Instantons and cosmologies in string theory

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Chapter 2

Instantons

In this chapter we will study the basics of instantons, heavily borrowing material from the classic textbooks by S. Coleman [14] and R. Rajaraman [15]. First, we will see their application to quantum mechanics, which is conceptually and technically the simplest framework to introduce the topic. Then, we will move on to quantum field theory, where the example of the Yang-Mills instanton will give us all the tools to understand these objects in generality. Finally, solitons will be briefly introduced, and we will see how sometimes an instanton in D Euclidean dimensions can correspond to a soliton in $D+1$ Lorentzian dimensions.

2.1 Introduction

2.1.1 An alternative to WKB

In quantum mechanics it is possible for a particle to penetrate a region of potential energy that is higher than the particle's own energy. This classically forbidden motion is known as *quantum tunneling* and, for a general potential barrier, one can compute the tunneling amplitude of a particle by means of the WKB approximation. The latter is a so-called *semiclassical* approximation, which means that it requires small $\hbar$. Let us see what happens in the case of a particle of unit mass in 1 + 1 dimensions, subject to some potential $V(x)$.

The Schrödinger equation reads:

$$\frac{d^2\psi}{dx^2} = \frac{2\,(V(x)-E)}{\hbar^2}\,\psi\,. \tag{2.1}$$

If $V(x) = constant$, then the solution would be a plain wave:

$$\psi \propto e^{-ikx}, \quad \text{where} \quad k \equiv \frac{\sqrt{2\,(E-V)}}{\hbar}\,. \tag{2.2}$$

In the case of quantum tunneling $V > E$, so the momentum becomes imaginary, and instead of a plain wave, we obtain an exponentially decreasing function:

$$\psi \propto e^{-\kappa x}, \quad \text{where} \quad \kappa \equiv \frac{\sqrt{2\,(V-E)}}{\hbar}\,. \tag{2.3}$$

Let us now take a non-constant potential but make the approximation that $V(x)$ varies slowly compared to the rate of decay κ of the wave function. Then, we can rewrite the Schrödinger equation as follows:

$$\frac{d\psi}{dx} = \pm\frac{\sqrt{2\,(V(x)-E)}}{\hbar}\,\psi\,, \tag{2.4}$$

Differentiating this equation yields the original Schrödinger (2.1) upon dropping a term proportional to $V'/\hbar^2\,\kappa$. The solution for a particle tunneling to the right is then:

$$\psi \propto \exp\left(-\frac{1}{\hbar}\int \sqrt{2\,(V(x)-E)}\,dx\right). \tag{2.5}$$

The amplitude for the particle to tunnel is then:

$$\exp\left(-\frac{1}{\hbar}\int_a^b \sqrt{2\,(V(x)-E)}\,dx\right), \tag{2.6}$$

where a and b are the beginning and endpoint of the tunneling trajectory.

The approximation we made is a semiclassical one in the sense that it requires that $\hbar$ be ‘small’. To see this, recall that differentiating the equation we actually solved (2.4) yielded the true Schrödinger (2.1) equation if we dropped a V' term. Comparing this term to the term that we did keep shows that the dimensionless quantity we are neglecting is $\hbar\,V'/(2\,(V-E))^{3/2}$, which is small in the semiclassical limit $\hbar \to 0$.

Now that we have obtained this result by using the WKB approximation, we will rederive it through a completely different method, which will be the subject of this chapter: the method of instantons.

Let us begin by rewriting (2.5) in a different way. First, we set the energy of the particle to zero (which can always be done via a suitable shift in the potential), $E = 0$. Then, we have:

$$\int_a^b \sqrt{2\,(V(x)-E)}\,dx = \int_a^b i\,p\,dx = \int_a^b i\,\frac{dx}{dt}\,dx\,, \tag{2.7}$$

where p is the momentum of the particle, and in the second equation we used the fact that the mass has been set to 1. If we perform a Wick rotation $t \to i\,\tau$ we can write this as follows:

$$\int_{\tau_a}^{\tau_b} p\,\dot{x}\,d\tau = \int_{\tau_a}^{\tau_b} \mathcal{L}_E\,d\tau = \mathcal{S}_E\,, \tag{2.8}$$

where $\mathcal{S}_E$ is the action of the classical particle in Euclidean spacetime with zero energy. This teaches us a new way to compute tunneling amplitudes. Simply compute the Euclidean action of the tunneling trajectory. To see where this comes from, let us turn to the language of path integrals.

Let us compute the tunneling amplitude for the same $(1+1)$-dimensional problem using path integrals. The amplitude is given by the following:

$$K(a,b;T) \equiv \langle x=a|e^{i\,H\,T/\hbar}|x=b\rangle = \int d[x(t)]\,e^{i\,S[x(t)]/\hbar} \tag{2.9}$$

$$\text{with} \qquad S \equiv \int_{t_a}^{t_b} \left(\tfrac{1}{2}\,(dx/dt)^2 - V(x)\right)\,dt\,, \tag{2.10}$$

