

University of Groningen

Statistical Auditing and the AOQL-method

Talens, Erik

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version

Publisher's PDF, also known as Version of record

Publication date:

2005

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Talens, E. (2005). *Statistical Auditing and the AOQL-method*. s.n.

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Chapter 5

Properties of AOQL

5.1 Introduction

AOQL is identified with a sampling method which guarantees that the quality of the population investigated meets certain minimum quality requirements after an inspection that allows for correction of errors found. Originally AOQL was designed for industrial purposes but nowadays this method is also used in auditing and in the control of administrative processes.

The Average Outgoing Quality Limit (AOQL) sampling system was developed by Dodge and Romig (1959). It was originally developed for industrial purposes. Applications in auditing can be found in for instance Arkin (1974), Cyert and Davidson (1962), and Kriens and Dekkers (1979). Veenstra and Kriens (1982) applied the method to the control of administrative processes. Van Batenburg and Kriens (1988) criticized the statistical derivation by Dodge and Romig and they presented a modification. This modified version of the AOQL-method is sometimes called the EOQL-method (Expected Outgoing Quality Limit). Simons et al. (1989) developed an algorithm for finding the sample size and the critical level of items in error in the sample for the EOQL-method. However, this algorithm uses a Poisson approximation for the hypergeometric distribution involved, which is allowed if the sample size is small compared to the population size. This chapter presents theory that fully exploits the underlying hypergeometric distribution and hence can always be applied. It also exploits the results we found in Chapter 4 on the hypergeometric distribution, which are essential to the effectiveness of our methods.

The following idea underlies the AOQL-method. Consider a population of size N consisting of good and bad items. The unknown number of bad items will be denoted by M . The fraction of errors before inspection is given by $p = \frac{M}{N}$. A random sample of size n is taken without replacement from this population and the items in this sample are inspected and corrected. If K , the number of bad items in the sample, exceeds a certain critical level k_0 , then all items in the population have to be inspected. Therefore, after inspection the quality of the population will really have improved, unless no errors are found in the sample. The expected fraction of remaining defects will be denoted by π . This sampling method requires n and k_0 to be determined cost effectively such that π does not exceed a certain predefined level P_l .

We assume that the inspection is perfect. This means that all errors are detected and that no correct items are marked as bad items, see e.g. Moors (2000), Raats and Moors (2000), and Raats and Moors (2004) for imperfect inspections.

Obviously, π depends on four parameters, namely k_0 , n , M and N . This chapter always assumes that $k_0, n, M \in \{0, 1, \dots, N\}$, where $N \geq 1$. The choice of k_0 will be an important feature. For the time being, k_0 will be considered fixed. We shall look at π as a function of n , M , and N for fixed k_0 . We denote the expected fraction of errors after inspection as $\pi(n, M, N)$.

We use the same extended definition of the binomial coefficients that we used in Chapter 4. Using this definition we do not have to incorporate the usual domain for k , the number of errors in the sample, namely $k = 0, \dots, n$ with the restriction that $k \geq n - (N - M)$ and $k \leq M$.

Since sampling is done without replacement, the number K of defective items in the sample follows a hypergeometric distribution with parameters n , M and N , thus $K \sim \mathcal{H}(n, M, N)$. Thus for non-negative integers k we have

$$\mathrm{P}\{K = k\} = \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}}. \quad (5.1.1)$$

We will also use $\Lambda(n, M, N)$ and $\lambda(n, M, N)$ in this chapter, especially in proving the theorems we will introduce. They were already defined in Chapter 4 in the following way

$$\Lambda(n, M, N) = \mathrm{P}\{K \leq k_0 | n, M, N\} = \sum_{k=0}^{k_0} \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}}, \quad (5.1.2)$$

and

$$\lambda(n, M, N) = \begin{cases} \frac{\Lambda(n, M+1, N)}{\Lambda(n, M, N)} > 0 & \text{if } M < N - n + k_0, \\ 0 & \text{if } N - n + k_0 \leq M \leq N - 1. \end{cases} \quad (5.1.3)$$

The fraction of errors after inspection $p_a(K)$ is given by

$$p_a(K) = \begin{cases} \frac{M-K}{N} & \text{if } K = 0, \dots, k_0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1.4)$$

From (4.1.1) and (5.1.4) it follows

$$\begin{aligned} \pi(n, M, N) &= E(p_a(K)) \\ &= E\left(\frac{M-K}{N} \mid K \leq k_0\right) \cdot \Lambda(n, M, N) \\ &= \sum_{k=0}^{\min(k_0, M)} \frac{M-k}{N} P\{K = k\}. \end{aligned} \quad (5.1.5)$$

Some obvious cases are:

(i) no inspection ($n = 0$):

$$\pi(0, M, N) = \frac{M}{N}, \quad \text{for } M \in \{0, \dots, N\};$$

(ii) no errors ($M = 0$):

$$\pi(n, 0, N) = 0, \quad \text{for } n \in \{0, \dots, N\};$$

(iii) complete inspection ($n = N$):

$$\pi(N, M, N) = 0, \quad \text{for } M \in \{0, \dots, N\}.$$

In less trivial cases, we can rewrite (5.1.5) in the following way.

Theorem 5.1.1. *For $M \in \{1, \dots, N\}$ and $n \in \{0, \dots, N-1\}$ the following alternative expression holds:*

$$\pi(n, M, N) = \frac{M}{N} \left(1 - \frac{n}{N}\right) \cdot \Lambda(n, M-1, N-1).$$

Proof. See Appendix 5.A

□

We have already mentioned that the statistical derivation of Dodge and Romig is disputable. They use

$$\begin{aligned}\pi(n, M, N) &= E\left(\frac{M - K}{N}\right) \cdot \Lambda(n, M, N) \\ &= \frac{M}{N}\left(1 - \frac{n}{N}\right) \cdot \Lambda(n, M, N)\end{aligned}$$

instead of

$$\begin{aligned}\pi(n, M, N) &= E\left(\frac{M - K}{N} \mid K \leq k_0\right) \cdot \Lambda(n, M, N) \\ &= \frac{M}{N}\left(1 - \frac{n}{N}\right) \cdot \Lambda(n, M - 1, N - 1).\end{aligned}$$

So, they use the unconditional expected value instead of the conditional one in the definition of π . Notice that the two are almost alike, except that we lower the arguments M and N of $\Lambda(n, M, N)$ by one.

If the original population does not contain many errors, then the sample will not contain many errors either. Only few errors will be corrected and π will increase as a function of M . If M increases further, then more errors will occur in the sample and therefore more errors will be corrected. The expected fraction of errors will still increase as a function of M , but not as much as for really small M . An even further increase of M will increase the probability of exceeding k_0 , and thus integral inspection of the population becomes more probable. A decrease of π as a function of M will be the result and finally π will become zero ($M = N$). Therefore, it can be expected that π is a unimodal function of M . We will call this the unimodality property of π . Of course, one would also expect that an increase in the sample size would decrease the expected fraction of errors after inspection. We will call this the monotonicity property of π . This monotonicity property also ensures that $\max_M \pi(n, M, N)$ decreases for increasing n . Using the Poisson approximation for the hypergeometric distribution it is easy to prove the unimodality and monotonicity properties described above. This chapter will establish these properties also when no approximation is used.

Using the AOQL-method, we have to find the smallest sample size for which the expected fraction of errors after inspection does not exceed a predefined value, P_l . Since we assume that we do not have any a priori knowledge of the amount of errors in the population before inspection, this has to hold for all

possible values of M . Checking for the least favourable case, $\max_M \pi(n, M, N)$ will suffice, because π is a unimodal function of M . Since $\max_M \pi(n, M, N)$ is a decreasing function of n , we have to find the smallest sample size for which

$$\max_M \pi(n, M, N) \leq P_l.$$

This smallest sample size is the optimal size.

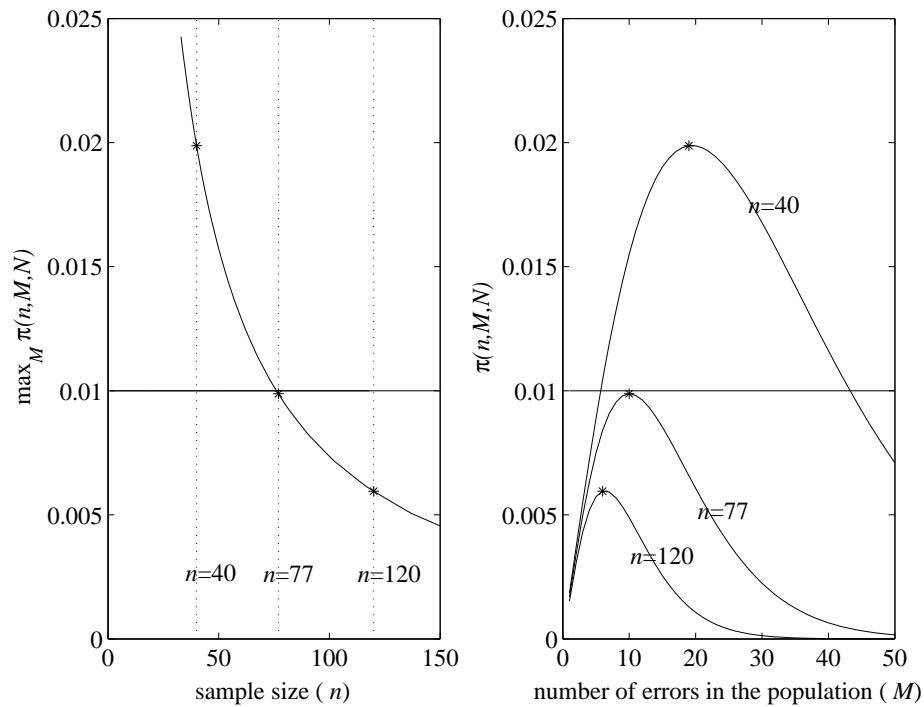


Figure 5.1. Plots of $\max_M \pi(n, M, N)$ and $\pi(n, M, N)$ for values of $n = 40$, $n = 77$, and $n = 120$ with $N = 500$, $k_0 = 1$, and $P_l = 0.01$. The optimal n equals 77.

