

University of Groningen

## Instantons and cosmologies in string theory

Collinucci, Giulio

**IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.**

*Document Version*

Publisher's PDF, also known as Version of record

*Publication date:*

2005

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Collinucci, G. (2005). *Instantons and cosmologies in string theory*. s.n.

### Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

### Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

# Appendix A

## 2-D Quantum Mechanics

In this appendix, we will study the point particle in  $(2 + 1)$ -dimensional quantum mechanics. In [44], Lee introduced this example as a toy model to show how the path integral of a positive-definite action can effectively be computed by finding the saddle points of an action that is not positive-definite. This toy model will allow us to understand why we are solving a system with a negative kinetic term for the axion in (3.5).

### A.1 Path integral for momentum eigenstates

Let us begin by defining the system and its path integral. We want to study quantum mechanics of a unit mass particle moving in two spatial directions, by using polar coordinates:  $r(t)$  and  $\theta(t)$ . The partition function and path integral between position eigenstates are defined and computed as follows:

$$\langle r_F, \theta_F | e^{-HT} | r_I, \theta_I \rangle = \int_{\text{b.c.}} (\Pi_r r(t')) d[r(t)] d[\theta(t)] \exp \left[ -\frac{1}{2} \int_{t_I}^{t_F} dt (\dot{r}^2 + r^2 \dot{\theta}^2) \right], \quad (\text{A.1})$$

where  $I$  and  $F$  stand for initial and final, respectively;  $T \equiv t_F - t_I$ ; and ‘b.c.’ stands for Dirichlet boundary conditions, i.e.  $r(t_{I,F}) = r_{I,F}$  and  $\theta(t_{I,F}) = \theta_{I,F}$ , respectively. The product in the integration measure is simply the Jacobian for polar coordinates. For convenience, we will omit the integration over  $r(t)$  and its kinetic term, and reinsert it when it is needed. As already mentioned in chapter 2, this partition function can, but need not be thought of as an imaginary time path integral. In this appendix, we will think of  $t$  as real time.

Suppose that we want to compute the partition function between initial and final angular momentum eigenstates  $|\ell\rangle$ , as opposed to angular position eigenstates  $|\theta\rangle$ . Using the following definition for the angular momentum states

$$|\ell\rangle \equiv \int d\theta e^{i\ell\theta} |\theta\rangle, \quad (\text{A.2})$$

we see that all we have to do is Fourier transform the path integral in (A.1) with respect to its boundary conditions:

$$\langle r_F, \ell_F | e^{-HT} | r_I, \ell_I \rangle = \int d\theta_I d\theta_F \exp(-i \ell_F \theta_F + i \ell_I \theta_I) \langle r_F, \theta_F | e^{-HT} | r_I, \theta_I \rangle . \quad (\text{A.3})$$

The Dirichlet path integral can be combined with the integral over boundary conditions to yield one path integral without boundary conditions:

$$\langle r_F, \ell_F | e^{-HT} | r_I, \ell_I \rangle = \int_{\text{no b.c.}} d[\theta(t)] \exp \left[ -\frac{1}{2} \int_{t_I}^{t_F} dt (\dot{r}^2 + r^2 \dot{\theta}^2) - i \ell_F \theta_F + i \ell_I \theta_I \right] , \quad (\text{A.4})$$

where this sums over all possible  $\theta(t)$  with arbitrary boundary values. If we try to compute this path integral via the standard saddle point approximation, the Euler-Lagrange variation of the action w.r.t.  $\theta$  will be the following:

$$\delta S = - \int_{t_I}^{t_F} dt \left[ \partial_t (r^2 \dot{\theta}) \delta\theta \right] + (r^2 \dot{\theta} - i \ell) \delta\theta \Big|_{t_I}^{t_F} . \quad (\text{A.5})$$

Notice we do not throw away the total derivative, because there are no boundary conditions. Since this must vanish for arbitrary variations  $\delta\theta$ , both terms must vanish independently. Hence, we get the following equations:

$$\begin{aligned} \partial_t (r^2 \dot{\theta}) &= 0 , \\ r^2 \dot{\theta} \Big|_{t_I, F} &= i \ell \Big|_{t_I, F} . \end{aligned} \quad (\text{A.6})$$

The first equation is a normal equation of motion for  $\theta$ ; however, the second is a constraint that is inconsistent with the assumption that  $\theta$ ,  $\ell$ , and  $t$  are real. Therefore, the path integral must be computed by means of a different method. In what follows, two methods for doing this will be presented.