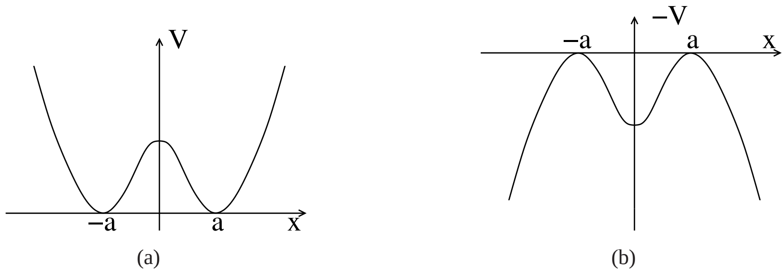

Figure 2.1: *Figure (a) depicts a double-well potential, while figure (b) depicts the inverted potential.*

where the path integral sums over all paths from $x = a$ to $x = b$, and $T \equiv t_b - t_a$. If we now analytically continue this to Euclidean spacetime (i.e. $t \to i\tau$), this becomes:

$$K_E(a, b; T) \equiv \langle x = a|e^{-HT/\hbar}|x = b\rangle = \int d[x(\tau)]\, e^{-S_E[x(\tau)]/\hbar} \tag{2.11}$$

$$\text{with} \qquad S_E \equiv \int_{\tau_a}^{\tau_b} \left(\tfrac{1}{2}(dx/d\tau)^2 + V(x)\right) d\tau\,. \tag{2.12}$$

There are basically two motivations to perform this Wick rotation: firstly, the Minkowskian path integral is rigorously speaking not well-defined. It is difficult to prove that the phases of trajectories that greatly differ from the classical path actually cancel out, in order to make the path integral convergent. However, since the partition function is an analytic function of time, one can properly define the path integral by Wick rotating into Euclidean signature, which yields a well-defined convergent object, and then Wick rotating physical results back to Minkowskian signature.

The second motivation is the fact that the partition function $\langle e^{-HT/\hbar}\rangle$, in the limit $T \to \infty$, projects the lowest energy eigenstates. This provides information about vacuum energy and the ground state wave function, as we will see later on. From this point of view, there is no need to think in terms of Euclidean time. The path integral for the partition function can be derived from first principles without use of the Wick rotation.

If we now take the limit $\hbar \to 0$, we see that the largest contribution to this path integral will come from a trajectory that minimizes the Euclidean action. If S_0 is the value of the action for such a trajectory, then, to leading order in $\hbar$, the Euclidean amplitude will go like $K_E \propto e^{-S_0/\hbar}$. The problem of extremizing the Euclidean action S_E is equivalent to that of extremizing the Minkowskian action of a particle subject to an inverted potential $-V(x)$. More explicitly, the variational equation of the Euclidean action (2.12),

$$\frac{d^2x}{d\tau^2} - \frac{dV}{dx} = 0\,, \tag{2.13}$$

looks just like the *classical* equation of motion of a particle in a potential $-V(x)$, as shown in figure 2.1(b). Solving this equation, we find that

$$\frac{dx}{d\tau} = \sqrt{2V}\,, \tag{2.14}$$

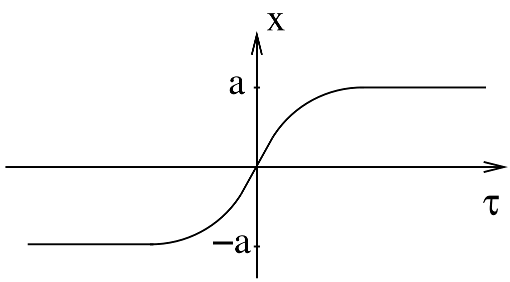

Figure 2.2: *The kink solution: a classically forbidden trajectory that interpolates between the two classical vacua of the double-well potential.*

and using this we can rewrite the action (2.12) as

$$S_0 = \int_{\tau_a}^{\tau_b} 2\,V\,d\tau = \int_{\tau_a}^{\tau_b} 2\,V \frac{d\tau}{dx}\,dx = \int_{\tau_a}^{\tau_b} \sqrt{2\,V}\,dx\,, \tag{2.15}$$

which matches our WKB calculation (for $E = 0$) (2.6). So, in order to compute a tunneling amplitude, instead of thinking of a classically forbidden trajectory where the particle goes through a potential barrier such as the one depicted in figure 2.1(a), we simply compute the action for a classically allowed trajectory where the particle rolls down from the top of the left-hand side hill and then up to the top of the right-hand side hill of the inverted potential in figure 2.1(b).

This classical trajectory $x_{cl}(\tau)$ will qualitatively have the shape depicted in fig 2.2. It is usually referred to as the *kink*. The precise shape of this trajectory is not important. What matters is that this function interpolates between the two constant functions, $x = -a$ and $x = a$, which are the two classical vacua of the double-well problem. It differs significantly from those two constant values only within a localized region in the range of τ, so the Lagrangian density is itself non-zero only in a finite region. It is because of this that the trajectory has finite action, giving rise to a non-zero contribution to the path integral.

2.1.2 A tool of the trade: The semiclassical approximation

Although the minimum of the Euclidean action gives the largest contribution to the path integral, it only constitutes a "point" of measure zero in the space of all trajectories we integrate over. This is emphatically stated and clearly explained in Coleman's work [14]. It is, therefore, a bit too brutal and incorrect to define the semiclassical approximation as a sum of contributions of the minimum (or minima) of the action. The semiclassical approximation consists in computing the path integral by approximating the *regions* around the local minima of the action with Gaussians. Although it is treated extensively in many standard QFT books such as [16], we will briefly go over it here. Let us start with by computing the following one-dimensional integral as a toy example:

$$I = \int_{-\infty}^{+\infty} \exp(-f(x)/\hbar)\,dx\,, \tag{2.16}$$

where we assume that $f(x)$ is bounded from below and has exactly one minimum at $x = x_0$. By expanding the function in its Taylor series around x_0, we can re-write the integral as follows:

$$I = \int_{-\infty}^{+\infty} dx \, \exp\left(-\frac{1}{\hbar}\left(f(x_0) + \tfrac{1}{2}(x-x_0)^2 f''(x_0) + O\left((x-x_0)^3\right)\right)\right), \tag{2.17}$$

$$= \exp(-\frac{1}{\hbar} f(x_0)) \int_{-\infty}^{+\infty} d\overline{x} \, \exp\left(\frac{1}{2\hbar} \overline{x}^2 f''(x_0)\right) h(\overline{x}), \tag{2.18}$$

where $\overline{x} = x - x_0$ and $h(\overline{x})$ contains the higher order terms. If we take the limit $\hbar \to 0$, it can be easily shown that the Gaussian in the integrand becomes a δ-function of strength $\sqrt{2\pi\hbar/f''(0)}$. Since $h(x_0) = 1$, we have the following result[1] for small $\hbar$:

$$I \approx \exp\left(-\frac{1}{\hbar} f(x_0)\right) \sqrt{\frac{\pi\hbar}{f''(x_0)}} \, (1 + O(\hbar)) \, . \tag{2.19}$$

Therefore, the semiclassical approximation does not only sum points of measure zero, it actually sums over the regions around minima. These regions have non-zero measure. This is reflected by the fact that the result (2.19) contains not only the value of the action minimum $f(x_0)$, but also the curvature around it $f''(x_0)$. In the case where $f(x)$ has many local minima one must approximate the calculation by summing over several Gaussian integrals, each centered at a local minimum.

In quantum mechanics, one performs an integral over the infinite dimensional space of paths $x(\tau)$, and the function f is replaced by the functional $S[x(\tau)]$, the action. If we discretize time, (i.e. $\tau = ..., \tau_{-i}, \tau_{-i+1}, ..., \tau_0, ...\tau_{i-1}, \tau_i, ...$), then the variables of the integral become the $x_i \equiv x(\tau_i)$. Let us rewrite our action as follows:

$$S[x(\tau)] = \int d\tau \, (-x\,\partial_\tau^2\, x + V(x)), \tag{2.20}$$

where we partially integrate the kinetic term. Notice that in a discrete time a derivative is simply a difference, i.e. $x'(\tau) \to x_{i+1} - x_i$; therefore, the kinetic term of the action can be represented by a matrix, $-x\,\partial^2\, x \to \sum_{i,j} x_i\, D_{ij}\, x_j$ for some symmetric D_{ij}. Hence, we can write the action as

$$S[x(\tau)] \to S(x_{0i}) = \sum_j \left(-\sum_k x_j\, D_{jk}\, x_k + V(x_j)\right) \tag{2.21}$$

for some proper choice of D_{jk}. Now, let us perform the semiclassical approximation by expanding the action around its minimum, x_{0i} (the classical path), and keeping only the quadratic terms:

$$S[x_i] = S[x_{0i}] + \sum_{jk} \overline{x}_j \, \frac{\partial^2 S[x_0]}{\partial x_j \partial x_k} \, \overline{x}_k \, , \tag{2.22}$$

$$= S_0 + \sum_{jk} \overline{x}_j \left(-D_{jk} + \frac{\partial^2 V(x_{0j})}{\partial x_j \partial x_k}\, \delta_{jk}\right) \overline{x}_k = S_0 + \sum_{jk} \overline{x}_j \left(A_{jk}\right) \overline{x}_k \, , \tag{2.23}$$

[1] Note that this requires $f'' \neq 0$.

where $S_0 \equiv S[x_{0i}]$, $\overline{x_i} \equiv x_i - x_{0i}$, and A_{jk} is some matrix. This form of the action now looks like the exponent of a multi-variable Gaussian. The result for a M-variable Gaussian integral with a generic matrix $\mathbf{A}$ is the following[2]:

$$\int_{\infty}^{+\infty} d\mathbf{x}\, \exp\left(-\frac{1}{2\hbar}\mathbf{x}^T\,\mathbf{A}\,\mathbf{x}\right) = \sqrt{\frac{(2\pi\hbar)^M}{\det\mathbf{A}}}\,, \tag{2.24}$$

where the determinant can be computed as a product of eigenvalues. In the continuum limit, the path integral defines determinants for operators. In the case at hand, it defines the following:

$$\int d[x(\tau)]\, \exp\left[-\frac{1}{2\hbar}\int d\tau\, x\left(-\partial_\tau^2 + V''(x_0(\tau))\right) x\right] = \frac{N}{\sqrt{\det\left(-\partial^2 + V''(x_0(\tau))\right)}}\,, \tag{2.25}$$

where N is a normalization constant, and the determinant can be computed by analogy with matrices, i.e. by finding the eigenfunctions of the operator $(-\partial^2 + V''(x_0))$ and then taking the product of their eigenvalues.

This is a natural point to give a definition of an instanton.
Definition: *An instanton is a solution to the Euclidean equations of motion with finite, non-zero action.* This definition ensures that the instanton is a saddle point that will contribute to a path integral.

