde{w} in (4.10) becomes $\widehat{w} := \mathbf{V}^{-1} \widetilde{w} = (0 \ 1 \ 0)^\top$.

Theorem 4.6. *Assume $n = 0$ and $m > 0$. Then Theorem 4.1 and Corollary 4.5 remain true provided in all 4×4 matrices the 2-nd row and the 2-nd column are deleted and in the formulas for B , \widehat{B} , \widetilde{S} , and $B^{(\alpha)}$ the space \mathbb{C}^4 and the entry h_3 are replaced by \mathbb{C}^3 and h_2 .*

The case $n > 0$ and $m = 0$: Now $c(z) = 1$, $q(z)$ is a nonconstant real polynomial, and the irreducible representation (1.2) becomes $\widehat{N}(z) = N_0(z) + q(z)$. From $\widetilde{N}(z) = \text{diag} \{N_0(z), q(z)\}$ it follows that $\mathcal{L}(\widetilde{N}) = \mathcal{L}(N_0) \oplus \mathcal{L}(q)$. With the vector function $v(z) = (1 \ 1)^\top$ the mapping $V : \mathcal{L}(\widetilde{N}) \rightarrow \mathcal{L}(\widehat{N})$ of multiplication by $v^\#(z) = (1 \ 1)$ is unitary.

Theorem 4.7. *Assume $n > 0$ and $m = 0$ and for $\alpha \in \mathbb{R}$ define the operators*

$$B^{(\alpha)} = \{ \{ \widetilde{f}, \widetilde{g} \} \in \mathcal{L}(\widetilde{N}) \mid \exists \mathbf{h} \in \mathbb{C}^2 : \widetilde{g}(\zeta) - \zeta \widetilde{f}(\zeta) = (\mathcal{A}_\alpha + \widetilde{N}(\zeta) \mathcal{B}) \mathbf{h} \},$$

where

$$\mathcal{A}_\alpha = \begin{pmatrix} 1 & 0 \\ -1 & \alpha \end{pmatrix}, \quad \mathcal{B} = - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) *Theorem 4.1 holds provided $B = B^{(0)}$, $\widehat{B} = A_{\widetilde{N}} = B^\infty$, and $\widetilde{S} = B \cap \widehat{B}$, which takes the form*

$$\widetilde{S} = \{ \{ \widetilde{f}, \widetilde{g} \} \in A_{\widetilde{N}} \mid \exists h \in \mathbb{C} : (\widetilde{g}(\zeta) - \zeta \widetilde{f}(\zeta)) = h (1 \ -1)^\top \}.$$

(ii) *Corollary 4.5 becomes: For $\alpha \in \mathbb{R}$,*

$$(B^{(\alpha)} - z)^{-1} = (A_{\tilde{N}} - z)^{-1} - \frac{1}{N_0(z) + q(z) - \alpha} \Gamma_{\tilde{N}z} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} E_z,$$

where $(A_{\tilde{N}} - z)^{-1}$ is the difference-quotient operator in the space $\mathcal{L}(\tilde{N})$,

$$\Gamma_{\tilde{N}z} = \text{diag} \{K_{N_0}(\cdot, z^*), K_q(\cdot, z^*)\},$$

and $E_z = (\Gamma_{\tilde{N}z^*})^*$ is point evaluation at z .

5. A decomposition of $\mathcal{L}(\tilde{N})$

In this section we choose a basis in $\mathcal{L}(q) \oplus \mathcal{L}(M)$ and determine the associated Gram matrix \tilde{G} ; see Lemma 5.1 below. In the next section we identify $\mathcal{L}(q) \oplus \mathcal{L}(M)$ with $\mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$ equipped with an inner product determined by \tilde{G} and exhibit the matrix representations of the operators $B^{(\alpha)}$ and the relation \widehat{B} . As in [12] we define in $\mathcal{L}(\tilde{N})$ the linearly independent elements

$$v_j = (B - z_0^*)^{j-1} \tilde{w}, \quad j = 1, \dots, m, m+1, \dots, m+n,$$

and with $\varphi(z) = \varphi(\cdot, z) = K_{\tilde{N}}(\cdot, z^*)v(z)$

$$u_j = (\widehat{B} - z_0)^{-j+1} \varphi(z_0) = \frac{1}{(j-1)!} \left(\frac{d}{dz} \right)^{j-1} \varphi(z) \Big|_{z=z_0}, \quad j = 1, \dots, m.$$

Here, on account of (4.6), $u_1 = \mathbf{V}^{-1}u$, where u is given by (4.5). Moreover, we introduce the three subspaces

$$\mathcal{L}^0 = \text{span} \{v_1, \dots, v_m\}, \quad \mathcal{L}' = \text{span} \{v_{m+1}, \dots, v_{m+n}\}, \quad \mathcal{M} = \text{span} \{u_1, \dots, u_m\}.$$

According to Theorem 2.1, the root space of \widehat{B} at ∞ is the direct sum $\mathcal{L}' \dot{+} \mathcal{L}^0$ and \mathcal{L}^0 is its isotropic part.

Lemma 5.1. *We have $\mathcal{L}(q) = \mathcal{L}'$ and $\mathcal{L}(M) = \mathcal{L}^0 \dot{+} \mathcal{M}$. The basis elements for $\mathcal{L}(q)$ and $\mathcal{L}(M)$ can be written more explicitly as*

$$v_{m+j}(\zeta) = (0 \quad (\zeta - z_0^*)^{j-1} \quad 0 \quad 0)^\top, \quad j = 1, \dots, n,$$

and

$$v_j(\zeta) = (0 \quad 0 \quad (\zeta - z_0^*)^{j-1} \quad 0)^\top, \quad j = 1, \dots, m,$$

$$u_j(\zeta) = (0 \quad 0 \quad R_{z_0}^j p_0(\zeta) \quad (\zeta - z_0)^{m-j})^\top, \quad j = 1, \dots, m,$$

where R_z stands for the difference-quotient operator at z . The Gram matrix associated with this basis for the space $\mathcal{L}(q) \oplus (\mathcal{L}^0 \dot{+} \mathcal{M})$ is given by

$$\tilde{G} = \begin{pmatrix} G_q & 0 & 0 \\ 0 & 0 & I_{\mathbb{C}^m} \\ 0 & I_{\mathbb{C}^m} & G_{p_0} \end{pmatrix},$$

in which $G_q = (q_{i,j})_{i,j=1}^n$ has entries

$$q_{i,j} = \langle v_{m+j}, v_{m+i} \rangle_{\mathcal{L}(\tilde{N})}$$

$$= \sum_{k=0}^{m+j-1} \sum_{l=0}^{m+i-1} \binom{m+j-1}{k} \binom{m+i-1}{l} (-z_0^*)^{m+j-1-k} (-z_0)^{m+i-1-l} s_{k+l},$$

where s_{k+l} are the moments of N in (2.2), and $G_{p_0} = (p_{i,j})_{i,j=1}^m$ has entries

$$p_{i,j} = \langle u_j, u_i \rangle_{\mathcal{L}(\tilde{N})} \tag{5.1}$$

$$= \frac{1}{(j-1)!} \frac{1}{(i-1)!} \left(\frac{d}{dz}\right)^{j-1} \left(\frac{d}{dw^*}\right)^{i-1} \frac{p_0(z) - p_0(w^*)}{z - w^*} \Big|_{z=w=z_0}.$$

The formulas for the basis element $u_j(\zeta)$ and the entry $p_{i,j}$ of the Gram matrix G_{p_0} in this lemma are independent of the way the polynomial $p_0(z)$ is written. If we write $p_0(\zeta) = \sum_{k=0}^{\ell_0} p_k(\zeta - z_0)^k$ and set $p_k = 0$ if $k > \ell_0$, then

$$R_{z_0}^j p_0(\zeta) = \sum_{k=j}^{\ell_0} p_k(\zeta - z_0)^{k-j} \text{ and straightforward calculations yield}$$

$$p_{i,j} = \sum_{k=i}^{2m-j} \binom{k-1}{i-1} p_{k+j-1}(z_0^* - z_0)^{k-i};$$

so, in particular, if $z_0 \in \mathbb{R}$ then $p_{i,j} = p_{i+j-1}$. After the proof of the lemma we give some other formulas for the Gram matrix G_q as well.

Proof of Lemma 5.1. Since $v_1 = \tilde{w}$, the element v_1 is of the given form. We calculate $v_j = (B - z_0^*)v_{j-1}$ for $j = 2, \dots, m+n$. Write Bv_{j-1} as

$$Bv_{j-1}(\zeta) = (f(\zeta) \quad a(\zeta) \quad b(\zeta) \quad d(\zeta))^\top \in \mathcal{L}(N_0) \oplus \mathcal{L}(q) \oplus \mathcal{L}(M).$$

Then, by (4.4), there exists a vector $\mathbf{h} = (h_1 \quad h_2 \quad h_3 \quad h_4)^\top \in \mathbb{C}^4$ such that

$$\begin{pmatrix} f(\zeta) \\ a(\zeta) \\ b(\zeta) - \zeta(\zeta - z_0^*)^{j-2} \\ d(\zeta) \end{pmatrix} = \begin{pmatrix} h_1 - N_0(\zeta)h_4 \\ h_2 - q(\zeta)h_4 \\ -c^\#(\zeta)(h_1 + h_2) - p_0(\zeta)h_3 \\ -c(\zeta)h_3 + h_4 \end{pmatrix}. \tag{5.2}$$

By Lemma 4.3, $h_1 = h_4 = 0$. Since d is a polynomial of degree less than m also $h_3 = 0$. Thus b is the first component of an element in $\mathcal{L}(M)$ whose second component $d = 0$, therefore, see Lemma 4.2, the degree of b is less than m . The equality between the third components of the vectors in (5.2) now reads as

$$b(\zeta) - \zeta(\zeta - z_0^*)^{j-2} = -c^\#(\zeta)h_2. \tag{5.3}$$

For $2 \leq j \leq m$, a comparison of the degrees of the polynomials on both sides, yields $h_2 = 0$. Thus $b(\zeta) = (\zeta - z_0^*)^{j-1} + z_0^*(\zeta - z_0^*)^{j-2}$ and hence we have

$$v_j(\zeta) = (0 \quad 0 \quad (\zeta - z_0^*)^{j-1} \quad 0)^\top, \quad j = 1, \dots, m.$$