## A.2 Computing the path integral: first method

In this section, we will present one of two methods for computing the path integral in (A.4). It involves splitting up the integration into bulk and then boundary values of  $\theta$  as in (A.3). We can easily compute the bulk integral using the usual methods of Euler-Lagrange variations with Dirichlet boundary conditions. Then, by Fourier transforming the result w.r.t. the boundary conditions, we obtain the final result.

We start by evaluating the angular part of (A.1), which has Dirichlet boundary conditions for  $\theta$ :

$$\langle r_F, \theta_F | e^{-HT} | r_I, \theta_I \rangle = \int_{\text{b.c.}} d[\theta(t)] \exp \left[ -\frac{1}{2} \int_{t_I}^{t_F} dt r^2 \dot{\theta}^2 \right] . \quad (\text{A.7})$$

This is easily done by finding a saddle point through the Euler-Lagrange variation, which yields the following equation:

$$\partial_t (r^2 \dot{\theta}) = 0 \Rightarrow r^2 \dot{\theta} = \ell_{cl} , \quad (\text{A.8})$$

where the constant  $\ell_{cl}$  is the classical angular momentum. The solution is the following:

$$\theta(t) = \ell_{cl} \int_{t_I}^t \frac{dt'}{r^2(t')} + \theta_I \quad \text{where} \quad \ell_{cl} = (\theta_F - \theta_I) \int_{t_I}^{t_F} \frac{dt'}{r^2(t')}. \quad (\text{A.9})$$

Defining  $I[r] \equiv \int_{t_I}^{t_F} dt' / r^2(t')$ , and substituting the solution into the action, we obtain the following:

$$-\frac{1}{2} \int_{t_I}^{t_F} dt r^2 \dot{\theta}^2 = -\frac{1}{2} \ell_{cl}^2 I[r] = -\frac{(\theta_F - \theta_I)^2}{2 I[r]}. \quad (\text{A.10})$$

Because the action is quadratic in  $\theta$ , the semiclassical approximation provides us with an exact result for the path integral. Hence, evaluating the action at this saddle point and computing the functional determinant (as we saw in chapter 2) is an exact evaluation of this part of the path integral. The functional determinant contains  $\det(\partial_t^2)$  and  $1/(\Pi_t r(t))$ , which cancels the Jacobian in the path integral over  $r(t)$ . Now, in order to finish the evaluation of (A.4) (or (A.3)), all we have to do is Fourier transform this result w.r.t. the boundary conditions  $\theta_{I,F}$ :

$$\begin{aligned} \int d\theta_I d\theta_F \exp\left(-\frac{(\theta_F - \theta_I)^2}{2 I[r]} - i\ell_F \theta_F + i\ell_I \theta_I\right) &= \int d\theta_I d\tilde{\theta} \exp\left(-\frac{\tilde{\theta}^2}{2 I[r]} - i\ell_F \tilde{\theta} + i\theta_I (\ell_I - \ell_F)\right) \\ &= \sqrt{2\pi I[r]} \delta(\ell_F - \ell_I) \exp\left(-\frac{\ell_F^2 I[r]}{2}\right), \end{aligned} \quad (\text{A.11})$$

where the  $\delta$ -function comes from the  $\theta_I$  integral, and the exponential comes from the integral over the shifted variable  $\tilde{\theta} \equiv \theta_F - \theta_I$ . The path integral enforces conservation of angular momentum. Substituting<sup>1</sup> this result into the full path integral, we are left with the following:

$$\langle r_F, \ell_F | e^{-H T} | r_I, \ell_I \rangle = \delta(\ell_F - \ell_I) \int \sqrt{2\pi I[r]} d[r(t)] \exp\left[-\frac{1}{2} \int_{t_I}^{t_F} dt \left(\dot{r}^2 + \frac{\ell_F^2}{r^2}\right)\right]. \quad (\text{A.12})$$