Let us now get back to our double-well problem. We set out to compute the tunneling amplitude $< -a\,|\,e^{-HT/\hbar}\,|\,a >$ with the path integral given in (2.11). To apply the semiclassical approximation, we need to find the configurations with minimal Euclidean action. The kink in figure 2.2 is the absolute minimum of the action, so we should compute the path integral by means of a Gaussian integral centered around the kink. However, the kink is not the only minimum, it is only the absolute one. The action (2.12) has several local minima which have to be summed over too. One can take a sequence of kinks and *anti*-kinks as shown in figure 2.3. Any alternating sequence will do as long as it satisfies the boundary conditions of the path integral.[3]

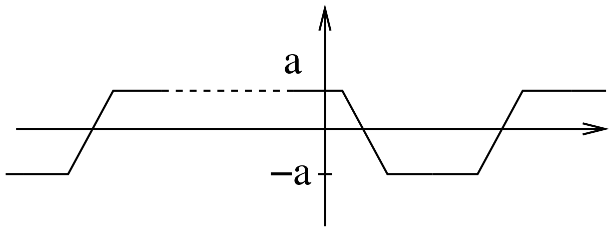

Figure 2.3: *An alternating sequence of kinks and anti-kinks. This interpolating trajectory is a local minimum of the Euclidean action.*

Another subtlety is that if the range T of τ is infinite, each kink or anti-kink can be displaced along the time axis by an arbitrary amount and yield a new trajectory whose action is equal to the

[2]In analogy with the one-dimensional case, this requires $\det(\mathbf{A}) \neq 0$

[3]Sequences of kinks and anti-kinks are only true stationary points in the limit where the range of Euclidean time $T \to \infty$, which is the limit we will always be interested in.

previous one. For instance, the one-kink trajectory can be centered around any value τ' and the value of its action will be independent of τ'. This means that we have to sum over the positions of the (anti-)kinks in each sector of the path integral. This is reflected in (2.25) by the fact that the operator $-\partial^2 + V''(x_0)$ will have some zero eigenvalues, or *zero modes*. This would *a priori* yield an infinite result for the amplitude calculation. Fortunately, there is a trick to "factor out" the infinity and cancel it against the N in (2.25). This is the Fadeev-Popov trick, which I will not derive here. For a pedagogical derivation of it, the reader is referred to [17].

The contribution to the amplitude from a single kink is the following:

$$\langle -a|e^{-HT/\hbar}|a\rangle_{(1)} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}\, K\, e^{-S_0/\hbar}\, T\,, \tag{2.26}$$

where $\omega \equiv V''(-a) = V''(a)$, and K is a constant which takes into account the calculation of the translational zero mode. Note that this is proportional to $e^{-S_0/\hbar}$, as expected. This is the biggest contribution to the tunneling process. Now we need to sum over all configurations with kink-anti-kink sequences. If we use the $T \to \infty$ approximation then, in most of the configurations, the kinks and antikinks will be far away from each other, in which case the action becomes additive, i.e. $S_{\text{kink+antikink}} = S_{\text{kink}} + S_{\text{antikink}} = 2\, S_{\text{kink}}$ [4]. Each (anti)kink also brings a power of K with it. In a tunneling trajectory from $-a$ to $+a$ there must always be one kink more than there are antikinks. Our task is then clear, the calculation and result are the following:

$$K(-a, a; T) = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{\text{odd } n} \frac{(K\, e^{-S_0/\hbar}\, T)^n}{n!} \tag{2.27}$$

$$= \frac{1}{2}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} \left[\exp\left(-\tfrac{1}{2}\,\omega T + K\, e^{-S_0/\hbar}\, T\right) + \exp\left(-\tfrac{1}{2}\,\omega T - K\, e^{-S_0/\hbar}\, T\right)\right]. \tag{2.28}$$

2.1.3 True vacua

Consider again the particle in $1 + 1$ dimensions subject to a double-well potential as depicted in fig 2.1(a). What is the vacuum structure of this problem?

If we neglected tunneling effects, our classical intuition would tell us that the ground state of the particle will be localized at one of the two wells. To find such a state, we would pick one of the wells (say, the one at $x = -a$), and approximate it with a parabolic or harmonic oscillator potential around its center,

$$V(x - a) = V(-a) + \frac{1}{2}\,\omega^2\, x^2 + O(x^3), \quad \text{where} \quad \omega^2 \equiv V''(-a)\,. \tag{2.29}$$

Then, we would solve the harmonic oscillator as usual, and do the same for the other well. This would lead us to conclude that the ground state is degenerate, namely, that there are two ground states, each localized at one well:

$$\psi_{-a}(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{\pi\hbar}\,(x + a)^2\right), \qquad E_{-a} = \tfrac{1}{2}\,\hbar\,\omega\,,$$

$$\psi_{a}(x) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{\pi\hbar}\,(x - a)^2\right), \qquad E_{a} = \tfrac{1}{2}\,\hbar\,\omega\,. \tag{2.30}$$

[4] an antikink has the same action as a kink

However, we know that a particle can tunnel from one well to the other, so these states we have constructed are not really stationary. This means that they are not energy eigenstates, and therefore, not vacuum states. A true vacuum state will have to be some linear combination of the two states we constructed in the naïve perturbative approach (2.30). As we will see next, instantons will give us all the information we need about this system.

Let us take a closer look at what the tunneling amplitudes we computed in the previous subsection tell us. Let $| E_n >$ be the set of true energy eigenstates of this system, then,

$$K(-a, a; T) \equiv \langle -a|e^{-HT/\hbar}|a\rangle \tag{2.31}$$

$$= \sum_n \langle -a|E_n\rangle\, \langle E_n|a\rangle\; e^{-E_n T/\hbar}\,, \tag{2.32}$$

which in the large T limit yields:

$$K(-a, a; T) = \sum_{\substack{\text{Lowest}\\ \text{energy states}}} \langle -a|E_n\rangle\, \langle E_n|a\rangle\; e^{-E_n T/\hbar}\,. \tag{2.33}$$

This provides us very valuable information. Comparing this to (2.28) we realize that the energies of the two lowest energy eigenstates are

$$E_\pm = \tfrac{1}{2}\,\hbar\,\omega \pm \hbar\,K\,e^{-S_0/\hbar}\,, \tag{2.34}$$

where E_- is the true ground state energy and E_+ is the energy of the second lowest level. Equation (2.33) also tells us what the wave functions of these states look like:

$$\langle -a|E_\pm\rangle\, \langle E_\pm|a\rangle = \langle -a|E_\pm\rangle\, \langle E_\pm|a\rangle = \mp\tfrac{1}{2}\left(\frac{\omega}{\pi\,\hbar}\right)^{1/2}\,. \tag{2.35}$$

The ground state wave function is spatially even and can be shown to coincide with an even linear combination of the two wave functions in (2.30) to leading order in the approximation of the potential. The next energy level is spatially odd.

