Instantons and cosmologies in string theory

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Chapter 3

Non-extremal D-instantons

3.1 Introduction

In the previous chapter, we studied instantons in quantum mechanics and quantum field theory. In this chapter we will be looking at instantons in gravitational theories. Instantons, as we have seen, are inherently linked to path integrals. However, a path integral formulation of quantum gravity is not as straight forward as one might wish. In an ideal world, we would simply write down the following:

$$\langle\, h_F \,|\, e^{-HT} \,|\, h_I \rangle = \int d[g] \exp\left(-\int d^D x\, R\right) , \tag{3.1}$$

where $h_{I,F}$ are the induced metrics on the initial and final spacelike hypersurfaces of spacetime, respectively, R is the Ricci scalar, and the path integral sums over all metrics satisfying the boundary condition that they asymptote to $h_{I,F}$ in the early past and late future, respectively. However, this path integral is not well-defined because the action is not bounded from below. In fact, even flat Euclidean space is not a minimum of the Einstein-Hilbert action. Suppose we wanted to perform a semiclassical approximation around the Wick rotated Minkowski spacetime, i.e. flat Euclidean space. There are infinitely many possible fluctuations around the flat metric, but let us restrict to summing over metrics that are related to flat space via a Weyl transformation; i.e. *conformally flat* metrics:

$$\tilde{g} = e^{2\sigma}\, \eta\, , \tag{3.2}$$

where η is the flat metric. Then, the action for $\tilde{g}$ will roughly go as follows:

$$\int d^D x R \sim -\int d^D x\, (\partial\sigma)^2 \ , \tag{3.3}$$

which means that the action can be made arbitrarily negative by quantum fluctuations, making flat spacetime a local maximum (or at best a saddle point), and making the whole path integral divergent. Fixing this problem requires a new formalism, which is developed in [23], but is not yet widely agreed upon. The idea is to first sum over conformal classes of metrics, and then, within each class of conformally related metrics, one rotates the contour of integration to imaginary conformal factors. In (3.3) this manifests itself in that only imaginary σ are allowed, thus keeping the action positive. We will not really be using any of this formalism in this thesis. The purpose of this paragraph was to show how severely different path integration becomes when dealing with gravity.

Despite difficulties with path integrals, gravitational instantons do exist and have been applied to many different problems in quantum gravity such as the renormalization of the constants of nature, the adjustment of the cosmological constant, spacetime topology fluctuations, and the creation of baby universes (see [24–28]).

In the field theory limit of string theory, instantons can give rise to non-perturbative effects (for an overview see [29]). The standard *D-instanton* is an instanton solution of type IIB supergravity, which was discovered in [30], and was later shown to give higher derivative correction terms, specifically R^4 terms, to the effective action of type IIB string theory [31]. The coefficient of such terms was conjectured to be an $\mathrm{SL}(2,\mathbb{Z})$ invariant modular function. In [32], the high-energy limit of this conjecture was tested. Other instantons have been obtained through dimensional reductions of supergravity by wrapping Euclidean D-branes around compact cycles of the internal space. This yields non-perturbative effects, which give rise to interesting lower-dimensional effective actions that have applications in cosmology [33].

The standard D-instanton is a solution of a truncation of type IIB supergravity with the metric, the dilaton, and the RR scalar known as *axion* as its field content. The solution has a flat Euclidean metric, preserves $1/2$ of the supersymmetry of the theory, and is characterized by the axion 'charge'[1]. The fact that it is 'charged' under a 0-form potential makes the D-instanton mathematically similar to p-branes. In this case it, could be thought of as a (-1)-brane, meaning it is localized in space *and* time. In this chapter, we will be studying solutions that generalize the standard D-instanton in many ways: their metrics will be non-trivial, and they will not preserve any supersymmetry. The solutions that will be presented are not new, but will be studied in a novel way. For earlier work on generalized D-instanton solutions see [25, 28, 34–41]

In this chapter, we will generalize the Lagrangian of type IIB supergravity to arbitrary dimensions, and arbitrary dilaton coupling. However, one important property of type IIB supergravity will be preserved: the scalars (dilaton and axion) are coupled in such a way that they parametrize an $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}(1,1)$ coset space. By conveniently reorganizing the fields into 2×2 matrices, the $\mathrm{SL}(2,\mathbb{R})$ symmetry will become manifest, and we will see that solutions to the field equations will have a 'conserved' *charge matrix* Q, as implied by Noether's theorem. This charge matrix Q transforms under the adjoint representation of $\mathrm{SL}(2,\mathbb{R})$, which means that its determinant is invariant under the symmetry. This implies that there are three families of solutions that are not related via $\mathrm{SL}(2,\mathbb{R})$, i.e. those with $\det Q > 0, = 0$ and < 0. This is analogous to the fact that Minkowski spacetime admits three families of vectors: Timelike, lightlike, and spacelike. In this chapter, we will see that all D-instanton solutions can be classified into three classes, whereby the standard D-instanton falls under the $\det Q = 0$ class.

A similar discovery was made in [42], where three classes of $\mathrm{SL}(2,\mathbb{R})$-unrelated seven-branes were found. Seven-branes can be seen as the magnetic duals of D-instantons. They are carried by the same fields; however, instead of being *electrically* charged under the axion, they

[1] We will give this 'charge' a physical interpretation later on.

are *magnetically* charged under it. This means that, in contrast to the D-instantons, seven-branes are not localized in spacetime. Given that seven-branes were shown to occupy all three possible *conjugacy* classes of $SL(2,\mathbb{R})$, it is natural to ask whether D-instantons do the same.

At the end of chapter 2 we saw that instantons in D Euclidean dimensions can sometimes be viewed as the spacelike sections of solitons in $D+1$ spacetime dimensions. In this chapter, we will show that the three $SL(2,\mathbb{R})$ classes of D-instantons can sometimes be seen as spacelike sections of electrically charged black holes, i.e. Reissner-Nordström black holes. As we will see, the three families of solutions, $\det Q > 0, = 0, < 0$, correspond to underextremal, extremal, and overextremal black holes (i.e. black holes with electric charges lower than, equal to, and greater than their masses). The condition for such a correspondence to hold will be worked out, and the correspondence will be extended to *uplift* the D-instantons to p-branes in higher dimensions.

This chapter is based on a collaboration with E. Bergshoeff, U. Gran, D. Roest, and S. Vandoren, entitled *Non-extremal D-instantons* [43]. It is organized as follows: in section 3.2, we will present the metric-scalar system we are interested in and discuss the realization of the $SL(2,\mathbb{R})$-duality group for the Euclidean case. In section 3.3, we will give the generalized instanton solutions mentioned above. At this point we only construct the bulk solutions without taking care of boundary terms and/or boundary conditions. Next, in section 3.4, we will discuss the relation to wormholes corresponding to non-extremal Reissner-Nordström black holes one dimension higher. In section 3.5 we will consider generalizations that uplift to non-extremal p-branes in $D+p+1$ dimensions. The application as true instantons of type IIB string theory will be investigated in section 3.6. Finally, we will discuss our results in section 3.7.

3.2 The system and its symmetries

3.2.1 Lagrangian

The system we will be interested in is described by the following Minkowskian Lagrangian density:

$$\mathcal{L}_M = \tfrac{1}{2}\sqrt{|g|}\,[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{b\phi}(\partial\chi)^2]\,, \tag{3.4}$$

where ϕ and χ are scalars. We will work in D arbitrary dimensions, and will keep the coupling b unspecified. This theory occurs, for example, as the scalar section of IIB supergravity in $D=10$ Minkowski spacetime with coupling parameter $b=2$. In this case, the scalar ϕ corresponds to the string theory *dilaton*, and the scalar χ is the Ramond-Ramond scalar known as the *axion*. Other values of b can arise when considering (truncations of) compactifications of IIB supergravity. For instance, in $D=3$ one has supersymmetry for $b=2, b=\sqrt{2}, b=\sqrt{4/3}$ and $b=1$. In order to study instanton solutions of this system we not only need to Wick rotate the theory, but we also need to change the sign of the axion kinetic term, yielding the following Euclidean Lagrangian:

$$\mathcal{L}_E = \tfrac{1}{2}\sqrt{|g|}\,[R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}e^{b\phi}(\partial\chi)^2]\,, \tag{3.5}$$

The effect of the Wick rotation on the scalar is a very subtle issue, which I will further develop in section 3.6. I will now summarize the three basic arguments to justify the sign change in the kinetic term:

- In the context of type IIB supergravity the axion is considered a *pseudo*scalar. In that case one could claim that the Wick rotation is the 'square root of time reversal', and hence a pseudoscalar should get multiplied by an 'i' upon transforming. This argument, however, is neither rigorous, nor widely agreed upon. Since we want to study D-instanton solutions in theories with arbitrary D and b that are not necessarily imbeddable in supergravity, we will not endorse this claim.

- A theory with a scalar is *dual* to a theory with a $(D-1)$-form field strength. Dual means that there exists a procedure to show that the path integrals of the two theories are equivalent. This procedure allows one to move back and forth from the one path integral to the other. In our case, the Lagrangian of the dual theory is the following:

$$\mathcal{L} = \tfrac{1}{2}\sqrt{|g|}\,[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot(D-1)!}e^{-b\phi}F_{D-1}^2]\,, \tag{3.6}$$

where F_{D-1} is a $(D-1)$-form field-strength. Contrary to common belief, the quantum mechanical dualization *starting from* the $(D-1)$-form theory *does not* yield a scalar theory with the wrong kinetic term sign, but a scalar theory with the normal sign. However, one quickly notices that the Euclidean scalar theory does not have any non-trivial *real* saddle points, so instead of performing the semiclassical approximation on the scalar theory, one does it on the dual $(D-1)$-form theory, which does have non-trivial real saddle points. After writing down the classical Euclidean equations of motion to do the semiclassical approximation one notices that, if one rewrites the $(D-1)$-form field-strength as the *Hodge*-dual of a 1-form field-strength as follows:

$$F_{D-1} = -e^{b\phi} * d\chi\,, \tag{3.7}$$

then the Euclidean equations of motion of the $(D-1)$-form look like the equations of motion of a would-be scalar theory with the wrong sign for the kinetic term. In other words, looking for the saddle points of the $(D-1)$-form theory is *effectively* the same as looking for the saddle points of (3.5). I would like to emphasize that quantum mechanical dualization and Hodge dualization are two different things.

- In a quantum field theory, imposing Dirichlet boundary conditions on the field yields a transition amplitude between eigenstates of the field operators. In our case, this means that the path integral is actually computing the following:

$$\langle\,\phi_F,\chi_F\,|\,e^{-HT}\,|\,\phi_I,\chi_I\rangle\ . \tag{3.8}$$

However, one can also compute a transition amplitude between axionic charge-eigenstates by means of Fourier transformation:

$$\begin{aligned}&\langle\,\phi_F,\pi_F\,|\,e^{-HT}\,|\,\phi_I,\pi_I\rangle = \\ &\int d[\chi_I]\,d[\chi_F]\,\exp\left(-i\int_{\Sigma_I}\pi_I\,\chi_I + i\int_{\Sigma_F}\pi_F\,\chi_F\right)\langle\,\phi_F,\chi_F\,|\,e^{-HT}\,|\,\phi_I,\chi_I\rangle\end{aligned} \tag{3.9}$$

where the path integral over $\chi_{I,F}$ runs over functions defined on the initial and final time hypersurfaces Σ_I and Σ_F, respectively; and $\pi_{I,F}$ are the time components of the conjugate momenta of the axion. This theory has no boundary conditions. The path integral (3.9) has no real saddle points. However, it can be computed in the semiclassical approximation; and it can be shown that the result of this path integration can also be obtained by looking for the saddle points of a would-be system with the wrong kinetic term sign (3.5). Effectively, it is as if we were looking for imaginary saddle points of the original system. This argument was first discovered by Lee in [44]. In [45–47] the argument was refined; however, the clearest and simplest explanation, in my view, can be found in [48].

In section 3.6.1, we will further develop the second method in order to evaluate the actions of our solutions, and in appendix A, a toy model will be used to illustrate the phenomenon of the 'wrong' sign in a simpler setting.