Performing the saddle point approximation on the remaining integration over  $r(t)$ , we find the following equations of motion:

$$\ddot{r} + \frac{\ell_F^2}{r^3} = 0. \quad (\text{A.13})$$

However, had we derived the normal Euler-Lagrange equations from the standard path integral with Dirichlet boundary conditions (A.1), these would have had a relative minus sign between these two terms. The result in (A.13) can also *effectively* be obtained by finding the saddle point of the following pseudo-action

$$S = \frac{1}{2} \int_{t_I}^{t_F} dt (\dot{r}^2 - r^2 \dot{\theta}^2) - \ell_F \theta_F + \ell_I \theta_I, \quad (\text{A.14})$$

and evaluating the action of the solution with it. The boundary conditions are then enforced by the surface term. The wrong sign in front of the kinetic term of  $\theta$  is analogous to the sign in front

<sup>1</sup>The fact that we have expressed  $\ell_{cl}$  as a functional of  $r(t)$  and the  $\theta$  boundary conditions means that this substitution is legal. What would be wrong, would be to explicitly keep  $\ell_{cl}$ , and subsequently treat it as a constant upon integrating over  $r(t)$ , which it is not.

of the axion kinetic term in (3.5). This is as though we had looked for imaginary saddle points of the action in (A.4). Some papers in the literature have gone as far as saying that one needs to rotate the contour of integration of  $\theta$  into the imaginary line in the complex plane. However, I would like to stay away from such an unnecessary and unnatural interpretation of what is really taking place in this calculation.

The method we have presented in this section makes use of the fact that we can easily express  $\ell_{cl}$  in terms of the boundary values of  $\theta$ . This is, however, due to the fact that we are doing  $(2 + 1)$ -dimensional quantum mechanics, or  $(0 + 1)$ -dimensional quantum field theory. In higher dimensions, this task becomes more difficult; and the definition of a boundary is no longer unique, which it was in this case. Therefore, we need a more covariant way to compute the path integral that can be applied to higher-dimensional field theory.

### A.3 Computing the path integral: second method

The second method we will be exploring involves the concept of dualization. The basic idea behind this is the realization that, if one wants to compute a path integral with initial and final *momentum* states, one should be working with momentum variables in the first place, as opposed to position variables.

We begin by rewriting (A.4) as follows:

$$\langle r_F, \ell_F | e^{-HT} | r_I, \ell_I \rangle = \int_{\text{b.c.}} d[\theta] d[\ell] \exp \left[ -\frac{1}{2} \int_{t_I}^{t_F} dt \left( \dot{r}^2 + \frac{\ell^2}{r^2} + 2i\theta \dot{\ell} \right) \right], \quad (\text{A.15})$$

where we have inserted an integration over a dummy variable  $\ell(t)$ . We impose Dirichlet boundary conditions on  $\ell$ , i.e.  $\ell(t_{I,F}) = \ell_{I,F}$ , and no boundary conditions on  $\theta$ . Let us first show how this reduces to (A.4) upon integrating  $\ell$  out. Integrating the last term by parts

$$- \int_{t_I}^{t_F} dt i \theta \dot{\ell} = \int_{t_I}^{t_F} dt i \ell \dot{\theta} - i(\theta_F \ell_F - \theta_I \ell_I). \quad (\text{A.16})$$

we recognize the surface term as the one in (A.4). The first term can be used to complete a square in the action as follows:

$$-\frac{\ell^2}{2r^2} + i\ell \dot{\theta} = -\frac{1}{2r^2} (\ell - ir^2 \dot{\theta})^2 - \frac{1}{2} r^2 \dot{\theta}^2. \quad (\text{A.17})$$