The lesson instantons teach us is that when the vacuum of a system is *classically* degenerate, tunneling effects lift the degeneracy, and the quantum mechanical vacuum state will be a linear combination of the naive wave functions that respects the symmetry of the potential. In the case of the double-well problem, the vacuum state turned out to be even, just like the potential.

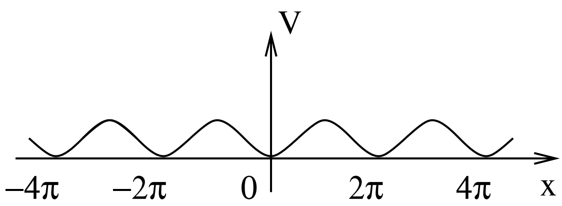

Figure 2.4: *The periodic potential.*

Let us now see what happens when the symmetry of the potential is larger than just $\mathbb{Z}_2$. Consider the periodic potential whose shape is depicted in figure 2.4. Again let us ask the question:

what is the vacuum structure of this system? Let us go through it as we did in the previous problem, starting from the naive approach. Naively, neglecting quantum tunneling effects, we would assume that the particle's wave function is centered around one of the infinitely many minima of the potential, say $x = 0$, thereby spontaneously breaking the $\mathbb{Z}$-symmetry of the system. At this point we would approximate the potential around $x = 0$ with a harmonic oscillator, and find the ground state wave function and energy. But in light of the above discussion, we are aware of tunneling effects. By computing the tunneling amplitude for the particle to go from one minimum $x = 2\pi N_1$ to another $x = 2\pi N_2$, and taking the limit $T \to \infty$, we will obtain information about the true vacuum states:

$$K(2\pi N_1, 2\pi N_2; T) = \sum_n \langle 2\pi N_1 | E_n \rangle \langle E_n | 2\pi N_2 \rangle \, e^{-E_n T/\hbar}, \tag{2.36}$$

$$\longrightarrow \sum_{\substack{\text{Lowest} \\ \text{energy states}}} \langle 2\pi N_1 | E_n \rangle \langle E_n | 2\pi N_2 \rangle \, e^{-E_n T/\hbar}, \tag{2.37}$$

namely, the lowest energy eigenvalues and their wave functions. To compute this amplitude, we again need to sum over the one-kink sector, and over all sequences with multiple kinks and antikinks. The one-kink contribution to the amplitude is the same as in that the in the double-well potential, namely equation (2.26), and the action is still additive, so the rules of the game are the same. The only difference is that, now, kinks do not have to be followed by antikinks and vice-versa, because the space where the particle moves has been enlarged to infinity. In other words, the instanton trajectories need not be confined to the interval $[2\pi N_1, 2\pi N_2]$, they just need to begin and end at $2\pi N_1$ and $2\pi N_2$ respectively. The sum is the following:

$$K(2\pi N_1, 2\pi N_2; T) = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n,\bar{n}} \frac{(K\, e^{-S_0/\hbar}\, T)^{n+\bar{n}}}{n!\,\bar{n}!}\, \delta_{N_2 - N_1 - n - \bar{n}}, \tag{2.38}$$

where the Kroenecker δ-function imposes the boundary conditions. This δ-function can be rewritten as follows:

$$\delta_{N_2 - N_1 - n + \bar{n}} = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(N_2 - N_1 - n + \bar{n})}. \tag{2.39}$$

By inserting this integral, the sums over n and $\bar{n}$ decouple. The result, which is also derived in Coleman's lectures [14] and in Rajaraman's book [15] is the following:

$$K(2\pi N_1, 2\pi N_2; T) = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(N_2 - N_1)} \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\tfrac{1}{2}\,\omega T + 2K\, e^{-S_0/\hbar}\, T\, \cos(\theta)\right). \tag{2.40}$$

Notice that the discrete sum over low energy states has become an integral over a continuum of energy states labeled by θ. The energy of a *theta*-state is then given by the following:

$$E_\theta = \hbar\left(\tfrac{1}{2}\,\omega T - 2K\, e^{-S_0/\hbar}\, T\, \cos(\theta)\right), \quad \text{where} \quad 0 \le \theta \le 2\pi, \tag{2.41}$$

where the state of lowest energy is the one with $\theta = 0$. These energy levels are reminiscent of the band structures exhibited by systems with periodic potentials. This is the limit where the number of minima of the potential goes to infinity (in other words, this is the limit where the periodic potential goes on forever). In this limit, the band of energy levels becomes continuous, yielding the energy formula (2.41). The double-well problem could be regarded roughly as the opposite limit, where the number of potential minima is two. In that case θ could only have two discrete values, 0 and π. We also have the following information about the wave function of the θ-state:

$$\langle 2\pi N_1|\theta\rangle\,\langle\theta|2\pi N_2\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{i\theta(N_2-N_1)}\,. \tag{2.42}$$

The wave function of a θ-state is quasi-periodic: Under a translation by 2π it gains a phase $e^{i\theta}$. So these states restore the symmetry of the system. In fact it can be shown that, to leading order in the approximation of the potential, the wave function of a θ-state is the following:

$$|\theta\rangle = \sum_N e^{i\theta N}\,|\psi_{2\pi N}\rangle\ , \tag{2.43}$$

where the $|\psi_{2\pi N}\rangle$ are the naively constructed harmonic oscillator ground states of each potential minimum, when tunneling effects are neglected. This is analogous to what we noted in the double-well case except that now, instead of just having two possible linear combinations of the naive states, we have a whole continuum of them.