3.2.2 SL(2, ℝ)-symmetry

The Lagrangian (3.5) has a manifest SL(2, ℝ) symmetry. In fact, in chapter 7 we will see that the scalar sector parametrizes a two-dimesional hyperboloid with Lorentzian signature; i.e. a dS_2 spacetime. The latter can be viewed as the following coset:

$$\frac{\mathrm{SO}(2.1)}{\mathrm{SO}(1,1)}\,, \tag{3.10}$$

where $\mathrm{SO}(2,1) \cong \mathrm{SL}(2,\mathbb{R})$. In this chapter, we will making the symmetry manifest by writing the Lagangian in a different form. Define the following matrix:

$$\mathcal{M} = e^{b\phi/2}\begin{pmatrix} \frac{1}{4}b^2\chi^2 - e^{-b\phi} & \frac{1}{2}b\chi \\ \frac{1}{2}b\chi & 1 \end{pmatrix}. \tag{3.11}$$

Now we can write (3.5) as follows:

$$\mathcal{L}_E = \frac{1}{2}\sqrt{g}\,[R + b^{-2}\mathrm{Tr}(\partial\mathcal{M}\,\partial\mathcal{M}^{-1})]\,. \tag{3.12}$$

It is clear that this is invariant under the following transformation:

$$\mathcal{M} \to \Omega\,\mathcal{M}\,\Omega^T \quad \text{with} \quad \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,. \tag{3.13}$$

The attentive reader will probably have noticed that any invertible matrix $\Omega \in \mathrm{GL}(2,\mathbb{R})$ will do. However, only elements of SL(2, ℝ) yield a transformed matrix $\mathcal{M}$ that is consistent with the scalar parametrization (3.11) of the coset space[2].

This symmetry, like any continuous symmetry, has Noether current:

$$J_\mu = (\partial_\mu\mathcal{M})\mathcal{M}^{-1} = \begin{pmatrix} j_\mu^{(3)} & j_\mu^{(+)} \\ -j_\mu^{(-)} & -j_\mu^{(3)} \end{pmatrix}, \tag{3.14}$$

[2]Throughout this chapter we assume that $b \neq 0$. Note that for $b = 0$ the Euclidean SL(2, ℝ) symmetry degenerates to an ISO(1, 1) symmetry, and the scalar coset becomes a two-dimensional Minkowski spacetime.

which is a current matrix, with the following components:

$$j_\mu^{(3)} = \tfrac{1}{2}\, e^{b\phi}\partial_\mu(e^{-b\phi} - \tfrac{1}{4}b^2\chi^2)\,, \qquad j_\mu^{(-)} = \tfrac{1}{2}be^{b\phi}\partial_\mu\chi\,,$$
$$j_\mu^{(+)} = -b\chi\, j_\mu^{(3)} + (e^{-b\phi} - \tfrac{1}{4}b^2\chi^2)\, j_\mu^{(-)}\,. \tag{3.15}$$

Althought this is a Euclidean theory, we can still regard this current as giving rise to 'charges' that are 'conserved' with respect to a Euclidean time direction. Throughout this section, we will choose it to be the radial direction. However, for a proper tunneling interpretation of the instantons, we will choose a Cartesian direction in subsection 3.6.2. For a spherical boundary defined by a radial normal unit vector n^μ, the conserved charge matrix is the following:

$$Q = \frac{(2\,(D-1)\,(D-2))^{-1/2}}{b\,\mathcal{V}ol(S^{D-1})} \int_{S^{D-1}} J_\mu n^\mu\,, \tag{3.16}$$

where the S^{D-1} is transverse to the unit vector. Under an SL(2, $\mathbb{R}$) transformation (3.13) the corresponding charge matrix transforms as

$$Q \to \Omega\, Q\, \Omega^{-1}\,. \tag{3.17}$$

Note that the determinant of Q is invariant under SL(2, $\mathbb{R}$). Thus, solutions with different values of det(Q) can never be related via SL(2, $\mathbb{R}$)-transformations. Hence, as discussed in the introduction the cases $\det(Q) = 0, \det(Q) > 0$ and $\det(Q) < 0$ define the three different 'conjugacy classes' of SL(2, $\mathbb{R}$).

3.3 The solutions and their geometries

In this section we will consider solutions to the bulk equations of motion of (3.5). Issues like boundary terms and the value of the action are postponed to section 6, where we will determine which solutions can be considered as instantons.

3.3.1 Solutions

We consider the Euclidean gravity-dilaton-axion system in $D \geq 3$ dimensions given by the Lagrangian (with arbitrary dilaton coupling parameter b)

$$\mathcal{L}_E = \tfrac{1}{2}\sqrt{g}\,[R - \tfrac{1}{2}\,(\partial\phi)^2 + \tfrac{1}{2}\, e^{b\phi}(\partial\chi)^2]\,, \tag{3.18}$$

and search for generalized D-instanton solutions with manifest SO(D) symmetry of the form[3]

$$ds^2 = e^{2\,B(r)}(dr^2 + r^2 d\Omega^2_{D-1})\,, \qquad \phi = \phi(r)\,, \qquad \chi = \chi(r)\,. \tag{3.20}$$

[3] Note that by using reparameterizations of r one can obtain different, but equivalent, forms of the metric in which the SO(D) symmetry is non-manifest, in particular

$$ds^2 = e^{2\,B(r)}(e^{-2\,f(r)}dr^2 + r^2 d\Omega^2_{D-1})\,, \tag{3.19}$$

in analogy to what we will encounter later, see (3.78). We choose to take as our starting point a conformally flat metric, i.e. $f(r) = 0$.

The standard D-instanton solution [30] is obtained for the special case where $B(r)$ is constant. In order to obtain an SO(D) symmetric generalized D-instanton solution, we allow for a non-constant $B(r)$ and solve the field equations following from the Euclidean action (3.18), which read

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\,\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\,e^{b\phi}\partial_\mu\chi\partial_\nu\chi\,, \\ 0 &= \partial_\mu\left(\sqrt{g}g^{\mu\nu}e^{b\phi}\partial_\nu\chi\right), \\ 0 &= \frac{b}{2}\,e^{b\phi}(\partial\chi)^2 + \frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right). \end{aligned} \qquad (3.21)$$

The expression for the Ricci tensor for the Ansatz (3.20) is given by

$$\begin{aligned} R_{rr} &= -(D-1)\left(B''(r) + \frac{B'(r)}{r}\right), \\ R_{\theta\theta} &= -e^{-2\,B(r)}\,g_{\theta\theta}\,[B''(r) + (D-2)\,B'(r)^2 + (2\,D-3)\,\frac{B'(r)}{r}]\,, \end{aligned} \qquad (3.22)$$

where the prime denotes differentiation with respect to r, and θ stands for all angular coordinates. In addition to the SL(2, $\mathbb{R}$) symmetry these field equations are invariant under a constant Weyl rescaling of the metric[4]

$$g_{\mu\nu} \to e^{2\omega}g_{\mu\nu}\,. \qquad (3.23)$$

However, this is only a symmetry of the field equations and not of the action. In our Ansatz (3.20), this has the effect of shifting B by a constant, i.e. $B \to B + \omega$.

In order to solve for $B(r)$, one can consider the angular component of the Einstein equation of (3.21). Having solved for $B(r)$ the expressions for the dilaton and axion scalars can be obtained from the remaining two equations of (3.21). We thus obtain the following solution[5] for $B(r)$, $\phi(r)$ and $\chi(r)$, which extends the solution given in [37] to arbitrary b:

$$\begin{aligned} e^{(D-2)\,B(r)} &= f_+(r)\,f_-(r)\,, \\ e^{b\phi(r)} &= \left(\frac{q_-}{2\,q}\,[e^{C_1}\,(f_+(r)/f_-(r))^{b\,c/2} - e^{-C_1}\,(f_+(r)/f_-(r))^{-b\,c/2}]\right)^2, \\ \chi(r) &= \frac{2}{b\,q_-}\,[q\left(\frac{e^{C_1}\,(f_+(r)/f_-(r))^{b\,c/2} + e^{-C_1}\,(f_+(r)/f_-(r))^{-b\,c/2}}{e^{C_1}\,(f_+(r)/f_-(r))^{b\,c/2} - e^{-C_1}\,(f_+(r)/f_-(r))^{-b\,c/2}}\right) - q_3]\,. \end{aligned} \qquad (3.24)$$

The solution is given in terms of the two flat-space harmonic functions

$$f_\pm(r) = 1 \pm \frac{q}{r^{D-2}} \qquad (3.25)$$

[4] The constant Weyl rescaling symmetry is broken by $O(\alpha')$ corrections.

[5] For practical purposes we omit an overall ± sign corresponding to the $\mathbb{Z}_2$ symmetry of the axion, which defines the difference between between the instanton and anti-instanton. This sign affects some signs in the SL(2, $\mathbb{R}$) charges of the solution, but does not change its conjugacy class.

and the four integration constants q, q_3, q_- and C_1. The integration constant q is defined as the square root of q^2, which is an integration constant that can be positive, zero or negative[6]. Finally, the constant c is given by

$$c = \sqrt{\frac{2(D-1)}{(D-2)}} \,. \tag{3.26}$$

Note that the metric, specified by $B(r)$ given in (3.24), only depends on the product of f_+ and f_-, whereas the scalars only depend on the quotient of f_+ and f_-. This reflects the presence of the scale symmetry (3.23), whose effect is to scale both $f_\pm$ with the same factor. The constants q^2 and q_- occur with inverse powers and have been taken non-zero in the above solution. Below, we will see that sending them to zero yields interesting limits.

The solution (3.24) carries electric $\mathrm{SL}(2,\mathbb{R})$ charges given by

$$Q_E = \begin{pmatrix} q_3 & q_+ \\ -q_- & -q_3 \end{pmatrix} , \tag{3.27}$$

where we have defined the dependent integration constant q_+ via

$$q^2 = -q_+ q_- + {q_3}^2 = -\det(Q_E) \,. \tag{3.28}$$

Thus, the solution (3.24) has general $\mathrm{SL}(2,\mathbb{R})$ charges (q_+, q_-, q_3).

The appearance of the four independent integration constants, q^2, q_-, q_3 and C_1, can be understood as follows. As can be inferred from the solution (3.24), the constant q_3 corresponds to the freedom to apply $\mathbb{R}$ transformations, which shift the axion. Similarly, the constant q_- corresponds to $\mathrm{SO}(1,1)$ transformations, which scale the axion and shift the dilaton. By applying such transformations one can shift q_3 with arbitrary numbers while q_- can be rescaled with a positive number. The constant C_1 is shifted as follows

$$C_1 \to C_1 - 2\,\lambda\, q \tag{3.29}$$

under the $\mathrm{SL}(2,\mathbb{R})$ transformation, with parameter λ, whose generator is given by the electric charge matrix:

$$\Omega_E = \exp(\lambda\, Q_E) \,. \tag{3.30}$$

Since Q_E is invariant under such transformations (see (3.17)), while C_1 is shifted, this explains why C_1 does not appear in (3.27). The remaining constant, q^2, is invariant under $\mathrm{SL}(2,\mathbb{R})$ and hence does not correspond to these symmetry transformations. Rather, this constant corresponds to the freedom to perform rescalings of the metric (3.23). To retain a metric that asymptotically goes to 1, this must be combined with an appropriate rescaling of r. The resulting effect of this transformation is a rescaling of q^2 with a positive number. One therefore always stays in the same conjugacy class under such transformations.

The solution (3.24) can be written in a more compact form by using, instead of the two functions f_+ and f_- which are harmonic over D-dimensional flat space, a function $H(r)$ which is

[6]Note that this implies that the solution (3.24) is not manifestly real, since q can be imaginary. Below, we discuss this issue separately for the three cases q^2 positive, negative or zero.

harmonic over a conformally flat space with the conformal factor specified by the function $B(r)$ given in (3.24), i.e.

$$\Box H(r) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial r}\left(r^{D-1} e^{(D-2)\,B(r)} \frac{\partial H(r)}{\partial r}\right) = 0\,. \tag{3.31}$$

The general solution to this equation is of the following form:

$$H(r) \propto \log(f_+(r)/f_-(r))\,. \tag{3.32}$$

We can, therefore, rewrite the solutions (3.24) as follows:

$$\boxed{\begin{aligned} ds^2 &= \left(1 - \frac{q^2}{r^{2\,(D-2)}}\right)^{2/(D-2)} (dr^2 + r^2 d\Omega^2_{D-1})\,,\\ e^{b\,\phi(r)} &= \left(\frac{q_-}{q}\ \sinh(H(r) + C_1)\right)^2\,,\\ \chi(r) &= \frac{2}{b\,q_-}\,(q\ \coth(H(r) + C_1) - q_3)\,, \end{aligned}} \tag{3.33}$$

where

$$H(r) = \frac{b\,c}{2}\ \log(f_+(r)/f_-(r))\,. \tag{3.34}$$

The solutions (3.33) are valid both for q^2 positive, negative and zero. Below, we will discuss the reality and validity of the solutions for each of these three cases. Note that we are using the Einstein frame.