The remaining integral over  $\ell$  is easy to perform:

$$\int d[\tilde{\ell}] \exp \left[ -\frac{1}{2} \int dt \left( \frac{\tilde{\ell}^2}{r^2} + r^2 \dot{\theta}^2 \right) + \text{surface term} \right], \quad (\text{A.18})$$

where we have used the shift invariance of the measure by setting  $\tilde{\ell} = \ell - ir^2 \dot{\theta}$ . Because the boundary values of  $\ell$  are fixed, they are not being integrated over, hence, they are not affected by this shift. This integral is simply a Gaussian.<sup>2</sup> The end result is the original path integral

<sup>2</sup>The result of the determinant is a factor  $\Pi_t r(t)$ , which can be absorbed with the other equal factor we saw in (A.1) in the measure  $r^2 d[r] \rightarrow d[r^3]$ . This is a bijective transformation of variables, and hence yields no problems in the extremization process.

(A.4) over  $\theta$ . It should not come as a big surprise that one can write a partition function in terms of a path integral over *both* a variable and its momentum conjugate. Usually, in deriving a path integral from first principles, one obtains such an integral and subsequently eliminates the momentum variable as we just did above.

Now that we have proven that the right-hand-side of (A.15) yields (A.4) upon integrating  $\ell$  out, let us change the order of integration, and eliminate  $\theta$  first. The integral over  $\theta$  is simply a  $\delta$ -functional:

$$\int d[\theta] \exp \left[ i \int dt \theta \dot{\ell} \right] = \delta[\dot{\ell}]. \quad (\text{A.19})$$

This simply imposes conservation of angular momentum. Hence, the path integral over  $\ell$  yields the following:

$$\int d[\ell] \delta[\dot{\ell}] \exp \left[ -\frac{1}{2} \int dt \left( \frac{\ell^2}{r^2} \right) \right] = \delta(\ell_F - \ell_I) \exp \left[ -\frac{1}{2} \int dt \left( \frac{\ell_F^2}{r^2} \right) \right]. \quad (\text{A.20})$$

Therefore, the final result is the following:

$$\langle r_F, \ell_F | e^{-HT} | r_I, \ell_I \rangle = \delta(\ell_F - \ell_I) \int (\Pi'_t r(t')) d[r(t)] \exp \left[ -\frac{1}{2} \int_{t_I}^{t_F} dt \left( \dot{r}^2 + \frac{\ell_F^2}{r^2} \right) \right], \quad (\text{A.21})$$

which is what we obtained with the previous method.

In terms of our dilaton-axion system in chapter 3, the radial coordinate  $r$  is roughly analogous to the dilaton  $\phi$ , and the angular coordinate  $\theta$  is analogous to the axion  $\chi$ . The angular momentum, which is the conjugate variable to  $\theta$ , is analogous to the  $(D-1)$ -form field-strength. There, the restriction that  $dF = 0$  implies that, locally,  $F = dC$ . Here, this translates to the constraint  $\dot{\ell} = 0$ , which implies that  $\ell$  is a constant. The conservation of angular momentum in the two-dimensional quantum mechanical system translates to the conservation of axion charge.

This method of dualization is preferable to the previous one, because it does not require an explicit choice of parametrization of the boundary, and does not require us to split up path integrals into bulk and boundary integrals. Hence, we will use this method in chapter 3.



# Appendix B

## Useful formulae in Riemannian geometry

In this appendix, I will spare the reader the annoying work of calculating the curvatures of metrics with spherical or hyperbolic symmetry that are relevant in this thesis. I will first write down some basic definitions for the sake of clarity, and to establish my conventions.

