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## Instantons and cosmologies in string theory

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# Chapter 7

## Link between D-instantons and Cosmology

Throughout this thesis, we have been studying two different kinds of scalar-gravity field configurations: D-instantons and cosmological solutions. We also briefly looked at solitons. We pointed out in chapter 2 that instantons and solitons can be equivalent, if certain conditions are met by the system they are in. In chapter 3, we studied the specific correspondence between black hole solutions and D-instantons, and the requirements for the correspondence to hold. But there is a strikingly simpler and more obvious fact that ties D-instantons, black holes, and cosmological solutions together. They all depend on one coordinate, be it space-like or time-like, and they all interpolate between ‘trivial’ configurations. More specifically, the wormhole geometry of the non-extremal D-instanton with  $q^2 > 0$  interpolates between two flat Euclidean spaces. Cosmological solutions such as the ones we studied in chapters 5 and 6 interpolate in time between power-law regimes, or between power-law and de Sitter spaces with non-trivial behavior in between, such as transient acceleration. In this chapter, we will pursue the similarity between D-instantons and cosmological solutions carried by scalar fields in detail. We will do so in two ways. In the first section, we will relate some of these solutions to each other via the Wick rotation. In the second section, we will take a different perspective on the degrees of freedom we are studying. We will view the D-instanton and cosmological solutions as trajectories in a scalar manifold, an abstract target space, if you will. That will enable us to present these solutions in a mathematically unified way. It will even suggest a way of pasting together a cosmological solution and an instanton solution, as though they were part of the same phenomenon.

### 7.1 Wick rotation

**WARNING:** The following section contains passages with explicit Wick rotations that may not be suitable for self-respecting mathematicians. Parental discretion is advised.

Let us begin by reviewing the non-extremal D-instanton solution from chapter 3, which I will

rewrite here for the reader's convenience. The Euclidean Lagrangian density is the following:

$$\mathcal{L} = R - \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} e^{b\phi} (\partial\chi)^2, \quad (7.1)$$

where,  $\phi$  are  $\chi$  scalars. The 'wrong' sign of the kinetic term for  $\chi$  is explained in subsection 3.6.1. We take the Ansatz of a conformally flat metric with maximal spherical symmetry. We also assume that all fields respect the spherical symmetry. This yields the following solution:

$$ds^2 = \left(1 - \frac{\mathbf{q}^2}{r^{2(D-2)}}\right)^{2/(D-2)} \left(dr^2 + r^2 d\Omega_{S^{D-1}}^2\right), \quad (7.2)$$

$$e^{b\phi(r)} = \left(\frac{q_-}{\mathbf{q}} \sinh(H(r) + C_1)\right)^2, \quad (7.3)$$

$$\chi(r) = \frac{2}{b q_-} (\mathbf{q} \coth(H(r) + C_1) - q_3). \quad (7.4)$$

with

$$H(r) = b c \operatorname{arctanh}\left(\frac{\mathbf{q}}{r^{D-2}}\right), \quad (7.5)$$

where  $\mathbf{q}^2, q_-, q_3$  and  $C_1$  are integration constants and

$$c = \sqrt{\frac{2(D-1)}{(D-2)}}. \quad (7.6)$$

As we saw previously,  $\mathbf{q}^2$  can be positive, negative or zero. Therefore, as we can see, everything depends on one coordinate  $r$ . At this point, the reader should feel the irresistible temptation to Wick rotate this solution to see if that yields a cosmological configuration. This has been explored in [137], but I will do it in my own notation here. The first step is to make ' $r$ ' timelike by letting  $r \rightarrow it$ . This takes care of the  $dr^2$  term in the metric, but messes up the spherical part by creating a minus sign. To fix this, let us rewrite the spherical metric as follows:

$$d\Omega_{S^{D-1}}^2 = d\theta^2 + \sin^2(\theta) d\Omega_{S^{D-2}}^2. \quad (7.7)$$