- $\mathbf{q^2 > 0}$:

  In this case q is real and the solution is given by (3.33) with all constants real. However, the metric poses a problem: it becomes imaginary for

  $$r^{D-2} < r_c^{D-2} = q\,. \tag{3.35}$$

  One can check that there is a curvature singularity at $r = r_c$. However, this curvature singularity happens at strong string coupling:

  $$e^{\phi(r)} \to \infty\,, \qquad\qquad r \to r_c\,. \tag{3.36}$$

  Between $r = r_c$ and $r = \infty$, H varies between ∞ and 0, and with an appropriate choice[7] of C_1, i.e. a positive value of C_1, the scalars have no further singularities in this domain. One might hope to have a modification of this solution by higher-order contributions to the effective action of IIB string theory [38]. Alternatively, one can consider the possible

[7] According to (3.29), the constant C_1 can be changed by an SL(2, $\mathbb{R}$) transformation, leading to singular scalars (but non-singular currents, which are independent of C_1). However, since these are related to regular scalars by a global SL(2, $\mathbb{R}$) transformation, this does not pose a problem.

resolution of this singularity upon uplifting. In the next section, we will see that this indeed happens for the special case of

$$b = \sqrt{\frac{2(D-2)}{D-1}}\,, \tag{3.37}$$

equivalent to $bc = 2$.

In the case with $q^2 > 0$, there is an interesting limit in which $q_- \to 0$. For generical values of the other three constants, this yields a non-sensible solution with infinite scalars. To avoid this, one must simultaneously impose

$$C_1 \to -\log(\frac{q_-}{2q})\,, \qquad q_3 \to q - \frac{q_+ q_-}{2q}\,, \qquad q_- \to 0\,. \tag{3.38}$$

This yields a well-defined limit, in which the scalars read

$$e^{\phi/c} = \frac{f_+}{f_-}\,, \qquad \chi = \frac{-q_+}{bq}\,, \tag{3.39}$$

while the metric is unaffected and given by (3.24). This solution can also be deduced by simply solving the equations of motion from scratch, with the constant axion Ansatz. Note that in this limit the dilaton becomes independent of b: when the axion is constant, the dilaton coupling drops out of the field equations. In this limit, one is left with two independent integration constants, q_+ and q^2. The range of validity of this solution is equal to that of the above solution with $q_- \neq 0$: it is well-defined for $r > r_c$, while at $r = r_c$ the metric has a singularity and the dilaton blows up. We will find that this singularity is resolved upon uplifting for all values of $bc \geq 2$.

- $\mathbf{q^2 = 0}$

  We now consider the limit $q^2 \to 0$ of the general solution (3.33). Taking this limit for generic values of C_1, one sees that $e^{\phi(r)} \to \infty$ for all r. The only way to avoid this bad behaviour is to have $C_1 \to 0$, as $q^2 \to 0$. Thus, to obtain a well-defined limit, we simultaneously take

$$C_1 \to g_s^{b/2}\,\frac{q}{q_-}\,, \qquad q^2 \to 0\,. \tag{3.40}$$

The constant g_s is assumed positive and will correspond to the value of $e^{\phi(r)}$ at $r = \infty$. Taking the limit (3.40) of the general solution (3.33) yields the extremal solution:

$$\boxed{ds^2 = dr^2 + r^2\, d\Omega^2_{D-1}\,, \qquad e^{b\,\phi(r)/2} = h \qquad \chi(r) = \frac{2}{b}\,(h^{-1} - \frac{q_3}{q_-})\,,} \tag{3.41}$$

where $h(r)$ is the harmonic function:

$$h(r) = g_s^{b/2} + \frac{b\,c\,q_-}{r^{D-2}}\,. \tag{3.42}$$

This is the extremal D-instanton solution of [30]. It can also be obtained by solving the equations from scratch with a flat metric in the Ansatz. This solution is regular over the range $0 < r < \infty$ provided one takes both g_s and $b\,c\,q_-$ positive; at $r = 0$ however, the harmonic function blows up and the scalars are singular. Again, string theory corrections may resolve these scalar singularities.

- $\mathbf{q^2 < 0}$:

  In this case q is imaginary. To obtain a real solution we must take C_1 to be imaginary. We therefore redefine

$$q \to i\,\tilde{q} \qquad\qquad C_1 \to i\,\tilde{C}_1\,, \tag{3.43}$$

such that $\tilde{q}$ and $\tilde{C}_1$ are real. One can now rewrite the solution (3.33) by using the relation[8]

$$\log(f_+/f_-) = 2\,\mathrm{arctanh}(q/r^{D-2})\,, \tag{3.44}$$

and, next, replacing the hyperbolic trigonometric functions by trigonometric ones in such a way that no imaginary quantities appear. We find that, for $q^2 < 0$, the general solution (3.33) takes the following form:

$$\boxed{\begin{aligned} ds^2 &= (1 + \frac{\tilde{q}^2}{r^{2\,(D-2)}})^{2/(D-2)}\,(dr^2 + r^2\,d\Omega^2_{D-1})\,,\\ e^{b\phi(r)} &= \left(\frac{q_-}{\tilde{q}}\,\sin(b\,c\,\arctan(\frac{\tilde{q}}{r^{D-2}}) + \tilde{C}_1)\right)^2\,,\\ \chi(r) &= \frac{2}{b\,q_-}\,(\tilde{q}\,\cot(b\,c\,\arctan(\frac{\tilde{q}}{r^{D-2}}) + \tilde{C}_1) - q_3)\,. \end{aligned}} \tag{3.45}$$

The metric and curvature are well behaved over the range $0 < r < \infty$. However, the scalars can only be non-singular over the same range by an appropriate choice of $\tilde{C}_1$ provided that $bc < 2$. This can be seen as follows: the arctan varies over a range of $\pi/2$ when r goes from 0 to ∞. Since it is multiplied by bc, the argument of the sine varies over a range of more than π if $bc > 2$. Therefore, for $bc > 2$ there is always a point r_c such that $\chi \to \infty$ as $r \to r_c$. Note that the breakdown of the solution occurs at weak string coupling: $e^\phi \to 0$ as $r \to r_c$. In the next section we will find that this singularity is not resolved upon uplifting and will correspond to a black hole with a naked singularity. The same holds for the limiting case of $bc = 2$. Therefore the case $q^2 < 0$ only yields regular instanton solutions for $bc < 2$, together with the condition that C_1 and $C_1 + bc\pi/2$ are on the same branch of the cotangent.

3.3.2 Wormhole geometries

It is known [30] that the standard D-instanton, i.e. $D = 10, b = 2$, in string frame has the geometry of a wormhole, i.e. it has two asymptotically flat regions connected by a neck, see figure 3.1. It will therefore be interesting to investigate whether there exist frames, in which the non-extremal instantons also have the geometries of wormholes.

[8] Here we have used the general relation $\log((1 + x)/(1 - x)) = 2\,\mathrm{arctanh}(x)$.

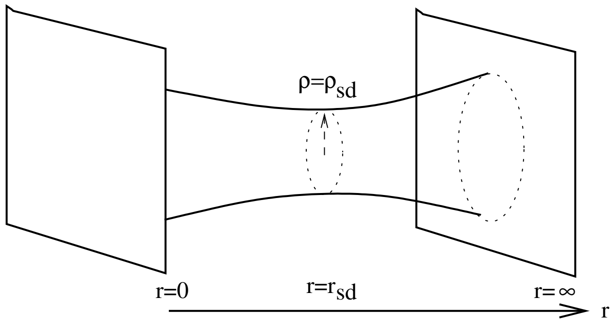

Figure 3.1: *The geometry of a wormhole. The two asymptotically flat regions at $r = 0$ and $r = \infty$ are connected via a neck with a minimal physical radius ρ_{sd} at the self-dual radius r_{sd}.*

We consider a general wormhole metric of the form

$$ds^2 = f(r)^{2/(D-2)}\,(dr^2 + r^2 d\Omega^2)\,, \qquad f(r) = \alpha + \beta r^{2-D} + \gamma r^{4-2D}\,, \tag{3.46}$$

where α, β and γ are constants. The metric has a $\mathbb{Z}_2$ isometry corresponding to the transformation $r^{D-2} \to \gamma\, r^{2-D}/\alpha$ which interchanges the two asymptotically flat regions. The physical radius ρ is the square root of the coefficient of the angular part of the metric, given by $\rho^{D-2} = f(r)r^{D-2}$. The minimum of this physical radius of the neck occurs at the fixed point of the transformation above, i.e. at the so-called self-dual radius $r_{\rm sd}^{D-2} = \sqrt{\gamma/\alpha}$, and is given by $\rho_{\rm sd}^{D-2} = 2\sqrt{\alpha\gamma} + \beta$. We will now study the three conjugacy classes in order to see for each case if there exists a frame[9] in which the metric takes the form (3.46).

- $\mathbf{q^2 > 0}$: As we will see in section 3.4, the appropriate frame in this case is the frame dual to the instanton, i.e. the $(D-3)$-brane frame, given by

$$g_{\mu\nu}^{\rm dual} = e^{b\phi/(D-2)}\, g_{\mu\nu}^{\rm E}\,. \tag{3.47}$$

  In the special case of $b\,c = 2$, the metric takes the form (3.46) in the dual frame with

$$f(r) = \frac{q_-}{q}\sinh(C_1) + 2q_-\cosh(C_1)r^{2-D} + q_- q\sinh(C_1)r^{4-2D}\,. \tag{3.48}$$

  This gives the self-dual radius $r_{\rm sd}$ and the minimal physical radius $\rho_{\rm sd}$

$$r_{\rm sd}^{D-2} = q\,, \qquad \rho_{\rm sd}^{D-2} = 2q_- e^{C_1}\,. \tag{3.49}$$

  Note that the self-dual radius $r_{\rm sd}$ coincides with the critical radius r_c of the previous section: the curvature singularity in Einstein frame becomes the center of the wormhole in

[9] In arbitrary dimension one can define three different frames as follows: in the Einstein frame, the Einstein-Hilbert term has no dilaton factor; in the string frame, the kinetic term for the axionic field strength comes without a dilaton factor (like all Ramond-Ramond field strengths); and in the dual frame, the Einstein-Hilbert term, the dilaton kinetic term and the kinetic term for the dual field strength (i.e. F^2_{D-p-2} for the frame dual to a p-brane) come with the same dilaton factor (see e.g. [49, 50] for a more detailed discussion).

the dual frame. The limit $q_- \to 0$, with appropriate scaling of C_1 as given in (3.38), yields $\rho_{\rm sd}^{D-2} = 4q$. For generic values of bc, the instanton metrics cannot be written in the form (3.46) in any frame.

- $\mathbf{q^2 = 0}$: It turns out that for any value of b the wormhole geometry is made manifest by going to the string frame

$$g^{\rm str}_{\mu\nu} = e^{2b\phi/(D-2)}\, g^{\rm E}_{\mu\nu}\,. \tag{3.50}$$

In this frame, the metric is given by (3.46) with

$$f(r) = g_s^b + 2bcq_- g_s^{b/2} r^{2-D} + (bcq_-)^2 r^{4-2D}\,. \tag{3.51}$$

This gives the self-dual and minimal physical radii

$$r_{\rm sd}^{D-2} = bcq_-/g_s^{b/2}\,, \qquad \rho_{\rm sd}^{D-2} = 4bcq_- g_s^{b/2}\,. \tag{3.52}$$

- $\mathbf{q^2 < 0}$: Here, the metric has the appropriate form already in Einstein frame, hence, from (3.45) we get, for any value of b,

$$r_{\rm sd}^{D-2} = \tilde{q}\,, \qquad \rho_{\rm sd}^{D-2} = 2\tilde{q}\,. \tag{3.53}$$

We thus see that for all three conjugacy classes there exists frames, in which the solutions have the geometries of wormholes.