If $j = m + 1$ then (5.3) implies $h_2 = 1$ and hence we find

$$v_{m+1}(\zeta) = (0 \quad 1 \quad 0 \quad 0)^\top.$$

Now the formula for v_j , $j = m + 2, \dots, n$, can be checked in a similar way as above. It is easy to see that the element $u_1 = K_{\tilde{N}}(\cdot, z_0^*)v(z_0)$ has the stated form. By (4.12), we have for $2 \leq j \leq m$,

$$u_j(\zeta) = (\widehat{B} - z_0)^{-1}u_{j-1}(\zeta) = (A_{\tilde{N}} - z_0)^{-1} \begin{pmatrix} 0 \\ 0 \\ R_{z_0}^{j-1}p_0(\zeta) \\ (\zeta - z_0)^{m-j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R_{z_0}^j p_0(\zeta) \\ (\zeta - z_0)^{m-j} \end{pmatrix}.$$

As to the Gram matrix \tilde{G} , the zeros come from the facts that $\mathcal{L}(q) \perp \mathcal{L}(M)$ and \mathcal{L}^0 is neutral. The formula for G_q follows from expanding

$$q_{i,j} = \langle (B - z_0^*)^{m+j-1}\tilde{w}, (B - z_0^*)^{m+i-1}\tilde{w} \rangle_{\mathcal{L}(\tilde{N})}$$

in terms of

$$\begin{aligned} & \langle B^l \tilde{w}, B^k \tilde{w} \rangle_{\mathcal{L}(\tilde{N})} \\ &= \langle (\widehat{A}_{\tilde{N}})^l 1, (\widehat{A}_{\tilde{N}})^k 1 \rangle_{\mathcal{L}(\tilde{N})} = \langle A_N^l N, A_N^k N \rangle_{\mathcal{L}(N)} = \langle A^l w, A^k w \rangle_{\mathcal{P}} = s_{k+l}. \end{aligned}$$

The entries $I_{\mathcal{C}^m}$ in \tilde{G} are obtained from the reproducing kernel property of $K_{\tilde{N}}(\zeta, z)$:

$$\begin{aligned} \langle u_j, v_i \rangle_{\mathcal{L}(\tilde{N})} &= \frac{1}{(j-1)!} \left(\frac{d}{dz} \right)^{j-1} \langle K_{\tilde{N}}(\cdot, z^*)v(z), v_i \rangle_{\mathcal{L}(\tilde{N})} \Big|_{z=z_0} \\ &= \frac{1}{(j-1)!} \left(\frac{d}{dz} \right)^{j-1} (z - z_0)^{i-1} \Big|_{z=z_0} = \delta_{ij}, \quad 1 \leq i, j \leq m. \end{aligned}$$

Finally, the formula for $p_{i,j}$ in the lemma readily follows from

$$\begin{aligned} \langle u_j, u_i \rangle_{\mathcal{L}(\tilde{N})} &= \frac{1}{(j-1)!} \frac{1}{(i-1)!} \\ &\times \left(\frac{d}{dz} \right)^{j-1} \left(\frac{d}{dw^*} \right)^{i-1} \langle K_{\tilde{N}}(\cdot, z^*)v(z), K_{\tilde{N}}(\cdot, w^*)v(w) \rangle_{\mathcal{L}(\tilde{N})} \Big|_{z=w=z_0}. \quad \square \end{aligned}$$

We claim that the Gram matrix $G_q = (\langle v_{m+j}, v_{m+i} \rangle_{\mathcal{L}(\tilde{N})})_{i,j=1}^n$ in Lemma 5.1 is lower diagonal with respect to the second diagonal. To see this we use the equality

$$\langle v_{m+j}, v_{m+i} \rangle_{\mathcal{L}(\tilde{N})} = \langle v_{m+j+1}, v_{m+i-1} \rangle_{\mathcal{L}(\tilde{N})} + (z_0^* - z_0) \langle v_{m+j}, v_{m+i-1} \rangle_{\mathcal{L}(\tilde{N})}, \quad (5.4)$$

which readily follows from the relation $v_{m+i} = (B - z_0^*)v_{m+i-1}$. Since $\mathcal{L}(q)$ is orthogonal to \mathcal{L}^0 the recurrence relation (5.4) implies

$$\langle v_{m+j}, v_{m+1} \rangle_{\mathcal{L}(\tilde{N})} = 0, \quad j = 1, \dots, n-1,$$

and hence, again with (5.4), also the lower triangular form of G_q . Furthermore, (5.4) also implies that the entries on the second diagonal: $\langle v_{m+d}, v_{m+n+1-d} \rangle_{\mathcal{L}(\tilde{N})}$

are independent of $d = 1, \dots, n$. Note that G_q is not a Hankel matrix in general, however, it is if $z_0 = z_0^*$.

In the following two propositions we present two other formulas for G_q . The first one is in terms of the real coefficients τ_j of q :

$$q(z) = \tau_n z^n + \tau_{n-1} z^{n-1} + \dots + \tau_1 z + \tau_0, \quad \tau_n \neq 0.$$

The Gram matrix S_q associated with the standard basis $\{1, \zeta, \dots, \zeta^{n-1}\}$ in $\mathcal{L}(q)$ is equal to

$$S_q = \begin{pmatrix} \tau_1 & \tau_2 & \dots & \tau_{n-1} & \tau_n \\ \tau_2 & \tau_3 & \dots & \tau_n & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \tau_n & 0 & \dots & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & \sigma_{n-1} \\ 0 & 0 & \dots & \sigma_{n-1} & \sigma_n \\ \vdots & \vdots & & \vdots & \vdots \\ \sigma_{n-1} & \sigma_n & \dots & \sigma_{2n-3} & \sigma_{2n-2} \end{pmatrix}$$

where the real numbers σ_j are defined by the expansion

$$\frac{1}{q(z)} = \frac{\sigma_{n-1}}{z^n} + \dots + \frac{\sigma_{2n-2}}{z^{2n-1}} + O\left(\frac{1}{z^{2n}}\right), \quad z = iy, \quad y \uparrow \infty \tag{5.5}$$

(see, for example, [10, Theorem 3.4]). Notice that τ_0 does not play a role. Since G_q is the Gram matrix of the basis $v_{m+1}, v_{m+2}, \dots, v_{m+n}$, we express this basis in terms of the standard basis via the $n \times n$ matrix \mathbb{H} :

$$(1 \quad \zeta - z_0^* \quad (\zeta - z_0^*)^2 \quad \dots \quad (\zeta - z_0^*)^{n-1}) = (1 \quad \zeta \quad \zeta^2 \quad \dots \quad \zeta^{n-1}) \mathbb{H}.$$

and then we have $G_q = \mathbb{H}^* S_q \mathbb{H}$. If $\mathbb{H} = (h_{i,j})_{i,j=1}^n$ then the entries in the j th column are given by

$$h_{i,j} = \begin{cases} \binom{j-1}{i-1} (-z_0^*)^{j-i}, & i = 1, \dots, j, \\ 0, & i = j+1, \dots, n. \end{cases}$$

The connection with the moments s_j for N given by (2.2) can be obtained from the fact that the asymptotics of $N(z) = -1/\widehat{N}(z)$ in Theorem 2.1(ii) is the same as the asymptotics of the function $-1/(c^\#(z)q(z)c(z))$: Write for $|z| > |z_0|$,

$$\frac{1}{c(z)} = \sum_{k=0}^{\infty} \frac{t_{m+k-1}}{z^{m+k}}, \quad t_{m+k-1} = z_0^k \binom{m+k-1}{m-1} \tag{5.6}$$

(note that $t_{m-1} = 1$) and set

$$\mathbb{T} = \begin{pmatrix} t_{m-1} & t_m & \dots & t_{m+n-3} & t_{m+n-2} \\ 0 & t_{m-1} & \dots & t_{m+n-4} & t_{m+n-3} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & t_{m-1} \end{pmatrix}.$$

Using (5.5) and (5.6) to calculate the asymptotics of $-1/(c^\#(z)q(z)c(z))$ and comparing it with (2.2) we find that

$$M_N = \begin{pmatrix} 0 & 0 & \cdots & 0 & s_{2m+n-1} \\ 0 & 0 & \cdots & s_{2m+n-1} & s_{2m+n} \\ \vdots & \vdots & & \vdots & \vdots \\ s_{2m+n-1} & s_{2m+n} & \cdots & s_{2m+2n-3} & s_{2m+2n-2} \end{pmatrix} = \mathbb{T}^* S_q \mathbb{T}.$$

Hence we have proved the following proposition.

Proposition 5.2. *With \mathbb{H} , S_q , \mathbb{T} and M_N as defined above we have*

$$G_q = \mathbb{H}^* \mathbb{T}^{-*} M_N \mathbb{T}^{-1} \mathbb{H} = \mathbb{H}^* S_q \mathbb{H}.$$

The first equality follows from the formula for G_q given in Lemma 5.1. The triangular forms of the matrices \mathbb{H} with 1 on the diagonal and S_q with $\sigma_{n-1} = 1/\tau_n$ on the second diagonal yield the triangular form of G_q with $1/\tau_n$ on the second diagonal.