An example of finding the optimal sample size can be found in Figure 5.1. For improved clarity the graphs in this paper are drawn continuously, but one should keep in mind the discrete nature of the parameters. To find the least favourable M for a certain n , we could start at the largest/smallest possible value of M and decrease/increase M by one until we find the maximum of π and use a bisection method to find the optimal sample size. However, this would be a rather

inefficient and time-consuming way to find the optimal sample size. We will find a very efficient and appealing method that fully exploits the underlying hypergeometric distribution. The theory needed for this method will be described in the subsequent sections.

Although the AOQL sample system guarantees that the expected number of bad items after inspection does not exceed a limit chosen on beforehand and minimizes the amount of work due to inspection, there still is a positive probability that the fraction of errors after inspection exceeds this limit. This probability can be sizable if the fraction of errors before inspection exceeds the limit chosen beforehand mildly (Kleijnen, Kriens, Lafleur and Pardoel, 1992).

It is sensible to divide the population in subpopulations and to apply the AOQL-method to each of the subpopulations. This offers protection against too much work if more than k_0 errors are found in the sample, because then only the subpopulation has to be fully inspected, instead of the entire population. Another advantage of splitting the population into subpopulations is that errors are discovered sooner. This means that measures can be taken to avoid these errors and to reduce the amount of work in the future. Kleijnen et al. (1992) showed by a simulation study that splitting up the population into subpopulations also reduces the probability that the fraction of errors after inspection exceeds the limit chosen beforehand. How the population has to be divided into subpopulations and how the size of the subpopulations has to be chosen depends on statistical as well as non-statistical reasons (Kleijnen et al., 1992; Kriens, 1988). If the expected fraction of errors in each subpopulation is smaller than P_l , then this obviously also holds for the whole population. Therefore, this paper only focuses on how we can determine the sample size n in one population. The condition that in all subpopulations π does not exceed P_l is a sufficient but not a necessary condition to ensure that π does not exceed P_l in the population. Other approaches are possible, for instance Klaassen (2001) describes a credit-based acceptance sampling system with $k_0 = 0$. His system loosens inspection as the total number of accepted items since the last rejection increases.

The following sections¹ study the behaviour of the expected fraction of errors after inspection, π , which, if $M \in \{1, \dots, N\}$ and $n \in \{0, \dots, N - 1\}$, can be

¹All proofs of these sections can be found in Appendix 5.A

written as

$$\pi(n, M, N) = \frac{M}{N} \left(1 - \frac{n}{N}\right) \cdot \Lambda(n, M - 1, N - 1). \quad (5.1.6)$$

Section 5.2 will prove the unimodality property of π with respect to M , the number of errors in the population, and the monotonicity property with respect to n , the sample size. We already mentioned these properties in this section. We will also establish some other properties of π .

Section 5.2 considers the expected fraction of errors π as a function of M . We are especially interested in its maximum $\pi^*(n, N)$. In order to obtain these maxima for various values of n and N we have to consider the values $M^*(n, N)$ at which $\pi(n, M, N)$ achieves this maximum. In Section 5.3 we derive various properties of $M^*(n, N)$ and we focus on its behaviour as a function of n for fixed values of N (and k_0).

Table 5.1. The triangular array of the values of $M^*(n, N)$ for $N, n \in \{0, \dots, 10\}$ and $k_0 = 1$.

$N \setminus n$	0	1	2	3	4	5	6	7	8	9	10
0	0	-	-	-	-	-	-	-	-	-	-
1	1	0	-	-	-	-	-	-	-	-	-
2	2	2	0	-	-	-	-	-	-	-	-
3	3	3	2	0	-	-	-	-	-	-	-
4	4	4	2	2	0	-	-	-	-	-	-
5	5	5	3	2	2	0	-	-	-	-	-
6	6	6	4	3	2	2	0	-	-	-	-
7	7	7	4	3	2	2	2	0	-	-	-
8	8	8	5	3	3	2	2	2	0	-	-
9	9	9	5	4	3	2	2	2	2	0	-
10	10	10	6	4	3	3	2	2	2	2	0

One may imagine that in order to generate tables we have to produce a triangular array of the values of $M^*(n, N)$ where $N = k_0 + 1, k_0 + 2, \dots$, and $n = k_0 + 1, \dots, N - 1$. Table 5.1 gives an example of such a triangular array. Section 5.3 develops theory that helps us to fill the row $M^*(k_0 + 1, N), \dots, M^*(N - 1, N)$ as efficiently as possible. We use this theory also in the algorithm of finding the optimal sample size in the EOQL-method for given N, k_0 , and P_l ,

see Chapter 6. Section 5.4 develops theory that will help us to fill the column $M^*(n, k_0 + 1), M^*(n, k_0 + 2), \dots$ of such a triangular array in an efficient way. Filling the triangular array of $M^*(n, N)$ can be done most efficiently by filling alternately columns and rows in a clever way, as we will see in Chapter 6.

5.2 Properties of $\pi(n, \cdot, N)$

We first present some properties that are very easy to prove and follow immediately from Theorem 4.1.1 and the expression for π in (5.1.6). Especially, we will look at the behaviour of π in a couple of special cases.

Property 5.2.1. *The function π has the following properties.*

1. $\pi(n, M, N) = 0$ if and only if $M > N - n + k_0$ or $M = 0$ or $n = N$.
2. $\pi(n, M, N) = \frac{M}{N}(1 - \frac{n}{N})$ if and only if $n \leq k_0$ or $M \leq k_0 + 1$ or $n = N$.
3. $\pi(n, N, N) = \begin{cases} 1 - \frac{n}{N} & \text{if } n \leq k_0, \\ 0 & \text{if } n > k_0. \end{cases}$
4. $\pi(N - 1, M, N) = \begin{cases} \frac{M}{N^2} & \text{if } M \leq k_0 + 1, \\ 0 & \text{if } M > k_0 + 1. \end{cases}$
5. If $n < N$, then $\pi(n, M, N) > 0$ for $M \in \{1, \dots, N - n + k_0\}$, or equivalently, $n \in \{1, \dots, N - M + k_0\}$.

As we discussed in Section 5.1, we are interested in finding the smallest value of n for which the maximum of π over M stays under a certain predefined value. Therefore, we are interested in the behaviour of $\pi^*(n, N) = \max_M \pi(n, M, N)$.

The following theorem will show that $\pi(n, M, N)$ is unimodal. The smallest value of M for which π achieves its maximum will be denoted by $M^*(n, N)$. An exception is the case $n = N$, then we have $\pi \equiv 0$ and hence we have $M^*(N, N) = 0$. If there is another solution then it is $M^*(n, N) + 1$, see e.g. example 5.2.1 where $\pi(n, M, N)$ achieves its maximum in two successive values of M .

In fact we would also like to construct a triangular array with entries for n and N that gives us $\pi^*(n, N)$ for fixed k_0 . Basically we have to find $M^*(n, N)$,

because $\pi^*(n, N) = \pi(n, M^*(n, N), N)$. In completing such an array we have to solve a lot of optimization problems. Hence, it pays to find efficient ways to obtain the $M^*(n, N)$.

Example 5.2.1. For $k_0 = 2$ the function $\pi(13, M, 46)$ achieves its maximum for $M = 7$ and $M = 8$; $M^*(13, 46) = 7$.

Theorem 5.2.1 (Unimodality property). Let $n \in \{0, \dots, N - 1\}$, then the function $\pi(n, M, N)$ is unimodal in M . Let $M^*(n, N)$ be the smallest solution to $\pi^*(n, N) = \max_M \pi(n, M, N)$, then $M^*(n, N)$ is the unique solution or $M^*(n, N) + 1$ the only possible other solution. If $n < N$, then $1 \leq M^*(n, N) \leq N - n + k_0$.

Proof. See Appendix 5.A □

Now, we look at the monotonicity property of π . The next theorem will show that the function π is non-increasing in n , and in most cases even decreasing.