In this section we have learned that the classical vacua of a system do not always correspond to the quantum mechanical ones. In basic quantum mechanics we learn that for "small" $\hbar$ a particle will tend to be "smeared" around its classical vacuum equilibrium point. The more orders of $\hbar$ we keep in our approximation, the better we know the shape of the wave function and its energy. Instantons tell us, however, that tunneling effects drastically modify this picture. The particle will actually tend to be "smeared" around all of its classical vacua, thereby restoring the symmetry of the theory. We could have never seen this effect in an order-by-order approximation of the wave function in $\hbar$. This effect is non-perturbative.

In the next section we will see that gauge theories can also have tunneling effects that modify the vacuum structure.

2.2 Yang-Mills instantons

Now that we have seen the basics about instantons through simple examples, we are ready to take a look at a more sophisticated example. We will study instantons in a quantum field theory; specifically Yang-Mills theory. Although everything we have seen up to now in this chapter were instantons in quantum mechanics, we will be able to generalize the knowledge we have gathered to field theories very easily, thanks to the wonderful language of path integrals. This section will not be as technical as the previous one, as it is only meant to illustrate how the *ideas* we have seen so far apply to Yang-Mills theory. For an introduction to Yang-Mills theory and a full derivation of the Yang-Mills instanton and all of its properties, the reader is again referred to [14] and [15].

The goal is to find the vacuum structure of the Yang-Mills quantum field theory. We will work specifically with the structure group SU(2), because the results can be generalized for SU(N) with arbitrary N. The action is the following:

$$S_{YM} = -\frac{1}{2g^2}\int d^4x\,Tr\left[F_{\mu\nu}F^{\mu\nu}\right]\ , \tag{2.44}$$

where g is the coupling constant of the theory. $F_{\mu\nu}$ is the field-strength defined as follows:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \tag{2.45}$$

and the conection $A_m(x)$ is a Lie algebra valued vector field:

$$A_\mu = g\, A^a_\mu\, T^a\,, \tag{2.46}$$

The T^a are the generators of SU(2), which can be expressed in terms of the Pauli matrices as $T^a = -i\,\sigma^a/2$. SU(2) is a connected manifold, so any group element can be written in terms of the Lie algebra as follows:

$$g(x) = \exp(\alpha^a(x)\, T^a)\,, \tag{2.47}$$

where the $\alpha^a(x)$ are arbitrary smooth functions. The trace in (2.44) runs over SU(2) indices. The action (2.44) is invariant under the following gauge transformations of A_μ:

$$A_\mu \to g\, A_\mu\, g^{-1} + g\, \partial_\mu g^{-1}\,, \tag{2.48}$$

under which the field-strength transforms as follows:

$$F_{\mu\nu} \to g\, F_{\mu\nu}\, g^{-1}\,. \tag{2.49}$$

There are two kinds of gauge transformations, which we must distinguish: "small" and "large" gauge transformations. "Small" gauge transformations are those that satisfy $\alpha(|\vec{x}| = \infty) = 0$. Those that do not satisfy this restriction are denominated "large" gauge transformations. The reader should note that the physical interpretation of a gauge symmetry is different from that of a global symmetry. A global symmetry relates physically inequivalent solutions of a system. In a gauge theory, however, one considers configurations that are related via "small" gauge transformations as being physically equivalent. In fact, the physical states (in the classical sense) are defined by the gauge equivalence classes (equivalence under "small" g. t.'s) of the solutions for the gauge field.

First things first, we need to understand the classical vacua of this system. To simplify the task we take the so-called *static* gauge $A_0 = 0$, which is left invariant by time-independent gauge transformations. Now we can rewrite the Lagrangian density for (2.44) as follows:

$$\mathcal{L} = \frac{1}{g^2}\, Tr\left(\tfrac{1}{2}\,(\partial_0 A_i)^2 - \tfrac{1}{4}\, F_{ij}\, F_{ij}\right)\,. \tag{2.50}$$

This looks like the kinetic term minus a potential for the A_i fields. So we immediately notice that the classical vacua of this action are the so-called *static pure gauges*. Static, means $A_i(x) = A_i(\vec{x})$, and pure gauge means gauge equivalent to $A_i = 0$. These configurations can be written as follows:

$$A_i(\vec{x}) = e^{-\alpha(\vec{x})}\partial_i\, e^{\alpha(\vec{x})} \quad \text{where} \quad \alpha(\vec{x}) = \alpha^a(\vec{x})\, T^a\,. \tag{2.51}$$