3.3.3 Instanton solutions with multiple dilatons

We will now consider extensions of the instanton solution described in the previous sections, which is carried by the SL(2, $\mathbb{R}$) scalars ϕ and χ. We will extend this system with n dilatons φ_α ($\alpha = 1, \ldots, n$), which are SL(2, $\mathbb{R}$) singlets and do not couple to the axion (this can always be achieved by field redefinitions provided one allows for an arbitrary dilaton coupling b to the original dilaton ϕ). We will call the corresponding solution a multi-dilaton instanton. The multi-dilaton action is given by

$$\mathcal{L}_E = \tfrac{1}{2}\sqrt{g}\,[R - \tfrac{1}{2}\sum_{\alpha=1}^{n}(\partial\varphi_\alpha)^2 - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}e^{b\phi}(\partial\chi)^2]\,, \tag{3.54}$$

with field equations (3.21) plus n equations, requiring φ_α to be harmonic in the curved space. The case of one extra dilaton was considered in [51].

The solution to this system has the same metric as given in (3.24), see also [51]. Then the extra dilatons φ_α satisfy a d'Alembertian equation in a conformally flat background specified by $B(r)$ as given in (3.24):

$$\frac{\partial}{\partial r}\left(r^{D-1} e^{(D-2)\,B(r)} \frac{\partial\varphi(r)}{\partial r}\right) = 0\,. \tag{3.55}$$

This equation is solved by the harmonic function as given in (3.32), yielding dilatons given by

$$\varphi_\alpha = \nu_\alpha + \mu_\alpha \log\left(\frac{f_+(r)}{f_-(r)}\right), \tag{3.56}$$

with $2n$ integrations constants ν_α and μ_α.

Of course, due to the presence of the extra dilatons φ_α, the Einstein equation in (3.21) is modified. It turns out that the contribution of φ_α to the energy-momentum tensor is cancelled by similar μ_α-dependent contributions of the dilaton ϕ and the axion χ to the energy-momentum tensor. Since all μ_α-dependent contributions of the dilatons and the axion to the energy-momentum tensor cancel each other, this extension allows for a μ_α-independent metric.

3.4 Uplift to black holes

In this section, we will find an explicit example of the soliton-instanton correspondence mentioned in chapter 2. We will show that a D-instanton can sometimes be viewed as a spacelike section of a charged black hole, and more generally a p-brane.

3.4.1 Kaluza-Klein reduction

In this section we consider the possible higher-dimensional origin of the Euclidean system (3.18) as a consistent truncation of the $(D+1)$-dimensional Lagrangian, defined over Minkowski space,

$$\mathcal{L}_{D+1} = \sqrt{-\hat{g}}\,[\hat{R} - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{4}e^{a\hat{\phi}}\hat{F}^2], \tag{3.57}$$

with the two-form field strength $\hat{F} = d\hat{A}$. It consists of an Einstein-Hilbert term (for a metric of Lorentzian signature), a dilaton kinetic term and a kinetic term for a vector potential with arbitrary dilaton coupling, parametrized by a. The corresponding Δ value [52] is given by

$$\Delta = a^2 + \frac{2(D-2)}{D-1}, \tag{3.58}$$

which characterizes the dilaton coupling in $D+1$ dimensions.

The reduction Ansatz over the time coordinate is

$$\hat{ds}^2 = e^{2\alpha\varphi}\,ds^2 - e^{2\beta\varphi}\,dt^2, \qquad \hat{A} = \chi\,dt, \qquad \hat{\phi} = \phi, \tag{3.59}$$

with the constants

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \qquad \beta = -(D-2)\,\alpha, \tag{3.60}$$

which are chosen such as to obtain the Einstein frame in the lower dimension with appropriate normalization of the dilaton φ. Note that the dilaton factor in front of the spatial part of the metric $\hat{g}_{\mu\nu}$ coincides, for $bc = 2$, with the dual frame defined in section 3.3.2.

With the above Ansatz, the Einstein-Maxwell-dilaton system reduces to the D-dimensional Euclidean system

$$\mathcal{L}_D = \sqrt{-g}\,[R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}e^{a\,\phi-2\beta\,\varphi}(\partial\chi)^2]. \tag{3.61}$$

Next, we perform a field redefinition corresponding to a rotation in the (ϕ, φ)-plane such that we obtain

$$\mathcal{L}_D = \sqrt{-g}\,[R - \tfrac{1}{2}(\partial\tilde{\phi})^2 - \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \tfrac{1}{2}e^{b\tilde{\phi}}(\partial\chi)^2]\,, \tag{3.62}$$

with dilaton coupling b given by

$$b = \sqrt{a^2 + \frac{2(D-2)}{D-1}}\,. \tag{3.63}$$

The corresponding value of Δ is equal to the original value (3.58). This system can be truncated to the one we are considering by setting $\tilde{\varphi} = 0$.

Therefore, the system that we consider in section 3.3 has a higher-dimensional origin if the dilaton coupling satisfies $bc \geq 2$ or

$$b \geq \sqrt{\frac{2(D-2)}{D-1}}\,. \tag{3.64}$$

The case which saturates the inequality, i.e. $a = 0$, can be uplifted to an Einstein-Maxwell system without the dilaton $\hat{\phi}$. For $bc > 2$ one needs to include an explicit dilaton $\hat{\phi}$ in the higher-dimensional system; i.e. one must consider the Einstein-Maxwell-dilaton system (3.57) with $a \neq 0$. Note that in string theory toroidal reductions, under which the combination Δ is preserved, only lead to values of b with $bc \geq 2$.

Since the Euclidean gravity-axion-dilaton system we are considering can be obtained as a consistent truncation of the higher-dimensional Minkowskian Einstein-Maxwell-dilaton system (3.57), it is natural to look for a higher-dimensional origin of the non-extremal instanton solutions within this system. In the following two sections we consider the cases $bc = 2$ and $bc > 2$ separately. The instantons with $bc < 2$ have no physical higher-dimensional origin from toroidal reduction.

3.4.2 Reissner-Nordström black holes: $bc = 2$

It is not difficult to see that for $bc = 2$ the generalized instanton solutions uplift to the $(D+1)$-dimensional Reissner-Nordström (RN) black hole solution

$$ds^2 = -g_+(\rho)\,g_-(\rho)\,dt^2 + \frac{d\rho^2}{g_+(\rho)\,g_-(\rho)} + \rho^2 d\Omega^2_{D-1}\,, \quad F_{t\rho} = -\partial_\rho A_t = (D-2)\,c\,\frac{Q}{\rho^{D-1}}\,, \tag{3.65}$$

where

$$g_\pm(\rho) = 1 - \frac{\rho_\pm^{D-2}}{\rho^{D-2}}\,, \qquad \rho_\pm^{D-2} = M \pm \sqrt{M^2 - Q^2}\,, \tag{3.66}$$

and Q and M are the charge and mass of the black hole, respectively. The RN black hole has naked singularities for $M^2 < Q^2$, while these are cloaked for $M^2 \geq Q^2$, yielding a physically acceptable spacetime. Note that the coordinate ρ coincides with the physical radius of the previous section, for which the angular part of the metric $d\Omega^2_{D-1}$ is multiplied by ρ^2.

In order to establish the precise relation between the charge Q and the mass M of the RN black hole and the SL$(2,\mathbb{R})$ charges of the $bc = 2$ instanton solutions given in (3.33) we must first cast the RN metric in isotropic form as follows:

$$ds^2 = -\frac{g(r)}{\rho(r)^{2(D-2)}}\, dt^2 + \frac{\rho(r)^2}{r^2}\,(dr^2 + r^2\, d\Omega^2_{D-1})\,, \tag{3.67}$$

where

$$\rho(r) = \left(r^{D-2} + M + \frac{M^2 - Q^2}{4\, r^{D-2}}\right)^{1/(D-2)}\,, \qquad g(r) = \left(r^{D-2} - \frac{M^2 - Q^2}{4\, r^{D-2}}\right)^2\,. \tag{3.68}$$

To relate the instanton and black hole solutions, we need to choose proper boundary conditions for the instanton solutions (3.33), which are implied by the boundary conditions of the RN black hole:

$$\begin{array}{lcl} \lim_{r\to\infty} g_{tt} = -1\,, & \Longleftrightarrow & \lim_{r\to\infty} e^{\phi} = 1\,, \\ \lim_{r\to\infty} A_t = 0\,, & & \lim_{r\to\infty} \chi = 0\,. \end{array} \tag{3.69}$$

This fixes the constants C_1 and one of the three SL$(2,\mathbb{R})$ charges q_3 in (3.33) as follows:

$$C_1 = \text{arcsinh}(\frac{q}{q_-})\,, \qquad q_3 = q\, \coth(C_1) = \sqrt{q^2 + q_-^2}\,. \tag{3.70}$$

The relation between the charge Q and the mass M of the RN black hole and the two unfixed SL$(2,\mathbb{R})$ charges q_- and q^2 is:

$$Q = -2\, q_-\,, \qquad M = 2\, \sqrt{q^2 + q_-^2}\,, \tag{3.71}$$

such that

$$q^2 = \frac{M^2 - Q^2}{4}\,. \tag{3.72}$$

From (3.72) we see that the physically acceptable non-extremal RN black holes with $M^2 \geq Q^2$ coincide with the uplifted instanton solutions in the $q^2 = 0$ and $q^2 > 0$ conjugacy classes:

$$\begin{array}{lcl} M^2 > Q^2 & \Leftrightarrow & q^2 > 0\,, \\ M^2 = Q^2 & \Leftrightarrow & q^2 = 0\,. \end{array} \tag{3.73}$$

More specifically, we find that the non-extremal (extremal) RN metric in isotropic coordinates (3.67) reduces to the $q^2 > 0$ ($q^2 = 0$) instanton solution in the dual frame metric (3.47). Note that the $q^2 > 0$ instanton has a wormhole geometry in the dual frame metric. It turns out that the minimal physical radius ρ_{sd} for this case is given by $\rho_{\text{sd}} = \rho_+$, where ρ_+ is the position of the outer event horizon given in (3.66).

3.4.3 Interpretation of instantons as BH wormholes

In the previous section we have seen that the non-extremal D-instanton solutions (3.33) in the dual frame metric (3.47) with $b\,c = 2$ and $M^2 \geq Q^2$ can be viewed as $t = constant$ space-like sections of the RN black hole metric (3.67). In the Kruskal-Szekeres-like extension of the RN black hole, the spatial part of the metric (3.67) has the geometry of an Einstein-Rosen bridge or wormhole, which connects two asymptotically flat regions of space (see [20] for a general introduction to black holes). Indeed, the spatial part of (3.67) has, for $M^2 > Q^2$, the $\mathbb{Z}_2$ isometry

$$r^{D-2} \to \frac{M^2 - Q^2}{4\,r^{D-2}}\,, \tag{3.74}$$

which relates each point on one side of the Einstein-Rosen bridge to a point on the other side.

It is instructive to consider the special case of the Schwarzschild black hole, (i.e. $Q = 0$). Due to (3.71), this corresponds to the uplift of instantons with $q_- = 0$, i.e. the solutions given in (3.38). As shown in figure 3.2, in the Kruskal-Szekeres extension of the Schwarzschild black hole, every $t = constant$ section of space time corresponds to a straight space-like line going through the origin of this coordinate system, with slope determined by the constant value of t.