Theorem 5.2.2 (Monotonicity property). Let $n, M \in \{0, \dots, N - 1\}$, then

$$\pi(n, M, N) \geq \pi(n + 1, M, N).$$

If $n \leq N - M + k_0$ and $M \neq 0$, then

$$\pi(n, M, N) > \pi(n + 1, M, N).$$

Proof. See Appendix 5.A □

According to Theorem 5.2.2, the graph of $\pi(n, \cdot, N)$ lies strictly above the graph of $\pi(n + 1, \cdot, N)$ for $M \in \{1, \dots, N - n + k_0\}$, see Figure 5.2. For other values of M , we have $\pi(n + 1, M, N) = \pi(n, M, N) = 0$. Hence, we have the following result.

Theorem 5.2.3. Let $n \in \{0, \dots, N - 1\}$, then the function $\pi^*(n, N)$ is decreasing in n , i.e.

$$\pi^*(n, N) > \pi^*(n + 1, N).$$

5.3 Properties of $M^*(\cdot, N)$

We have proved the unimodality and monotonicity properties of π and could use the method for finding the optimal sample size as described in Section 5.1. This method is rather naive in finding M^* . To find M^* , we either start at $M = 1$ and increase M by one until $\pi(n, M+1, N) < \pi(n, M, N)$ or start at $M = N - n + k_0$ and decrease M by one until $\pi(n, M-1, N) < \pi(n, M, N)$. Of course, we can find $M^*(n, N)$ in a more efficient way. The results also help to complete the rows and columns of the triangular array of $M^*(n, N)$. The remaining part of this section will focus on this issue. Can we tighten the interval for M where we have to search for $M^*(n, N)$? Do we need to calculate $M^*(n, N)$ for every n , keeping N fixed? How does $M^*(n, N)$ change if we increase n or N ? But first we consider some special cases where we have explicit solutions. As the next theorem will show, we can find $M^*(n, N)$ analytically in the case of $k_0 = 0$.

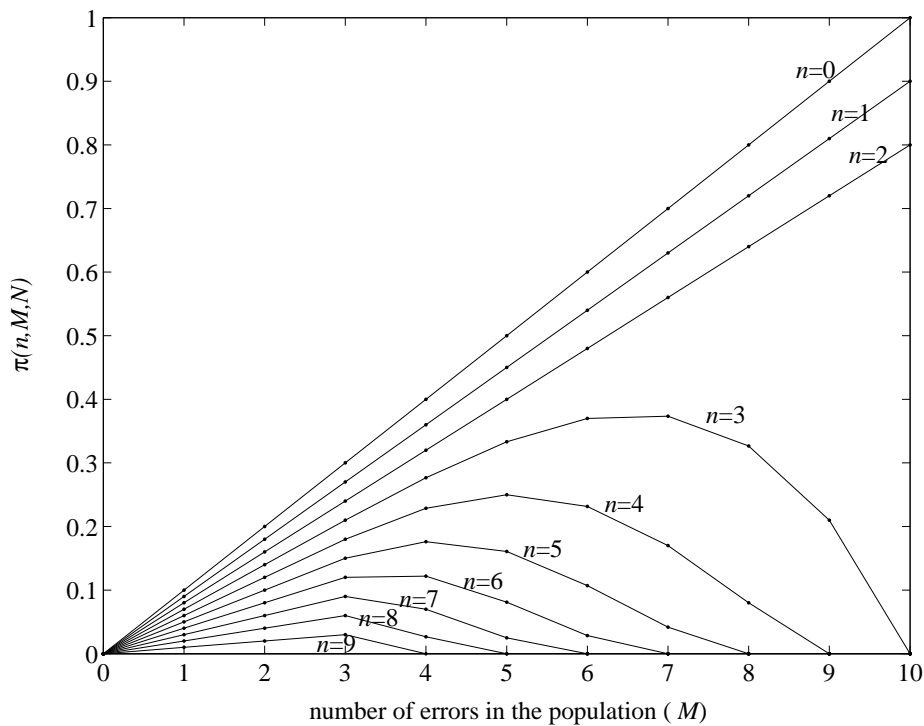


Figure 5.2. Plot of $\pi(n, M, N)$ for $n = 0, \dots, 9$, $N = 10$ and $k_0 = 2$.

Theorem 5.3.1. *If $k_0 = 0$, then*

$$M^*(n, N) = \left\lceil \frac{N - n}{n + 1} \right\rceil.$$

($\lceil x \rceil$ is the smallest integer $\geq x$).

Proof. See Appendix 5.A □

For other values of k_0 , we cannot find the value of $M^*(n, N)$ in such an easy way and we have to look for other ways. However, there are some special cases collected in Property 5.3.1 below.

Property 5.3.1. *Some special cases for π^* are:*

1. $\pi^*(n, N) = 0$ if and only if $n = N$, and by definition $M^*(N, N) = 0$.
2. If $n \leq k_0$, then $M^*(n, N) = N$ and $\pi^*(n, N) = 1 - \frac{n}{N}$.
3. If $N \geq k_0 + 1$, then $M^*(N - 1, N) = k_0 + 1$ and $\pi^*(N - 1, N) = \frac{k_0 + 1}{N^2}$.

We have already seen that the graph of $\pi(n, \cdot, N)$ lies above the one of $\pi(n + 1, \cdot, N)$, but the graph of relative changes of $\pi(n, \cdot, N)$ also lies above the graph of relative changes of $\pi(n + 1, \cdot, N)$ as M changes to $M + 1$, see Figure 5.3. We prove this property in the following theorem.

Theorem 5.3.2. *Let $n \in \{0, \dots, N - 2\}$. Then for $M = 1, \dots, N - n + k_0 - 1$*

$$\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} \geq \frac{\pi(n + 1, M + 1, N)}{\pi(n + 1, M, N)}.$$

Proof. See Appendix 5.A □

Note that in Figure 5.3 the cases $n = 0, 1, 2$ lead to the same graph, because then $n \leq k_0 = 2$ and $\lambda(n, M - 1, N - 1) = 1$ according to Theorem 4.1.2, part 1. From Theorem 5.3.2 we may expect that $\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} - 1$ will not change sign before its counterpart for $n + 1$ changes sign. This can for example be observed in Figure 5.3. Hence, $\pi(n, M, N)$ will not achieve its maximum before $\pi(n + 1, M, N)$ will achieve its maximum. We now prove that indeed $M^*(n, N)$ is non-increasing in n .

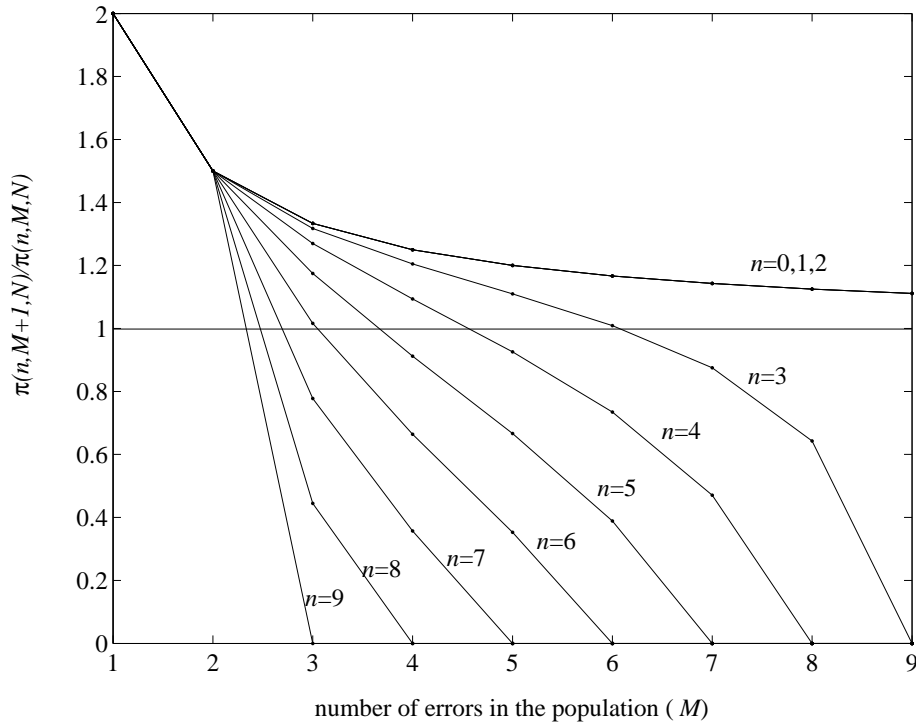


Figure 5.3. Plot of $\frac{\pi(n, M+1, N)}{\pi(n, M, N)}$ for $n = 0, \dots, 9$, $N = 10$ and $k_0 = 2$.

Theorem 5.3.3. Let $n \in \{0, \dots, N - 1\}$, then

$$M^*(n, N) \geq M^*(n + 1, N).$$

Proof. See Appendix 5.A □

Note that Theorem 5.3.3 already helps us to tighten the interval for possible candidates $M^*(n + 1, N)$ once we have $M^*(n, N)$. As a consequence we have the following lower bound for $M^*(n, N)$.