The Wick rotation created an overall sign in front of this metric, so to fix it, we let  $\theta \rightarrow i\psi$ :

$$d\theta^2 + \sin^2(\theta) d\Omega_{S^{D-2}}^2 \rightarrow -d\psi^2 - \sinh^2(\psi) d\Omega_{S^{D-2}}^2 = -d\mathbf{H}_{D-1}^2, \quad (7.8)$$

where  $\mathbf{H}$  stands for a hyperbolic space. The end result is the following metric:

$$ds^2 = \left(1 - \frac{\tilde{\mathbf{q}}^2}{t^{2(D-2)}}\right)^{2/(D-2)} \left(-dt^2 + t^2 d\mathbf{H}_{D-1}^2\right), \quad (7.9)$$

where  $\tilde{\mathbf{q}}^2 = (-1)^{(D-2)} \mathbf{q}^2$ . This is indeed a cosmological solution. Specifically, it is an FLRW metric with  $k = -1$ . But what about  $\phi$  and  $\chi$ ? Those are less straightforward to study, but as we will see we can already gather one qualitative piece of information from them. Let us first write down the Lagrangian density for the Lorentzian system to which this cosmological solution belongs:

$$\mathcal{L} = R - \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} e^{b\phi} (\partial\chi)^2. \quad (7.10)$$

Notice that we are now using the ‘normal’ sign for the kinetic term for  $\chi$ . Hence, in order to establish a relation between an instanton and a cosmology, we must effectively multiply  $\chi$  by  $i$ . If we take the metric (7.9) as an Ansatz for this system and fill it into the time component of the Einstein equation, we get the following:

$$R_{tt} = \frac{4(D-1)(D-2)}{r^{2(D-1)} \left(1 - \frac{\tilde{\mathbf{q}}^2}{r^{2(D-2)}}\right)^2} \tilde{\mathbf{q}}^2 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} e^{b\phi} \dot{\chi}^2. \quad (7.11)$$

The right-hand side is positive definite, therefore,  $\tilde{\mathbf{q}}^2$  must be positive. Hence, not all three classes of D-instantons (i.e.  $\mathbf{q}^2$  positive, negative and zero) can be Wick rotated to a cosmological solution. Not even the extremal D-instanton has a cosmological partner. The only class that can be Wick rotated is the one with  $\mathbf{q}^2 (-)^{D-2} > 0$ . This is obviously a dimension-dependent condition. The actual process of Wick rotating the solutions for the scalars is less obvious. The idea is that  $\chi$  has to get multiplied by an  $i$ , whereas the dilaton should remain unaffected. This is accomplished by letting  $C_1 \rightarrow C_1 + i\pi/2$ , and  $q_- \rightarrow iq_-$ . The result is the following:

$$e^{b\phi(r)} = \left( \frac{q_-}{\tilde{\mathbf{q}}} \cosh(H(r) + C_1) \right)^2, \quad (7.12)$$

$$\chi(r) = \frac{2}{b q_-} (\tilde{\mathbf{q}} \tanh(H(r) + C_1) - q_3), \quad (7.13)$$

with

$$H(r) = b c \operatorname{arctanh} \left( \frac{\tilde{\mathbf{q}}}{r^{D-2}} \right). \quad (7.14)$$

## 7.2 Target space interpretation

In this section, we are going to investigate another parallelism between *axionic* instantons and cosmologies. The idea is to regard all fields, including the non-constant part of the metric, as coordinates in a fictitious *target space*. Because instantons and cosmologies both depend on only one parameter, they will be interpreted as trajectories of a particle in the target space.