To derive yet another formula for G_q , we identify the elements $v_{m+j} \in \mathcal{L}(\tilde{N})$ with the functions $(\zeta - z_0^*)^{j-1} \in \mathcal{L}(q)$, $j = 1, \dots, n$. Also for later use, we introduce the vector polynomial

$$\mathbf{s}_q(z) = (s_i(z))_{i=1}^n, \quad s_i(z) = R_{z_0^*}^i q(z), \quad (5.7)$$

where R_z is the difference-quotient operator at z . The kernel $K_q(\cdot, z)$ can be expressed in the basis $\{v_{m+1}, \dots, v_{m+n}\}$ as

$$K_q(\cdot, z) = \sum_{i=1}^n v_{m+i} s_i(z^*). \quad (5.8)$$

For this, write q as $q(z) = \sum_{k=0}^n q_k (z - z_0^*)^k$, then $s_i(z) = \sum_{k=i}^n q_k (z - z_0^*)^{k-i}$ and hence

$$\begin{aligned} K_q(\zeta, z) &= \sum_{k=1}^n q_k \frac{(\zeta - z_0^*)^k - (z^* - z_0^*)^k}{\zeta - z^*} = \sum_{k=1}^n q_k \sum_{i=1}^k (\zeta - z_0^*)^{i-1} (z^* - z_0^*)^{k-i} \\ &= \sum_{i=1}^n (\zeta - z_0^*)^{i-1} \sum_{k=i}^n q_k (z^* - z_0^*)^{k-i} = \sum_{i=1}^n v_{m+i}(\zeta) s_i(z^*). \end{aligned}$$

For the next proposition, we choose n distinct points $z_1, \dots, z_n \in \mathbb{C}$, denote by \mathbb{V} the $n \times n$ Vandermonde matrix

$$\mathbb{V} = \begin{pmatrix} 1 & z_1 - z_0^* & \cdots & (z_1 - z_0^*)^{n-1} \\ 1 & z_2 - z_0^* & \cdots & (z_2 - z_0^*)^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_n - z_0^* & \cdots & (z_n - z_0^*)^{n-1} \end{pmatrix},$$

and define the $n \times n$ matrix $\mathbb{S} = (s_{i,j})_{i,j=1}^n$ by $s_{i,j} = s_i(z_j^*)$.

Proposition 5.3. *With the above notation $G_q = \mathbb{S}^{-*} \mathbb{V}$.*

Proof. If $\mathbb{S}^{-1} = (t_{i,j})_{i,j=1}^n$, then on account of (5.8), $v_{m+i} = \sum_{k=1}^n K_q(\cdot, z_k)t_{k,i}$, $i, j = 1, \dots, n$, and by the reproducing kernel property,

$$\begin{aligned} \langle v_{m+j}, v_{m+i} \rangle_{\mathcal{L}(q)} &= \sum_{k=1}^n t_{k,i}^* \langle v_{m+j}, K_q(\cdot, z_k) \rangle_{\mathcal{L}(q)} = \sum_{k=1}^n t_{k,i}^* v_{m+j}(z_k) \\ &= \sum_{k=1}^n (\mathbb{S}^{-*})_{i,k} (z_k - z_0^*)^{j-1} = (\mathbb{S}^{-*}\mathbb{V})_{i,j}. \quad \square \end{aligned}$$

6. Minimal models in the space $(\mathcal{K}; G)$

In this section we construct minimal models for the functions N and \widehat{N} in the orthogonal sum $\mathcal{K} = \mathcal{L}(N_0) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$, where $\mathcal{L}(N_0)$ is the reproducing kernel Hilbert space associated with the Nevanlinna function N_0 in the representation (4.1) of \widehat{N} . The inner product on \mathbb{C}^m will be denoted by $(\mathbf{x}, \mathbf{y})_m = \mathbf{y}^* \mathbf{x}$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$; the index m in the inner product will be omitted when it is clear from the context. We denote by $(\mathcal{K}; G)$ the linear space \mathcal{K} equipped with the indefinite inner product $\langle G \cdot, \cdot \rangle_{\mathcal{K}}$ defined by the Gram matrix

$$G = \begin{pmatrix} I_{\mathcal{L}(N_0)} & 0 & 0 & 0 \\ 0 & G_q & 0 & 0 \\ 0 & 0 & 0 & I_{\mathbb{C}^m} \\ 0 & 0 & I_{\mathbb{C}^m} & G_{p_0} \end{pmatrix}, \tag{6.1}$$

where G_q and G_{p_0} are given in Lemma 5.1.

Because of Lemma 4.3 we have $N_0 \notin \mathcal{L}(N_0)$. But since the element

$$R_{w_0} N_0 = K_{N_0}(\cdot, w_0^*), \quad w_0 \in \mathcal{D}(N_0),$$

belongs to $\mathcal{L}(N_0)$, we see that N_0 is a generalized element belonging to the space $\mathcal{L}(N_0)_{-1}$ defined in the Introduction. Thus the pairing $\langle f_0, N_0 \rangle$ between an element $f_0 \in \text{dom } A_{N_0}$ and N_0 is well defined:

$$\langle f_0, N_0 \rangle = \langle (A_{N_0} - w_0) f_0, K_{N_0}(\cdot, w_0) \rangle_{\mathcal{L}(N_0)} = g_0(w_0) - w_0 f_0(w_0), \tag{6.2}$$

where $g_0 = A_{N_0} f_0$ and the right-hand side is independent of $w_0 \in \mathcal{D}(N_0)$. In this connection we write χ_{-1} for $N_0 \in \mathcal{L}(N_0)_{-1}$ and we define

$$\varphi_0(z) = (A_{N_0} - z)^{-1} \chi_{-1} = K_{N_0}(\cdot, z^*), \quad z \in \mathcal{D}(N_0).$$

By Theorem 3.1, the triplet $(A_{N_0}, \varphi_0(z), S_{N_0})$ is a minimal model for N_0 in $\mathcal{L}(N_0)$.

To keep the formulation of the next theorem short, we introduce the following notation. We write

$$q(z) = \sum_{j=0}^n q_j (z - z_0^*)^j, \quad p_0(z) = \sum_{j=0}^{\ell_0} p_k (z - z_0)^j, \tag{6.3}$$

set $p_k = 0$ for $k > \ell_0$, and define the column vectors

$$\mathbf{q} = (q_0 \cdots q_{n-1})^\top \in \mathbb{C}^n, \quad \mathbf{p} = (p_0 \cdots p_{m-1})^* \in \mathbb{C}^m.$$

We use the column vector polynomials

$$\mathbf{s}_q(z) = (s_j(z))_{j=1}^n, \quad \mathbf{t}_{p_0}(z) = (t_j(z))_{j=1}^m, \quad \mathbf{r}_{1p_0}(z) = (r_{1j}(z))_{j=1}^m, \quad \mathbf{r}_{2p_0}(z) = (r_{2j}(z))_{j=1}^m.$$

Here $\mathbf{s}_q(z)$ is as in (5.7), $t_j(z) = R_{z_0}^j p_0(z)$, and the entries of the last two vectors are the coefficients in the expansions

$$R_z R_{z_0}^m p_0(\zeta) = \sum_{j=1}^m r_{1j}(z) (\zeta - z_0^*)^{j-1},$$

and

$$\sum_{j=m+1}^{\ell_0} R_{z_0}^j p_0(\zeta) (z - z_0)^{j-1} = \sum_{j=1}^m r_{2j}(z) (\zeta - z_0^*)^{j-1}.$$

Furthermore, we write

$$\begin{aligned} \mathbf{b}(z) &= ((z - z_0^*)^{m-1} \quad (z - z_0^*)^{m-2} \quad \cdots \quad 1)^\top, \\ \mathbf{d}(z) &= (1 \quad (z - z_0) \quad \cdots \quad (z - z_0)^{m-1})^\top. \end{aligned}$$

We denote by $J_m(z_0)$ the $m \times m$ Jordan block matrix at z_0 with $J_m(z_0)\mathbf{e}_{m,1} = z_0\mathbf{e}_{m,1}$, where for $j = 1, 2, \dots, m$, $\mathbf{e}_{m,j}$ stands for the j th element in the standard orthogonal basis of \mathbb{C}^m .

Theorem 6.1. *Assume the conditions (1)–(4).*

(i) *For $\alpha \in \mathbb{R}$, let $C^{(\alpha)}$ be the set of all pairs of the form*

$$\left\{ \left(\begin{array}{c} f_0 + \lambda\chi_0 \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{d} \end{array} \right), \left(\begin{array}{c} g_0 + w_0\lambda\chi_0 \\ -\langle f_0, \chi_{-1} \rangle \mathbf{e}_{n,1} - \lambda(N_0(w_0)\mathbf{e}_{n,1} + \mathbf{q}) + J_n(z_0)^*\mathbf{a} + (\mathbf{b}, \mathbf{e}_{m,m})\mathbf{e}_{n,1} \\ -\lambda G_{p_0} \mathbf{e}_{m,m} + J_m(z_0)^*\mathbf{b} + (\mathbf{d}, \alpha\mathbf{e}_{m,1} - \mathbf{p})\mathbf{e}_{m,1} \\ \lambda\mathbf{e}_{m,m} + J_m(z_0)\mathbf{d} \end{array} \right) \right\},$$

with $\chi_0 = \varphi_0(w_0)$, $\{f_0, g_0\} \in A_{N_0}$, $\lambda \in \mathbb{C}$, $\mathbf{a} \in \mathbb{C}^n$ such that $a_n = \lambda q_n$, and $\mathbf{b}, \mathbf{d} \in \mathbb{C}^m$, where w_0 is a fixed point in $\mathcal{D}(N_0)$.

Then $C^{(\alpha)}$ is the graph of a self-adjoint operator (also denoted by $C^{(\alpha)}$) in the space $(\mathcal{K}; G)$.

(ii) *Let \widehat{C} be the set of all pairs of the form*

$$\left\{ \left(\begin{array}{c} f_0 + \lambda\chi_0 \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{d} \end{array} \right), \left(\begin{array}{c} g_0 + w_0\lambda\chi_0 \\ -\langle f_0, \chi_{-1} \rangle \mathbf{e}_{n,1} - \lambda(N_0(w_0)\mathbf{e}_{n,1} + \mathbf{q}) + J_n(z_0)^*\mathbf{a} + (\mathbf{b}, \mathbf{e}_{m,m})\mathbf{e}_{n,1} \\ -\lambda G_{p_0} \mathbf{e}_{m,m} + J_m(z_0)^*\mathbf{b} + \mu\mathbf{e}_{m,1} \\ \lambda\mathbf{e}_{m,m} + J_m(z_0)\mathbf{d} \end{array} \right) \right\},$$

with $\chi_0 = \varphi_0(w_0)$, $\{f_0, g_0\} \in A_{N_0}$, $\lambda, \mu \in \mathbb{C}$, $\mathbf{a} \in \mathbb{C}^n$ such that $a_n = \lambda q_n$, $\mathbf{b} \in \mathbb{C}^m$, and $\mathbf{d} \in \mathbb{C}^m$ such that $d_1 = 0$, where w_0 is a fixed point in $\mathcal{D}(N_0)$.