It can be shown that it is enough to restrict our search to configurations that satisfy $e^\alpha = \mathbb{I}$ at spatial infinity $|\vec{x}| = \infty$. Since α tends toward the same value in any direction at spatial infinity, we can actually identify all of spatial infinity to a point. In other words, we can reformulate the problem of finding the static pure gauges with $\alpha \to 0$ for $|\vec{x}| \to \infty$ as the problem of finding maps from S^3 into SU(2). As a manifold, SU(2) is diffeomorphic to S^3; hence we are looking for maps $\alpha : S^3 \to S^3$. Maps that are homotopic (can be continuously deformed into each other) correspond to field configurations that are related by "small" gauge transformations. Hence, the vacua can be classified in homotopy classes. In this case, the homotopy group is $\Pi_3(S^3) \cong \mathbb{Z}$. To each homotopy class we can associate an integer, which counts the number of times S^3 is "wrapped" around S^3 by the map α. Given such a map, its homotopy class is determined by computing the following:

$$N = \frac{1}{24\pi^2} \int_{S^3} d^3x\, \epsilon_{ijk}\, Tr[(e^{-\alpha}\,\partial_i\, e^{\alpha})\,(e^{-\alpha}\,\partial_j\, e^{\alpha})\,(e^{-\alpha}\,\partial_k\, e^{\alpha})]\,. \tag{2.52}$$

This is called the Pontryagin index, it literally yields the integer representing the homotopy class of the vacuum configuration. Because a homotopy class is invariant under continuous deformations one usually calls these configurations *topological* vacua. The classical N-vacuum can be thought of as the analogue of the $x = 2\pi N$ vacuum in the periodic potential problem. They are physically inequivalent because no "small" gauge transformation can relate them. However, they can be related via "larger" gauge transformations, just like $x = 2\pi N$ is related to $x = 2\pi(N+1)$ via a 2π shift. From the classical N-vacuum, one can build a naive perturbative quantum state $|N\rangle$, just as we did with in the previous examples, and deduce that the vacuum is infinitely degenerate. However, Yang-Mills theory also has instantons, and tunneling between the different $|N\rangle$ states takes place. By computing tunneling amplitudes in analogy with the periodic potential problem, one sees that the true low energy eigenstates form a band parametrized by an angle θ; and in terms of the $|N\rangle$, a θ-state is given by the following:

$$|\theta\rangle = \sum_N e^{i\theta N}\; |N\rangle\ , \tag{2.53}$$

which restores the symmetry under "large" gauge transformation. This is analogous to the restoration of the $\mathbb{Z}$-symmetry by the θ-vacua of the periodic potential system. One other important property of these θ-states is that they can never talk to each other. In other words, there can never be a physical transition from one such state to another. For any gauge invariant operator B, it can be shown that

$$\langle\theta\,|\,B\,|\,\theta'\rangle = 0\,, \tag{2.54}$$

for any choice of θ and θ'. Therefore, we can make a paradigm shift and consider each $|\theta\rangle$ as the vacuum of a separate theory. For each value of θ we have a theory whose *unique* vacuum state is $|\theta\rangle$. In quantum field theory, one is interested in the vacuum-to-vacuum amplitude $\langle 0\,|\,e^{-HT/\hbar}\,|\,0\rangle$, also known as the partition function Z. In this case, to compute the partition function we have to choose a theory by choosing a value of θ and use its vacuum state. Then, we can write Z as follows:

$$Z = \langle\theta\,|\,e^{-HT/\hbar}\,|\,\theta\rangle = \sum_{N,Q} e^{-i\theta Q}\;\langle N+Q\,|\,e^{-HT/\hbar}\,|\,N\rangle\ , \tag{2.55}$$

using the fact that $\langle N+Q\,|\,e^{-HT/\hbar}\,|\,N\rangle$ is independent of N[5] , we write

$$Z = K\sum_Q e^{-i\theta Q}\int_Q d[A_\mu]\,e^{-S_E}\,, \tag{2.56}$$

[5]The amplitude is invariant under all gauge transformations, "small" and "large", because the Yang-Mills action is. Since N can be changed to any value via a "large" gauge transformation, the amplitude must be independent of N

where S_E is the Euclidean version of (2.44), and the subscript Q indicates that the path integral corresponds to a tunneling amplitude between two topological states whose Pontryagin indices differ by Q. K is just an normalization constant encoding the infinity coming from the summation over N. It is not physically relevant, as all quantum field theoretic amplitudes are normalized by dividing by Z.

Let us summarize what we have learned so far. Yang-Mills theory for SU(2) has classical vacua, which are classified by the third homotopy group of the 3-sphere $\Pi_3(S^3)$. Each class consists of static pure gauge field configurations, which are related by "small" gauge transformations, and it is labeled by the Pontryagin index N. For each N, we have a topological naive vacuum, which can tunnel into another topological naive vacuum, and, just as in the case of the periodic potential, the true energy eigenstates are combinations of the $|N\rangle$, labeled by an angle θ. Since different θ-vacua can never physically interact, we consider θ as a parameter labeling a theory, whose *unique* vacuum is $|\theta\rangle$. To compute the partition function of the theory, we have to sum over all possible tunneling amplitudes, weighing each by $e^{-i\theta Q}$. However, this whole language of topological $|N\rangle$ states is not gauge invariant. It only works in the static gauge. Therefore, the partition function as we wrote it in (2.56) is not gauge invariant. Fortunately, there is a way to remedy this.