Corollary 5.3.1. Let $n \in \{0, \dots, N - 1\}$. If $N \geq k_0 + 1$, then $M^*(n, N) \geq k_0 + 1$.

Proof. See Appendix 5.A □

The intuitive algorithm for finding M^* can be improved upon if we can reduce the interval of feasible values of M . The following theorem gives a lower and upper bound for M^* .

Theorem 5.3.4. *Let n , k_0 and N be given, such that $N \geq k_0 + 1$ and $n \in \{k_0 + 1, \dots, N - 1\}$, then*

$$\begin{aligned} \max \left\{ k_0 + 1, \left\lceil \frac{N - n}{n + 1} \right\rceil \right\} &\leq M^*(n, N) \\ &\leq \min \left\{ N - n + k_0, \left\lceil \frac{N + 1}{n + 1} \cdot k_0 + \frac{N - n}{n + 1} \right\rceil \right\}. \end{aligned}$$

Proof. See Appendix 5.A □

Remark 5.3.1. *Notice that if we insert $k_0 = 0$ into Theorem 5.3.4, then Theorem 5.3.1 follows as a special case.*

Now we have found lower and upper bounds for M^* . This makes the algorithm of finding the optimal sample size for given k_0 and N more efficient. But it appears that we can find some other properties of M^* , which will improve the algorithm even further and these properties will also help us to fill the rows of the triangular array of $M^*(n, N)$. We will prove that if we know the values of $M^*(n, N)$ for which $k_0 + 1 \leq n \leq M^*(n, N)$, then we shall also get to know the values of $M^*(n, N)$ for which $n > M^*(n, N)$ in an easy way. In proving this the behaviour of $M^*(M^*(n, N), N)$ plays a key role. We use an example with $N = 17$ and $k_0 = 2$ to illustrate the theorems that we deduce. In this example we suppose that the values of $M^*(n, N)$ for $n \leq M^*(n, N)$ are known and we illustrate the effect of the theorems on the values of $M^*(n, N)$ for which $n > M^*(n, N)$. We already know that $M^*(n, N) = N$ for $n \leq k_0$, $M^*(N - 1, N) = k_0 + 1$ and $M^*(N, N) = 0$.

Example 5.3.1. *Let $N = 17$ and $k_0 = 2$.*

n	≤ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$M^*(n, 17)$	17	11	8	7	6										3	0

A first observation is that $3 \leq M^*(n, 17) \leq 6$ for $n = 7, \dots, 15$. This follows from Theorem 5.3.3. More strongly, we will prove that if we would know the value of $M^*(n, N)$, then we would immediately know that the value of $M^*(M^*(n, N), N)$ does not exceed n .

Example 5.3.2. *Let $N = 17$ and $k_0 = 2$. If we apply the statement above to Example 5.3.1, then we get*

n	≤ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$M^*(n, 17)$	17	11	8	7	6	≤ 5	≤ 4			3	3	3	3	3	3	0

We have $M^*(7, 17) = M^*(M^*(5, 17), 17) \leq 5$, and so forth. Notice that we actually get $M^*(11, 17) = M^*(M^*(3, 17), 17) \leq 3$. But since we know that $M^*(n, N) \geq k_0 + 1$, we also know by Theorem 5.3.3 that $M^*(n, 17)$ equals 3 for $n \geq 11$.

Theorem 5.3.5. Let $n \in \{0, \dots, N\}$, then

$$M^*(M^*(n, N), N) \leq n.$$

Proof. See Appendix 5.A □

For $n \geq k_0$ we will show that if $M^*(n, N)$ is at least equal to n , then $M^*(M^*(n, N), N)$ is always larger than or equal to n .

Example 5.3.3. Let $N = 17$ and $k_0 = 2$. If we apply the statement above to Example 5.3.1, then we get

n	≤ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$M^*(n, 17)$	17	11	8	7	6	≥ 5	≥ 4			≥ 3					3	0

We have $M^*(7, 17) = M^*(M^*(5, 17), 17) \geq 5$, and so forth.

Theorem 5.3.6. Let $k_0 + 1 \leq n \leq M^*(n, N)$, then

$$M^*(M^*(n, N), N) \geq n.$$

Proof. See Appendix 5.A □

Example 5.3.4. Let $N = 17$ and $k_0 = 2$. If we combine Examples 5.3.2 and 5.3.3, then we get

n	≤ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$M^*(n, 17)$	17	11	8	7	6	5	4			3	3	3	3	3	3	0

Looking at Example 5.3.4, we see that $M^*(M^*(n, N), N) = n$ for $k_0 + 1 \leq n \leq M^*(n, N)$. Of course this is an immediate consequence of combining Theorem 5.3.5 and Theorem 5.3.6.

Corollary 5.3.2. *Let $k_0 + 1 \leq n \leq M^*(n, N)$, then*

$$M^*(M^*(n, N), N) = n.$$

We shall prove that if we know the value of $M^*(n, N)$, then we also know that the value of $M^*(M^*(n, N) - 1, N)$ is always larger than n . If $k_0 + 1 \leq n \leq M^*(n, N)$, then from Theorem 5.3.3 and Theorem 5.3.6 it follows that $M^*(M^*(n, N) - 1, N) \geq n$. In Theorem 5.3.7 we will have a stronger result: the condition on n can be deleted and the inequality is strict.

Example 5.3.5. *Let $N = 17$ and $k_0 = 2$. If we apply the statement above to Example 5.3.4, then we get*

n	≤ 2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$M^*(n, 17)$	17	11	8	7	6	5	4	4	4	3	3	3	3	3	3	0

Theorem 5.3.7 tells us that $M^*(M^*(3, 17) - 1, 17) = M^*(10, 17) \geq 4$. Using Theorem 5.3.3 and $M^*(8, 17) = 4$, we know that $M^*(n, 17) = 4$ for $n = 8, 9, 10$. Thus, apparently to fill the entire row it suffices to calculate the values of $M^*(n, 17)$ for $n = 3, \dots, 6$ and with the help of the theorems above the rest of the values of $M^*(n, 17)$ will follow from these initial values.

Theorem 5.3.7. *Let $n \in \{0, \dots, N - 1\}$, then*

$$M^*(M^*(n, N) - 1, N) \geq n + 1.$$

Proof. See Appendix 5.A □

Example 5.3.5 suggests that if we have calculated the values of $M^*(n, N)$ with $n \leq M^*(n, N)$, then we will know the value of $M^*(n, N)$ for all possible sample sizes. The following theorem gives the essential ingredients for proving this and will help us in Chapter 6 to develop an algorithm that fills the row $M^*(k_0 + 1, N), \dots, M^*(N - 1, N)$ as efficiently as possible. Notice that from Property 5.3.1 follows that for values of $N \leq k_0 + 2$ we already can complete the rows entirely. For values of $N \geq k_0 + 3$, we can use Theorem 5.3.8.

Theorem 5.3.8. *Let $N \geq k_0 + 3$ and let n_l be the largest value of n for which $n \leq M^*(n, N)$, then $M^*(\cdot, N)$ has the following properties.*

1. If $n = k_0 + 1, \dots, n_l$, then $M^*(n - 1, N) > M^*(n, N)$.
2. If $M^*(n - 1, N) > M^*(n, N)$ for some $n \in \{k_0 + 1, \dots, N - 1\}$, then $M^*(j, N) = n$ for $j = M^*(n, N), M^*(n, N) + 1, \dots, M^*(n - 1, N) - 1$.
3. The largest value of n for which $n \leq M^*(n, N)$ equals n_l , if and only if $M^*(n_l, N)$ is equal to n_l or $n_l + 1$.

Proof. See Appendix 5.A □

Table 5.2 can be used to provide extra insight in the properties of Theorem 5.3.8.

Table 5.2. Some values of $M^*(n, N)$ for $N \in \{15, \dots, 30\}$, $n \in \{0, \dots, 12\}$ and $k_0 = 1$.

