Instantons and cosmologies in string theory

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Chapter 5

Scalar Cosmologies I: A simple case

5.1 Introduction

The discovery that the universe may currently be in a phase of accelerated expansion [79, 80] has led to strong interest in finding de Sitter solutions or more general accelerating cosmologies from M-theory, see [33, 83, 87–97] and references therein.

A simple way to study accelerating cosmologies is to consider models containing just gravity and a number of scalars with a potential. This method has a long history and has resulted in models for inflation [98], describing the early universe, and for quintessence [99], describing the present universe. The potentials for the scalar fields give rise to a small effective cosmological constant. Multi-exponential potentials comprise a specific class of potentials, which have been frequently studied, and these are of interest for two reasons: first, they can arise from M-theory in many ways; e.g. via compactifications on product spaces possibly with fluxes [100–103], and second, the equations of motion can be written as an autonomous system. This approach allows for an algebraic determination of power-law and de Sitter solutions, which are viewed as critical points that can correspond to early- and late-time asymptotics of general solutions. Many authors have made use of this fact, see [95, 104–110] and references therein.

The purpose of this chapter is to investigate the possibility of transient acceleration for the class of cosmologies whose solutions are described by a metric and N scalars, with a scalar potential given by a single exponential. The consequence of this is that, effectively, the scalar potential depends on only one scalar. All other $N-1$ scalars are represented by their kinetic terms only. Since the metric cannot distinguish between these different $N-1$ scalars, there is no qualitative difference between the $N = 2$ scalar cosmology and the $N > 2$ scalar cosmologies. We therefore only consider the one-scalar ($N = 1$) and two-scalar ($N = 2$) cosmologies. We will be studying these models purely from the four-dimensional point of view, without reference to possible higher dimensional origins. This chapter can be considered as a warm up for the next chapter, where we will study scalar cosmologies with multi-exponential potentials. In that case, things will be much more complicated, as there it will no longer be possible to reduce a multi-scalar system into a 2-scalar model.

The cosmological solutions discussed in this paper have been given sometime ago [111, 112].

The fact that these cosmologies, for particular cases at least, exhibit a period of acceleration, was noted recently in [83] where a specific class of solutions was obtained by compactification over a compact hyperbolic space (for earlier discussions, see [90, 101, 112–114]). The relation with S-branes was subsequently noted in [91, 92, 97] (for general literature on S-brane solutions, see [84, 94, 100–102, 114–121]).

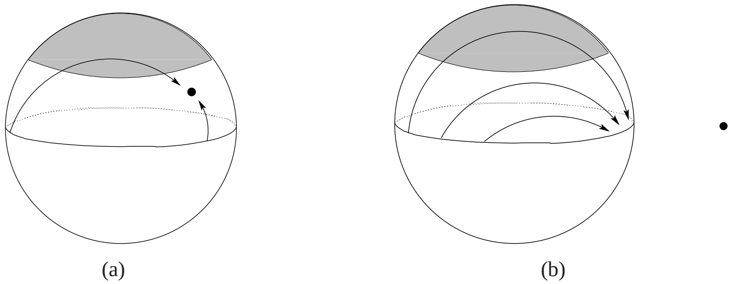

Figure 5.1: *Each cosmological solution is represented by a curve on the sphere. In figure (a) “Rome", represented by the dot, is on the sphere and each curve is directed from the equator towards “Rome", which corresponds to a power-law solution to the equations of motion. In figure (b) “Rome" is not on the sphere and each curve, again being directed towards “Rome", begins and ends on the equator. In this case “Rome" is not a solution. The accelerated expansion of the solution occurs whenever the curve lies within the “arctic circle". This region is shown by the shaded area.*

In this work we will discuss systematically the accelerating phases of all 2-scalar cosmologies with a single exponential potential by associating to each solution a trajectory on a 2-sphere. It turns out that all trajectories have the property that, when projected onto the equatorial plane, they reduce to straight lines which are directed towards a point that we will call “Rome". Depending on the specific dilaton coupling of the potential, this point can be either on the sphere or not. In the former case, it corresponds to a power-law solution for the scale factor, whereas in the latter case, it is not a solution. We find that the accelerating phase of a solution is represented by the part of the trajectory that lies within the “arctic circle" on the sphere, see figure 5.1. This enables us to calculate the expansion factors in a straightforward way for each of the solutions.

This chapter is based on a collaboration with E. Bergshoeff, U. Gran, M. Nielsen, and D. Roest, entitled *Transient quintessence from group manifold reductions or how all roads lead to Rome* [95]. It is organized as follows: in sections 5.2-5.4 we present, under the assumptions stated, the most general N-scalar accelerating cosmology in 4 dimensions. The accelerating phases of these cosmologies are discussed in section 5.5. Their equations of state and the one-scalar truncations are discussed in sections 5.6 and 5.7, respectively.