Then \widehat{C} is a self-adjoint relation in $(\mathcal{K}; G)$.

(iii) *The minimal models of N and \widehat{N} in the space $(\mathcal{K}; G)$ are given by the triplets*

$$(C, N(z)\Gamma_z, S) \quad \text{and} \quad (\widehat{C}, \Gamma_z, S),$$

where $C = C^{(0)}$, $S = C \cap \widehat{C}$, and

$$\Gamma_z = \begin{pmatrix} \varphi_0(z)c(z) \\ \mathbf{s}_q(z)c(z) \\ \mathbf{r}_{2p_0}(z) + \mathbf{b}(z)c(z)(N_0(z) + q(z)) \\ \mathbf{d}(z) \end{pmatrix}.$$

(iv) *The family of all self-adjoint extensions of S in $(\mathcal{K}; G)$ is given by $C^{(\alpha)}$, $\alpha \in \mathbb{R}$, and \widehat{C} . Moreover, $\widehat{C} = C^\infty$, the limit in the resolvent sense of $C^{(\alpha)}$ as $\alpha \rightarrow \infty$.*

Note that $C^{(\alpha)}$ is of the form (2.3):

$$C^{(\alpha)} = C + \alpha \left\langle \cdot, \begin{pmatrix} 0 \\ 0 \\ \mathbf{e}_{m,1} \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 0 \\ \mathbf{e}_{m,1} \\ 0 \end{pmatrix}.$$

Note also \widehat{C} is multi-valued: $(0 \ 0 \ \mathbf{e}_{m,1} \ 0)^\top \in \widehat{C}(0)$, S can also be written as

$$S = C^{(\alpha)} \cap \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathbf{e}_{m,1} \\ 0 \end{pmatrix} \right\} \right\}^*,$$

and that

$$C^{(\alpha)} = S + \text{span} \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{e}_{m,1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ (\alpha - \widehat{N}_0(z_0))\mathbf{e}_{m,1} \\ z_0\mathbf{e}_{m,1} \end{pmatrix} \right\} \right\},$$

in particular, the domain of $C^{(\alpha)}$ is dense and independent of $\alpha \in \mathbb{R}$.

In the proof of Theorem 6.1 we identify – according to the basis discussed in Section 5 – the space $\mathcal{L}(\widetilde{N})$ with $\mathcal{K} = \mathcal{L}(N_0) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$ and the relations $B^{(\alpha)}$, \widehat{B} , and \widetilde{S} with $C^{(\alpha)}$, \widehat{C} , and S defined in the theorem. To explain the identification, let the vector function $\tilde{f} \in \mathcal{L}(\widetilde{N})$ be given as

$$\tilde{f}(\zeta) = (f(\zeta) \ a(\zeta) \ b(\zeta) \ d(\zeta))^\top,$$

where

$$\mathcal{L}(q) \ni a(\zeta) = \sum_{j=1}^n a_j(v_{m+j}(\zeta))_2 = \sum_{j=1}^n a_j(\zeta - z_0^*)^{j-1} \quad (6.4)$$

and

$$\begin{aligned} \mathcal{L}^0 \dot{+} \mathcal{M} \ni \begin{pmatrix} b(\zeta) \\ d(\zeta) \end{pmatrix} \\ = \sum_{j=1}^m b_j (v_j(\zeta))_{3,4} + d_j (u_j(\zeta))_{3,4} = \begin{pmatrix} \sum_{j=1}^m d_j R_{z_0}^j p_0(\zeta) + b_j (\zeta - z_0^*)^{j-1} \\ \sum_{j=1}^m d_j (\zeta - z_0)^{m-j} \end{pmatrix}. \end{aligned}$$

Then \tilde{f} will be identified with the element

$$\begin{pmatrix} f \\ \mathbf{a} \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} \in \mathcal{K},$$

where $\mathbf{a} = (a_1 \ \dots \ a_n)^\top$, etc. We write $\tilde{f} \simeq (f \ \mathbf{a} \ \mathbf{b} \ \mathbf{d})^\top$. For example, for \tilde{w} in (4.10) we have $\tilde{w} \simeq (0 \ 0 \ \mathbf{e}_{m,1} \ 0)^\top$.

Proof of Theorem 6.1. Identify the vector functions $\tilde{f}, \tilde{g} \in \mathcal{L}(\tilde{N})$ given by

$$\tilde{f}(\zeta) = (f(\zeta) \ a_1(\zeta) \ b_1(\zeta) \ d_1(\zeta))^\top, \quad \tilde{g}(\zeta) = (g(\zeta) \ a_2(\zeta) \ b_2(\zeta) \ d_2(\zeta))^\top$$

with the elements

$$\tilde{f} \simeq (f \ \mathbf{a}_1 \ \mathbf{b}_1 \ \mathbf{d}_1)^\top, \quad \tilde{g} \simeq (g \ \mathbf{a}_2 \ \mathbf{b}_2 \ \mathbf{d}_2)^\top$$

in \mathcal{K} . Here for $i = 1, 2$, the entries of the vectors $\mathbf{a}_i = (a_{i,1} \ \dots \ a_{i,n})^\top$, etc., appear as coefficients in the representations

$$a_i(\zeta) = \sum_{j=1}^n a_{i,j} (\zeta - z_0^*)^{j-1}, \quad \begin{pmatrix} b_i(\zeta) \\ d_i(\zeta) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^m (d_{i,j} R_{z_0}^j p_0(\zeta) + b_{i,j} (\zeta - z_0^*)^{j-1}) \\ \sum_{j=1}^m d_{i,j} (\zeta - z_0)^{m-j} \end{pmatrix}.$$

According to Theorem 4.1 we have

$$\{\tilde{f}(\zeta), \tilde{g}(\zeta)\} \in B^{(\alpha)} \iff \exists (h_1 \ h_2 \ h_3 \ h_4)^\top \in \mathbb{C}^4 :$$

$$\tilde{g}(\zeta) - \zeta \tilde{f}(\zeta) = \begin{pmatrix} h_1 - N_0(\zeta) h_4 \\ h_2 - q(\zeta) h_4 \\ -c^\#(\zeta)(h_1 + h_2) + (\alpha - p_0(\zeta)) h_3 \\ -c(\zeta) h_3 + h_4 \end{pmatrix}. \quad (6.5)$$

Comparison of the fourth components on both sides of this equality yields $h_3 = d_{1,1}$ and

$$d_{2,j} = z_0 d_{1,j} + d_{1,j+1}, \quad j = 1, \dots, m-1; \quad d_{2,m} = z_0 d_{1,m} + h_4.$$

In the same way the second components in (6.5) yield $h_4 = a_{1,n}/q_n$ and

$$\begin{aligned} a_{2,1} &= z_0^* a_{1,1} + h_2 - q_0 \frac{a_{1,n}}{q_n}, \\ a_{2,j} &= z_0^* a_{1,j} + a_{1,j-1} - q_{j-1} \frac{a_{1,n}}{q_n}, \quad j = 2, \dots, n. \end{aligned}$$

We now consider the third components in (6.5). Inserting the expressions for $d_{2,j}$ already obtained we find that

$$\sum_{j=1}^m d_{2,j} \sum_{k=j}^{\ell_0} p_k (\zeta - z_0)^{k-j} - \sum_{j=1}^m \sum_{k=j}^{\ell_0} d_{1,j} p_k \left((\zeta - z_0)^{k-j+1} + z_0 (\zeta - z_0)^{k-j} \right)$$

reduces to

$$\sum_{j=1}^m d_{1,j} p_{j-1} - h_3 p_0(\zeta) + h_4 \sum_{k=m}^{\ell_0} p_k (\zeta - z_0)^{k-m}.$$

In the last term we may replace ℓ_0 by $2m - 1$, because $\ell_0 < 2m$ and $p_k = 0$ if $k > \ell_0$. Using the relation (5.1) the last term can now be rewritten as $h_4 \sum_{j=1}^m \langle u_m, u_j \rangle_{\mathcal{L}(\tilde{N})} (\zeta - z_0^*)^{j-1}$. Then in the third components there only remain powers of $\zeta - z_0^*$ and comparing these we find $h_1 + h_2 = b_{1,m}$ and

$$\begin{aligned} b_{2,1} &= z_0^* b_{1,1} + \alpha d_{1,1} - \sum_{j=1}^m p_{j-1} d_{1,j} - h_4 \langle u_m, u_1 \rangle_{\mathcal{L}(\tilde{N})}, \\ b_{2,j} &= z_0^* b_{1,j} + b_{1,j-1} - h_4 \langle u_m, u_j \rangle_{\mathcal{L}(\tilde{N})}, \quad j = 2, \dots, m. \end{aligned}$$

If we rewrite $N_0(\zeta)$ in the first component of (6.5) as $(\zeta - w_0)K_{N_0}(\zeta, w_0^*) + N_0(w_0)$ we find

$$\underbrace{(g(\zeta) - w_0 h_4 K_{N_0}(\zeta, w_0^*))}_{=:g_0(\zeta)} - \zeta \underbrace{(f(\zeta) - h_4 K_{N_0}(\zeta, w_0^*))}_{=:f_0(\zeta)} = h_1 - h_4 N_0(w_0).$$

So $\{f_0, g_0\} \in A_{N_0}$ and with $\lambda = a_{1,n}/q_n = h_4$ we have

$$\begin{cases} f &= f_0 + \lambda K_{N_0}(\cdot, w_0^*), \\ g &= g_0 + w_0 \lambda K_{N_0}(\cdot, w_0^*). \end{cases} \quad (6.6)$$

Because $N_0 \notin \mathcal{L}(N_0)$, we have that $K_{N_0}(\cdot, w_0^*) \notin \text{dom } A_{N_0}$, hence the decomposition (6.6) is unique. Furthermore, we have that $h_1 = \langle f_0, N_0 \rangle + \lambda N_0(w_0)$. Indeed, since $\{f_0, g_0\} \in A_{N_0}$, the difference $g_0(\zeta) - \zeta f_0(\zeta)$ is identically equal to a constant and hence, on account of (6.2),

$$\begin{aligned} h_1 - \lambda N_0(w_0) &= h_1 - h_4 N_0(w_0) \\ &= g_0(\zeta) - \zeta f_0(\zeta) = g_0(w_0) - w_0 f_0(w_0) = \langle f_0, N_0 \rangle. \end{aligned}$$

Hence $h_2 = b_{1,m} - h_1 = b_{1,m} - \langle f_0, N_0 \rangle - \lambda N_0(w_0)$. Together these formulas show that $\{\tilde{f}(\zeta), \tilde{g}(\zeta)\} \in B^{(\alpha)}$ can be identified with a pair of elements in \mathcal{K} of the form described in the theorem. Hence under the identification $B^{(\alpha)}$ coincides with $C^{(\alpha)}$.