$N \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	12
10	10	10	6	4	3	3	2	2	2	2	0	-	-
11	11	11	6	5	4	3	2	2	2	2	2	0	-
12	12	12	7	5	4	3	3	2	2	2	2	2	0
13	13	13	8	5	4	3	3	3	2	2	2	2	2
14	14	14	8	6	5	4	3	3	2	2	2	2	2
15	15	15	9	6	5	4	3	3	3	2	2	2	2
16	16	16	9	7	5	4	4	3	3	2	2	2	2
17	17	17	10	7	6	5	4	3	3	3	2	2	2
18	18	18	11	8	6	5	4	4	3	3	3	2	2
19	19	19	11	8	6	5	4	4	3	3	3	2	2
20	20	20	12	8	7	5	5	4	3	3	3	3	2
21	21	21	12	9	7	6	5	4	4	3	3	3	2
22	22	22	13	9	7	6	5	4	4	3	3	3	3
23	23	23	13	10	8	6	5	5	4	4	3	3	3
24	24	24	14	10	8	7	6	5	4	4	3	3	3
25	25	25	15	11	8	7	6	5	4	4	4	3	3

Theorem 5.3.8 helps us to efficiently calculate $M^*(n, N)$ for fixed N . Only values $N \geq k_0 + 1$ and $n = k_0, \dots, N - 1$ are sensible. Therefore, we start with $M^*(k_0, N) = N$, then we calculate $M^*(k_0 + 1, N)$, and so forth. Hence $M^*(n, N)$ is calculated for increasing values of n . We stop if we obtained

$M^*(n, N) = n$ or $n + 1$. This value is called n_l in Theorem 5.3.8. The theorem tells us for $N \geq k_0 + 3$ that $M^*(n, N)$ is strictly decreasing on $\{k_0, \dots, n_l\}$. The values of $M^*(n, N)$ on $\{n_l + 1, \dots, N - 1\}$ are obtained as follows. The sequence

$$M^*(n_l, N), M^*(n_l, N) + 1, \dots, M^*(k_0, N) - 1$$

is strictly increasing with a difference of unit one between the elements and with $M^*(n_l, N) = n_l$ or $n_l + 1$ and $M^*(k_0, N) = N$. So, we have

$$\begin{aligned} \{n_l + 1, \dots, N - 1\} &\subset \{M^*(n_l, N), M^*(n_l, N) + 1, \dots, M^*(k_0, N) - 1\} \\ &= \bigcup_{k=k_0+1}^{n_l} \{M^*(k, N), M^*(k, N) + 1, \dots, M^*(k - 1, N) - 1\}. \end{aligned}$$

For all elements in the subset $\{M^*(k, N), M^*(k, N) + 1, \dots, M^*(k - 1, N) - 1\}$ we have $M^*(\cdot, N) = k$. Theorem 5.3.8, part 1 tells us that for $k = k_0 + 1, \dots, n_l$ the ‘integer interval’ $\{M^*(k, N), M^*(k, N) + 1, \dots, M^*(k - 1, N) - 1\}$ is well-defined and consists of at least one element. Moreover, these intervals are disjoint. Hence at the right-hand side we have a disjoint union and it gives a partition of the ‘integer interval’ $\{n_l + 1, \dots, N - 1\}$. From this observation we have the following result:

Theorem 5.3.9. *If $N \geq k_0 + 3$ and $n = n_l + 1, \dots, N - 1$, then*

$$M^*(n - 1, N) - 1 \leq M^*(n, N) \leq M^*(n - 1, N).$$

Proof. See Appendix 5.A □

Remark 5.3.2. *Notice that if $N \geq k_0 + 3$, then we know the values of $M^*(n, N)$ for $n = n_l + 1, \dots, N - 1$ as soon as we have calculated the values of $M^*(n, N)$ for $n = k_0 + 1, \dots, n_l$. The converse is also true.*

5.4 Properties of $M^*(n, \cdot)$

In order to fill the columns of the triangular array of $M^*(n, N)$ in an efficient way, we have to know the behaviour of $M^*(n, N)$ as N increases. We will establish that an increment of N with one unit leads to the same M^* or to $M^* + 1$. Therefore, it is important to study the relative change of the curve $\pi(n, \cdot, N)$ as N changes to $N + 1$.

Theorem 5.4.1. *Assume that $M \in \{1, \dots, N - n + k_0 - 1\}$. Then*

$$\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} > \frac{\pi(n, M + 2, N + 1)}{\pi(n, M + 1, N + 1)}, \quad (5.4.1)$$

and if $M \geq k_0 + 1$ and $n > k_0$, then

$$\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} < \frac{\pi(n, M + 1, N + 1)}{\pi(n, M, N + 1)}. \quad (5.4.2)$$

Now, we are ready to prove that the optimal value M^* remains unchanged or increases by one unit if N is increased by one unit and n remains unchanged.

Theorem 5.4.2. *Let $n \in \{0, \dots, N - 1\}$, then*

$$M^*(n, N) \leq M^*(n, N + 1) \leq M^*(n, N) + 1.$$

Proof. See Appendix 5.A □

The elements in the columns of the triangular array $M^*(n, N)$ can be calculated in an efficient way by using Theorem 5.4.2. According to this theorem,

$$\pi^*(n, M, N) = \max(\pi(n, M^*(n, N - 1), N), \pi(n, M^*(n, N - 1) + 1, N)).$$

If we know $M^*(n, N - 1)$, then $M^*(n, N - 1)$ and $M^*(n, N - 1) + 1$ are the only two possible solutions for $M^*(n, N)$. It is possible that $M^*(n, N - 1)$ and $M^*(n, N - 1) + 1$ give the same optimal solution of $\pi^*(n, N)$ (see Example 5.2.1), in this case by the definition of M^* we have $M^*(n, N) = M^*(n, N - 1)$.

We can also show that if N is increased by one, then the largest value of n which is equal to or exceeded by M^* remains unchanged or increases by one unit.

Corollary 5.4.1. *Let $n_l(N)$ be the largest value of n for which $n \leq M^*(n, N)$, then*

$$n_l(N) \leq n_l(N + 1) \leq n_l(N) + 1.$$

Moreover, if $M^(n_l(N), N)$ equals $n_l(N)$, then $n_l(N + 1)$ equals $n_l(N)$. If $M^*(n_l(N), N) = n_l(N) + 1$ and $M^*(n_l(N), N + 1) = n_l(N) + 1$, then $n_l(N + 1) = n_l(N)$, else $n_l(N + 1) = n_l(N) + 1$.*

Proof. See Appendix 5.A □

The theory we developed in this chapter will enable us to produce a triangular array of $M^*(n, N)$ where $N = k_0 + 1, k_0 + 2, \dots$ and $n = k_0 + 1, \dots, N - 1$ in a very efficient way. The theory helps us to fill in the row $M^*(k_0 + 1, N), \dots, M^*(N - 1, N)$ and will be used in Chapter 6 to construct an algorithm of finding the optimal sample size in the EOQL-method for given N, k_0 , and P_l . We also developed theory that enables us to fill the column $M^*(n, k_0 + 1), M^*(n, k_0 + 2), \dots$ of such a triangular array. Chapter 6 introduces an very appealing and efficient algorithm to generate triangular arrays in which we will combine the row- and column filling theory we developed in this chapter.

5.A Proofs of chapter 5

Proof of Theorem 5.1.1

From (5.1.5) we obtain

$$\begin{aligned}
 \pi(n, M, N) &= \sum_{k=0}^{\min(k_0, M-1)} \frac{M-k}{N} \cdot \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}} \\
 &= \sum_{k=0}^{\min(k_0, M-1)} \frac{M(M-1)!}{k!(M-1-k)!} \cdot \frac{n!(N-n)(N-1-n)!}{N^2(N-1)!} \times \\
 &\hspace{20em} \times \binom{N-M}{n-k} \\
 &= \frac{M}{N} \cdot \frac{N-n}{N} \cdot \sum_{k=0}^{k_0} \frac{\binom{M-1}{k} \binom{N-M}{n-k}}{\binom{N-1}{n}} \\
 &= \frac{M}{N} \left(1 - \frac{n}{N}\right) \cdot \Lambda(n, M-1, N-1).
 \end{aligned}$$

Proof of Theorem 5.2.1 (Unimodality property)

According to Property 5.2.1, part 5, for all $M \in \{1, \dots, N - n + k_0\}$ the function $\pi(n, M, N)$ is positive. If $N - n + k_0 = 1$ or 2, then the result is trivial. Now suppose that $N - n + k_0$ equals at least 3. The function $\pi(n, M, N)$ is a unimodal

function if we can show that for $M \in \{1, \dots, N - n + k_0\}$,

$$\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} = \frac{M + 1}{M} \cdot \lambda(n, M - 1, N - 1)$$

is decreasing in M . Because we cannot exclude the possibility that this ratio can be equal to 1 for one specific value of M , it is possible that we have two optimal solutions of π for succeeding values of M . Combining Theorem 4.1.2, part 4 with the fact that $1 + \frac{1}{M}$ is decreasing in M concludes the proof.

Proof of Theorem 5.2.2 (Monotonicity property)

The case $M = 0$ is trivial. Therefore, we may assume that $M \neq 0$. We have to prove that

$$\begin{aligned} \pi(n, M, N) &= \frac{M}{N} \left(1 - \frac{n}{N}\right) \Lambda(n, M - 1, N - 1) \\ &\geq \frac{M}{N} \left(1 - \frac{n + 1}{N}\right) \Lambda(n + 1, M - 1, N - 1) = \pi(n + 1, M, N). \end{aligned}$$

This holds on account of Theorem 4.1.1, part 6. If $n \leq N - M + k_0$, then Theorem 4.1.1, part 6 provides the strict inequality $\Lambda(n, M - 1, N - 1) > \Lambda(n + 1, M - 1, N - 1)$. From this the strict inequality of the theorem immediately follows.