5.2 Setup: Lagrangian and Ansatz

Our starting point is gravity coupled to N scalars [122] which we denote by $(\varphi, \vec{\phi})$. We assume that the scalar potential consists of a single exponential term:

$$\mathcal{L} = \sqrt{-g}\left[R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\vec{\phi})^2 - V(\varphi, \vec{\phi})\right], \qquad V(\varphi, \vec{\phi}) = \Lambda\, \exp(-\alpha\varphi - \vec{\beta}\cdot\vec{\phi})\,, \tag{5.1}$$

where we restrict[1] to $\Lambda > 0$. To characterize the potential we introduce the following parameter:

$$\Delta \equiv \alpha^2 + |\vec{\beta}|^2 - \frac{2\,(D-1)}{(D-2)} = \alpha^2 + |\vec{\beta}|^2 - 3 \quad \text{for} \quad D = 4. \tag{5.2}$$

This parameter, first introduced in [52], is invariant under toroidal reductions.

The kinetic terms of the dilatons are invariant under $SO(N)$-rotations of $(\varphi, \vec{\phi})$. However, in the scalar potential the coefficients α and $\vec{\beta}$ single out one direction in N-dimensional space. Therefore the Lagrangian (5.1) is only invariant under $SO(N-1)$. The remaining generators of $SO(N)$ can be used to set $\vec{\beta} = 0$, in which case only the scalar φ appears in the scalar potential. Such a choice of basis leaves Δ invariant.

Motivated by observational evidence, we choose a flat FLRW Ansatz. This basically means a spatially flat metric that can only contain time-dependent functions. One can always perform a reparametrization of time to bring the metric to the following form:

$$ds^2 = -a(u)^{2\delta}\, du^2 + a(u)^2\, dx_3^2\,, \tag{5.3}$$

for some δ. In this paper we will choose δ as follows[2]:

$$\text{Cosmic time:} \qquad \delta = 0\,, \qquad u = \tau\,, \quad \frac{da}{d\tau} = \dot{a}\,, \tag{5.4}$$

$$\text{Non-cosmic time:} \quad \delta = 3\,, \qquad u = t\,, \quad \frac{da}{dt} = a'\,. \tag{5.5}$$

As a part of the Ansatz, we also assume:

$$\varphi = \varphi(u)\,, \qquad \vec{\phi} = \vec{\phi}(u)\,. \tag{5.6}$$

For this Ansatz one can reduce the $N-1$ scalars $\vec{\phi}$ that do not appear in the potential to one scalar by using their field equations as follows:

$$\frac{d^2\vec{\phi}}{du^2} = (\delta - 3)\,\frac{d\,\log a}{du}\,\frac{d\vec{\phi}}{du} \qquad \Rightarrow \qquad \frac{d\vec{\phi}}{du} = \vec{c}\,a^{\delta-3}\,, \tag{5.7}$$

where $\vec{c}$ is some constant vector. The only influence of the $N-1$ scalars comes from their total kinetic term:

$$\left|\frac{d\vec{\phi}}{du}\right|^2 = |\vec{c}|^2\, a^{2\delta-6}\,. \tag{5.8}$$

[1] We make this choice in order to obtain dark energy and therefore accelerating solutions.

[2] The non-cosmic time corresponds to the gauge in which the lapse function $N \equiv \sqrt{-g_{tt}}$ is equal to the square root of the determinant γ of the spatial metric, i.e. $N = \sqrt{\gamma}$, whereas cosmic time corresponds to $N = 1$. We thank Marc Henneaux for a discussion on this point.

Therefore, from the metric point of view, there is no difference between $N = 2$ and $N > 2$ scalars (under the restriction of a single exponential potential). The truncation of the system (5.1) to one scalar corresponds to setting $\vec{c} = 0$.

To summarize, we will be using the following Lagrangian:

$$\mathcal{L} = \sqrt{-g}\left[R - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}(\partial\phi)^2 - V(\varphi)\right], \qquad V(\varphi) = \Lambda\ \exp(-\alpha\varphi)\,, \tag{5.9}$$

with $\Lambda > 0$ and we choose the convention $\alpha \geq 0$. From now on we will use $\Delta = \alpha^2 - 3$ instead of α.

In the next two subsections we will first discuss the critical points corresponding to the system (5.9) and then the solutions that interpolate between these critical points. We will use cosmic time (5.4) when discussing the critical points in section 5.3 and non-cosmic time (5.5) when dealing with the interpolating solutions in section 5.4.

5.3 Critical points

It is convenient to choose a basis for the fields, such that they parametrize a 2-sphere. In this basis, we will be able to regard our system as an autonomous one, and we will find that all constant configurations (critical points) correspond to power-law solutions for the scale factor $a(\tau) \sim \tau^p$ for some p. By studying the stability of these critical points [104,107] one can deduce that there exist interpolating solutions which tend to these points in the far past or the distant future. We will actually be able to draw these interpolating solutions without having to do any stability analysis.