(ii) That \widehat{C} can be identified with \widehat{B} can be proved in a similar way and therefore the details are omitted.

(iii) In the identification between $\mathcal{L}(\widehat{N})$ and \mathcal{K} , the Γ -field $K_{\widehat{N}}(\cdot, z^*)v(z)$ in Theorem 4.1 coincides with the Γ -field Γ_z in (iii). \square

Example. Consider $\widehat{N}(z) = (z^2 + 1)N_0(z) + \gamma_3z^3 + \gamma_2z^2 + \gamma_1z + \gamma_0$, where N_0 is a Nevanlinna function satisfying (1.3) and the γ_j 's are real numbers with $\gamma_3 \neq 0$. We rewrite \widehat{N} in the form (4.1) and (6.3):

$$\widehat{N}(z) = (z + i)\{N_0(z) + q_1(z + i) + q_0\}(z - i) + p_1(z - i) + p_0$$

with $q_1 = \gamma_3$, $q_0 = \gamma_2 - i\gamma_3$, $p_1 = \gamma_1 - \gamma_3$ and $p_0 = \widehat{N}(i) = \gamma_0 - \gamma_2 + i(\gamma_1 - \gamma_3)$. Then $m = 1$, $n = 1$, $\widehat{N} \in \mathcal{N}_\kappa^\infty$, where $\kappa = 2$ if $\gamma_3 < 0$ and $\kappa = 1$ if $\gamma_3 > 0$. The state space for N and \widehat{N} is $\mathcal{K} = \mathcal{L}(N_0) \oplus \mathbb{C} \oplus \mathbb{C}^2$ equipped with the indefinite inner product $\langle G \cdot, \cdot \rangle_{\mathcal{K}}$ in which the Gram matrix is given by

$$G = \text{diag} \left(I_{\mathcal{L}(N_0)}, \frac{1}{\gamma_3}, \begin{pmatrix} 0 & 1 \\ 1 & \gamma_1 - \gamma_3 \end{pmatrix} \right).$$

We take $w_0 = i$, and recall $\chi_0(z) = K_{N_0}(\cdot, z^*) \in \mathcal{L}(N_0)$ and $\chi_{-1} = N_0 \in \mathcal{L}(N_0)_{-1}$. We find that the graph of the self-adjoint operator $C^{(\alpha)}$ is the set of all elements of the form

$$\left\{ \begin{pmatrix} f_0 + \lambda\chi_0 \\ \lambda\gamma_3 \\ b \\ d \end{pmatrix}, \begin{pmatrix} g_0 + i\lambda\chi_0 \\ -\langle f_0, \chi_{-1} \rangle - \lambda(N_0(i) + \gamma_2) + b \\ -\lambda(\gamma_1 - \gamma_3) - ib + (\alpha - \widehat{N}(i))d \\ \lambda + id \end{pmatrix} \right\}$$

with $\{f_0, g_0\} \in A_{N_0}$, $\lambda, b, d \in \mathbb{C}$. The self-adjoint relation \widehat{C} is the set of all elements of the form

$$\left\{ \begin{pmatrix} f_0 + \lambda\chi_0 \\ \lambda\gamma_3 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} g_0 + i\lambda\chi_0 \\ -\langle f_0, \chi_{-1} \rangle - \lambda(N_0(i) + \gamma_2) + b \\ \mu \\ \lambda \end{pmatrix} \right\}$$

with $\{f_0, g_0\} \in A_{N_0}$, $\lambda, b, \mu \in \mathbb{C}$.

From Corollary 4.5 we obtain the following theorem. We set $S_1(z; z_0) = 0$ and if $n \geq 2$,

$$S_n(z; z_0) = \begin{pmatrix} 0 & 1 & (z - z_0) & \dots & (z - z_0)^{n-2} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & (z - z_0) \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

and we recall that the definitions of the matrix functions $K_\alpha(z)$ and $\widehat{K}(z)$ are given just before Corollary 4.5.

Theorem 6.2. *The resolvents of the operators $C^{(\alpha)}$, $\alpha \in \mathbb{R}$, and the relation \widehat{C} in \mathcal{K} are given by*

$$\begin{aligned} (C^{(\alpha)} - z)^{-1} &= \text{diag} \left((A_{N_0} - z)^{-1}, S_n(z; z_0^*), \begin{pmatrix} S_m(z; z_0^*) & \mathbf{r}_{1p_0}(z) \mathbf{b}^\#(z) \\ 0 & S_m^\#(z; z_0^*) \end{pmatrix} \right) \\ &+ \frac{N(z)}{1 + \alpha N(z)} \text{diag} \left(\varphi_0(z), \mathbf{s}_q(z), \begin{pmatrix} \mathbf{r}_{2p_0}(z) & \mathbf{b}(z) \\ \mathbf{d}(z) & 0 \end{pmatrix} \right) K_\alpha(z) \\ &\times \text{diag} \left(\langle \cdot, \varphi_0(z^*) \rangle_{\mathcal{L}(N_0)}, \mathbf{d}^\#(z), \begin{pmatrix} \mathbf{d}^\#(z) & \mathbf{t}_{p_0}(z)^\top \\ 0 & \mathbf{b}^\#(z) \end{pmatrix} \right), \end{aligned}$$

where $\varphi_0(z) = K_{N_0}(\cdot, z^*)$, and

$$\begin{aligned} (\widehat{C} - z)^{-1} &= \text{diag} \left((A_{N_0} - z)^{-1}, S_n(z; z_0^*), \begin{pmatrix} S_m(z; z_0^*) & \mathbf{r}_{1p_0}(z) \mathbf{b}^\#(z) \\ 0 & S_m^\#(z; z_0^*) \end{pmatrix} \right) \\ &+ \text{diag} \left(\varphi_0(z), \mathbf{s}_q(z), \begin{pmatrix} \mathbf{r}_{2p_0}(z) & \mathbf{b}(z) \\ \mathbf{d}(z) & 0 \end{pmatrix} \right) \widehat{K}(z) \\ &\times \text{diag} \left(\langle \cdot, \varphi_0(z^*) \rangle_{\mathcal{L}(N_0)}, \mathbf{d}^\#(z), \begin{pmatrix} \mathbf{d}^\#(z) & \mathbf{t}_{p_0}(z)^\top \\ 0 & \mathbf{b}^\#(z) \end{pmatrix} \right). \end{aligned}$$

As to the proof of the theorem we only mention the following identifications between the elements and operators in $\mathcal{L}(q)$ and \mathbb{C}^n :

$$\begin{aligned} K_q(\cdot, z^*) &\simeq \mathbf{s}_q(z) &= (J_n(z_0)^* - z)^{-1} (\mathbf{q} - q(z) \mathbf{e}_{n,1}), \\ \langle \cdot, K_q(\cdot, z) \rangle &\simeq (G_q \cdot, \mathbf{s}_q(z^*)) &= \mathbf{d}^\#(z), \\ (A_q - z)^{-1} &\simeq S_n(z; z_0^*) &= (I + (z_0^* - z) J_n(0))^{-1} J_n(0). \end{aligned}$$

The first identification follows from (5.7) and (5.8) which show that $\mathbf{s}_q(z)$ is the vector representation of the element $K_q(\cdot, z^*) \in \mathcal{L}(q)$ relative to the basis $\{v_{m+i}\}_{i=1}^n$. The linear functional $\langle \cdot, K_q(\cdot, z^*) \rangle_{\mathcal{L}(q)}$ on $\mathcal{L}(q)$ can be identified with $(G_q \cdot, \mathbf{s}_q(z^*)) = \mathbf{d}^\#(z)$ viewed as a mapping from \mathbb{C}^n to \mathbb{C} . This also follows from (6.4). In the theorem $\mathbf{r}_{1p_0}(z) \mathbf{b}^\#(z)$ is an $m \times m$ matrix polynomial of rank 1. Also, the matrix

$$\begin{pmatrix} \mathbf{r}_{2p_0}(z) & \mathbf{b}(z) \\ \mathbf{d}(z) & 0 \end{pmatrix}$$

whose entries are column m -vectors should be viewed as a mapping from \mathbb{C}^2 to \mathbb{C}^{2m} , whereas the matrix

$$\begin{pmatrix} \mathbf{d}^\#(z) & \mathbf{t}_{p_0}(z)^\top \\ 0 & \mathbf{b}^\#(z) \end{pmatrix}$$

whose entries are row m -vectors should be seen as a mapping from \mathbb{C}^{2m} to \mathbb{C}^2 . The matrices are related via the formula

$$\begin{pmatrix} \mathbf{r}_{2p_0}(z) & \mathbf{b}(z) \\ \mathbf{d}(z) & 0 \end{pmatrix}^\# \begin{pmatrix} 0 & I_{\mathbb{C}^m} \\ I_{\mathbb{C}^m} & G_{p_0} \end{pmatrix} = \begin{pmatrix} \mathbf{d}^\#(z) & \mathbf{t}_{p_0}(z)^\top \\ 0 & \mathbf{b}^\#(z) \end{pmatrix},$$

which corresponds to a part of the identity $\Gamma_{Nz^*}^* = E_z$.