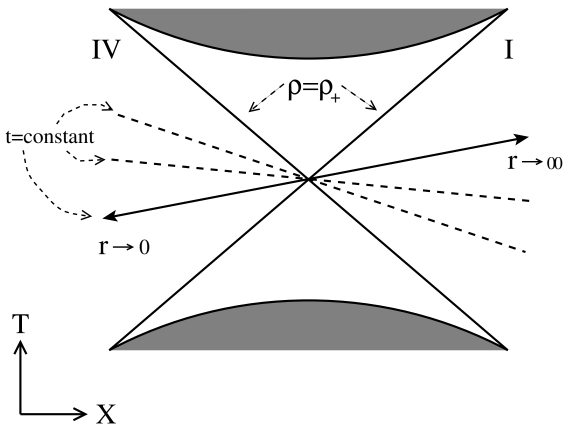

Figure 3.2: *Schwarzschild black hole in Kruskal-Szekeres coordinates. Spatial sections with $t = constant$ are space-like lines through the origin, going from region IV to region I. T and X are the Kruskal-Szekeres time-like and space-like directions respectively. The horizons are at $\rho = \rho_+$, which coincides with the minimal physical radius at the center $\rho = \rho_{\rm sd}$.*

Notice that on each line, the coordinate r from (3.67) runs from $r = 0$ at the spatial infinity on the left-hand-side, to $r = \infty$ on the right-hand-side. The fixed point of the $\mathbb{Z}_2$-isometry (3.74) (now with $Q = 0$) is positioned at the center of figure 3.2. The value of r at this fixed point and the corresponding minimal physical radius are given by

$$r_{\rm sd}^{D-2} = \tfrac{1}{2}M\,, \qquad\qquad \rho_{sd}^{D-2} = 2M\,. \tag{3.75}$$

Note that this value of the physical radius corresponds to the horizon of the black hole, as can also be seen from figure 3.2. One can make the wormhole geometry visible by associating to

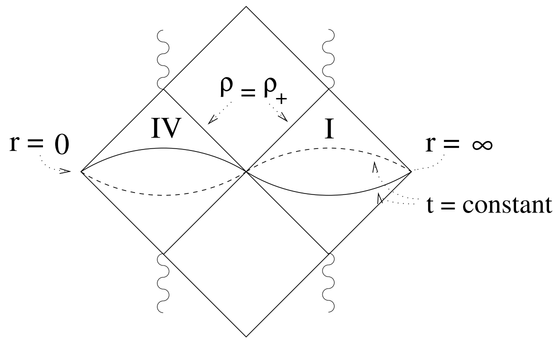

Figure 3.3: *Carter-Penrose diagram of RN black hole. The lines with $\rho = \rho_+$ are the horizons, which coincide with the minimal physical radius $\rho = \rho_{sd}$ in the center.*

every value of r a $(D-1)-$sphere. Representing every $(D-1)-$sphere by a circle one obtains the wormhole picture of figure 3.1.

In the more general case (i.e. $Q \neq 0$), the $t = constant$ sections are still paths connecting two regions of the RN black hole. To see what these regions correspond to, it is helpful to draw a Carter-Penrose diagram, see figure 3.3. The wormhole geometry is qualitatively the same as in the Schwarzschild case. The position of the wormhole neck and the value of the minimal physical radius are given by

$$r_{\mathrm{sd}}^{D-2} = \tfrac{1}{4}(M^2 - Q^2)\,, \qquad\qquad \rho_{sd}^{D-2} = M + \sqrt{M^2 - Q^2}\,, \tag{3.76}$$

which again coincide with the horizon at $\rho = \rho_+$. The curvature singularity of the D-instanton solutions with $q^2 > 0$ (3.33) at $r_c = (q)^{1/D-2}$ are resolved in this uplifting and can now be understood as the usual coordinate singularity of the RN black hole outer event horizons (i.e. $\rho = \rho_+$, or $r^{2\,(D-2)} = (M^2 - Q^2)/4$).

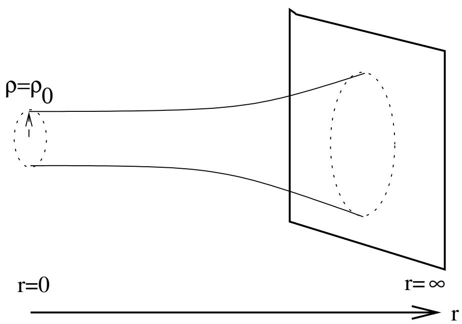

Figure 3.4: *The geometry of the extremal black hole as a "one-sided" wormhole with minimal physical radius ρ_0.*

The extremal RN black hole (i.e. $M^2 = Q^2$) is qualitatively different from the other cases. As one can see from (3.74), the $\mathbb{Z}_2$-isometry is gone. By taking the limit $M^2 \to Q^2$ of a non-extremal black hole we see that the wormhole stretches to an infinitely long neck. The fixed point of the isometry goes to spatial infinity at $r = 0$. This means that the extremal black hole has a "one-sided" wormhole with a minimal physical radius $\rho_0^{D-2} = M$, and the full Kruskal-like extension is geodesically complete without need for a region IV. This situation is illustrated in figure 3.4.

3.4.4 Dilatonic black holes: $bc > 2$

The instantons with $bc > 2$ uplift to non-extremal *dilatonic* black holes, i.e. black hole solutions carried by a metric, a vector and a dilaton. In fact, the uplift is identical to a version of the black hole solution presented in [53]. To be more precise, the non-extremal dilatonic black hole solutions of [53] contain an extra parameter μ. For generic values of this parameter the black hole solution is singular[10]. One only obtains a regular solution if[11] $\mu \sim q$.

The uplift of the $bc > 2$ instantons equals the $\mu \to 0$ limit of the non-extremal black hole solutions of [53]. Therefore, in contrast to the $bc = 2$ case, we obtain a singular black hole solution. This singularity can only be avoided in two limiting cases. The singularity disappears both in the extremal limit (3.40) when $q^2 \to 0$ and in the Schwarzschild limit (3.39) when $q_- \to 0$, where the dilaton decouples.

3.5 Uplift to *p*-branes

In section 4 we have discussed the uplift of the instantons of section 3 to higher-dimensional black hole solutions. It is therefore natural to consider the uplift to higher-dimensional *p*-branes. To this end, it will be useful to first introduce the following nomenclature.

Non-extremal deformations of general *p*-branes have been considered in [53, 55]. These are solutions of the $(D + p + 1)$-dimensional Lagrangian, defined over Minkowski space,

$$\mathcal{L}_{D+p+1} = \sqrt{-\hat{g}}\,[\hat{R} - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2\,(p+1)!}\,e^{a\hat{\phi}}\,\hat{G}^2_{(p+2)}]\,, \tag{3.77}$$

with the rank-$(p + 2)$ field strength $\hat{G}_{(p+2)} = d\hat{C}_{(p+1)}$. For a *p*-brane in $D + p + 1$ dimensions the metric (in Einstein frame) is of the form

$$ds^2 = e^{2A}(-e^{2f}dt^2 + dx_p{}^2) + e^{2B}(e^{-2f}dr^2 + r^2 d\Omega_{D-1}{}^2)\,, \tag{3.78}$$

where A, B and f are functions that depend on the radial coordinate r only. It is convenient to introduce the quantity

$$X = (p + 1)A + (D - 3)B\,. \tag{3.79}$$

The extremal *p*-brane solutions with equal mass and charge, preserving half of the supersymmetry, are obtained by taking $X = f = 0$.

Assuming that $D \geq 3$ there exist two types of non-extremal *p*-brane solutions in the literature. Following [53], we will call them type 1 and type 2 non-extremal *p*-branes:

---

[10] These (singular) solutions are a generalization of the (regular) black holes of [54].
[11] The parameter q^2 can be identified with the parameter k of [53].

- **Type 1 non-extremal p-branes:** $X = 0$ and $f \neq 0$.

  These are the non-extremal black branes of [55, 56]. The deformation function f is given by

$$e^{2f} = 1 - \frac{k}{r^{D-2}}\,, \tag{3.80}$$

  where k is the deformation parameter. In a different coordinate frame, with radial coordinates ρ, these branes can be expressed in terms of the two harmonic functions

$$f_\pm(\rho) = 1 - \left(\frac{\rho_\pm}{\rho}\right)^{D-2} \,. \tag{3.81}$$

  Physical branes without a naked singularity have more mass than charge, which corresponds to $\rho_+ > \rho_-$ or $k > 0$. For this type of non-extremal deformation, the dilaton $\hat{\phi}$ is proportional to A and B, which are linearly related since $X = 0$.

- **Type 2 non-extremal p-branes:** $X \neq 0$ and $f = 0$.

  These are the non-extremal black branes of [53]. The deformation function X reads

$$e^{X} = 1 - \frac{k}{r^{2(D-2)}}\,, \tag{3.82}$$

  where k is the deformation parameter. The absence of naked singularities requires k to be positive. In this case, the dilaton $\hat{\phi}$ is not proportional to A or B, which are not linearly related.

  The non-extremal D-instanton solutions (3.33) fit exactly in this chain of non-extremal p-branes for $p = -1$. Although the type 2 non-extremal p-branes are defined in Minkowski space, we find that one can extend the formulae of [53] to $p = -1$ branes in Euclidean space, i.e. generalized D-instantons, by taking $f = 0$ and $B \neq 0$.

Both types of non-extremal p-branes break supersymmetry. A special case is $p = 0$, for which the regular type 1 and type 2 non-extremal 0-branes are equivalent up to a coordinate transformation in r. From the form of the metric (3.78), which has different world-volume isometries for $f = 0$ and $f \neq 0$, it is clear that this is not the case for $p > 0$.

To relate the (multi-dilaton) instanton solutions of section 3 to the non-extremal p-branes, it is instructive to reduce the p-branes over their $(p+1)$-dimensional world-volume, including time. In complete analogy with the reduction over time of section 4.1, this will give rise to $p+1$ dilatons from the world-volume of the p-brane. However, these are not all unrelated: for one thing, the dilatons corresponding to the spatial world-volume will be proportional to each other, and can therefore be truncated to a single dilaton. We will denote the dilaton from the spatial metric components by φ, while the time-like component of the metric gives rise to $\tilde{\varphi}$. In general, the reduction of non-extremal p-branes will therefore give rise to a multi-instanton solution with three different dilatons, including the explicit dilaton ϕ:

$$\hat{g}_{tt} \to \tilde{\varphi}\,, \qquad \hat{g}_{xx} \to \varphi\,, \qquad \hat{\phi} \to \phi\,. \tag{3.83}$$

For the two types of non-extremal deformations considered here, however, there is always a relation between the three dilatons, allowing a truncation to two dilatons[12]. For the type 1 deformations the dilatons ϕ and φ are related, as can be seen from the metric with $X = 0$. Similarly, the type 2 deformations yield a relation between φ and $\tilde{\varphi}$ since $f = 0$. Therefore, these non-extremal p-branes reduce to multi-dilaton instanton solutions with two inequivalent dilatons. Conversely, two-dilaton instanton solutions can uplift to either types of non-extremal p-branes, by embedding these dilatons in different ways in the higher-dimensional metric and dilaton.

It is interesting to investigate when these two dilatons can be related or reduced to one, therefore corresponding to our explicit $\mathrm{SL}(2,\mathbb{R})$ instanton solution (3.24) with only one dilaton. For the type 1 deformations, this is only possible for the special case with $p = 0$ and $a = 0$. For these values, the dilatons ϕ and φ vanish, leaving one with only $\tilde{\varphi}$. The constraint on a implies $bc = 2$ which, as discussed in section 3, gives rise to the Reissner-Nordström black hole.

For the type 2 deformations there are more possibilities to eliminate the dilaton ϕ. It can be achieved by requiring $a = 0$, as we did for the uplift to black holes. For general p, this leads to the following constraint on b:

$$b = \sqrt{\frac{2(p+1)(D-2)}{D+p-1}}\,. \tag{3.84}$$

Note that this yields $bc = 2$ for black holes with $p = 0$. For these values of b, the instanton solution (3.24) can be uplifted to regular non-extremal non-dilatonic p-branes. For higher values of b, the instanton solution uplifts to singular non-extremal dilatonic p-branes. For these solutions to become regular, one must take either $q^2 \to 0$ or $q_- \to 0$, exactly like we found in the $bc > 2$ discussion of section 4.3.

The uplift of the $\mathrm{SL}(2,\mathbb{R})$ instanton solution (3.24) to p-branes is therefore very similar to the uplift to black holes. There is one value of b (3.84) for which the instanton solution can be uplifted to a regular non-extremal non-dilatonic p-brane of type 2. For higher values of b one can obtain singular non-extremal dilatonic p-branes of type 2, which only become regular on either of the limits $q^2 \to 0$ and $q_- \to 0$. By adding an extra dilaton to the instanton solution one can also make a connection to the regular type 1 and type 2 non-extremal dilatonic p-branes.

3.6 Instantons

In the previous section we focused on the bulk behavior of the three conjugacy classes of instanton-like solutions. In this section we will investigate which of these solutions can be interpreted as instantons. Instantons, as we have seen in chapter 2 are defined to be solutions of the Euclidean equations of motion with finite, non-zero value of the action. They have a tunneling interpretation, and generically contribute to certain correlation functions in the path integral with terms that are exponentially suppressed by the instanton action. These correlation functions then induce new interactions in the effective action, and for the extremal, 1/2 BPS, D-instantons in type IIB in $D = 10$, these effects are captured by certain $\mathrm{SL}(2,\mathbb{Z})$ modular functions that

[12] This seems to indicate a generalization of the non-extremal deformations with both $X \neq 0$ and $f \neq 0$, reducing to a three-dilaton instanton.

multiply higher derivative terms such as R^4 and their superpartners [31]. Before we study correlation functions and effective interactions induced by non-extremal D-instantons, we must first discuss the properties and show the finiteness of the non-extremal instanton action. We will do this using a method that will allow us to recover the special case of extremal D-instantons easily.