Proof of Theorem 5.3.1

We only have to look at $\pi(n, M, N) - \pi(n, M - 1, N)$ for $M = 1, \dots, N - n$. If this difference changes sign from positive to negative only once, then the maximum is achieved for the largest value of M such that this difference is positive. By Theorem 5.1.1 it is straightforward to see that the sign of $\pi(n, M, N) - \pi(n, M - 1, N)$ is the same as the sign of

$$\begin{aligned} &M \cdot \frac{(N - M)!}{(N - M - n)!} - (M - 1) \cdot \frac{(N - M + 1)!}{(N - M - n + 1)!} \\ &= \frac{(N - M)!}{(N - M - n + 1)!} \cdot [M(N - M - n + 1) - (M - 1)(N - M + 1)] \\ &= \frac{(N - M)!}{(N - M - n + 1)!} \cdot [N + 1 - M(n + 1)] \end{aligned}$$

Indeed this difference changes sign only once and for non-integer values of $\frac{N+1}{n+1}$ the maximum will be achieved for the largest integer smaller than or equal to $\frac{N+1}{n+1}$ or, equivalently, the smallest integer larger than $\frac{N+1}{n+1} - 1$, i.e. $M^*(n, N) = \lceil \frac{N-n}{n+1} \rceil$. For integer values of $\frac{N+1}{n+1}$, we have two successive values of M for which the maximum is achieved. The smaller of the two values is the optimal solution, i.e. $M^*(n, N) = \frac{N+1}{n+1} - 1 = \frac{N-n}{n+1}$.

Proof of Theorem 5.3.2

According to Theorem 5.1.1 and (4.1.2) it suffices to prove

$$\lambda(n, M - 1, N - 1) \geq \lambda(n + 1, M - 1, N - 1).$$

This is true on account of Theorem 4.1.2, part 5.

Proof of Theorem 5.3.3

Proof. For $n \leq k_0$ the proof immediately follows from Property 5.3.1, part 2. For $n = N - 1$ the proof follows from Property 5.3.1, part 1. Let $n < N - 1$. If $M^*(n, N) = N - n + k_0$, then the result follows because $M^*(n + 1, N) \in \{1, \dots, N - n + k_0 - 1\}$ according to Theorem 5.2.1. Now let us assume that $n \in \{0, \dots, N - 2\}$ and $M^*(n, N) \in \{1, \dots, N - n + k_0 - 1\}$ and we apply Theorem 5.3.2. The unimodality property and this theorem show us that

$$\begin{aligned} 1 &\geq \frac{\pi(n, M^*(n, N) + 1, N)}{\pi(n, M^*(n, N), N)} \\ &\geq \frac{\pi(n + 1, M^*(n, N) + 1, N)}{\pi(n + 1, M^*(n, N), N)}. \end{aligned}$$

Using the unimodality property again we find $M^*(n, N) \geq M^*(n + 1, N)$. \square

Proof of Corollary 5.3.1

According to Property 5.3.1, part 3, we have

$$M^*(n, N) \geq M^*(N - 1, N) = k_0 + 1.$$

Proof of Theorem 5.3.4

If $M \leq k_0 + 1$, then we know from Property 5.2.1, part 2 that $\pi(n, M, N)$ increases. For $M > N - n + k_0$, we also know that $\pi(n, M, N) = 0$. Therefore, we know that $k_0 + 1 \leq M^*(n, N) \leq N - n + k_0$. The other bounds can be established as follows. For $M \in \{k_0 + 2, \dots, N - n + k_0 - 1\}$ and $k \in \{0, \dots, k_0\}$ the k -th term of $\pi(n, M, N)$ is called $\pi_k(n, M, N)$. Look at the term-wise difference $D_k(n, M, N)$ between $\pi_k(n, M, N)$ and $\pi_k(n, M - 1, N)$. Using Theorem 5.1.1, we find

$$\begin{aligned} D_k(n, M, N) &= \pi_k(n, M, N) - \pi_k(n, M - 1, N) \\ &= \frac{M}{N} \left(1 - \frac{n}{N}\right) \frac{\binom{M-1}{k} \binom{N-M}{n-k}}{\binom{N-1}{n}} + \\ &\quad - \frac{M-1}{N} \left(1 - \frac{n}{N}\right) \frac{\binom{M-2}{k} \binom{N-M+1}{n-k}}{\binom{N-1}{n}} \\ &= C_k(n, N, M) \left(\frac{M}{M-k-1} - \frac{N-M+1}{N-M-n+k+1} \right) \end{aligned}$$

with

$$C_k(n, N, M) = \frac{(N-n)(M-1)!(N-M)!n!(N-n-1)!}{N \cdot k!(n-k)!(M-k-2)!N!(N-M-n+k)!}.$$

Therefore, $D_k(n, M, N)$ is positive for integers M for which

$$\frac{M}{M-k-1} - \frac{N-M+1}{N-M-n+k+1} > 0.$$

This holds if $M < \frac{(N+1)(k+1)}{n+1}$ or, equivalently, if $M \in \{0, \dots, M_k^*(n, N)\}$, where

$$M_k^*(n, N) = \left\lceil \frac{(N+1)(k+1)}{n+1} - 1 \right\rceil = \left\lceil \frac{N+1}{n+1} \cdot k + \frac{N-n}{n+1} \right\rceil.$$

Note that $M_k^*(n, N)$ is increasing in k . Since π is a unimodal function of M , we have $M^*(n, N) \geq M_0^*(n, N)$, because for $M \leq M_0^*(n, N)$ all terms $D_k(n, M, N)$ are positive and then π increases. From this it follows that $M_0^*(n, N)$ is a lower bound for $M^*(n, N)$. For $M > M_{k_0}^*(n, N)$ all terms $D_k(n, M, N)$ are non-positive and then π is non-increasing. Therefore, $M_{k_0}^*(n, N)$ is an upper bound for $M^*(n, N)$.

Proof of Theorem 5.3.5

In case $n = 0$ we have

$$M^*(M^*(0, N), N) = M^*(N, N) = 0,$$

and in case $n = N$ we have

$$M^*(M^*(N, N), N) = M^*(0, N) = N.$$

Now, we assume $n \in \{1, \dots, N - 1\}$. In order to guarantee later on in this proof that $\pi(M^*(n, N), n + 1, N) > 0$, we have to check the case $M^*(n, N) = N - n + k_0$ separately. Suppose $M^*(n, N) = N - n + k_0$, then

$$M^*(M^*(n, N), N) = M^*(N - n + k_0, N) \leq N - (N - n + k_0) + k_0 = n.$$

Hence, we are now allowed to assume $M^*(n, N) < N - n + k_0$ and hence we have $\pi(M^*(n, N), n + 1, N) > 0$. Due to the unimodality property of π it suffices to prove

$$1 \leq \frac{\pi(M^*(n, N), n, N)}{\pi(M^*(n, N), n + 1, N)} = \frac{n}{n + 1} \cdot \frac{\Lambda(M^*(n, N), n - 1, N - 1)}{\Lambda(M^*(n, N), n, N - 1)}.$$

By using Theorem 4.1.1, part 4 we can write this as

$$\frac{n}{n + 1} \cdot \left(1 + \frac{M^*(n, N)}{N - 1} \cdot \frac{\mathbb{P}\{K = k_0 | M^*(n, N) - 1, n - 1, N - 2\}}{\Lambda(M^*(n, N), n, N - 1)} \right) \geq 1,$$

or, equivalently,

$$\frac{\mathbb{P}\{K = k_0 | M^*(n, N) - 1, n - 1, N - 2\}}{\Lambda(M^*(n, N), n, N - 1)} \geq \frac{N - 1}{n \cdot M^*(n, N)}. \quad (5.A.1)$$

We know that

$$1 \leq \frac{\pi(n, M^*(n, N), N)}{\pi(n, M^*(n, N) + 1, N)} = \frac{M^*(n, N)}{M^*(n, N) + 1} \cdot \frac{\Lambda(n, M^*(n, N) - 1, N - 1)}{\Lambda(n, M^*(n, N), N - 1)}.$$

By using Theorem 4.1.1, part 4 we can write this as

$$\frac{M^*(n, N)}{M^*(n, N) + 1} \cdot \left(1 + \frac{n}{N - 1} \cdot \frac{\mathbb{P}\{K = k_0 | n - 1, M^*(n, N) - 1, N - 2\}}{\Lambda(n, M^*(n, N), N - 1)} \right) \geq 1,$$

or, equivalently,

$$\frac{\mathbb{P}\{K = k_0 | n - 1, M^*(n, N) - 1, N - 2\}}{\Lambda(n, M^*(n, N), N - 1)} \geq \frac{N - 1}{n \cdot M^*(n, N)},$$

which shows that (5.A.1) holds true. This completes the proof.