We now consider the analogs of Theorem 6.1 in the cases $n = 0$ and $m = 0$.
 The case $n = 0$ and $m > 0$: Here $\mathcal{K} = \mathcal{L}(N_0) \oplus \mathbb{C}^m \oplus \mathbb{C}^m$ and the Gram matrix takes the form

$$G = \begin{pmatrix} I_{\mathcal{L}(N_0)} & 0 & 0 \\ 0 & 0 & I_{\mathbb{C}^m} \\ 0 & I_{\mathbb{C}^m} & G_{p_0} \end{pmatrix}.$$

Theorem 6.3. *Assume $n = 0$ and $m > 0$. Then Theorem 6.1 holds if we delete the second component in all 4-vectors in the formulas, omit in (i) and (ii) the statement “ $\mathbf{a} \in \mathbb{C}^n$ such that $a_n = \lambda q_n$ ”, add in (i) and (ii) the statement “ $b_m = \langle f_0, \chi_{-1} \rangle + \lambda(N_0(w_0) + q_0)$ ”, and set $q(z) = q_0$ in Γ_z in (iii).*

The case $n > 0$ and $m = 0$: Now $\mathcal{K} = \mathcal{L}(N_0) \oplus \mathbb{C}^n$ and $G = \text{diag} \{I_{\mathcal{L}(N_0)}, G_q\}$.

Theorem 6.4. *Assume $n > 0$ and $m = 0$. Then Theorem 6.1 holds if $C^{(\alpha)}$ is the set of all pairs of the form*

$$\left\{ \begin{pmatrix} f_0 + \lambda \chi_0 \\ \mathbf{a} \end{pmatrix}, \begin{pmatrix} g_0 + w_0 \lambda \chi_0 \\ -\langle f_0, \chi_{-1} \rangle \mathbf{e}_{n,1} - \lambda(N_0(w_0) \mathbf{e}_{n,1} + \mathbf{q} + \alpha \mathbf{e}_{n,1}) + J_n(z_0)^* \mathbf{a} \end{pmatrix} \right\} \quad (6.7)$$

with $\chi_0 = \varphi_0(w_0)$, $\{f_0, g_0\} \in A_{N_0}$, $\lambda \in \mathbb{C}$, and $\mathbf{a} \in \mathbb{C}^n$ such that $a_n = \lambda q_n$, where w_0 is a fixed point in $\mathcal{D}(N_0)$, if

$$\widehat{C} = A_{N_0} \oplus \{ \{ \mathbf{a}, J_n(z_0)^* \mathbf{a} + \mu \mathbf{e}_{n,1} \} \mid \mathbf{a} \in \mathbb{C}^n, a_n = 0, \mu \in \mathbb{C} \},$$

and if $\Gamma_z = (\varphi_0(z), \mathbf{s}_q(z))^\top$.

Next we give the formulas for the compressions of the resolvents $(C^{(\alpha)} - z)^{-1}$, $\alpha \in \mathbb{R}$, and $(\widehat{C} - z)^{-1}$ to the subspaces $\mathcal{L}(N_0)$ and $\mathcal{L}(N_0) \oplus \mathbb{C}^n$ of $(\mathcal{K}; G)$ in the case $n > 0$ and $m > 0$; similar formulas can be obtained in the other two cases. We denote by P_0 and P_1 the orthogonal projections in $(\mathcal{K}; G)$ onto $\mathcal{L}(N_0)$ and $\mathcal{L}(N_0) \oplus \mathbb{C}^n$.

Theorem 6.5.

(i) For $\alpha \in \mathbb{R}$,

$$P_0(C^{(\alpha)} - z)^{-1}|_{\mathcal{L}(N_0)} = (A_{N_0} - z)^{-1} - \frac{1}{N_0(z) + T_\alpha(z)} \langle \cdot, \varphi_0(z^*) \rangle_{\mathcal{L}(N_0)} \varphi_0(z)$$

with parameter $T_\alpha(z) = q(z) + \frac{p_0(z) - \alpha}{c^\#(z)c(z)}$, and for $\alpha = \infty$,

$$P_0(\widehat{C} - z)^{-1}|_{\mathcal{L}(N_0)} = (A_{N_0} - z)^{-1}.$$

(ii) For $\alpha \in \mathbb{R}$,

$$P_1(C^{(\alpha)} - z)^{-1}|_{\mathcal{L}(N_0) \oplus \mathbb{C}^n} = \begin{pmatrix} (A_{N_0} - z)^{-1} & 0 \\ 0 & S_n(z; z_0^*) \end{pmatrix} - \frac{1}{N_0(z) + T_\alpha(z)} \begin{pmatrix} \langle \cdot, \varphi_0(z^*) \rangle_{\mathcal{L}(N_0)} \varphi_0(z) & \varphi_0(z) \mathbf{d}^\#(z) \\ \langle \cdot, \varphi_0(z^*) \rangle_{\mathcal{L}(N_0)} \mathbf{s}_q(z) & \mathbf{s}_q(z) \mathbf{d}^\#(z) \end{pmatrix},$$

and for $\alpha = \infty$,

$$P_1(C^\infty - z)^{-1}|_{\mathcal{L}(N_0) \oplus \mathbb{C}^n} = \begin{pmatrix} (A_{N_0} - z)^{-1} & 0 \\ 0 & S_n(z; z_0^*) \end{pmatrix}.$$

The resolvent formula

$$(A_\tau - z)^{-1} = (A_{N_0} - z)^{-1} - \frac{1}{N_0(z) + \tau} \langle \cdot, \varphi_0(z^*) \rangle \varphi_0(z)$$

describes the family of all self-adjoint extensions A_τ of S_{N_0} in $\mathcal{L}(N_0)$ in terms of the parameter $\tau \in \mathbb{R} \cup \{\infty\}$. If the number τ is replaced by a generalized Nevanlinna function $\tau(z)$ the formula describes the minimal self-adjoint extensions which act in spaces containing $\mathcal{L}(N_0)$. This is called Krein’s resolvent formula. In part (i) the parameter describing $C^{(\alpha)}$ with $\alpha \in \mathbb{R}$ is explicitly given by $\tau(z) = T_\alpha(z) \in \mathcal{N}_\kappa$. Related to Krein’s formula here are the references [21, Theorem 4.7], [13, Theorem 4.2], and [7, Theorem 3.5].

It is of interest to compare the compressed resolvents of $C^{(\alpha)}$ and \widehat{C} with the compressions of these operators/relation themselves. Recall Stenger’s lemma (see [15, Theorem 3.3 and a remark after the theorem]) that if A is a self-adjoint relation with $\rho(A) \neq \emptyset$ in a Pontryagin space $\widetilde{\mathcal{H}}$ and \mathcal{H} is a Hilbert or Pontryagin subspace of $\widetilde{\mathcal{H}}$, such that $\dim \widetilde{\mathcal{H}} \ominus \mathcal{H} < \infty$, then the compression of A to \mathcal{H} , that is, the linear relation

$$P_{\mathcal{H}}A|_{\mathcal{H}} = \{ \{f, P_{\mathcal{H}}g\} \mid \{f, g\} \in A, f \in \mathcal{H} \},$$

where $P_{\mathcal{H}}$ is the orthogonal projection in $\widetilde{\mathcal{H}}$ onto \mathcal{H} , is self-adjoint in \mathcal{H} . In our case $\mathcal{L}(N_0)$ is a Hilbert subspace and $\mathcal{L}(N_0) \oplus \mathbb{C}^n$ is a Pontryagin subspace of the Pontryagin space \mathcal{K} and both have a finite codimension. Therefore the compressions just mentioned are self-adjoint. The following theorem follows directly from Theorem 6.1 and its versions for the special cases $n = 0$ or $m = 0$.

Theorem 6.6.

(i) If $n > 0$ and $m \geq 0$, then

$$P_0C^{(\alpha)}|_{\mathcal{L}(N_0)} = P_0\widehat{C}|_{\mathcal{L}(N_0)} = A_{N_0},$$

and if $n = 0$ and $m > 0$, then (in graph notation)

$$\begin{aligned} P_0C^{(\alpha)}|_{\mathcal{L}(N_0)} &= P_0\widehat{C}|_{\mathcal{L}(N_0)} \\ &= \{ \{f_0 + \lambda\chi_0, g_0 + w_0\lambda\chi_0\} \mid \{f_0, g_0\} \in A_{N_0}, \langle f_0, \chi_{-1} \rangle + \lambda(N_0(w_0) + q_0) = 0 \}. \end{aligned}$$

(ii) If $n > 0$ and $m > 0$, then

$$P_1C^{(\alpha)}|_{\mathcal{L}(N_0) \oplus \mathbb{C}^n} = P_1\widehat{C}|_{\mathcal{L}(N_0) \oplus \mathbb{C}^n}$$

and their graphs coincide with the set of all pairs of the form (6.7) with $\alpha = 0$.