3.6.1 Instanton action

The first thing we notice is that the action (3.18), evaluated on *any* solution of (3.21) vanishes. What is also bothersome about the Euclidean action (3.18) is that it is not bounded from below, not even in the scalar sector. Such actions cannot be used in a path integral, since fluctuations around the instanton will diverge. However, this should not be a surprise at all. After all, the Lagrangian (3.5), whose equations of motion we have been solving, is not the true Lagrangian of the full quantum field theoretic system, but an *effective* Lagrangian that is only meant to be used for finding 'saddle points'[13]. It was never meant to appear in a path integral. In order to evaluate the true value of the action of the non-extremal D-instanton we will use the dualization procedure and replace this dilaton-axion system with a system containing the dilaton and a $(D-1)$-form field-strength, which *does* have true saddle points. This procedure was briefly mentioned at the beginning of this chapter. We will now fully develop it here. For a toy model illustration of this procedure, see appendix A.

The goal is to prove that two different systems can be regarded as the effective path integrals of one and only one common parent path integral. Let us first write down the Euclidean path integral for a dilaton coupled to a $(D-1)$-form field-strength[14], subject to the constraint of being a closed form, i.e. $dF_{D-1} = 0$:

$$\int d[F_{D-1}]\, d[\lambda]\, \exp\left(\int_{\mathcal{M}} -\tfrac{1}{2}\left(d\phi \wedge *d\phi + e^{-b\phi}\, F_{D-1} \wedge *F_{D-1}\right) + i\,\lambda\, dF_{D-1}\right), \tag{3.85}$$

where λ is *real* and acts as a quantum Lagrange multiplier that imposes the constraint $dF_{D-1} = 0$, by means of the following identity:

$$\int d[\lambda]\, \exp\,[i\,\lambda\, G] = \delta[G]\,, \qquad \text{for any function} \quad G\,, \tag{3.86}$$

where the $\delta[]$ stands for δ-function*al*. Notice that we are treating the field-strength as fundamental, not the gauge potential. This path integral is defined with 'Dirichlet' boundary conditions on F_{D-1}, i.e. some of the components of F_{D-1} are fixed on the boundary. The constraint that the former be closed implies that it is locally exact, i.e. locally, $F_{D-1} = dC_{D-2}$, for some C_{D-2}. The path integral (3.85) is well-defined because the action is positive-definite, and it is straightforward to find its saddle points, by treating C_{D-2} as fundamental, and deriving the usual higher-dimensional Maxwell equations.

Let us now change the order of integration and perform the path integral over F_{D-1} first. In order to do this, we need to rewrite the action in such a way that the field-strength appears

[13] They are not the true saddle points of the scalar system, but they still provide a semiclassical approximation of the path integral.

[14] We will not worry about the gravitational sector in the following derivation, since it is not relevant. The integration over the dilaton is also omitted.

without derivatives acting on it:

$$
\begin{aligned}
\mathcal{S}_E &= \int_{\mathcal{M}} \tfrac{1}{2}\,(d\phi \wedge *d\phi + e^{-b\phi}\, F_{D-1} \wedge *F_{D-1} - 2\, i\, \lambda\, dF_{D-1}) \\
&= \int_{\mathcal{M}} \tfrac{1}{2}\,\Big[d\phi \wedge *d\phi + e^{-b\phi}\, \Big(F_{D-1} + i\, e^{b\phi} * d\lambda\Big) \wedge * \Big(F_{D-1} + i\, e^{b\phi} * d\lambda\Big) \\
&\qquad + e^{b\phi}\, d\lambda \wedge *d\lambda - 2\, i\, d\,(\lambda\, F_{D-1})\Big]\,,
\end{aligned} \tag{3.87}
$$

where we have used partial integration and the fact that, in a Euclidean space, $**A_p = (-)^{(D-1)p} A_p$, where A_p is a p-form. The last term in (3.87) is a surface term, and since boundary conditions have been imposed on the field-strength, it will not participate in the path integral over the latter. The term can be interpreted as an external current J^μ. Defining $*J = F_{D-1}$, we have

$$
\int_{\mathcal{M}} d\,(\lambda\, F_{D-1}) = \int_{\partial\mathcal{M}} \lambda\, *\, J = \int_{\partial\mathcal{M}} \lambda\, J_\mu\, n^\mu\,, \tag{3.88}
$$

where n^μ is an outward normal vector.

To integrate F_{D-1} in (3.87), we first perform the following shift of integration variables:

$$
F_{D-1} \to \bar{F}_{D-1} + i\, e^{b\phi} * d\lambda\,. \tag{3.89}
$$

We are allowed to do this even though λ is real. This is *not* a rotation of the contour of integration, it is just a shift in the imaginary direction. The resulting integration over $\bar{F}_{D-1}$ is nothing other than the plain old Gaussian integral, yielding a determinant $e^{b\phi/2}$ in the path integral. We can absorb the latter in the measure of the dilatonic path integral by changing variables as follows:

$$
e^{b\phi/2}\, d[\phi] = 2/b\, d[e^{b\phi/2}]\,. \tag{3.90}
$$

This means we are treating the exponential of the dilaton as fundamental. As long as we only sum over positive values of the exponential, this does not affect anything. The change of variables is valid because the exponential is a strictly monotonic function of the dilaton, and hence injective. When the smoke clears, we are left with the following system:

$$
\int d[\lambda]\, \exp\left(- \int_{\mathcal{M}} \tfrac{1}{2}\, \Big[d\phi \wedge *d\phi + e^{b\phi}\, d\lambda \wedge *d\lambda\Big] + i \int_{\partial M} *J\,\lambda\right), \tag{3.91}
$$

where no boundary conditions are imposed on λ. The constraint $dF_{D-1} = 0$ translates to $d * J = 0$, i.e. the external current must be divergenceless. The important thing to notice is that the kinetic term of λ has the 'normal' sign. Contrary to common belief, a quantum mechanical dualization does *not* yield a negative action scalar. The boundary term in this path integral corresponds to the two surface[15] terms in (3.9). This boundary term, combined with the fact that boundary values of λ are being integrated over, plays the role of a Fourier transformation of the boundary states. The path integral does not compute a transition amplitude between field eigenstates $|\,\lambda\,\rangle$, but between momentum eigenstates $|\,\pi\,\rangle \equiv \int d[\lambda]\, \exp(i\,\pi\,\lambda)\,|\,\lambda\,\rangle$.

[15] There is ambiguity in defining the boundary at infinity of a manifold. Although the surface terms in (3.9) are only defined on disconnected 'initial' and 'final' hypersurfaces, I believe that defining a single, connected, radial boundary at $r = \infty$ leads to equivalent results.

Note that the shift $\mathbb{R}$-symmetry of the axion is now broken to a $\mathbb{Z}$-symmetry by the surface term:

$$\lambda \to \lambda + \frac{2\pi n}{c}\,, \qquad \text{where} \quad c \equiv \int_{\partial M} *J\,, \qquad \text{and} \quad c \in \mathbb{Z}\,. \tag{3.92}$$

In theories where λ is periodically identified, the single-valuedness of the path integral imposes a quantization condition on c. String theory effects are expected to induce such a quantization, [57, 58]

Let us naïvely try to approximate (3.91) by means of the saddle point approximation. Because there are no boundary conditions on λ, variations need not vanish on the boundary. The Euler-Lagrange variation of the action then yields

$$\delta S = \int_{\mathcal{M}} d\left(e^{b\phi} * d\lambda\right) \delta\lambda - \int_{\partial\mathcal{M}} \left(e^{b\phi} * d\lambda - i * J\right) \delta\lambda\,. \tag{3.93}$$

For arbitrary $\delta\lambda$, this imposes a rather normal equation of motion for the axion in the bulk

$$d\left(e^{b\phi} * d\lambda\right) = 0\,. \tag{3.94}$$

However, it also imposes the following boundary condition on the current of the axion shift symmetry:

$$e^{b\phi}\, d\lambda\big|_{\partial\mathcal{M}} = i\,J\,. \tag{3.95}$$

This constraint is rather strange, as it would imply that the saddle point approximation requires λ to be imaginary. Hence, the path integral has no real saddle points. However, it is possible to perform a semiclassical approximation of it in two ways: the first method consists in using the fact that this path integral is at most quadratic in λ to compute it. The idea is that one can split up the integral into an integration over bulk fields with Dirichlet boundary conditions followed by one over the boundary fields. The former can be evaluated in the usual way by using the variational principle, since it is just a Gaussian. Then, by performing the integral over the boundary fields, one is basically Fourier transforming this result. However, this method is very cumbersome, as it requires an explicit choice of the boundary. The second method relies on the dualization procedure we described. This is a far simpler and more covariant approach, and we will be using it to evaluate the actions of our solutions. The idea is that, since the axion path integral (3.91) and the field-strength path integral (3.85) are equal to each other, instead of trying to evaluate the former, which has no real saddle points, one can just evaluate the latter, which does have saddle points. This indirectly yields a semiclassical approximation of the axion theory.

If we use the constraint dF_{D-1}, we can treat the $(D-1)$-form as locally exact; i.e $F_{D-1} = dC_{D-2}$. Then, we can derive the following equation of motion:

$$d(e^{-b\phi} * F_{D-1}) = 0\,, \tag{3.96}$$

which means that, locally, one can rewrite the field-strength as follows:

$$F_{D-1} = e^{b\phi} * d\chi\,, \tag{3.97}$$

where χ is a scalar. The equation of motion of the dilaton is the following:

$$d * d\phi + \frac{b}{2} e^{-b\phi} F \wedge *F = 0\,. \tag{3.98}$$

Substituting the definition of χ into this yields the following:

$$d * d\phi + \frac{b}{2}\, d\chi \wedge *d\chi = 0\,. \tag{3.99}$$

This equation of motion has the wrong sign in front of the χ term. One can similarly show that the Einstein equation also 'sees' a dilaton with the wrong sign. Hence, the remaining equations of motion of the resulting system are the ones we have been solving in this chapter; i.e. those of a system with a wrong sign kinetic term for the axion. At the end of the day, the result of solving the F_{D-1} equations and substituting the solution into (3.85) is effectively the same as performing a saddle point approximation of a 'would-be' imaginary scalar field χ with the following action:

$$S = \int_{\mathcal{M}} \tfrac{1}{2} \left[d\phi \wedge *d\phi - e^{b\phi}\, d\chi \wedge *d\chi + 2\, d\left(\chi\, e^{b\phi}\, * d\chi\right)\right], \tag{3.100}$$

and with the following Neumann boundary conditions for the axion current:

$$e^{b\phi}\, d\chi\big|_{\partial M} = J\,. \tag{3.101}$$

where J is the external current in (3.91) and the Hodge dual of the boundary value of F_{D-1} in (3.85). The equations of motion of the would-be scalar field χ seem to imply that J is divergenceless, which is equivalent to the constraint $dF_{D-1} = d * J = 0$. Therefore, the path integral yields a selection rule that enforces momentum conservation.

From now on, we will use the F_{D-1} action in (3.85) to evaluate the action of the non-extremal D-instanton, and the on-shell duality relation (3.97) to translate our 'electric' axionic solutions into dual 'magnetic' solutions.

It is now easy to show that this action satisfies a Bogomol'nyi bound [31]. We can rewrite the action as follows:

$$\mathcal{S}_E = \int_{\mathcal{M}} \tfrac{1}{2} \left(d\phi \wedge *d\phi + e^{-b\phi}\, F_{D-1} \wedge *F_{D-1}\right), \tag{3.102}$$

$$= \int_{\mathcal{M}} \tfrac{1}{2}[(d\phi \pm e^{-b\phi/2} * F_{D-1}) \wedge *(d\phi \pm e^{-b\phi/2} * F_{D-1}) \mp (-)^D \tfrac{4}{b} d(e^{-b\phi/2} F_{D-1})]\,, \tag{3.103}$$

where we have used the fact that $dF_{D-1} = 0$. Since the first term is positive semi-definite $\mathcal{S}_E$ is bounded from below by a topological surface term given by the last term in (3.103). The bound is saturated when the Bogomol'nyi equation

$$*F_{D-1} = \mp e^{b\phi/2} d\phi\,, \tag{3.104}$$

is satisfied. The $\mp$ distinguishes instantons from anti-instantons, and for simplicity, we will use the upper sign from now on. Using (3.97), one can write the Bogomol'nyi equation as

$$d\chi = -e^{-b\phi/2} d\phi\,, \tag{3.105}$$

and one can check explicitly that the instanton solutions with $q^2 = 0$, given in (3.41), satisfy this bound. They are therefore rightfully called extremal. The instanton action can then easily be evaluated, and has only a contribution from the boundary at infinity,

$$\mathcal{S}^{\infty}_{inst} = \frac{4}{b^2}(D-2)\mathcal{V}ol(S^{D-1})\frac{|bcq_-|}{g_s^{b/2}}\,, \tag{3.106}$$

while the contribution from $r = 0$ vanishes.