Proof of Theorem 5.3.6

If $n = k_0 + 1$, then we know from Theorem 5.3.5 that $M^*(M^*(k_0 + 1, N), N) \leq k_0 + 1$, but from Theorem 5.3.4 we know that $M^*(M^*(k_0 + 1, N), N) \geq k_0 + 1$, because $M^*(k_0 + 1, N) \geq k_0 + 1$. Hence $M^*(M^*(k_0 + 1, N), N) = k_0 + 1$. This establishes the theorem for $n = k_0 + 1$. Therefore we assume $n > k_0 + 1$. For notational purposes we will write M^* instead of $M^*(n, N)$ in the remainder of this proof. If $M^* = n$, then the result is obvious. Hence, we assume $M^* > n$ in the sequel. By the unimodality property we know that

$$\pi(n, M^* - 1, N) < \pi(n, M^*, N),$$

which can be simplified to

$$(M^* - 1) \cdot \Lambda(n, M^* - 2, N - 1) < M^* \cdot \Lambda(n, M^* - 1, N - 1).$$

Using Theorem 4.1.1, part 4 we can rewrite this as

$$(M^* - 1) \cdot \left(\Lambda(n, M^* - 1, N - 1) + \frac{\binom{M^*-2}{k_0} \binom{N-M^*}{n-k_0-1}}{\binom{N-1}{n}} \right) < M^* \cdot \Lambda(n, M^* - 1, N - 1),$$

or, equivalently,

$$\Lambda(n, M^* - 1, N - 1) > \frac{(M^* - 1) \cdot \binom{M^*-2}{k_0} \binom{N-M^*}{n-k_0-1}}{\binom{N-1}{n}}.$$

Applying Theorem 4.1.1, part 4 again, this gives

$$\Lambda(n, M^*, N - 1) > \frac{(M^* - 1) \cdot \binom{M^*-2}{k_0} \binom{N-M^*}{n-k_0-1} - \binom{M^*-1}{k_0} \binom{N-M^*-1}{n-k_0-1}}{\binom{N-1}{n}}. \quad (5.A.2)$$

The unimodality property ensures that it is sufficient to prove that

$$\pi(M^*, n - 1, N) < \pi(M^*, n, N).$$

Using the same reasoning as above and by changing the role of M^* and n we can rewrite this to

$$\Lambda(M^*, n, N - 1) > \frac{(n - 1) \cdot \binom{n-2}{k_0} \binom{N-n}{M^*-k_0-1} - \binom{n-1}{k_0} \binom{N-n-1}{M^*-k_0-1}}{\binom{N-1}{M^*}}. \quad (5.A.3)$$

By Theorem 4.1.1, part 1, (5.A.2) and (5.A.3) it is sufficient to prove that

$$\begin{aligned} & \frac{(M^* - 1) \cdot \binom{M^*-2}{k_0} \binom{N-M^*}{n-k_0-1} - \binom{M^*-1}{k_0} \binom{N-M^*-1}{n-k_0-1}}{\binom{N-1}{n}} \\ & \geq \frac{(n-1) \cdot \binom{n-2}{k_0} \binom{N-n}{M^*-k_0-1} - \binom{n-1}{k_0} \binom{N-n-1}{M^*-k_0-1}}{\binom{N-1}{M^*}}. \end{aligned}$$

By writing out and cancelling common factors we get

$$\begin{aligned} & n \cdot \left(\frac{N - M^*}{(n - k_0 - 1)(N - n - M^* + k_0 + 1)} - \frac{1}{(M^* - k_0 - 1)(n - k_0 - 1)} \right) \\ & \geq M^* \cdot \left(\frac{N - n}{(M^* - k_0 - 1)(N - n - M^* + k_0 + 1)} + \right. \\ & \quad \left. - \frac{1}{(M^* - k_0 - 1)(n - k_0 - 1)} \right). \end{aligned}$$

Since all terms in the denominator are positive this is equivalent to

$$(M^* - n) (N - n - M^* + k_0 + 1 - M^* n + N(k_0 + 1)) \geq 0.$$

Since we assumed $M^* > n$, it remains to show that

$$M^* \leq \frac{N+1}{n+1} \cdot k_0 + \frac{N-n}{n+1} + \frac{N+1}{n+1}.$$

But Theorem 5.3.4 tells us that

$$M^* \leq \left\lceil \frac{N+1}{n+1} \cdot k_0 + \frac{N-n}{n+1} \right\rceil,$$

and since $\frac{N+1}{n+1} > 1$ this completes the proof.

Proof of Theorem 5.3.7

Suppose $M^*(n, N) = 1$. Since $M^*(0, N) = N$, we have that

$$M^*(M^*(n, N) - 1, N) = N \geq n + 1.$$

Let us now consider the case that $M^*(n, N) \geq 2$. We know that

$$1 > \frac{\pi(n, M^*(n, N) - 1, N)}{\pi(n, M^*(n, N), N)} = \frac{M^*(n, N) - 1}{M^*(n, N)} \cdot \frac{\Lambda(n, M^*(n, N) - 2, N - 1)}{\Lambda(n, M^*(n, N) - 1, N - 1)}.$$

Using Theorem 4.1.1, part 4 we can write this as

$$\frac{M^*(n, N) - 1}{M^*(n, N)} \cdot \left(1 + \frac{n}{N - 1} \cdot \frac{P\{K = k_0 | n - 1, M^*(n, N) - 2, N - 2\}}{\Lambda(n, M^*(n, N) - 1, N - 1)} \right) < 1,$$

or, equivalently,

$$\frac{P\{K = k_0 | n - 1, M^*(n, N) - 2, N - 2\}}{\Lambda(n, M^*(n, N) - 1, N - 1)} < \frac{N - 1}{n \cdot (M^*(n, N) - 1)}, \quad (5.A.4)$$

The unimodality property ensures that we only have to prove that

$$1 > \frac{\pi(M^*(n, N) - 1, n, N)}{\pi(M^*(n, N) - 1, n + 1, N)} = \frac{n}{n + 1} \cdot \frac{\Lambda(M^*(n, N) - 1, n - 1, N - 1)}{\Lambda(M^*(n, N) - 1, n, N - 1)}.$$

holds. By using Theorem 4.1.1, part 4 we can write this as

$$\frac{n}{n + 1} \cdot \left(1 + \frac{M^*(n, N) - 1}{N - 1} \cdot \frac{P\{K = k_0 | M^*(n, N) - 2, n - 1, N - 2\}}{\Lambda(M^*(n, N) - 1, n, N - 1)} \right) < 1,$$

or, equivalently,

$$\frac{P\{K = k_0 | M^*(n, N) - 2, n - 1, N - 2\}}{\Lambda(M^*(n, N) - 1, n, N - 1)} < \frac{N - 1}{n \cdot (M^*(n, N) - 1)},$$

which holds because of (5.A.4).

Proof of Theorem 5.3.8

Since $n \geq k_0 + 1$, we have by Corollary 5.3.2 and Theorem 5.3.5

$$M^*(M^*(n, N), N) = n > n - 1 \geq M^*(M^*(n - 1, N), N),$$

which implies $M^*(n, N) \neq M^*(n - 1, N)$. Since $M^*(n, N)$ is non-increasing in n , it follows $M^*(n - 1, N) > M^*(n, N)$, which concludes the proof of part 1.

Now we prove part 2. From the assumption it follows that $M^*(n, N) \leq M^*(n - 1, N) - 1$ for some $n \in \{k_0 + 1, \dots, N - 1\}$. From Theorem 5.3.5 we know that $M^*(M^*(n, N), N) \leq n$. Moreover, we have $M^*(M^*(n - 1, N) - 1, N) \geq n$ on account of Theorem 5.3.7. Because $M^*(j, N)$ is non-increasing in j , we have $M^*(j, N) = n$ for $j = M^*(n, N), \dots, M^*(n - 1, N) - 1$.

To prove part 3 first note that $M^*(k_0, N) = N$ and $M^*(N - 1, N) = k_0 + 1$. Since $M^*(n, N)$ is non-increasing in n and $k_0 + 1 \leq M^*(k_0 + 1, N)$, there is a

largest integer $n_l \in \{k_0 + 1, \dots, N - 1\}$ such that $M^*(n_l, N) \geq n_l$. According to Theorem 5.3.7 we have

$$M^*(M^*(n_l, N) - 1, N) \geq n_l + 1 \quad (5.A.5)$$

By definition of n_l we have

$$M^*(n_l + 1, N) < n_l + 1. \quad (5.A.6)$$

Since M^* is non-increasing in n and because of (5.A.5) and (5.A.6), we get

$$M^*(n_l, N) - 1 < n_l + 1 \quad (5.A.7)$$

If we rewrite (5.A.7) in $M^*(n_l, N) \leq n_l + 1$, then, because we know that $M^*(n_l, N) \geq n_l$, we may conclude that $M^*(n_l, N)$ is equal to n_l or $n_l + 1$.