We begin by choosing the flat FLRW Ansatz (5.3) in cosmic time:

$$ds^2 = -d\tau^2 + a(\tau)^2\,(dx^2 + dy^2 + dz^2)\,. \tag{5.10}$$

The Einstein equations for the system (5.9) with this Ansatz become:

$$H^2 = \tfrac{1}{12}\,(\dot\varphi^2 + \dot\phi^2) + \tfrac{1}{6}\,V\,, \tag{5.11}$$

$$\dot H = -\tfrac{1}{4}\,(\dot\varphi^2 + \dot\phi^2)\,, \tag{5.12}$$

where $H \equiv \dot a/a$ is the Hubble parameter and the dot denotes differentiation w.r.t. τ. Equations (5.11) and (5.12) are usually referred to as the Friedmann equation and the acceleration equation, respectively. The scalar equations are:

$$\ddot\varphi = -3\,H\,\dot\varphi + \sqrt{\Delta + 3}\ V\,, \qquad \ddot\phi = -3\,H\,\dot\phi\,. \tag{5.13}$$

We define the following three variables:

$$x = \frac{\dot\varphi}{\sqrt{12}\,H}\,, \qquad y = \frac{\dot\phi}{\sqrt{12}\,H}\,, \qquad z = \frac{\sqrt{V}}{\sqrt{6}\,H}\,. \tag{5.14}$$

In these variables the Friedmann equation (5.11) becomes the defining equation of a 2-sphere [107, 123]:

$$x^2 + y^2 + z^2 = 1\,. \tag{5.15}$$

This means that we can think of solutions as points or trajectories on a globe. It turns out that cosmological solutions are either eternally expanding (i.e. $H > 0$) or eternally contracting ($H < 0$), but cannot have an expanding phase and then a contracting phase (or vice-versa). Since we are only interested in expanding universes, we will only be concerned with the upper hemisphere (i.e. $z > 0$). In terms of x and y the scalar equations become:

$$\frac{\dot{x}}{H} = -3\,z^2\,(x - \sqrt{1 + \Delta/3})\,, \tag{5.16}$$

$$\frac{\dot{y}}{H} = -3\,z^2\,y\,. \tag{5.17}$$

We can rewrite the acceleration equation (5.12) as follows:

$$\frac{\dot{H}}{H^2} = -3\,(x^2 + y^2)\,. \tag{5.18}$$

If we now solve for the critical points ($\dot{x} = 0, \dot{y} = 0$), we can then integrate (5.18) twice and obtain the following power-law solutions for $a(\tau)$ [122]:

$$a(\tau) \sim \tau^p\,, \quad \text{where} \quad p = \frac{1}{3\,(x_c^2 + y_c^2)}\,, \tag{5.19}$$

and the following solutions for the scalars:

$$\varphi = \sqrt{12}\,p\,x_c\,\log(\tau) + \text{constant}\,. \tag{5.20}$$

We thus find the following critical points:

- **Equator**:

$$z = 0, \quad x^2 + y^2 = 1. \tag{5.21}$$

  Every point on the equator of the sphere is a critical point with power-law behaviour $a \sim \tau^{1/3}$.

- **“Rome"**:

$$x = \sqrt{1 + \Delta/3}\,, \quad y = 0\,, \quad z = \sqrt{-\Delta/3}\,. \tag{5.22}$$

  This critical point yields a power-law behaviour of the form (we ignore here irrelevant constants that rescale time)

$$a \sim \tau^{1/(\Delta+3)} \;\text{ for }\; -3 < \Delta < 0\,, \qquad a \sim e^{\tau} \;\text{ for }\; \Delta = -3. \tag{5.23}$$

  Note that the greater Δ is, the further “Rome" gets pushed towards the equator, and for $\Delta = 0$ it is on the equator.

Although the equatorial points (a.k.a. kinetic-dominated solutions) do solve (5.15)-(5.18) as critical points, they are not proper solutions of (5.11)-(5.13) in terms of the fundamental fields, since $z = 0$ would imply that $V = 0$, which is impossible for $\Lambda \neq 0$ unless φ is infinite at all times. However, these points will be interesting to us, as they will provide information about the asymptotics of the interpolating solutions.

In contrast to the equator, the "Rome" critical point is a physically acceptable solution of the system, provided it is well defined on the globe (i.e. $\Delta < 0$). In the case where $\Delta = -3$ it becomes De Sitter (i.e. $a \sim e^{\tau}$), as one would expect, since $V = \Lambda$.