For $D = 10$ and $b = 2$, this value of the instanton action precisely coincides with [30]. For other values of b, we notice the dependence of g_s on b. In ten dimensions, the only possible value for b compatible with maximal supersymmetry is $b = 2$. One then finds that the instanton action depends linearly on the inverse string coupling constant. In lower dimensions this is not necessarily so, and more values for b are possible, depending on whether χ comes from the RR sector or from the NS sector. This would imply different kinds of instanton effects, with instanton actions that depend on different powers of the string coupling constant. This indeed happens for instance in four dimensions, after compactifying type IIA strings on a Calabi-Yau threefold. There are D-instantons coming from wrapping (Euclidean) D2 branes around a supersymmetric three-cycle, and there are NS5-brane instantons coming from wrapping the NS5-brane around the entire Calabi-Yau. As explained in [59], such instanton effects are weighted with different powers of g_s in the instanton action. This was also explicitly demonstrated in [60–62]. In our notation, they correspond[16] to $b = 1$ and $b = 2$. Our results in (3.106) are consistent with these observations.

Notice also that the instanton action is proportional to q_-. For extremal instantons, this is precisely the mass of the corresponding black hole one dimension higher, see (3.71). This is the generic characteristic of the instanton-soliton correspondence that we explained in subsection 2.3.2. There, the Euclidean action of the instanton in D dimensions equals the mass or Hamiltonian of the black hole soliton in $D+1$ dimensions. It is interesting to note that this also happens for theories with gravity.

We now turn to the case of non-extremal instantons, and focus first on the case of $q^2 > 0$. The solutions (3.33) for the dilaton and axion fields can be written as

$$d\phi = \frac{2}{b}\coth(H + C_1)dH\,, \qquad e^{-b\phi/2}F_{D-1} = \frac{2}{b}\frac{*dH}{\sinh(H + C_1)}\,, \tag{3.107}$$

and do not satisfy the Bogomol'nyi equation (3.104). To evaluate the action on this non-extremal instanton solution, we substitute these expressions into the bulk action (3.102), and find

$$\mathcal{S}_{\text{scalars}} = \frac{2}{b^2}\int d\Big(\{H - 2\coth(H + C_1)\} * dH\Big)\,, \tag{3.108}$$

which is a total derivative term. Evaluating the Ricci scalar on the solution in (3.33) we find the following:

$$S_R = -\int_{\mathcal{M}} R = -\frac{2}{b^2}\int_{\mathcal{M}} d(H * dH)\,, \tag{3.109}$$

which precisely cancels the first term of the scalar action (3.108). Hence, the bulk action is given by the following:

$$S_R + S_{\text{scalars}} = -\frac{4}{b^2}\int d\left(\coth(H + C_1)\, * dH\right)\,, \tag{3.110}$$

[16] This corrects a minor mistake in the previous version and in the version published in *JHEP*. In our conventions, the $D = 4$ dilaton is related to the $D = 10$ string dilaton by a factor of 2, see [63] for further details and implications of this correction.

which is again a total derivative. In fact, had we used the pseudo action in (3.100), we would have also ended up with a total derivative of the form $d\left(\chi\, e^{b\,\phi}\; * d\chi\right)$, which would have yielded the same result.

Using Stokes theorem, we only pick up contributions from the boundaries. Since the $q^2 > 0$ instantons have a curvature singularity at $r = r_c$ (see section 3.1), one can take these boundaries at $r = \infty$ and at $r = r_c$. In terms of the variable H, this corresponds to $H = 0$ and $H = \infty$ respectively[17]. We stress again that we have taken C_1 to be positive, in order to avoid further singularities in the scalar sector when $H + C_1 = 0$.

Besides the bulk action, one also needs to include the Gibbons-Hawking term [64], to make the action consistent with the Einstein equations:

$$\mathcal{S}_{GH} = -2 \int_{\partial\mathcal{M}} (K - K_0)\,, \tag{3.111}$$

where $\mathcal{M}$ is the D-dimensional Euclidean space and $\partial\mathcal{M}$ is the boundary. In the second term, K is the trace of the extrinsic curvature of the boundary and K_0 the extrinsic curvature one would find for flat space, which is subtracted to normalize the value of the action. The extrinsic curvature is defined in terms of a unit vector n^μ that is normal to the boundary as follows:

$$K \equiv h_\mu{}^\nu\, \nabla_\nu\, n^\mu\,, \tag{3.112}$$

where $h_\mu{}^\nu$ is the tensor that projects components onto the boundary.

Let us now evaluate the total action at both $r = \infty$ and $r = r_c$: we first discuss the boundary at $r = \infty$. The contribution from (3.111) vanishes, while (3.110) yields a contribution

$$\begin{aligned}\mathcal{S}^{\infty}_{inst} &= \frac{4}{b^2}\,(D-2)\mathcal{V}ol(S^{D-1})\, b\, c\Big(q\, \coth C_1\Big)\,,\\ &= \frac{4}{b^2}\,(D-2)\,\mathcal{V}ol(S^{D-1})\, b\, c\Big(\sqrt{q^2 + \frac{q_-^2}{g_s^b}}\Big)\,.\end{aligned} \tag{3.113}$$

In the second line, we have used the relation between C_1 and the asymptotic value of the dilaton, $g_s^b = (q_-/q)^2\, \sinh^2 C_1$.

For $q^2 = 0$, (3.113) precisely yields back the result for the extremal instanton, see (3.106). There we made the relation between the instanton action and the black hole mass one dimension higher. Also for the non-extremal instanton, such a relation seems to hold. Indeed, from the mass formula for the non-extremal black hole in terms of the instanton parameters, one has that $q\, \coth C_1 = \sqrt{q^2 + q_-^2}$, and the string coupling constant is set to unity. One therefore sees that the contribution to the instanton action from the boundary at infinity is proportional to the black hole mass one dimension higher.

The boundary at $r = r_c$ receives contributions from both integrals (3.110) and (3.111), which add up to

$$\mathcal{S}^{r_c}_{inst} = \frac{4}{b^2}\,(D-2)\mathcal{V}ol(S^{D-1})\, b\, c\Big(q\,(\frac{bc}{2} - 1)\Big)\,. \tag{3.114}$$

[17] Without loss of generality, we can choose $q > 0$.

Because the dilaton *and* the curvature blow up at r_c, the supergravity approximation and string perturbation theory both break down. Hence, it is not clear whether this contribution is meaningful. One might take the point of view that string theory corrections, which are expected to take over at r_c, would actually smooth the singularities out. In that case, there would be no need to consider this point as a boundary, and no need to take this contribution into account. It is also plausible, however, that string theory corrections completely modify the geometry, 'opening up' a wormhole that leads into a whole new space. In that case, a second boundary would exist, but the values of the fields might be different there.

Note that this contribution vanishes for the case $bc = 2$, while it is positive for $bc > 2$. However, as discussed above, it is not at all clear whether this contribution to the integrals (3.110) and (3.111) should be included in the instanton action, since it is calculated in a region of space where the supergravity approximation is no longer valid.

We now turn to the case of $q^2 < 0$, or with $q = i\tilde{q}$, a positive $\tilde{q}^2 > 0$. A similar calculation as for $q^2 > 0$ shows that, for the solution (3.45), we have

$$d\phi = \frac{2}{b}\cot(\tilde{H} + \tilde{C}_1)d\tilde{H}\,, \qquad e^{-b\phi/2}F_{D-1} = \frac{2}{b}\frac{*d\tilde{H}}{\sin(\tilde{H} + \tilde{C}_1)}\,, \tag{3.115}$$

where

$$\tilde{H} = bc\arctan(\frac{\tilde{q}}{r^{D-2}})\,, \tag{3.116}$$

is a harmonic function over the geometry given by the metric in (3.45). Plugging in these expressions into the bulk action (3.102), we find

$$\mathcal{S}_{inst} = -\frac{2}{b^2}\int d\Big(\{\tilde{H} + 2\cot(\tilde{H} + \tilde{C}_1)\} * d\tilde{H}\Big)\,. \tag{3.117}$$

Since this is a total derivative, we can use Stokes theorem again to reduce it to an integral over the boundaries. These boundaries are at $r = \infty$ and $r = 0$, where we required that $bc < 2$, as discussed in section 3.1. In contrast to the discussion of the $r = r_c$ boundary for $q^2 > 0$, the instanton solution is perfectly regular everywhere, in particular at both boundaries. Therefore the contribution from the boundary at $r = 0$ can also be trusted.

In addition to the above action, one also needs to include the gravitational contribution (3.111). Similar to the case of $q^2 > 0$, the first term of (3.117) is cancelled by the contribution from the Ricci scalar. We anticipate the Gibbons-Hawking term not to contribute, since the two asymptotic geometries at $r = 0$ and $r = \infty$ are equivalent due to the $\mathbb{Z}_2$-symmetry (3.74). Hence, their contributions should cancel.

Therefore the $q^2 < 0$ instanton action has contributions only from the second term of (3.117) from both boundaries at $r = 0$ and $r = \infty$:

$$\begin{aligned}
\mathcal{S}^{\infty}_{inst} &= \frac{4}{b^2}(D-2)\mathcal{V}ol(S^{D-1})\,b\,c\,\tilde{q}\Big(\cot\tilde{C}_1\Big)\,,\\
\mathcal{S}^{0}_{inst} &= \frac{4}{b^2}(D-2)\mathcal{V}ol(S^{D-1})\,b\,c\,\tilde{q}\Big(-\cot(\tilde{C}_1 + bc\frac{\pi}{2})\Big)\,.
\end{aligned} \tag{3.118}$$

Due to the fact that $\tilde{C}_1$ and $\tilde{C}_1 + bc\pi/2$ are on the same branch of the cotangent (due to the restriction of regular scalars for $0 < r < \infty$, which can only be achieved for $bc < 2$, see section 3.1), the total instanton action is manifestly positive definite. In the neighborhood of $bc \approx 2$, the instanton action becomes very large, and the limit to the extremal point where $bc = 2$, is discontinuous. This shows that this instanton is completely disconnected from the extremal D-instanton.

Using the asymptotic value of the dilaton in (3.45), we have $g_s^b = (q_-/\tilde{q})^2 \sin^2 \tilde{C}_1$, and therefore $\tilde{q}^2 < q_-^2/g_s^b$. Assuming that $\cot \tilde{C}_1 > 0$, the contribution from infinity is positive and can be rewritten as

$$\mathcal{S}_{inst}^{\infty} = \frac{4}{b^2}(D-2)\,\mathcal{V}ol(S^{D-1})\,b\,c\,\sqrt{\frac{q_-^2}{g_s^b} - \tilde{q}^2}\,, \tag{3.119}$$

which is the analytic continuation of the result with $q^2 > 0$.

3.6.2 Tunneling interpretation

The reader may wonder what the tunneling interpretation of a D-instanton is. In a standard non-gravitational QFT, the metric is fixed and one always knows what the Euclidean time direction is, because one knows how the theory was Wick rotated in the first place. In a theory where the metric is dynamical, however, this is not straightforward at all. Since the Euclidean spacetime is not part of the input, but rather the outcome of the equations of motion, which direction is viewed as time-like is not determined *a priori*. For our solutions, one might be tempted to think of r as the Euclidean time parameter, since all fields depend on it. However, this wouldn't lead to the tunneling interpretation we are after. Take for instance the case $q^2 = 0$, which has a flat space. Let us Wick rotate this back to Lorentzian signature taking the r direction to be time:

$$dr^2 + r^2\, d\Omega^2_{S^{D-1}} \rightarrow -dt^2 + t^2\, d\mathbb{H}^2_{D-1}\,. \tag{3.120}$$

See chapter 7 for a derivation of this Wick rotation. The initial slice $t = 0$ is singular, and the later slices are hyperbolic spaces. These are not the initial and final states one would like to have for a tunneling interpretation. The more natural Wick rotation takes place in Cartesian coordinates. Letting $r = (x_0^2 + ... + x_{D-1}^2)^{1/2}$, and rotating $x_0 \rightarrow i\,t$.