7. Block operator matrix models in the space $(\mathcal{K}; H)$

Changing the basis we have considered in the previous sections we can write $C^{(\alpha)}$ and \widehat{C} in Theorem 6.1 with $w_0 = z_0^*$ in a block operator matrix form, which is not possible with the basis used so far. Set

$$T = \begin{pmatrix} I_{\mathcal{L}(N_0)} & \frac{1}{q_n}(\cdot, \mathbf{e}_{n,n})\chi_0 & 0 & 0 \\ 0 & I_{\mathbb{C}^n} & 0 & 0 \\ 0 & 0 & I_{\mathbb{C}^m} & 0 \\ 0 & 0 & 0 & I_{\mathbb{C}^m} \end{pmatrix},$$

where $\chi_0 = \varphi_0(z_0^*)$, and define the operators $D^{(\alpha)} = T^{-1}C^{(\alpha)}T$, $\alpha \in \mathbb{R}$, the linear relation $\widehat{D} = T^{-1}\widehat{C}T$, and the Gram matrix

$$H = T^*GT = \begin{pmatrix} I_{\mathcal{L}(N_0)} & \frac{1}{q_n}(\cdot, \mathbf{e}_{n,n})\chi_0 & 0 & 0 \\ \frac{1}{q_n}\langle \cdot, \chi_0 \rangle_{\mathcal{L}(N_0)}\mathbf{e}_{n,n} & \frac{h_0}{q_n^2}(\cdot, \mathbf{e}_{n,n})\mathbf{e}_{n,n} + G_q & 0 & 0 \\ 0 & 0 & 0 & I_{\mathbb{C}^m} \\ 0 & 0 & I_{\mathbb{C}^m} & G_{p_0} \end{pmatrix},$$

where $h_0 = \langle \chi_0, \chi_0 \rangle_{\mathcal{L}(N_0)} = K_{N_0}(z_0, z_0)$. Since $T^{-1}\tilde{w} = \tilde{w}$, we have

$$D^{(\alpha)} = D + \alpha \langle H \cdot, \tilde{w} \rangle_{\mathcal{K}} \tilde{w}, \quad D = T^{-1}CT.$$

The space \mathcal{K} equipped with the indefinite inner product $\langle H \cdot, \cdot \rangle_{\mathcal{K}}$ will be denoted by $(\mathcal{K}; H)$. Clearly, $D^{(\alpha)}$ and \widehat{D} are self-adjoint in $(\mathcal{K}; H)$. The relation \widehat{D} can be obtained via infinite coupling of D and \tilde{w} , that is, as limit of $D^{(\alpha)}$ in the resolvent sense by letting $\alpha \rightarrow \infty$. The following theorem shows that $D^{(\alpha)}$ and \widehat{D} can be expressed by means of block operator matrices. We use the notation explained directly above Theorem 6.1.

Theorem 7.1. *Let $D^{(\alpha)}$, $\alpha \in \mathbb{R}$, and \widehat{D} be as defined above. Then:*

- (i) $\text{dom } D^{(\alpha)} = (\text{dom } A_{N_0}) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^m$ and on this domain $D^{(\alpha)}$ has the block matrix form

$$D^{(\alpha)} = \begin{pmatrix} A_{N_0} & D_{12} & 0 & 0 \\ D_{21} & D_{22} & (\cdot, \mathbf{e}_{m,m})\mathbf{e}_{n,1} & 0 \\ 0 & -\frac{1}{q_n}(\cdot, \mathbf{e}_{n,n})G_q\mathbf{e}_{m,m} & J_m(z_0)^* & (\cdot, \alpha\mathbf{e}_{m,1} - \mathbf{p})\mathbf{e}_{m,1} \\ 0 & \frac{1}{q_n}(\cdot, \mathbf{e}_{n,n})\mathbf{e}_{m,m} & 0 & J_m(z_0) \end{pmatrix},$$

where with $\chi_0 = \varphi_0(z_0)$

$$D_{12} = \left(\frac{q_{n-1}}{q_n^2}(\cdot, \mathbf{e}_{n,n}) - \frac{1}{q_n}(\cdot, \mathbf{e}_{n,n-1}) \right) \chi_0, \quad D_{21} = -\langle \cdot, \chi_{-1} \rangle \mathbf{e}_{n,1},$$

and

$$D_{22} = J_n(z_0)^* - \frac{1}{q_n}(\cdot, \mathbf{e}_{n,n})(N_0(z_0^*)\mathbf{e}_{n,1} + \mathbf{q}).$$

(ii) $\text{dom } \widehat{D} = (\text{dom } A_{N_0}) \oplus \mathbb{C}^n \oplus \mathbb{C}^m \oplus (\mathbb{C}^m \ominus \{\mathbf{e}_{m,1}\})$ and

$$\widehat{D} = \{ \{ \tilde{f}, \tilde{g} \} \mid \tilde{f} \in \text{dom } \widehat{D}, \exists \mu \in \mathbb{C} : \tilde{g} = D\tilde{f} + \begin{pmatrix} 0 & 0 & \mu \mathbf{e}_{m,1} & 0 \end{pmatrix}^\top \},$$

where $D = D^{(0)}$.

(iii) The triplets $(D, N(z)\tilde{\Gamma}_z, \tilde{S})$, $(\widehat{D}, \tilde{\Gamma}_z, \tilde{S})$ are minimal models for N and \widehat{N} in $(\mathcal{K}; H)$, where $\tilde{S} = D \cap \widehat{D}$ and

$$\tilde{\Gamma}_z = \begin{pmatrix} (\varphi_0(z) - \varphi_0(z_0))c(z) \\ \mathbf{s}_q(z)c(z) \\ \mathbf{r}_{2p_0}(z) + \mathbf{b}(z)c(z)(N_0(z) + q(z)) \\ \mathbf{d}(z) \end{pmatrix}.$$

The theorem follows directly from Theorem 6.1 with $w_0 = z_0^*$, the definitions of $D^{(\alpha)}$ and \widehat{D} , and $\tilde{\Gamma}_z = T^{-1}\Gamma_z$. Note that $\tilde{\Gamma}_{z_0} = \begin{pmatrix} 0 & 0 & 0 & \mathbf{e}_{m,1} \end{pmatrix}^\top$.

From the theorems in Section 6 one can easily obtain formulas for the resolvents, the compressions of the resolvent and the compressions of $D^{(\alpha)}$ and \widehat{D} . We leave the details to the reader.

8. Examples

We give two examples and discuss an approximation problem taken from [11], [17], and [14] to which we refer for details and proofs. They are related to the Bessel differential expression

$$\ell_\nu y(x) = -y''(x) + \frac{\nu^2 - 1/4}{x^2}y(x)$$

on $(0, 1]$ with a self-adjoint boundary condition at the regular endpoint $x = 1$ and on $(0, \infty)$, which is limit point at $x = \infty$. We recall the series expansion of the Bessel function

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \nu + 1)} \left(\frac{z}{2}\right)^{2k}, \quad (8.1)$$

where the series on the right converges absolutely, and uniformly in any bounded domain of z and ν . We denote by

$$\begin{aligned} Y_\nu(z) &= \frac{1}{\sin(\nu\pi)} (J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)), \\ H_\nu^{(1)}(z) &= \frac{1}{i\sin(\nu\pi)} (J_{-\nu}(z) - J_\nu(z)e^{-i\nu\pi}), \\ K_\nu(z) &= i^{\frac{\pi}{2}} e^{i\frac{\pi}{2}\nu} H_\nu^{(1)}(iz) \end{aligned}$$

the Neumann function of order ν , the first Hankel function of order ν , and the Basset (or MacDonald) function of order ν , respectively; see, for example, [18].

Example. See [11]. We consider ℓ_ν on the interval $(0, 1]$ and impose the boundary condition $y(1) = 0$ at the regular endpoint $x = 1$ on all functions y .

(i) First assume $0 < \nu < 1$. Then the minimal realization S of ℓ_ν in $\mathcal{H}_0 = L^2(0, 1)$ is a symmetric operator with defect indices $(1, 1)$. We denote by A_0 the self-adjoint operator extension of S with graph

$$A_0 = \{ \{y, \ell_\nu y\} \in S^* \mid \lim_{x \downarrow 0} x^{\nu-1/2} y(x) = 0 \}. \tag{8.2}$$

The function

$$\widehat{\varphi}(x, z) = -\frac{\pi}{2 \sin \pi \nu} z^{\nu/2} x^{1/2} \left(\frac{J_{-\nu}(\sqrt{z})}{J_\nu(\sqrt{z})} J_\nu(x\sqrt{z}) - J_{-\nu}(x\sqrt{z}) \right) \tag{8.3}$$

belongs to $\ker(S^* - z)$ and is a defect function for S and A_0 with corresponding Q -function

$$\widehat{N}(z) = -\frac{\pi}{2 \sin \pi \nu} z^\nu \frac{J_{-\nu}(\sqrt{z})}{J_\nu(\sqrt{z})}. \tag{8.4}$$

Thus the following relations hold:

$$\widehat{\varphi}(z) = (I + (z - z_0)(A_0 - z)^{-1})\widehat{\varphi}(z_0), \quad \frac{\widehat{N}(z) - \widehat{N}(w)^*}{z - w^*} = \langle \widehat{\varphi}(z), \widehat{\varphi}(w) \rangle_0. \tag{8.5}$$

(ii) Now assume $\nu > 1, \nu \neq 2, 3, \dots$. Then the results are quite different from those in (i): The minimal realization of ℓ_ν in \mathcal{H}_0 is self-adjoint, the function $\widehat{\varphi}(\cdot, z)$ in (8.3) is well defined but it does not belong to \mathcal{H}_0 , and the function \widehat{N} in (8.4) is now a generalized Nevanlinna function with $\kappa = [(\nu + 1)/2]$ negative squares. Thus the model for this function involves a self-adjoint operator or relation in a Pontryagin space with κ negative squares. In [11] we show that $\widehat{N} \in \mathcal{N}_\kappa^\infty$: Let $z_n, n = 1, 2, \dots$, be the enumeration of the zeros of the function $z^{-\nu/2} J_\nu(\sqrt{z})$ in increasing order. (The zeros are positive and, since the function is entire, countable.) Then \widehat{N} admits the decomposition

$$\widehat{N}(z) = z^{2\kappa} (N_0(z) + q_0) + p_0(z),$$

where

$$N_0(z) = \sum_{n=1}^\infty \left(\frac{1}{z_n - z} - \frac{z_n}{z_n^2 + 1} \right) \frac{2z_n^{\nu-2\kappa}}{J'_\nu(z_n)^2},$$

$$q_0 = \frac{1}{(2\kappa)!} \widehat{N}^{(2\kappa)}(0) - \sum_{n=1}^\infty \frac{2z_n^{\nu-2\kappa-1}}{(z_n^2 + 1) J'_\nu(z_n)^2},$$

and

$$p_0(z) = \sum_{j=0}^{2\kappa-1} p_j z^j, \quad p_j = \frac{1}{j!} \widehat{N}^{(j)}(0).$$

The Nevanlinna function N_0 satisfies the relations (1.3) and hence $\widehat{N} \in \mathcal{N}_\kappa^\infty$. Theorem 6.3 with $m = \kappa > 0, z_0 = 0$, and $G_{p_0} = (p_{i+j-1})_{i,j=1}^\kappa$ yields the description of the models for \widehat{N} and $N = -1/\widehat{N}$.