The Christoffel symbols are defined as follows:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \right). \quad (\text{B.1})$$

In this thesis, I have used the following definition for the Ricci tensor:

$$R_{\mu\nu} = \partial_{\rho} \Gamma^{\rho}_{\mu\nu} - \partial_{\mu} \Gamma^{\rho}_{\nu\rho} + \Gamma^{\rho}_{\mu\nu} \Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\sigma\mu} \Gamma^{\sigma}_{\rho\nu}. \quad (\text{B.2})$$

The convention for metrics with Lorentzian signature is mostly plus, i.e.  $(-, +, \dots, +)$ . For Euclidean metrics the convention is, well ... all plus.

The general Ansatz that encompasses all instanton and cosmological metrics that have been used in this thesis can be written as follows:

$$ds_D^2 = \epsilon e^{2A(r)} dr^2 + e^{2B(r)} r^2 d\Sigma_{k,D-1}^2, \quad (\text{B.3})$$

where  $\epsilon = \pm 1$  depending on the desired signature, and the two functions  $A(r)$  and  $B(r)$  are undetermined. In the second term,  $d\Sigma_{k,D-1}^2$  is the line element of a  $(D-1)$ -dimensional sphere, plane or hyperbolic space for  $k = 1, 0$ , and  $-1$  respectively:

$$d\Sigma_{k,D-1}^2 = \frac{d\rho^2}{1 - k\rho^2} + r^2 d\Omega_{S^{D-2}}^2. \quad (\text{B.4})$$

This can also be written as follows:

$$d\Sigma_{k,D-1}^2 = d\psi^2 + f^2(\psi) d\Omega_{S^{D-2}}^2, \quad (\text{B.5})$$



where

$$f(\psi) = \begin{cases} \sin(\psi) & \text{if } k = +1 \\ \psi & \text{if } k = 0 \\ \sinh(\psi) & \text{if } k = -1 \end{cases}. \quad (\text{B.6})$$

I will now write down the radial component of the Ricci tensor as  $R_{rr}$  and will denote transverse components by  $R_{\theta\theta}$ :

$$R_{rr} = -(D-1) \left( B'' + B'^2 - A' B' + 2 \frac{B'}{r} - \frac{A'}{r} \right), \quad (\text{B.7})$$

$$R_{\theta\theta} = -\epsilon g_{\theta\theta} e^{-2A} \left( B'' + (D-1) B'^2 - A' B' + 2(D-1) \frac{B'}{r} - \frac{A'}{r} + \frac{(D-2)}{r^2} \right) \quad (\text{B.8})$$

$$+ k g_{\theta\theta} e^{-2B} \frac{(D-2)}{r^2}. \quad (\text{B.9})$$

All other components vanish. The non-vanishing Christoffel symbols are the following:

$$\begin{aligned} \Gamma^r_{rr} &= A', \\ \Gamma^r_{\theta\theta} &= -\epsilon g_{\theta\theta} e^{-2A} \left( B' + \frac{1}{r} \right), \\ \Gamma^\theta_{\theta r} &= B' + \frac{1}{r}, \end{aligned} \quad (\text{B.10})$$

where no sum over  $\theta$  is intended in the last component.

Sometimes one needs to compute the Ricci tensor of a metric that is related via a Weyl rescaling to another metric whose Ricci tensor is already known. There is a very useful identity, which can save time in this situation. Let  $g_{\mu\nu}$  be the components of a metric with Ricci tensor  $R_{\mu\nu}$ , and let  $\tilde{g}_{\mu\nu}$  be the components of a metric that is related to the first metric as follows:

$$\tilde{g}_{\mu\nu} = e^{2\sigma(x)} g_{\mu\nu}, \quad (\text{B.11})$$

where  $\sigma(x)$  is some function on the manifold. Define the tensor  $B_\mu{}^\kappa$  as follows:

$$B_\mu{}^\kappa = -\partial_\mu \sigma \partial^\kappa \sigma + \frac{1}{2} (\partial\sigma)^2 \delta_\mu{}^\kappa + \nabla_\mu (\partial^\kappa \sigma), \quad (\text{B.12})$$

where the covariant derivative is defined in terms of the metric  $g_{\mu\nu}$ . Then, the Ricci tensor  $\tilde{R}_{\mu\nu}$  of  $\tilde{g}_{\mu\nu}$  is related to  $R_{\mu\nu}$  as follows:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} B_\lambda{}^\lambda - (D-2) B_{\nu\mu}. \quad (\text{B.13})$$