Another reason not to pick r as a time direction is the fact that, for our solutions, the axion current $e^{b\,\phi}\,\partial\chi$ would be conserved in the r direction, since our Ansatz is such that the axion equation of motion is the following:

$$\partial_\mu \left(e^{b\,\phi}\,\nabla^\mu \chi\right) = \partial_r \left(e^{b\,\phi(r)}\,\nabla^r \chi(r)\right) \sim \delta(r)\,. \tag{3.121}$$

This means that, in the r direction, there would be no charge conservation violation due to tunneling, and hence no interesting tunneling effect in any way. If we pick x_0, however, then the point $r = 0$ will act as a source-like singularity[18] (for the cases with $q^2 \geq 0$) and will generate a charge difference between the initial and final states. See figures 3.5(a) and 3.5(b). One can calculate that this difference will be $\sim q_-$ for our solutions. Classically one could say that the δ-function in the equations of motion for χ can be reproduced by adding a source term in the action of the form $\chi\,\delta(r)$. From the point of view of the path integral in (3.91), one should

[18]This is basically because $\Box\, H(r) \sim \delta(r)$ for our harmonic functions.

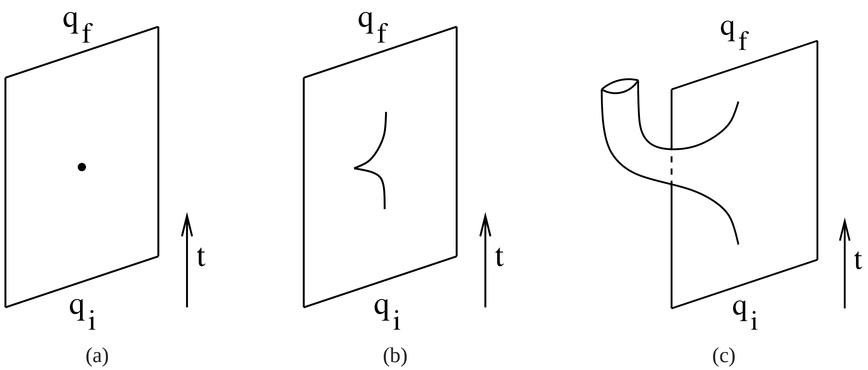

Figure 3.5: *The tunneling interpretation. Figures (a), (b), and (c) depict the* $q^2 = 0, > 0$, *and* < 0 *respectively. The first two solutions have a charge conservation violation because they have electric source-like singularities at the origin* $r = 0$*. The worhole (c) conserves its total charge, but splits up into two disconnected spaces* $\mathbb{R}^{D-1} \oplus S^{D-1}$*, so that an observer on* $\mathbb{R}^{D-1}$ *will see a charge loss.*

add a term of the form $i\,\lambda\,\delta(r)$ to the action; and in the path integral in (3.85) this corresponds to adding $i\,dF_{D-1}\,\delta(r)$. This will supplement the charge conservation or closedness constraint, respectively:

$$dF_{D-1} + \delta(r) = d * J + \delta(r) = 0\,. \tag{3.122}$$

Such a term is a local operator insertion or vertex operator in the path integral, i.e. $\langle \exp(i\,\lambda(r_0))\rangle$.

In the case of the wormhole, the point $r = 0$ is not included in the manifold, and there is no δ-function-like singularity. In order to find the tunneling interpretation, one must first cut the wormhole in half at its neck, and suggestively redraw the shape of the remaining geometry as in figure 3.5(c), as was done in [28]. The axion charge is globally conserved, but the manifold splits up into two disconnected spaces as follows:

$$\mathbb{R}^{D-1} \to \mathbb{R}^{D-1} \oplus S^{D-1}\,. \tag{3.123}$$

Although the total charge is conserved, an observer who stays on the $\mathbb{R}^{D-1}$ will see a charge loss, because the S^{D-1} *baby universe* will carry off some charge with it. From the string theory point of view, it is possible that, for the case $q^2 > 0$, string theory corrections will change the singular geometry into a smooth one, perhaps by 'opening up' a wormhole-like geometry where the singularity was. This would restore global axion charge conservation.

3.6.3 Correlation functions

Once the instanton solutions are established, one would like to study their effect in the path integral. As for D-instantons in ten-dimensional IIB, they contribute to certain correlation functions via the insertion of fermionic zero modes. For the D-instanton, which is 1/2 BPS, there are sixteen fermionic zero modes. These are solutions for the fluctuations that satisfy the linearized Dirac equation in the presence of the instanton. All of these zero modes can be generated by acting with the broken supersymmetries on the purely bosonic instanton solution. For the non-extremal instantons, no supersymmetries are preserved, so there are more fermionic zero modes. Let us focus for simplicity on ten-dimensional type IIB. Since all supercharges are broken, one can generate 32 fermionic zero modes. The path integral measure contains an integration over these fermionic collective coordinates, and to have a non-vanishing result, one must therefore insert 32 dilatinos in the path integral. Based on this counting argument of fermionic zero modes, a 32-point correlator of dilatinos would be non-zero, and induce new terms in the effective action, containing 32 dilatinos. In the full effective action, such terms are related to higher curvature terms like e.g. certain contractions of R^8. An explicit instanton calculation should be done to determine the non-perturbative contribution to the function that multiplies R^8. As for the D-instanton, we expect that the contributions of the instantons with different q^2-values build up a modular form with respect to $SL(2,\mathbb{Z})$, possibly after integrating over q^2.

These issues, though important, lie beyond the scope of this chapter, and are left open for investigation.

3.7 Discussion

In this chapter we investigated non-extremal instantons in string theory that are solutions of a gravity-dilaton-axion system with dilaton coupling parameter b. In particular, we constructed an $SL(2,\mathbb{R})$ family of spherically symmetric instanton-like solutions in all conjugacy classes labelled by q^2. Among these is the (anti-)D-instanton solution with $q^2 = 0$. For special values of the dilaton coupling parameter this solution is half-supersymmetric. The instanton solutions in the other two conjugacy classes, with $q^2 > 0$ and $q^2 < 0$, are non-supersymmetric and can be viewed as the non-extremal versions of the (anti-)D-instanton. This view is confirmed by the property that instantons in these two conjugacy classes, for $bc \geq 2$ with c defined in (3.26), can be uplifted to non-extremal black holes.

We stressed the wormhole nature of the instanton solutions. We found that each conjugacy class leads to a wormhole geometry provided the corresponding instanton is given in a particular metric frame:

$$\begin{aligned} q^2 > 0 \quad &\leftrightarrow \quad \text{dual frame metric (only for } bc = 2 \text{ or } q_- = 0) \\ q^2 = 0 \quad &\leftrightarrow \quad \text{string frame metric} \\ q^2 < 0 \quad &\leftrightarrow \quad \text{Einstein frame metric} \end{aligned} \tag{3.124}$$

For all these cases the metric takes the form (3.46), with the specific values given in section 3.2.

Not all instanton solutions we constructed are regular and not all can be uplifted to black holes. The non-extremal instantons in the $q^2 > 0$ conjugacy class all have a curvature singularity at $r = r_c$, see (3.35). Only the $bc = 2$ instanton can be uplifted to a regular non-extremal RN black hole with the singularity being resolved as a coordinate singularity at the outer event horizon of the RN black hole. The singularity remains for $bc > 2$ and in that case can be resolved by adding an extra dilaton to the original system [51]. Two exceptions are the limits $q^2 \to 0$ or $q_- \to 0$, which correspond to the extremal and Schwarzschild black hole solutions, respectively. Finally, the instantons in the $q^2 < 0$ conjugacy class are only regular for $bc < 2$. These instantons can never be uplifted to black holes.

We have also considered the uplift of our instanton solutions to p-branes. It turns out that an instanton can only be uplifted over a $(p+1)$-torus to a p-brane provided the dilaton coupling satisfies (following from (3.84))

$$b\,c \geq \sqrt{\frac{4(p+1)(D-1)}{D+p-1}}\,. \tag{3.125}$$

For the case that saturates this bound, the instanton with $q^2 \geq 0$ uplifts to a regular non-dilatonic p-brane. For larger values of b, the instanton solution (3.24) with $q^2 > 0$ uplifts to a singular limit of the dilatonic p-branes of [53]. These solutions only become regular in the limit $q^2 \to 0$ or $q_- \to 0$. A summary of the possible regular solutions is given in table 3.1. Alternatively, we have discussed the possibility of adding an extra dilaton to the instanton solution [51], which allows for the uplift to the regular dilatonic p-branes of both type 1 and type 2.

bc	Dimension	Regular solutions
< 2	D	Instantons with $q^2 \leq 0$, see (3.45)
$= 2$	$D+1$	RN black holes with $q^2 \geq 0$, see (3.67), or Schwarzschild black holes with $q^2 > 0$, $q_- = 0$
> 2	$D+1$	Dilatonic black holes with $q^2 = 0$ or Schwarzschild black holes with $q^2 > 0$, $q_- = 0$
$=$ in (3.125)	$D+p+1$	Non-dilatonic p-branes with $q^2 \geq 0$
$>$ in (3.125)	$D+p+1$	Dilatonic p-branes with $q^2 = 0$ or $q^2 > 0$, $q_- = 0$

Table 3.1: *The regular instanton, black hole and p-brane solutions that are obtained, depending on the dilaton coupling parameter b, the conjugacy class q^2 and the charge q_-.*

For the particular value $b = 2$, corresponding to $\Delta = 4$, there is another higher-dimensional origin. In this special case, the D-dimensional extremal instanton can be uplifted to a gravitational wave in $D+2$ dimensions [35]. Similarly, the other two conjugacy classes uplift to purely gravitational solutions in $D+2$ dimensions which we denominate “non-extremal waves”. The terminology is slightly misleading since the uplift only leads to a time-independent solution. Whether this solution can be extended to a time-dependent wave-like solution remains to be seen. It is also interesting to note the following curiosity: the source term for a pp-wave is a massless particle, i.e. a particle with a null-momentum vector: $p^2 = 0$. This suggests that we associate the source terms for the other two conjugacy classes with massive particles ($p^2 > 0$) and tachyonic particles ($p^2 < 0$). We leave this for future investigation.

In the second part of this chapter, we investigated whether the non-extremal instantons might contribute to certain correlation functions in string theory. For this application, it is a prerequisite that there be a well-defined and finite instanton action. Mimicking the calculation of the standard D-instanton action, we found that for $q^2 > 0$ the contribution from infinity to the instanton action, for all values of b, is given by the elegant formula (3.113). This action reduces to the standard D-instanton action for $q^2 = 0$. Having a finite action, the non-extremal instantons might contribute to certain correlation functions. In the case of type IIB string theory, we conjectured that non-extremal instantons contribute to the R^8 terms in the string effective action in the same way that the extremal D-instantons contribute to the R^4 terms in the same action. Whether the fact that all supersymmetries are broken by the non-extremal instantons poses problems remains to be seen. An explicit instanton calculation should decide whether our conjecture is correct. We leave this for future investigation.

Finally a few comments on some work in progress [65]. A natural and very interesting generalization to the solutions in this chapter can be achieved by adding a negative cosmological constant in the action. Just as the solutions we have studied here are asymptotically flat, solutions in a system with a cosmological constant are asymptotically *anti-de Sitter* or AdS. Asymptotically AdS spaces are particularly interesting in light of Maldacena's breakthrough in [1], where he conjectured that type IIB string theory in an $AdS_5 \times S^5$ background is completely equivalent to $\mathcal{N} = 4, d = 4$ super-Yang-Mills theory. The stronger version of his conjecture states that string theory on an asymptotically $AdS_5 \times S^5$ background is dual to some deformation of super-Yang-Mills. This duality has been used to show that the extremal D-instanton of type IIB supergravity corresponds to the super-Yang-Mills self-dual instanton [66–70]. It would be interesting to see what the field theory dual of a non-extremal D-instanton is. Perhaps it contains information about non-self-dual Yang-Mills instantons.

This concludes the first part of this thesis, which covered the topic of instantons. In the next two chapters, we will look at a different kind of scalar-gravity solutions that also have interpolating behavior: cosmological solutions. These are solutions of the Einstein equations that also depend on only one parameter, however, that parameter is Lorentzian time.