Besides these critical points there are other solutions, which are not points but rather trajectories. In fact, we can already determine their shapes. Dividing (5.16) and (5.17) we obtain the following:

$$\frac{dy}{dx} = \frac{y}{x - \sqrt{1 + \Delta/3}}\,. \tag{5.24}$$

Integrating this we get the following relation between x and y:

$$y = C\,(x - \sqrt{1 + \Delta/3})\,, \tag{5.25}$$

where C is an arbitrary constant[3]. This relation tells us that if we project the upper hemisphere onto the equatorial plane, in other words, if we view the sphere from above, any solution to the equations of motion must trace out a straight line that lies within the circle defined by $x^2 + y^2 = 1$ and has a y-intercept at ($x = \sqrt{1 + \Delta/3}, y = 0$). From now on, we will refer to that point as "Rome"[4]. Notice that all lines intersect at "Rome" independently of whether it is on the globe ($\Delta < 0$), right on the equator ($\Delta = 0$) or off the globe ($\Delta > 0$). These lines can only have critical points as end-points. So each line is a solution, which interpolates between two power-law solutions. In a similar, yet physically inequivalent context, such a line was found in [105].

Now that we know the shapes of the trajectories, let us figure out their time-orientations. By looking at (5.16) we realize that the time derivative of x is positive when $x < \sqrt{1 + \Delta/3}$ and negative when $x > \sqrt{1 + \Delta/3}$. This tells us that *all roads lead to Rome*. Figure 5.2 illustrates this for the cases where "Rome" is off the globe, right on the equator or on the globe.

One can also determine the orientations of the trajectories by analysing the stability of the critical points. One will find that whenever "Rome" is on the globe (i.e. $\Delta = 0$ and $\Delta < 0$), it is stable (i.e. an attractor), and the points on the equator are all unstable (i.e. repellers), except for "Rome" when $\Delta = 0$. In the case where "Rome" is off the globe (i.e. $\Delta > 0$), the equator splits up into a repelling and an attracting region. The attracting region turns out to be the portion of the equator that can "see" "Rome". In other words, any point on the equator that can be joined to "Rome" by a straight line such that the line does not intersect the equator again before reaching "Rome" is attracting. To summarize, for $\Delta > 0$, all points on the equator with $x > \sqrt{3/(\Delta + 3)}$ are attracting, and the rest are repelling. In the first illustration of figure 5.2, the attracting portion of the equator is depicted by the thick arc.

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[3] Since C is finite one might think that this excludes the line defined by $x = \sqrt{1 + \Delta/3}$. However, that line can be obtained by taking the inverse of (5.24) and solving for x as a function of y.

[4] Note that we have extended our definition of "Rome": only if "Rome" is on the globe ($\Delta < 0$) is it equal to the critical point discussed before.

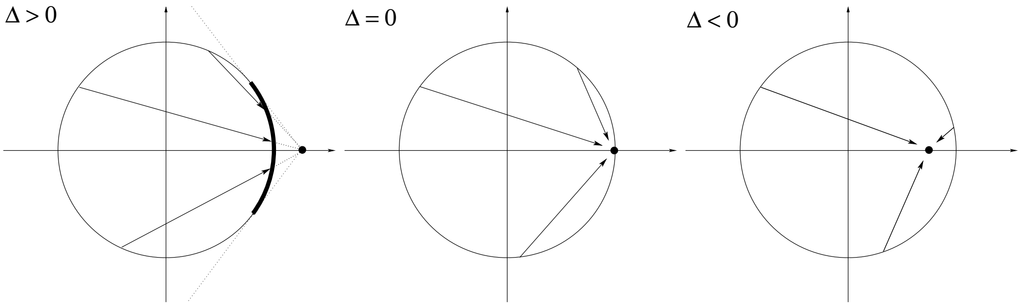

Figure 5.2: *The solutions represented as straight lines in the* (x, y)*-plane for* $\Delta > 0$ *where "Rome" is not on the sphere,* $\Delta = 0$ *where "Rome" is on the equator and* $-3 \leq \Delta < 0$ *where "Rome" is on the sphere. The thick arc in the left figure represents the attracting portion of the equator [124].*

5.4 Interpolating solutions

To solve the equations of motion, it is convenient to use the FLRW Ansatz (5.3) in non-cosmic time:

$$ds^2 = -a(t)^6 \, dt^2 + a(t)^2 \, dx_3^2 \, . \tag{5.26}$$

Substituting this Ansatz in the Einstein equations yields

$$F^2 = \tfrac{1}{3} F' + \tfrac{1}{12} (\phi'^2 + \varphi'^2) \, , \tag{5.27}$$

$$F' = \tfrac{1}{2} V a^6 \, , \tag{5.28}$$

where $F = a'/a$ is a Hubble parameter-like function, and the prime denotes differentiation w.r.t. t. The equations for the scalars are:

$$\phi'' = 0 \, , \qquad \varphi'' = \sqrt{\Delta + 3} \, V a^6 \, . \tag{5.29}$$

Combining (5.29) and (5.28) gives the following solutions for the scalars:

$$\varphi = 2 \sqrt{\Delta + 3} \, \log(a) + a_1 \, t + b_1 \, , \qquad \phi = a_2 \, t + b_2 \, . \tag{5.30}$$