This section is based on a collaboration with E. Bergshoeff, D. Roest, J. Russo, and P.K. Townsend, entitled *Cosmological D-instantons and Cyclic Universes*, [138]. It is organized as follows: first, we will present the general system and Ansatz we want to solve and dimensionally reduce it to one dimension. In subsection 7.2.2, we will introduce the ‘Liouville’ gauge, in which will allow us to view our solutions as trajectories in two-dimensional target spaces defined by the two scalar fields. In subsection 7.2.3, we will introduce the ‘Milne-Rindler’ gauge, which will also view the non-trivial part of the metric as a target space coordinate. In this new three-dimensional target space, we will be able to present instantons and cosmologies in a unified way, as trajectories of a particle.

### 7.2.1 Ansatz and reduction to one dimension

The Lagrangians we will be studying can be summarized as follows:

$$\mathcal{L} = R - \frac{1}{2} (\partial\phi)^2 + \epsilon \frac{1}{2} e^{b\phi} (\partial\chi)^2, \quad (7.15)$$

with  $\epsilon = \pm 1$  for Euclidean and Lorentzian signature respectively. To investigate cosmological solutions of our model, or to find instanton solutions of its Euclidean version, we make the Ansatz

$$ds^2 = \epsilon (e^{\alpha\varphi} f)^2 d\lambda^2 + e^{2\alpha\varphi/(d-1)} d\Sigma_k^2, \quad \phi = \phi(\lambda), \quad \chi = \chi(\lambda), \quad (7.16)$$

where  $f$  is an arbitrary function of  $\lambda$ , and

$$\alpha = \sqrt{\frac{d-1}{2(d-2)}}. \quad (7.17)$$

The  $(d-1)$ -metric  $d\Sigma_k^2$  is (at least locally) a maximally symmetric space of positive ( $k = 1$ ), negative ( $k = -1$ ) or zero ( $k = 0$ ) curvature. One can choose coordinates such that

$$d\Sigma_k^2 = (1 - kr^2)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \quad (7.18)$$

where  $d\Omega_{d-2}^2$  is an  $SO(d-1)$ -invariant metric on the unit  $(d-2)$ -sphere. This Ansatz constitutes a consistent reduction of the original degrees of freedom to a three-dimensional subspace, the ‘augmented target space’, with coordinates  $(\varphi, \phi, \chi)$ . The full equations of motion reduce to a set of equations that can themselves be derived by variation of the time-reparametrization invariant effective action

$$I = \frac{1}{2} \int d\lambda \left\{ f^{-1} \left( \epsilon \dot{\phi}^2 - \epsilon \dot{\chi}^2 + e^{b\phi} \dot{\chi}^2 \right) + 2k(d-1)(d-2) f e^{\varphi/\alpha} \right\}, \quad (7.19)$$

where the overdot indicates differentiation with respect to  $\lambda$ . For  $\epsilon = -1$  we can interpret  $\lambda$  as a time coordinate related to the time  $t$  of FLRW cosmology in standard coordinates by

$$dt \propto e^{\alpha\varphi} f d\lambda. \quad (7.20)$$

For  $\epsilon = 1$  the metric has Euclidean signature and we can interpret  $\lambda$  as imaginary time.

If we interpret all fields as being coordinates of a particle’s world-line in some target space, then we notice that the scalars  $\phi$  and  $\chi$  parametrize a two-dimensional hyperbolic space:

$$ds_T^2 = -\epsilon d\phi^2 + e^{b\phi} d\chi^2. \quad (7.21)$$

For  $\epsilon = -1$ , this is  $\mathbb{H}_2$ , the two-sheeted hyperboloid with Euclidean signature, in Poincaré coordinates, which are globally defined. For  $\epsilon = +1$ , however, this is a Lorentzian one-sheeted hyperboloid  $dS_2$  in Poincaré coordinates, which are *not* globally defined. They only cover half of the surface. To treat both signatures on equal footing, it is therefore convenient to switch to coordinates of the target space that are globally defined. This is done by defining new scalar field variables  $(\psi, \theta)$  by

$$\begin{aligned} e^{(b/2)\phi} &= e^\psi \cos^2(\theta/2) - \epsilon e^{-\psi} \sin^2(\theta/2), \\ e^{(b/2)\phi} \chi &= b^{-1} \left( e^\psi + \epsilon e^{-\psi} \right) \sin \theta, \end{aligned} \quad (7.22)$$

which yields the following target space metrics:

$$ds_T^2 = \begin{cases} -d\psi^2 + \cosh^2(\psi) d\theta^2 & \text{for } \epsilon = 1 \\ +d\psi^2 + \sinh^2(\psi) d\theta^2 & \text{for } \epsilon = -1 \end{cases}. \quad (7.23)$$