Conversely, if $M^*(n_l, N) = n_l$ or $n_l + 1$, then

$$M^*(n, N) \geq M^*(n_l, N) \geq n_l \geq n, \text{ for } n = k_0 + 1, \dots, n_l.$$

If $M^*(n_l, N) = n_l$, then

$$M^*(n, N) \leq M^*(n_l, N) = n_l < n, \text{ for } n = n_l + 1, \dots, N - 1.$$

If $M^*(n_l, N) = n_l + 1$, then

$$M^*(n, N) \leq M^*(n_l + 1, N) = M^*(M^*(n_l, N), N) = n_l < n, \\ \text{for } n = n_l + 1, \dots, N - 1.$$

Proof of Theorem 5.3.9

For $k \in \{k_0 + 1, \dots, n_l\}$ define

$$I_k = \{M^*(k, N), M^*(k, N) + 1, \dots, M^*(k - 1, N) - 1\}.$$

This is well-defined as we remarked after the formulation of Theorem 5.3.8. Moreover, if $j \in I_k$, then $M^*(j, N) = k$. Let $n \in \{n_l + 1, \dots, N - 1\}$, then for some unique $k \in \{k_0 + 1, \dots, n_l\}$ we have $n \in I_k$ and hence $M^*(n, N) = k$. If we look at $n - 1$, then there are two possibilities:

- $n - 1 \in I_k$ which implies $M^*(n - 1, N) = k$;

- $n - 1 \in I_{k+1}$ and this implies $M^*(n - 1, N) = k + 1$.

Special attention has to be paid to the situation $k = n_l$, because then the possibility of $n - 1 \in I_{k+1}$ has to be excluded, since the properties of the I_k can only be used for $k \in \{k_0 + 1, \dots, n_l\}$. Suppose that $n \geq n_l + 1$ and that

$$M^*(n_l, N) \leq n \leq M^*(n_l - 1, N) - 1.$$

If $M^*(n_l, N) = n_l$, then $n \in I_{n_l}$ also implies $n - 1 \in I_{n_l}$. If $M^*(n_l, N) = n_l + 1$ and $n \geq n_l + 2$, then $n \in I_{n_l}$ also implies $n - 1 \in I_{n_l}$. We have to check separately the case $n = n_l + 1$ and $M^*(n_l, N) = n_l + 1$. By Corollary 5.3.2 we have

$$M^*(n_l + 1, N) = M^*(M^*(n_l, N), N) = n_l,$$

hence

$$M^*(n_l, N) - 1 = M^*(n_l + 1, N) < M^*(n_l, N).$$

Proof of Theorem 5.4.1

First we prove (5.4.2). According to Theorem 5.1.1 and (4.1.2) it suffices to prove that

$$\lambda(n, M - 1, N - 1) < \lambda(n, M - 1, N).$$

From Theorem 4.1.2, part 6 it follows that this holds for $M \geq k_0 + 1$, $n > k_0$ and $N \geq n + M - k_0$.

Secondly we prove (5.4.1). We observe that

$$\frac{\pi(n, M + 1, N)}{\pi(n, M, N)} > \frac{\pi(n, M + 2, N + 1)}{\pi(n, M + 1, N + 1)}$$

if and only if

$$\frac{M + 1}{M} \cdot \lambda(n, M - 1, N - 1) > \frac{M + 2}{M + 1} \cdot \lambda(n, M, N).$$

Since $\frac{M+1}{M} > \frac{M+2}{M+1}$ it suffices to prove that

$$\lambda(n, M - 1, N - 1) \geq \lambda(n, M, N). \quad (5.A.8)$$

For $M \leq k_0$ or $n \leq k_0$ this follows from Theorem 4.1.2, part 1, because then $\lambda(n, M - 1, N - 1) = 1$. Now, we assume $M > k_0$ and $n > k_0$. We use the

following approach. From Theorem 4.1.2, part 3 we can deduce that if $N \geq n + M - k_0$, then (5.A.8) is equivalent to

$$1 - \frac{1}{g(n, M-1, N-1)} \geq 1 - \frac{1}{g(n, M, N)}.$$

By writing out g this gives

$$\frac{\binom{N-1}{n} \Lambda(n, M-1, N-1)}{\binom{M-1}{k_0} \binom{N-M-1}{n-k_0-1}} \geq \frac{\binom{N}{n} \Lambda(n, M, N)}{\binom{M}{k_0} \binom{N-M-1}{n-k_0-1}},$$

or, equivalently,

$$\sum_{k=0}^{k_0} \frac{\binom{M-1}{k} \binom{N-M}{n-k}}{\binom{M-1}{k_0}} \geq \sum_{k=0}^{k_0} \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{M}{k_0}}.$$

Finally, we rewrite this to

$$\sum_{k=0}^{k_0} \frac{\binom{M-1}{k} \binom{N-M}{n-k}}{\binom{M-1}{k_0}} \geq \sum_{k=0}^{k_0} \frac{M-k_0}{M-k} \cdot \frac{\binom{M-1}{k} \binom{N-M}{n-k}}{\binom{M-1}{k_0}}.$$

This inequality holds, since $\frac{M-k_0}{M-k} \leq 1$.

Proof of Theorem 5.4.2

For $n \leq k_0$ the proof immediately follows from Property 5.3.1, part 2. In the following we assume $n > k_0$. Hence $N \geq k_0 + 1$. By Corollary 5.3.1, $M^*(n, N) \geq k_0 + 1$. If $M^*(n, N) = k_0 + 1$, then we have $M^*(n, N) = k_0 + 1 \leq M^*(n, N + 1)$. Suppose now that $M^*(n, N) - 1 \geq k_0 + 1$. Moreover, $M^*(n, N) - 1 \leq N - n + k_0 - 1$. By using Theorem 5.2.1 and (5.4.2) of Theorem 5.4.1, we derive that

$$\begin{aligned} 1 &< \frac{\pi(n, M^*(n, N), N)}{\pi(n, M^*(n, N) - 1, N)} \\ &< \frac{\pi(n, M^*(n, N), N + 1)}{\pi(n, M^*(n, N) - 1, N + 1)}. \end{aligned}$$

Because of the unimodality property, we have $M^*(n, N) \leq M^*(n, N + 1)$, since $M^*(n, N)$ improves $M^*(n, N) - 1$.

Finally we consider the inequality on the right-hand side. Suppose that $M^*(n, N + 1) = k_0 + 1$, then $M^*(n, N + 1) \leq M^*(n, N) + 1$. Therefore,

we may assume that $M^*(n, N + 1) - 1 \geq k_0 + 1 \geq 1$. Suppose we have $M^*(n, N) = N - n + k_0$. Since $M \in \{1, \dots, N - n + k_0 + 1\}$ is the support of $\pi(n, M, N + 1)$, we have $M^*(n, N + 1) \leq N - n + k_0 + 1 = M^*(n, N) + 1$. Therefore, in this case the right-hand side holds. Now we consider the situation that $M^*(n, N) \leq N - n + k_0 - 1$. Note that $M^*(n, N + 1) - 1 \geq M^*(n, N) - 1 \geq k_0 + 1$. By using Theorem 5.2.1 and (5.4.1) of Theorem 5.4.1, we find that

$$\begin{aligned} 1 &\geq \frac{\pi(n, M^*(n, N) + 1, N)}{\pi(n, M^*(n, N), N)} \\ &> \frac{\pi(n, M^*(n, N) + 2, N + 1)}{\pi(n, M^*(n, N) + 1, N + 1)}. \end{aligned}$$

Now, we observe that $M^*(n, N) + 2$ is worse than $M^*(n, N) + 1$. Hence, $M^*(n, N) + 1$ should be at least equal to or larger than $M^*(n, N + 1)$ because of the unimodality property.

Proof of Corollary 5.4.1

Let $n_l(N)$ be the largest value of n for which $n \leq M^*(n, N)$, then Theorem 5.4.2 tells us that $M^*(n_l(N), N + 1)$ either equals $M^*(n_l(N), N)$ or $M^*(n_l(N), N) + 1$. We know by Theorem 5.3.8, part 3 that $M^*(n_l(N), N)$ either equals $n_l(N)$ or $n_l(N) + 1$.

Suppose that $M^*(n_l(N), N) = n_l(N)$, then $M^*(n_l(N), N + 1) = n_l(N)$ or $n_l(N) + 1$. From Theorem 5.3.8, part 3 it is clear that $n_l(N + 1) = n_l(N)$.

Now suppose that $M^*(n_l(N), N) = n_l(N) + 1$, then $M^*(n_l(N), N + 1) = n_l(N) + 1$ or $n_l(N) + 2$. In case of $M^*(n_l(N), N + 1) = n_l(N) + 1$ it is clear from Theorem 5.3.8, part 3 that $n_l(N + 1) = n_l(N)$. In case $M^*(n_l(N), N + 1) = n_l(N) + 2$ it follows from Theorem 5.3.8, part 3 that $n_l(N + 1) \geq n_l(N) + 1$. By Theorem 5.3.8, part 1 it then follows that

$$M^*(n_l(N) + 1, N + 1) < M^*(n_l(N), N + 1) = n_l(N) + 2.$$

This implies

$$M^*(n_l(N) + 1, N + 1) \leq n_l(N) + 1 \leq n_l(N + 1).$$

This is only possible if $n_l(N) + 1 \geq n_l(N + 1)$. Therefore we may conclude $n_l(N + 1) = n_l(N) + 1$ in this case.