By substituting this into equation (5.25) we can deduce that the slope of the line is given by $C = a_2/a_1$. Substituting the scalars into (5.27) and (5.28) we are now left with the following two equations:

$$F' = -\Delta F^2 - \sqrt{\Delta + 3} \, a_1 F - \tfrac{1}{4} (a_1^2 + a_2^2) \tag{5.31}$$

$$= \tfrac{1}{2} \Lambda \, e^{-\sqrt{\Delta+3} \, (b_1 + a_1 t)} \, a^{-2\Delta} \, . \tag{5.32}$$

Keeping in mind that F' must be positive due to (5.32) we can now solve for F in the three different cases where Δ is positive, zero and negative. We can then easily find $a(t)$. We will choose b_1 (the constant part of φ) such that all solutions for $a(t)$ have a proportionality constant of 1, which does not affect the cosmological properties of the solutions. The integration constants appearing in the solutions are defined as follows:

$$c_1 = \frac{-\sqrt{\Delta+3}\,a_1}{2\,\Delta}\,, \quad c_2 = \frac{\sqrt{3\,a_1^2 - \Delta\,a_2^2}}{2}\,, \quad d_1 = -\frac{a_1^2 + a_2^2}{4\,\sqrt{3}\,a_1}\,, \quad d_2 = -\sqrt{3}\,a_1\,. \tag{5.33}$$

Below we present the solutions [111, 112] and their late- and early-time asymptotic behaviours (we give the latter without any irrelevant constants that rescale time):

1. $\underline{\Delta > 0}$:

$$a(t) = e^{c_1\,t}\,\cosh\left(c_2\,t\right)^{1/\Delta}\,, \qquad \text{for} \quad -\infty < t < +\infty\,. \tag{5.34}$$

The positivity of F' requires a_1 to be negative, and it also imposes the following constraint:

$$\left(\frac{a_2}{a_1}\right)^2 < \frac{3}{\Delta}\,. \tag{5.35}$$

This solution corresponds to a generic line on the first illustration in figure 5.2. It starts on the equator somewhere to the left of $x = \sqrt{3/(\Delta+3)}$, then moves in the direction of "Rome", but ends on the equator on the right-hand side. Note that the constraint (5.35) is simply the requirement that the slope of the line is bounded from above and from below such that the line actually intersects the sphere. We can confirm this asymptotic behaviour of the solution by converting to cosmic time (5.4) for $t \to -\infty$ and $t \to +\infty$ with the relation $a(t)^3\,dt = d\tau$:

$$\begin{aligned} &t \to -\infty\,, && \tau \to 0\,, && a \to e^t \sim \tau^{1/3}\,, \\ &t \to +\infty\,, && \tau \to +\infty\,, && a \to e^t \sim \tau^{1/3}\,. \end{aligned} \tag{5.36}$$

2. $\underline{\Delta = 0}$:

$$a(t) = e^{d_1\,t}\,\exp\left(e^{d_2\,t}\right)\,, \qquad \text{for} \quad -\infty < t < +\infty\,. \tag{5.37}$$

The positivity of F' requires a_1 to be negative. This corresponds to a line on the second illustration in figure 5.2. It starts on the equator and reaches "Rome"[5], which is also on the equator. Its asymptotic behaviour goes as follows:

$$\begin{aligned} &t \to -\infty\,, && \tau \to 0\,, && a \to e^t \sim \tau^{1/3}\,, \\ &t \to +\infty\,, && \tau \to +\infty\,, && a \to e^{e^t} \sim \tau^{1/3}\,. \end{aligned} \tag{5.38}$$

To find the late-time behaviour of a in cosmic time one must realize the following two facts: First, $a(t) \sim \exp(e^t)$ for $t \to \infty$. Second, in this limit, $a' \sim a$ and therefore a behaves like a normal exponential.

[5]In this case, "Rome" is again attracting, however to see that, one must perform the stability analysis by going to second order perturbation. The first order vanishes, which means that the interpolating trajectory approaches "Rome" more slowly than in the cases where $\Delta < 0$.

3. $\underline{-3 \leq \Delta < 0}$:

$$a(t) = e^{c_1 t} \sinh(-c_2 t)^{1/\Delta}, \qquad \text{for} \quad -\infty < t < 0. \tag{5.39}$$

This solution corresponds to any line on the third illustration in figure 5.2. It starts at any point on the equator and ends at “Rome". This is reflected in the asymptotics as follows:

$$\begin{aligned} t &\to -\infty, & \tau &\to 0, & a &\to e^t \sim \tau^{1/3}, \\ t &\to 0, & \tau &\to +\infty, & a &\to (-t)^{1/\Delta} \sim \tau^{1/(\Delta+3)} \quad \text{for} \quad \Delta > -3, \\ & & & & &\qquad\qquad \sim e^{\tau} \quad \text{for} \quad \Delta = -3. \end{aligned} \tag{5.40}$$

There is one more solution for $-3 \leq \Delta < 0$. If we set $a_1 = a_2 = 0$ we find:

$$a(t) = (-t)^{1/\Delta} \qquad \text{for} \quad -\infty < t < 0. \tag{5.41}$$

This solution corresponds to the “Rome" solution itself. For $-3 < \Delta < 0$ the conversion to cosmic time is the following:

$$a \sim \tau^{1/(\Delta+3)}. \tag{5.42}$$

Notice, however, that in the case where $\Delta = -3$, the “Rome" solution (5.41) and therefore the late-time asymptotics of (5.39) have a different conversion to cosmic time, namely:

$$a \sim (-t)^{1/\Delta} \sim e^{\tau}, \tag{5.43}$$

which we recognize as the De Sitter solution, in agreement with the fact that we have $V = \Lambda$.