We can now recognize the first metric as that of a  $dS_2$ , by comparison with the  $dS_3$  metric (4.41) we introduced in chapter 4. The second metric is the usual one for a two-dimensional hyperboloid. The new effective action is

$$I = \frac{1}{2} \int d\lambda \left\{ \frac{4}{b^2} f^{-1} \left[ \frac{b^2}{4} \epsilon \dot{\psi}^2 - \epsilon \dot{\psi}^2 + \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 \right] + 2k(d-1)(d-2) f e^{\varphi/\alpha} \right\}. \quad (7.24)$$

Introducing the new scale-factor variable  $\eta$  by

$$\eta^\gamma = 2\gamma(d-1) e^{\varphi/(2\alpha)}, \quad (7.25)$$

where

$$\gamma = 1/(b\alpha), \quad (7.26)$$

we arrive at the action

$$I = \frac{1}{2} \int d\lambda \left\{ \frac{4}{b^2} f^{-1} \left[ \epsilon (\dot{\eta}/\eta)^2 - \epsilon \dot{\psi}^2 + \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 \right] + \frac{b^2}{4} k f \eta^{2\gamma} \right\}. \quad (7.27)$$

We remark, for future reference, that the Ansatz (7.16) leads to  $\gamma = 2/3$  for  $d = 10$  IIB supergravity.

Because of the time-reparametrization invariance, we are free to choose the function  $f$ ; each choice of  $f$  corresponds to some choice of time parameter. There are two choices that are particularly convenient, and we now consider them in turn.

### 7.2.2 The ‘Liouville’ gauge

The simplest way to proceed for general  $b$  is to make the gauge choice

$$f = 4/b^2. \quad (7.28)$$

From (7.27) one sees that the effective Lagrangian in this gauge is

$$L = \frac{1}{2} \left[ -\epsilon \dot{\psi}^2 + \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 \right] + \frac{1}{2} \left[ \epsilon (\dot{\eta}/\eta)^2 + k \eta^{2\gamma} \right]. \quad (7.29)$$

Apart from the constraint, the dynamics of the motion on the target space, which is manifestly geodesic, is now separated from the dynamics of the scale factor, which is determined by a equation of Liouville-type; for this reason we will call this choice of gauge the ‘Liouville gauge’.

As  $SL(2; \mathbb{R})$  is the isometry group of both  $\mathbb{H}_2$  (the target space of the Lorentzian action) and  $dS_2$  (the target space of the Euclidean action), there is a conserved  $SL(2; \mathbb{R})$  ‘momentum’  $\ell^\mu$ , and the geodesics are such that

$$\dot{\psi}^2 - \epsilon \frac{1}{4} (e^\psi + \epsilon e^{-\psi})^2 \dot{\theta}^2 = \ell^2. \quad (7.30)$$

The constraint ( $f$  equation of motion) is

$$(\dot{\eta}/\eta)^2 = \ell^2 + k \epsilon \eta^{2\gamma}. \quad (7.31)$$

We now present the solutions of the equations of motion of (7.29) subject to the constraint (7.30) and (7.31), first for the target space fields and then for the scale factor.