Example. See [17]. We now consider ℓ_ν on $(0, \infty)$. The endpoint $x = \infty$ is limit point so we do not need to impose a condition at this endpoint. Where possible we use the same notation as in the previous example.

(i) First assume $0 < \nu < 1$. Then the minimal realization S of ℓ_ν in $\mathcal{H}_0 = L^2(0, \infty)$ is a symmetric operator with defect indices $(1, 1)$. The self-adjoint extension A_0 of S defined by formula (8.2) is uniquely determined by the facts that its spectrum $\sigma(A_0) = [0, \infty)$ is absolutely continuous and that the functions $y(x, \lambda) = c(\lambda)x^{1/2}J_\nu(x\sqrt{\lambda})$, $\lambda \in [0, \infty)$, form a complete set of generalized eigenfunctions of A_0 , where $c(\lambda)$ is some normalizing factor. The function

$$\widehat{\varphi}(x, z) = \sqrt{x}(-z)^{\frac{\nu}{2}}K_\nu(x\sqrt{-z}) \tag{8.6}$$

belongs to $\ker(S^* - z)$ and is a defect function for S and A_0 with corresponding Q -function

$$\widehat{N}(z) = -\frac{\pi}{2 \sin \pi\nu} (-z)^\nu. \tag{8.7}$$

Thus the relations (8.5) are also valid in this case. It follows from (8.7) that \widehat{N} is a Nevanlinna function, which satisfies the limit conditions in (1.3).

(ii) Now assume $\nu > 1$ and $\nu \neq 2, 3, \dots$. Then, as in the previous example, the minimal realization of ℓ_ν in \mathcal{H}_0 is self-adjoint, the function $\widehat{\varphi}(\cdot, z)$ in (8.6) does not belong to \mathcal{H}_0 , and the function \widehat{N} in (8.7) is a generalized Nevanlinna function with $\kappa = [(\nu + 1)/2]$ negative squares. Here the branch of $(-z)^\nu$ is chosen so that $(-z)^\nu = r^\nu e^{i\nu(\theta - \pi)}$ if $z = re^{i\theta}$, $0 < \theta < 2\pi$. For any $z_0 \in (-\infty, 0)$ the function \widehat{N} admits the decomposition

$$\widehat{N}(z) = (z - z_0)^{2\kappa}(N_0(z) + q_0) + p_0(z),$$

where

$$N_0(z) = \int_0^\infty \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) \frac{t^\nu}{2(t - z_0)^{2\kappa}} dt, \quad q_0 = -\frac{\pi}{4 \sin \frac{\pi\nu}{2}},$$

and

$$p_0(z) = \sum_{j=0}^{2\kappa-1} p_j(z - z_0)^j, \quad p_j = \frac{1}{j!} \widehat{N}^{(j)}(z_0) = \frac{\pi(-1)^{j+1}}{2 \sin \pi\nu} \binom{\nu}{j} (-z_0)^{\nu-j}.$$

Since N_0 is a Nevanlinna function which satisfies the relations (1.3), we have that $\widehat{N} \in \mathcal{N}_\kappa^\infty$. Hence Theorem 6.3 with $m = \kappa > 0$ applies and provides the description of the models for $\widehat{N}(z)$ and $N(z) = -1/\widehat{N}(z)$. Since z_0 is real, the Gram matrix G_{p_0} is given by $G_{p_0} = (p_{i+j-1})_{i,j=1}^\kappa$.

Inspired by [29] and [28], we discuss an approximation problem in $\mathcal{N}_\kappa^\infty$; for details we refer to the paper [14] in preparation. In the context of the discussion around (1.4), the problem is to approximate strongly singular perturbations by smoother perturbations. Consider a function $\widehat{N} \in \mathcal{N}_\kappa^\infty$ ($\kappa > 0$) with irreducible representation (4.1):

$$\widehat{N}(z) = (z - z_0)^{2\kappa}(N_0(z) + q_0) + p_0(z),$$

and a sequence of functions $\widehat{N}_j \in \mathcal{N}_\kappa^\infty$ with irreducible representation (4.1):

$$\widehat{N}_j(z) = N_{0j}(z) + q_j(z), \quad j = 1, 2, \dots,$$

where N_0 and all N_{0j} are Nevanlinna functions satisfying (1.3), z_0 is a real number belonging to their common domain \mathcal{D} of holomorphy, $q_0 \in \mathbb{R}$, $p_0(z)$ and $q_j(z)$ are real polynomials with $\deg p_0 \leq 2\kappa - 1$, and $n = \deg q_j$ is either 2κ or $2\kappa \pm 1$: if n is odd and the leading coefficient of $q_j(z)$ is negative, then $n = 2\kappa - 1$; if n is odd and the leading coefficient of $q_j(z)$ is positive, then $n = 2\kappa + 1$. Assume that, as $j \rightarrow \infty$, \widehat{N}_j converges to \widehat{N} uniformly on compact subsets of \mathcal{D} . The approximation problem with variable spaces then is to describe this convergence in terms of the models of \widehat{N}_j and \widehat{N} and of the corresponding state spaces. We rewrite \widehat{N}_j in the following form

$$\widehat{N}_j(z) = (z - z_0)^{2\kappa} (M_{0j}(z) + q_{0j}) + p_{0j}(z) + q_{j,2\kappa+1}(z - z_0)^{2\kappa+1}, \quad (8.8)$$

where $M_{0j}(z)$ is a Nevanlinna function, $p_{0j}(z)$ is a real polynomial with $\deg p_{0j} \leq 2\kappa - 1$, and $q_{0j}, q_{j,2\kappa+1} \in \mathbb{R}$ with $q_{j,2\kappa+1} \geq 0$ (if $q_{j,2\kappa+1} > 0$ then it is the leading coefficient of $q_j(z)$). The convergence assumption is equivalent to the convergence of $M_{0j}(z) + q_{0j}$ to $N_0(z) + q_0$ uniformly on compact subsets of \mathcal{D} , the pointwise convergence of the polynomials $p_{0j}(z)$ to $p_0(z)$, and the convergence $q_{j,2\kappa+1} \rightarrow 0$, as $j \rightarrow \infty$. The representation (8.8) of $\widehat{N}_j(z)$ need not be irreducible, and so models will have to be constructed, which fall outside the scope of this paper. Approximation of operators with variation of the space in which they act has been considered in [22, pp. 512, 513]; for such approximations in an indefinite setting, see [27] and [26].

The application we have in mind is related to ℓ_ν and the last example. In [14] we show that the function \widehat{N} in (8.7) with $\nu > 1$, $\nu \neq 2, 3, \dots$, can be approximated by functions of the form

$$\widehat{N}^\delta(z) = N_0^\delta(z) + q^\delta(z) \quad (8.9)$$

by letting $\delta \downarrow 0$. Here $q^\delta(z)$ is some real polynomial of degree $[\nu]$ with coefficients depending on δ , which we will not further specify here, and the function N_0^δ is obtained as follows. Consider the family of regularized differential expressions

$$l_{\nu,\delta}y(x) = -y''(x) + \frac{\nu^2 - 1/4}{(x+\delta)^2}y(x)$$

on $(0, \infty)$, where the parameter δ varies over some interval $(0, \delta_0)$, $\delta_0 > 0$. Let S_δ be the minimal operator associated with $l_{\nu,\delta}$ in the Hilbert space $\mathcal{H}_0 = L^2(0, \infty)$; it is symmetric and its defect indices are $(1, 1)$. Each self-adjoint extension of S_δ can be obtained as the restriction of the maximal operator S_δ^* by the boundary condition $y'(0) = \alpha y(0)$ with $\alpha \in \mathbb{R} \cup \{\infty\}$. We denote by A_δ the extension corresponding to $\alpha = \infty$. The function

$$\varphi_\delta(x, z) = \gamma(x+\delta)^{1/2} \frac{K_\nu((x+\delta)\sqrt{-z})}{\delta^\nu K_\nu(\delta\sqrt{-z})}, \quad \gamma = 2^{\nu-1}\Gamma(\nu),$$

is a defect function for S_δ and A_δ . The function considered in (8.9) is by definition the function

$$N_0^\delta(z) = \gamma^2 \delta^{1-2\nu} \sqrt{-z} \frac{K'_\nu(\delta\sqrt{-z})}{K_\nu(\delta\sqrt{-z})}. \quad (8.10)$$

It satisfies the relation

$$\frac{N_0^\delta(z) - N_0^\delta(w)^*}{z - w^*} = \langle \varphi_\delta(z), \varphi_\delta(w) \rangle_0$$

and hence is a Q -function for S_δ and A_δ . It follows that N_0^δ is a Nevanlinna function with integral representation

$$N_0^\delta(z) = \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma_\delta(t) + \operatorname{Re} N_0^\delta(i),$$

where, for $t \geq 0$,

$$d\sigma_\delta(t) = \frac{1}{\pi} \operatorname{Im} N_0^\delta(t+i0) dt = \frac{2\gamma^2}{\pi^2 \delta^{2\nu}} \frac{1}{J_\nu^2(\delta\sqrt{t}) + Y_\nu^2(\delta\sqrt{t})} dt. \quad (8.11)$$

It will be shown (in [14]) that the function \widehat{N}^δ in (8.9) belongs to $\mathcal{N}_\kappa^\infty$ with $\kappa = [(\nu+1)/2]$ and, if $\delta \downarrow 0$, converges to \widehat{N} in (8.7) uniformly on compact subsets of $\mathcal{D}(\widehat{N})$. Note that the representation (8.9) of \widehat{N}^δ is irreducible and corresponds to (4.1) with $m = 0$. This in contrast with the limit function \widehat{N} whose irreducible representation corresponds to (4.1) with $m = \kappa > 0$.

We conclude the paper with a final remark.

Remark 8.1. From the beginning up to and including Section 7 we may replace the factor $c(z) = (z - z_0)^m$ by $c(z) = (z - z_1) \cdots (z - z_m)$ with $z_j \in \mathcal{D}(N_0)$ to obtain similar but more general models as in [13, Sections 6 and 7].

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