The interpolating solutions above are given in non-cosmic time, which as mentioned is related to cosmic time by

$$d\tau = a(t)^3\, dt. \tag{5.44}$$

Integrating this equation yields hypergeometric functions for a generic interpolating solution, which we cannot invert to get the scale factor as a function of cosmic time. However, it is possible to get interpolating solutions in cosmic time for negative Δ when the following constraint on the constants holds:

$$\left(\frac{a_2}{a_1}\right)^2 = 12\, \frac{\Delta + \frac{9}{4}}{(2\Delta+3)^2}, \tag{5.45}$$

which can only be fulfilled for $-9/4 \leq \Delta < 0$. The relation between the two time coordinates is

$$\tau = \frac{2^{-3/\Delta}}{2c_2}\, \frac{\Delta}{3+\Delta}\, (e^{2c_2 t} - 1)^{(3+\Delta)/\Delta}, \tag{5.46}$$

and the scale factor in cosmic time becomes

$$a(\tau) = \left(k_1\, \tau^{3/(3+\Delta)} + k_2\, \tau\right)^{1/3}, \tag{5.47}$$

where $k_1 = (2/c_1)^{3/\Delta}$ and $k_2 = k_1\, c_1\, (2\Delta+3)/(18+6\Delta)$. From this solution, the asymptotic power-law behaviours are easily seen. The special one-scalar case, corresponding to $\Delta = -9/4$, was found in [125].

5.5 Acceleration

In this section we will investigate under which conditions “Rome" and the interpolating solutions represent an accelerating universe. This can be given a nice pictorial understanding in terms of the 2-sphere. We will show that acceleration takes place when the trajectory enters the region bounded by an “arctic circle". This is summarized in figure 5.3.

An accelerating universe is defined by $\ddot{a}/a > 0$. The existence of the “arctic circle" in connection to acceleration can now easily be determined. Assuming an expanding universe and using

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 \,, \tag{5.48}$$

as well as (5.18), we see that the condition for acceleration is equivalent to[6]

$$z^2 > \frac{2}{3} \,, \quad \text{i.e.} \quad x^2 + y^2 < \frac{1}{3} \,, \tag{5.49}$$

which exactly yields an “arctic circle" as the boundary of the region of acceleration. The straight line representing the exact solution is parametrized by the constants a_1 and a_2 as found in the previous section. From (5.49) and (5.25) it then easily follows that the condition for acceleration leads to the following restriction for the slope of the line:

$$\Big(\frac{a_2}{a_1}\Big)^2 (2 + \Delta) < 1 \,. \tag{5.50}$$

This condition is always fulfilled when $\Delta \leq -2$ and otherwise there is an interval of values for a_2^2/a_1^2 yielding an accelerating universe. This can easily be understood from figure 5.3. In general, a solution will only have transient acceleration. The only exception is when “Rome" lies within or on the “arctic circle", corresponding to $\Delta \leq -2$. Then, from the moment the line crosses the “arctic circle", there will be eternal acceleration [89] towards “Rome". When $\Delta = -2$, there will only be eternal acceleration when “Rome" is approached from the left. The possibilities of acceleration can be summarized as:

- $\Delta > -2$: A phase of transient acceleration is possible ,
- $\Delta = -2$: A phase of eternal acceleration is possible ,
- $-3 \leq \Delta < -2$: Always a phase of eternal acceleration .

The phase of eternal acceleration can also be understood from the power-law behaviour of the “Rome" solution, i.e. $a(\tau) \propto \tau^{1/(3+\Delta)}$. We have asymptotic acceleration when $1/(3 + \Delta) > 1$, i.e. $\Delta < -2$. In the limiting case $\Delta = -3$, corresponding to “Rome" being on the North Pole, the interpolating solution will asymptote to De Sitter.

[6] A similar inequality was given in [105] for the one-scalar case, and in terms of the scalars and the potential in [112] for the multi-scalar case.