### Target space geodesics

Geodesics on the  $\mathbb{H}_2$  ( $\epsilon = -1$ ) or  $dS_2$  ( $\epsilon = 1$ ) target space are solutions of the field equations of (7.29) for  $\psi$  and  $\theta$  subject to (7.30) and can be classified as follows, according to whether  $\ell^2$  is positive, negative or zero:

- $\ell^2 > 0$ . For  $\epsilon = 1$  the solution is

$$\begin{aligned}\sinh \psi &= \pm \sqrt{1 + \frac{q_-^2}{\ell^2}} \sinh [\ell (\lambda - \lambda_0)] \\ \tan (\theta - \theta_0) &= \pm \frac{q_-}{\ell} \tanh [\ell (\lambda - \lambda_0)] ,\end{aligned}\quad (7.32)$$

for constants  $\lambda_0$ ,  $\theta_0$  and  $q_-$  (this being the integration constant for the super-extremal D-instanton of [43]). For  $\epsilon = -1$  the solution is

$$\begin{aligned}\cosh \psi &= \sqrt{1 + \frac{q_-^2}{\ell^2}} \cosh [\ell (\lambda - \lambda_0)] \\ \tan(\theta - \theta_0) &= \pm \frac{q_-}{\ell} \coth [\ell (\lambda - \lambda_0)] .\end{aligned}\quad (7.33)$$

In the special case that  $q_- = 0$  these solutions simplify, for either choice of the sign  $\epsilon$ , to

$$\psi = \pm \ell (\lambda - \lambda_0), \quad \theta = \theta_0, \quad (\epsilon = \pm 1). \quad (7.34)$$

- $\ell^2 < 0$ . In this case only  $\epsilon = 1$  is possible, and the solution is

$$\begin{aligned}\sinh \psi &= \pm \sqrt{\frac{q_-^2}{(-\ell^2)} - 1} \sin \left[ \sqrt{-\ell^2} (\lambda - \lambda_0) \right] \\ \tan(\theta - \theta_0) &= \pm \frac{q_-}{\sqrt{-\ell^2}} \tan \left[ \sqrt{-\ell^2} (\lambda - \lambda_0) \right] .\end{aligned}\quad (7.35)$$

- $\ell^2 = 0$ . The only solution for  $\epsilon = -1$  in this case is the trivial one for which both  $\psi$  and  $\theta$  are constant. For  $\epsilon = 1$  the solution is

$$\sinh \psi = \pm q_- (\lambda - \lambda_0), \quad \tan(\theta - \theta_0) = \pm q_- (\lambda - \lambda_0) . \quad (7.36)$$

It should be noted that, in each case, the  $\pm$  signs for  $\psi$  and  $\theta$  can be chosen independently. We should note that, for  $\epsilon = 1$  ( $dS_2$ ), there are three classes of solutions, with  $\ell^2$  positive, zero and negative, whereas for  $\epsilon = -1$ , there is only one class with  $\ell^2 > 0$ . This is because  $dS_2$ , being Minkowskian, has a light-cone structure. It admits, space-like, light-like and time-like solutions. We can interpret  $\ell^2$  as the momentum squared (i.e.  $-m^2$ ), of the particle.  $\mathbb{H}_2$ , on the other hand, does not have a light-cone structure.

One interesting consequence of writing our instanton solutions in terms of these global coordinates, is that the singularities of the dilaton and axion that we previously encountered in

chapter 3 go away. In this target space interpretation, those singularities are mere coordinate singularities, signaling that the particle's world-line has departed from the Poincaré coordinate patch. It has gone over to the half of the hyperboloid that is not covered by these coordinates. The true physical meaning of this resolution of the singularities, however, remains to be discovered.