This formula is derived in [140]. Note that it applies to any metric, and does not require any Ansatz for either the metric or the function  $\sigma(x)$ .

# Appendix C

## Homogeneous spaces and group manifolds

In this appendix, I will define homogeneous spaces and isotropy, giving some examples. Then, I will define group manifolds and illustrate with one example.

### C.1 Homogeneous spaces

This section is based on a section in the book by Nakahara [140]. I will assume that the reader is familiar with Lie groups.

Let us begin by defining the action of a group on a manifold.

**Definition:** Given a Lie group  $G$  and a differentiable manifold  $M$ , we define an *action* of  $G$  on  $M$  to be a differentiable map  $\sigma : G \times M \rightarrow M$ , which satisfies the following conditions:

- (i)  $\sigma(e, p) = p$  for any  $p \in M$ ,
- (ii)  $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$  for any  $g_1, g_2 \in G$  and any  $p \in M$ ,

where  $e$  is the identity element of the group. The first condition needs no explanation, and the second one just means that the group action has to respect the group multiplication. Notice that each group element  $g$  defines a diffeomorphism from the manifold to itself as follows:

$$\sigma(g, \cdot) : M \rightarrow M. \tag{C.1}$$

From basic Physics we already know many examples of groups acting on spaces. The classic example is  $SO(3)$  acting on  $\mathbb{R}^3$  as the group of rotations. More generally, whenever a group  $G \subset GL(n, \mathbb{F})$  acts on an  $n$ -dimensional vector space  $V$  over some field  $\mathbb{F}$ , we call that specific action of  $G$  on  $V$  an  $n$ -dimensional representation of the group  $G$ .

An action  $\sigma$  of group  $G$  on a manifold  $M$  automatically induces an action  $\sigma^*$  on the tangent space  $T_p M$  of any point  $p$  on the manifold. I will not state the mathematical definition here,

but I will briefly give an intuitive picture of it. Given a vector  $V \in T_p M$ , tangent to  $p$ , one can always draw *integral curves*, i.e. curves passing through  $p$  that are tangent to  $V$ . Take one such integral curve  $c(t)$ , and let  $\sigma(g, \cdot)$ , for some  $g \in G$ , act on it point by point. This will yield a curve  $\tilde{c}(t) \equiv \sigma(g, c(t))$  that passes through  $q \equiv \sigma(g, p)$ . Define  $W \equiv \sigma_{g,p}^*(V)$  as the vector in  $T_q M$  that is tangent to  $\tilde{c}(t)$  at  $q$ . This defines what is called an *induced action* of  $G$  on  $TM$ .

We also need to define the following properties for group actions:

**Definition:** Let  $G$  be a Lie group that acts on a manifold  $M$  by  $\sigma : G \times M \rightarrow M$ . The action  $\sigma$  is said to be

- (a) *transitive* if, for any  $p_1, p_2 \in M$ , there exists an element  $g \in G$  such that  $\sigma(g, p_1) = p_2$ ;
- (b) *free* if every non-trivial element  $g \neq e$  of  $G$  has no fixed points in  $M$ . In other words, given an element  $g \in G$ , if there exists an element  $p \in M$  such that  $\sigma(g, p) = p$ , then  $g$  must be the identity element  $e$ .