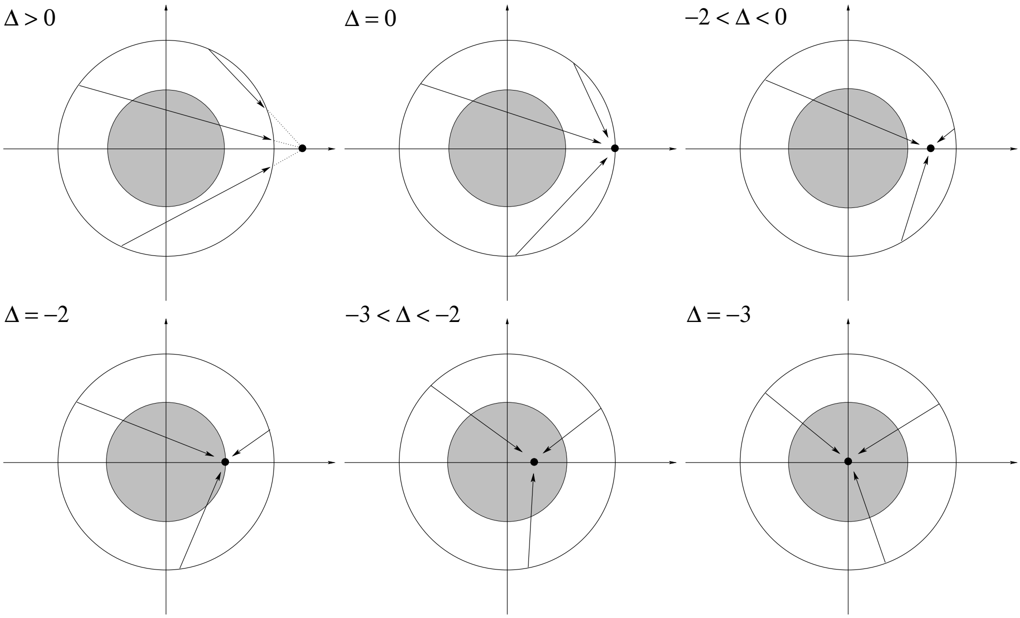

Figure 5.3: *The solutions represented as straight lines in the* (x, y)*-plane for* $\Delta > 0$ *where "Rome" is not on the sphere,* $\Delta = 0$ *where "Rome" is on the equator and* $-3 \leq \Delta < 0$ *where "Rome" is on the sphere. The inner circle corresponds to the "arctic circle", and solutions are accelerating when they enter the shaded area. The lower part of the figure corresponds to the cases where "Rome" is lying on the "arctic circle",* $\Delta = -2$*, inside the "arctic circle",* $-3 < \Delta < -2$ *and on the North Pole,* $\Delta = -3$*.*

5.6 Equation of state

In a cosmological setting, one often writes the matter part of the equations in terms of a perfect fluid, which is described by its pressure p and energy density ρ. These two variables are then assumed to be related via the equation of state:

$$p = \kappa \rho \,. \tag{5.51}$$

As is well known in standard cosmology, $\kappa = 0$ corresponds to the matter dominated era, $\kappa = 1/3$ to the radiation dominated era and $\kappa = -1$ to an era dominated by a pure cosmological constant. Quintessence is a generalization of the latter with $-1 \leq \kappa < -1/3$.

In our case, the matter is given by the two scalar fields, and thus p and ρ are given by the difference and sum of the kinetic terms and the potential, respectively:

$$p = \tfrac{1}{2}(\dot{\varphi}^2 + \dot{\phi}^2) - V \,, \quad \rho = \tfrac{1}{2}(\dot{\varphi}^2 + \dot{\phi}^2) + V \,. \tag{5.52}$$

Writing the above in terms of x, y and z, we see that the scalars describe a perfect fluid with an equation of state given in terms of the parameter:

$$\kappa = 1 - 2z^2 \,. \tag{5.53}$$

Hence, κ varies from 1 on the equator to -1 on the North Pole, and we need $\kappa < -1/3$ for quintessence. For the interpolating solutions, which are given as curves on the sphere, κ will depend on time, but it will be constant for the critical points with the following values [89]:

- Equator : $\kappa = 1$,
- "Rome" : $\kappa = 1 + \frac{2}{3}\Delta$. (5.54)

5.7 One-scalar truncations

The analysis has so far been done for two scalars, and as such it also contains the truncation to a system with one scalar with a potential, corresponding to $\phi = 0$. Here we will summarize the results of the previous sections in this truncation. On the sphere this yields $y = 0$, and for the solutions it corresponds to $a_2 = b_2 = 0$.