### The scale factor

We next turn to the constraint (7.31). Given  $\ell^2$ , this determines  $\eta$  as follows

- $\ell^2 > 0$ .

$$\eta^{2\gamma} = \eta_0^{2\gamma} \exp(\pm 2\ell\gamma\lambda), \quad (k = 0), \quad (7.37)$$

$$\eta^{2\gamma} = \frac{\ell^2}{\sinh^2(\ell\gamma\lambda)}, \quad (k\epsilon = 1), \quad (7.38)$$

$$\eta^{2\gamma} = \frac{\ell^2}{\cosh^2(\ell\gamma\lambda)}, \quad (k\epsilon = -1), \quad (7.39)$$

for some constant  $\eta_0$ . Note that all  $k = \pm 1$  trajectories are asymptotic to some  $k = 0$  trajectory near  $\eta = 0$ , as expected since the  $\sigma$ -model matter satisfies the strong energy condition.

- $\ell^2 < 0$ . In this case there is a solution only for  $k = \epsilon = 1$ :

$$\eta^{2\gamma} = \frac{-\ell^2}{\sin^2(\gamma\sqrt{-\ell^2}\lambda)}, \quad (k = \epsilon = 1). \quad (7.40)$$

- $\ell^2 = 0$ . In this case there is a solution only for  $k\epsilon \geq 0$ . If  $k = 0$  then  $\eta$  is constant. Otherwise

$$\eta^{2\gamma} = 1/(\gamma\lambda)^2, \quad (k\epsilon = 1). \quad (7.41)$$

For  $\epsilon = k = 1$  these solutions yield the super-extremal ( $\ell^2 > 0$ ), sub-extremal ( $\ell^2 < 0$ ) and extremal ( $\ell^2 = 0$ ) D-instantons of [43]. For  $\epsilon = -1$  they yield FLRW cosmologies; from (7.20) we see that the standard FLRW time  $t$  is related to the parameter  $\lambda$  by

$$dt \propto \eta^{2\gamma\alpha^2} d\lambda. \quad (7.42)$$

Given one of above solutions for  $\eta^{2\gamma}$  as a function of  $\lambda$  we can determine  $\lambda$  as a function of  $t$  and hence  $\eta$  as a function of  $t$ . Of most interest here is the behaviour near  $\eta = 0$ . For example, for  $\ell^2 > 0$  we have

$$\eta \sim \eta_0 e^{-\ell\lambda}, \quad (7.43)$$

for  $\lambda \rightarrow \infty$ , as  $\eta \rightarrow 0$ . This yields (for a choice of integration constant such that  $t \rightarrow 0$  as  $\lambda \rightarrow \infty$ )

$$-t \propto e^{-2\gamma\alpha^2\ell\lambda}. \quad (7.44)$$



Given that we start with a cosmological solution for negative  $t$ , this shows that a big-crunch singularity will be approached as  $t \rightarrow 0$ . By considering the behaviour as  $\lambda \rightarrow -\infty$  we may similarly deduce that a cosmological solution for positive  $t$  must have had a big-bang singularity at  $t = 0$ . In other words, cosmologies with  $\ell^2 > 0$  are incomplete in the sense that they have a beginning or an end (or both) at finite  $t$ . We shall make a suggestion in the following subsection, of how they can be completed.

### 7.2.3 The ‘Milne-Rindler’ gauge

We will now upgrade the approach we took in the previous subsection by augmenting the two-dimensional target space to a three-dimensional one. We will do so by considering the single degree of freedom of the spacetime metric as another target space coordinate.

Returning to (7.27), we make the new gauge choice

$$f = \frac{4}{b^2 \eta^2}. \quad (7.45)$$

As the possible choices of  $f$  are related by a redefinition of the independent variable, we will need to distinguish the independent variable in this gauge from the parameter  $\lambda$  previously used. Let us call the new independent variable  $\tau$ ; it is related to  $\lambda$  through the differential equation

$$d\tau = \eta^2(\lambda) d\lambda, \quad (7.46)$$

which can be solved for  $\tau(\lambda)$  given any of the scale factor solutions  $\eta(\lambda)$  presented above.