Now we are ready to define a homogeneous space. A manifold  $M$  is said to be homogeneous, if there exists a Lie group  $G$  that acts *transitively* on  $M$ . For instance, Lie groups act transitively on themselves via the group multiplication. The  $n$ -sphere is homogeneous because its group of rotations  $SO(n+1)$  acts transitively on it. It is tempting to think that one can then simply identify a manifold with the group that acts transitively on it, by choosing a base point  $p$  on the manifold, which one would identify with  $e$ , and identifying all other points with the group elements required to go from  $p$  to them. In general, however, given any two points  $p_1$  and  $p_2$  on a homogeneous manifold, there could be several group elements that connect them. For instance, given two points on a sphere, there are infinitely many rotations that can bring one point to the other. One can easily show that this implies that for any point, one can find rotations that leave it fixed. More generally this means that the action of the group is not *free*. This leads to the concept of *isotropy group*:

**Definition:** Let  $G$  be a Lie group that acts on a manifold  $M$ . The *isotropy group* of  $p \in M$  is a subgroup of  $G$  defined by

$$H(p) = [g \in G | \sigma(g, p) = p]. \quad (\text{C.2})$$

In other words,  $H(p) \subset G$  is the group of elements that leave  $p$  fixed. This is also called the *little group* or *stabilizer* of  $p$ . If  $G$  acts transitively on  $M$ , one can show that the isotropy groups of all points in  $M$  are isomorphic to each other. Let us take the example of the 2-sphere. Given a point  $p$ , we see that any rotation along the axis passing through  $p$  will leave the point fixed. So the isotropy of  $S^2$  is  $SO(2)$ .

There is a remarkable theorem that states that, under certain conditions, if one has a homogeneous manifold  $M$  with the group  $G$  acting on it and with isotropy group  $H$ , then the coset space  $G/H$  is a manifold (i.e. it has a differentiable structure), and it is diffeomorphic to  $M$ , i.e.  $G/H \cong M$ . The following are a few of the classic examples:

$$\begin{aligned} SO(n+1)/SO(n) &\cong S^n, \\ O((n+1)/O(n) &\cong S^n, \\ U(n+1)/U(n) &\cong S^{2n+1}, \\ O(n+1)/[O(1) \times O(n)] &\cong S^n/\mathbb{Z}_2 \cong \mathbb{R}P^n. \end{aligned} \quad (\text{C.3})$$

As the reader may have noticed, a manifold can have more than one representation as a coset space. It is this fact that will be exploited in the next section about group manifolds.

In section 4 we saw an intuitive definition of an *isotropic* manifold. We are now ready to give a more mathematical one:

**Definition:** Let  $M$  be a manifold with a group  $G$  acting on it via  $\sigma$  (not necessarily transitively), and let  $H(p)$  be the isotropy group of some point  $p \in M$ .  $M$  is *isotropic at  $p$* , if for any two vectors  $V_1$  and  $V_2$  in  $T_p M$ , there exists an element  $h \in H(p)$  such that  $\sigma_{h,p}^*(V_1) = V_2$ . In other words,  $M$  is isotropic at  $p$  if all tangent vectors at  $p$  can be rotated into each other by elements of the isotropy group of  $p$ . This matches our intuition that isotropy means that a space ‘looks’ the same in every direction, because all directions are related via a symmetry transformation. It can be shown, that if a manifold is isotropic at every point, then it is also homogeneous. Spaces that are homogeneous and isotropic are said to be *maximally symmetric*.

None of the definitions and concepts we have defined so far have required us to define a metric on the manifold in question. But when dealing with general relativity, there is always a metric at hand. So, all of these definitions must be slightly altered from the physicist’s point of view. Namely, every manifold must be endowed with a metric, and every group  $G$  acting as a group of diffeomorphisms on the manifold must leave the metric invariant, i.e. it must be a group of *isometries*. This means for instance, that the  $n$ -sphere will only be considered homogeneous, if its *isometry group* acts transitively on it. If  $S^n$  is endowed with the standard metric, then it will be homogeneous, since its isometry group  $SO(n+1)$  acts transitively on it. If, however, it is endowed with a metric that has, for instance, no isometries whatsoever, then it will not be called homogeneous. The concept of isotropy also changes in that the isotropy group has to be a subgroup of the isometry group. Again, a manifold can have a larger or smaller isotropy, depending on the metric defined on it. A manifold will be called *maximally symmetric* if it is homogeneous and isotropic under the physicist’s definitions of these two concepts.