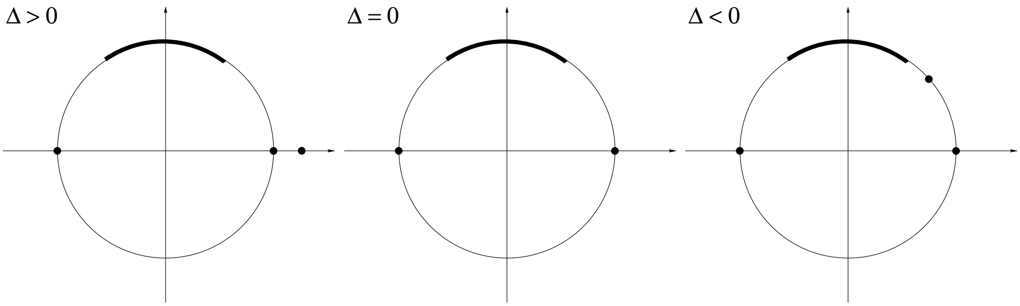

Figure 5.4: *The 2-dimensional (x, z) space and the critical points for the one-scalar truncations. The thick curve is the accelerating region. The two points on the x-axis are the equatorial critical points. The third point is "Rome". Note that in the middle illustration "Rome" coincides with the equatorial critical point $x = 1$.*

Since we only have one scalar, the Friedmann equation will define a circle when written in terms of x and z [105, 109]:

$$x^2 + z^2 = 1\,. \tag{5.55}$$

The critical points are [104]:

- Equatorial : $z = 0$, $x^2 = 1$,
- "Rome" $(-3 \leq \Delta < 0)$: $z = \sqrt{-\Delta/3}$, $x = \sqrt{1 + \Delta/3}$. (5.56)

The full circle is shown in figure 5.4, including the critical points, and is just the vertical slice of the two-sphere including the North Pole. The equator therefore becomes two points, and the region bounded by the "arctic circle" now becomes the part of the circle[7] corresponding to $x^2 < 1/3$.

[7] This is equivalent to the accelerating region of [126, 127].

The exact solutions can be obtained from the previous section by setting $a_2 = b_2 = 0$. For $\Delta \geq 0$ the solutions correspond to the curves starting at $x = -1$ and ending at $x = 1$, whereas for $-3 \leq \Delta < 0$ the curves start at either one of the equatorial points and end at “Rome". In all cases where the curve starts at $x = -1$, the corresponding solution will give rise to acceleration. For this reason, interpolating solutions with $\Delta \geq 0$ will always give rise to a period of acceleration. This is in clear contrast to the two-scalar case, where it is possible to avoid acceleration (see figure 5.3). As for the 2-scalar case, if “Rome" lies in the “arctic" region, the solution will be eternally accelerating from the moment it enters this region.

One can also consider the truncation to zero scalars. However, from the scalar field equations, it is seen that this is only consistent if $\Delta = -3$, and this corresponds to the De Sitter solution with $V = \Lambda$.

A comment on the relevance of the interpolating solutions to inflation would be in order. In this context the number of e-foldings is crucial. As mentioned already, it is defined by $N_e = \log(a(\tau_2)/a(\tau_1))$ with τ_1 and τ_2 the start and end times of the accelerating period. These times can easily be found in our approach as the points where the straight lines intersect the “arctic circle". The number of e-foldings is required to be of the order of 65 to account for astronomical data. For the interpolating solutions with $\Delta > -2$, which is a necessary requirement to have a finite period of acceleration, one finds $N_e \lesssim 1$ [93,97,120] for all values of a_1, a_2 and Λ. The only exception to this behaviour is when $\Delta \to -2$, where N_e blows up. For the required 65 e-foldings one needs to take $\Delta + 2 \sim 10^{-60}$. As an example, for a compactification over an m-dimensional hyperbolic space, leading to $\Delta = -2 + 2/m$, this translates into $m \sim 10^{60}$. Thus, it seems that the e-foldings requirement for inflation cannot be met by a single exponential potential emanating from a dimensional reduction from the effective action of string/M-theory. Such a potential may, however, be relevant for describing present day acceleration. This does not exclude, however, that potential with Δ close enough to 2 for inflation might arise in a string theory scenario that takes other string theory effects into account, such as in [33].

In this chapter we introduced the scalar-gravity model in the FLRW context. We also introduced the language of autonomous systems and their fruitful application to cosmology. We learned that it is not necessary to find explicit solutions to the Einstein equations in order to get important qualitative information about our system. Although we were fortunate enough to write down the solutions explicitly, just by reasoning in terms of critical points and stability, we realized that power-law and de Sitter solutions are not the only kind of cosmology. We found solutions that interpolate between those two basic cases, some of which showed periods of transient acceleration. Transient acceleration is phenomenologically more interesting because a realistic model of cosmology should dynamically bring inflation to an end. It is also useful to consider scenarios with transient acceleration simply because we do not know whether present day acceleration will last forever.

So far, we have specialized in the case where the scalar potential consists of one exponential term. This lead to the huge simplification of being able to redefine our fields such that only one scalar appears in the exponent, no matter how many scalars were present in it to begin with. This was a particularly simple prelude to what we are about to do in the next chapter, where we will deal with the *most* general multi-exponential potential. We will rely entirely on the language of dynamical systems, as explicit interpolating solutions will become virtually impossible to find. By looking for critical points in such systems we will discover that intricate multi-exponential potentials do not simply accumulate the effects of a single exponential potential, but actually lead new de Sitter solutions that had not been seen before.