In the gauge (7.45) the action is

$$I = \int d\tau L_\tau, \quad (7.47)$$

where<sup>1</sup>

$$L_\tau = \frac{1}{2}\epsilon \left(\frac{d\eta}{d\tau}\right)^2 + \frac{1}{2}\eta^2 \left[-\epsilon \left(\frac{d\psi}{d\tau}\right)^2 + \frac{1}{4}(e^\psi + \epsilon e^{-\psi})^2 \left(\frac{d\theta}{d\tau}\right)^2\right] + \frac{k}{2}\eta^{2\gamma-2}. \quad (7.48)$$

We observe that for  $\epsilon = -1$  the kinetic term is that of a particle in a 3-dimensional Minkowski spacetime in Milne coordinates. However, for  $\epsilon = 1$ , this kinetic term is again that of a particle in 3-dimensional Minkowski spacetime, only this time in Rindler coordinates. See 4.1.3 for a discussion on those two coordinates systems for four-dimensional Minkowski. We will call this choice of gauge the ‘Milne-Rindler’ gauge. We can unify both actions ( $\epsilon = \pm 1$ ), by going to Cartesian coordinates, since the latter are globally defined in Minkowski spacetime. The new field variables  $X_\mu$  ( $\mu = 0, 1, 2$ ) are

$$\begin{aligned} X_0 &= \pm \frac{1}{2}\eta (e^\psi - \epsilon e^{-\psi}), \\ X_1 &= \pm \frac{1}{2}\eta (e^\psi + \epsilon e^{-\psi}) \cos \theta, \\ X_2 &= \pm \frac{1}{2}\eta (e^\psi + \epsilon e^{-\psi}) \sin \theta. \end{aligned} \quad (7.49)$$

<sup>1</sup>Note that  $L_\tau d\tau = L d\lambda$ , where  $L$  is the lagrangian in the gauge used previously.

Note that

$$X^2 \equiv -X_0^2 + X_1^2 + X_2^2 = \epsilon \eta^2 . \quad (7.50)$$

Since  $\eta^2$  is positive, it follows that  $X^2 < 0$  when  $\epsilon = -1$ , and  $X^2 > 0$  when  $\epsilon = 1$ . The  $X^2 < 0$  region is the Milne region of Minkowski space and cosmological solutions are trajectories in this space. Generic trajectories reach  $\eta = 0$  at finite FLRW time, corresponding to a cosmological singularity. However, the hypersurface  $\eta = 0$  is just the Milne horizon, and the singularity at the Milne horizon disappears in the cartesian coordinates  $X_\mu$ . The trajectory can therefore be smoothly continued through the Milne horizon *in cartesian coordinates* into the Rindler region, in which  $X^2 > 0$ , where we need  $\epsilon = 1$ . Thus, on passing through the Milne horizon, a cosmological trajectory becomes an instanton (and vice-versa).

The Milne-Rindler gauge Lagrangian  $L_\tau$  in cartesian coordinates is

$$L_\tau = \frac{1}{2} \left[ (dX/d\tau)^2 + k (\epsilon X^2)^{\gamma-1} \right] . \quad (7.51)$$

The constraint is now

$$(dX/d\tau)^2 = k (\epsilon X^2)^{\gamma-1} . \quad (7.52)$$

We thus have a problem analogous to that of a particle of zero energy in a central potential, with conserved  $SL(2; \mathbb{R})$  ‘‘angular momentum’’

$$\ell^\mu = \epsilon^{\mu\nu\rho} X_\nu (dX_\rho/d\tau) . \quad (7.53)$$

The target space and the scale factor solutions given previously can now be combined into a single solution for  $X_\mu$ . For example, for  $\ell^2 > 0$ , the solutions are