## C.2 Group manifolds

In this section, we will take the *physics* definitions of homogeneity and isotropy.

A *group manifold* is a Lie group endowed with a metric that makes it homogeneous. In other words, it is a manifold that is diffeomorphic to a Lie group and it has a metric, such that its isometry group acts transitively on the manifold. One simple example is the  $S^3$  with the standard metric

$$ds^2 = d\psi^2 + \sin^2(\psi) d\Omega_{S^2}^2. \quad (\text{C.4})$$

This manifold is diffeomorphic to the group  $SU(2)$ . Endowed with this metric, it has  $SO(4)$  isometry and  $SO(3)$  isotropy, so we can write it as the quotient  $SO(4)/SO(3)$ . It is maximally symmetric.

However, as we mentioned before, there are more ways to identify a manifold with a quotient of groups. We could endow it with a metric that has less isometries, and hence less isotropy. The manifold  $S^3$  can be regarded as a  $U(1)$  principal bundle over  $S^2$  known as the *Hopf fibration*. This means that *locally*

$$S^3 \cong S^2 \times S^1. \quad (\text{C.5})$$

So we can in principle write down a locally defined metric for  $S^3$  that has  $SO(3) \times SO(2)$  as its isometry group, and  $SO(2)$  as its isotropy group. I will not write down the explicit formulae because they are not clarifying, but they can be found in [95]. Therefore, we can rewrite our manifold as the following coset:

$$S^3 \cong \frac{SO(3) \times SO(2)}{SO(2)}, \quad (\text{C.6})$$

where the  $SO(2)$  in the denominator is a subgroup of the  $SO(3)$  in the numerator, i.e. the quotient  $SO(3)/SO(2)$  forms the  $S^2$  factor of the Hopf fibration. This manifold is no longer maximally symmetric, it is *anisotropic*.

We can even go further and write down a metric with the least amount of isometry that can still act transitively on the manifold. The isometry group must then be at least three-dimensional. Such a metric will then have no isotropy group left. In that case, we will be writing our manifold as follows:

$$S^3 \cong \frac{SO(3)}{\cdot}, \quad (\text{C.7})$$

where the ‘ $\cdot$ ’ represents the trivial group. This space is totally anisotropic. All of these statements are valid only locally. Globally, of course,  $S^3 \cong SU(2)$ , and  $SO(3) \cong SU(2)/\mathbb{Z}_2$ .

To summarize, we have written our manifold as three different quotients in the order of decreasing isometry and isotropy:

$$S^3 \cong \frac{SO(4)}{SO(3)} \cong \frac{SO(3) \times SO(2)}{SO(2)} \cong \frac{SO(3)}{\cdot}. \quad (\text{C.8})$$

The first two forms are referred to as the *round* and the *squashed* 3-sphere respectively. In general relativity, one sees group manifolds as generalizations of maximally symmetric spaces, in that they are homogeneous but potentially completely anisotropic. In the standard terminology, which I personally find confusing, one names the group manifold after its isometry group. In the case of the 3-sphere one would call the cases in (C.8) the  $SO(4)$ -manifold, the  $SO(3) \times SO(2)$ -manifold, and the  $SO(3)$ -manifold respectively.

All three-dimensional group manifolds were completely classified by Bianchi [141]. In [95] they were used as internal spaces to compactify seven-dimensional pure gravity. This yielded four-dimensional theories with gravity and scalars, with interesting exponential potentials, which could be used to obtain cosmological solutions with periods of transient acceleration. In [128] some of those theories were studied as autonomous systems to find solutions with periods of acceleration.