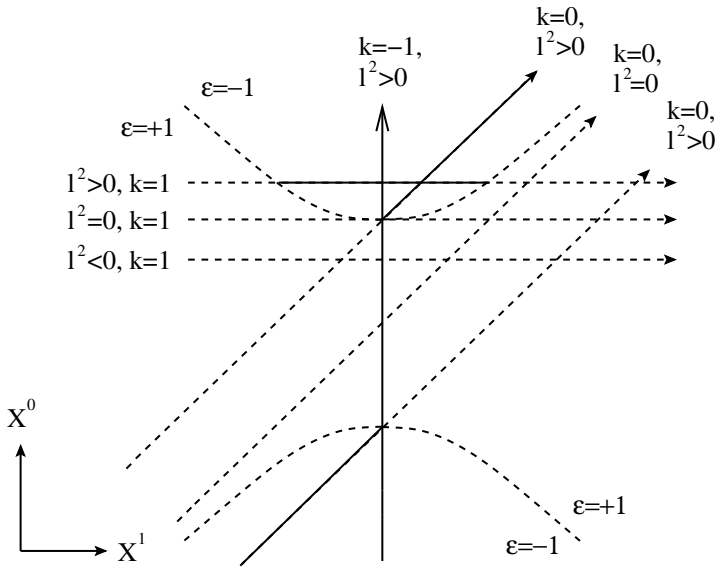
$$X_\mu = \begin{cases} \pm \eta \left( s_\mu \sinh(\ell\lambda) + c_\mu \cosh(\ell\lambda) \right) , & \epsilon = 1 , \\ \pm \eta \left( s_\mu \cosh(\ell\lambda) + c_\mu \sinh(\ell\lambda) \right) , & \epsilon = -1 , \end{cases} \quad (7.54)$$

where

$$\begin{aligned} s_0 &= \sqrt{1+a^2} \cosh(\ell\lambda_0) , & a &\equiv q_-/\ell , \\ c_0 &= -\sqrt{1+a^2} \sinh(\ell\lambda_0) , \\ c_1 &= \cosh(\ell\lambda_0) \cos(\theta_0) + a \sinh(\ell\lambda_0) \sin(\theta_0) , \\ s_1 &= -\sinh(\ell\lambda_0) \cos(\theta_0) - a \cosh(\ell\lambda_0) \sin(\theta_0) , \\ c_2 &= -a \sinh(\ell\lambda_0) \cos(\theta_0) + \cosh(\ell\lambda_0) \sin(\theta_0) , \\ s_2 &= a \cosh(\ell\lambda_0) \cos(\theta_0) - \sinh(\ell\lambda_0) \sin(\theta_0) . \end{aligned} \quad (7.55)$$

Note that  $(c_\mu \pm s_\mu)$  is null.

In [138], this coordinate system is used to ‘continue’ cosmological solutions into instanton solutions by passing through the target space Milne horizon. See figure 7.1, for the case where  $\gamma = 1$ . In this case, the Lagrangian simplifies tremendously, as all trajectories become geodesics in the three-dimensional Minkowski spacetime. The  $\gamma \neq 1$  have a central potential, which complicates the picture. An interesting idea that has not been experimented with, would be to eliminate such a potential by augmenting the target space to a four-dimensional one with non-trivial curvature. In [135, 139], this idea was applied in the context of cosmological solutions by



**Figure 7.1:** Instantons and cosmologies as geodesics in a 2-D projection of Minkowski spacetime onto a plane,  $X_2 \neq 0$ , that does not pass through the origin. The solid (dashed) lines are cosmologies (instantons), which are separated by the light-cone. The dotted hyperbola with  $X_0^2 - X_1^2 = X_2^2$  is the projection of the light-cone onto the 2-D plane.

reinterpreting the derivatives of scalar potentials in equations of motion as Christoffel symbols of an augmented target space.

The hope behind the idea of patching cosmological solutions with instanton geometries is to find a mechanism, by means of which the Big Bang singularity can be ‘smoothed out’. The Big Bang of the universe would actually be preceded by another cosmological solution that underwent a Big Crunch. The two cosmologies would be ‘connected’ by an instanton phase. The full solution, although singular in spacetime, is singularity-free in the target space. For details on the patching of specific cases, the reader is referred to [138].