Instead of classifying classical vacua, let us classify instantons; i.e. finite action Euclidean configurations. In order for a field configuration to have finite action, its Lagrangian density must be non-zero only in a localized area and vanish at the boundary of Euclidean spacetime. The Euclidean version of the Yang-Mills action (2.44) is positive-definite:

$$S_E = -\frac{1}{2g^2}\int d^4x\, Tr\left[F_{\mu\nu}F^{\mu\nu}\right] = -\frac{1}{2g^2}\int d^4x\, Tr\left[F_{\mu\nu}F_{\mu\nu}\right]. \tag{2.57}$$

The minus sign is due to the fact that the SU(2) trace is negative in the basis we have chosen. Hence, in order for a configuration to have $\mathcal{L}$ vanish at infinity it must be pure gauge at infinity. It must satisfy the following:

$$A_\mu(x) \to g^{-1}(x)\,\partial_\mu\, g(x)\,, \tag{2.58}$$

$$\text{as} \quad |x| \to \infty\,. \tag{2.59}$$

If we define the boundary of R^4 as a 3-sphere[6] whose radius is taken to infinity, then we can think of instanton configurations as maps $g : S^3 \to \mathrm{SU}(2) = S^3$. These are again classified by $\Pi_3(S^3) = \mathbb{Z}$. The Pontryagin index can be computed using formula (2.52), but this time integrating over the S^3 that represents the spacetime boundary. Since we are integrating over the boundary, we can use Stokes' theorem and rewrite the formula as a total derivative:

$$Q = -\frac{1}{24\pi^2}\int_{\partial\mathbb{R}^4} d^3x\,\epsilon^{\nu\rho\sigma}\, Tr[A_\nu A_\rho A_\sigma] = -\frac{1}{24\pi^2}\int_{\mathbb{R}^4} d^4x\,\epsilon^{\mu\nu\rho\sigma}\, Tr[\partial_\mu A_\nu A_\rho A_\sigma]\,, \tag{2.60}$$

which can be shown to be equivalent to

$$Q = -\frac{1}{16\pi^2}\int_{R^4} d^4x\,\epsilon^{\mu\nu\rho\sigma}\, Tr[F_{\mu\nu}F_{\rho\sigma}] = -\frac{1}{16\pi^2}\int_{R^4} d^4x\, Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]\,, \tag{2.61}$$

[6]A note of caution: the S^3 we previously considered was a one-point compactification of the *space* R^3, which we used in order to classify the state of the system at a certain point in time. The S^3 we are considering now is the boundary of Euclidean space-*time* $\mathbb{R}^4$, which we are using in order to classify instanton configurations.

where $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the Hodge dual of the field-strength. This expression is manifestly gauge invariant. It is also known as the second Chern class, due to its interpretation as the characteristic class of an SU(2)-principal bundle over the base manifold S^4.

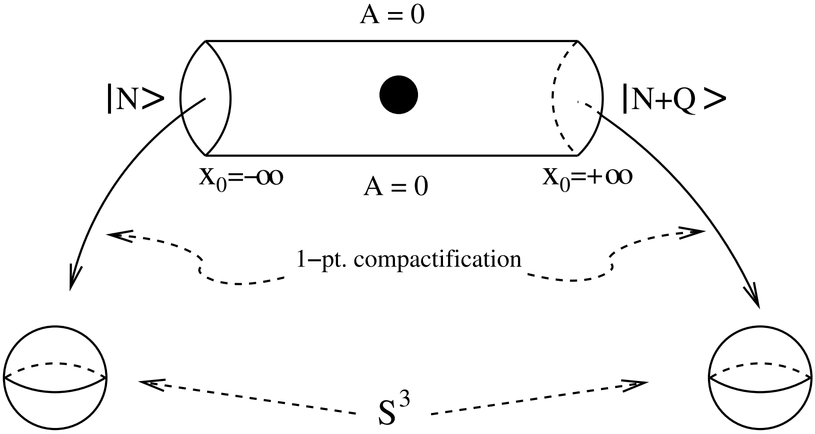

Figure 2.5: *The boundary of spacetime as a time-like cylinder $\mathbb{R} \times S^2$, with one suppressed dimension. The initial and final topological states reside at the caps of the cylinder. The latter, which are two D^3, are compactified to two S^3 to determine their topological indices N and $N + Q$, respectively. The black filled circle represents the localization of the instanton.*

This topological term classifies the boundary conditions of all instanton configurations in a gauge invariant way. However, any such configuration with second Chern class Q can be interpreted as a tunneling process from a topological state $|N\rangle$ to a state $|N + Q\rangle$ by performing a gauge transformation to go to the static gauge. In the static gauge, if we view the boundary of Euclidean spacetime as a generalized cylinder $\mathbb{R} \times S^2$ as in figure 2.5, where $\mathbb{R}$ is the Euclidean time range, then the only contribution to (2.61) will come from the two 3-discs at $x_0 = \pm\infty$ (i.e. the caps of the cylinder):

$$Q = -\frac{1}{16\pi^2} \int_{D^3} d^3x \, \epsilon^{\nu\rho\sigma} \; Tr[A_\nu A_\rho A_\sigma]\Big|_{x=-\infty}^{x=\infty} , \tag{2.62}$$

$$= (N + Q) - N \, . \tag{2.63}$$

Hence, the second Chern class computes the change in N of the tunneling process. We can now finally rewrite the partition function (2.56) in a gauge invariant way:

$$Z = \int_{\text{all } Q} d[A_\mu] \, \exp\left[-S_E - i \frac{\theta}{16\pi^2} \int d^4x Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]\right] . \tag{2.64}$$

The θ-term has a physical effect on the theory. It breaks parity. This actually makes θ a physically measurable quantity in gauge theories.

2.3 Solitons vs. instantons

Having studied the mathematics and physics of instantons, we should also look at a special class of solutions to classical equations of motion called *solitons*. These will be interesting to us for a number of reasons: first of all, they have a similar mathematical structure to instantons in that they are *topologically non-trivial*. They too, are in some sense interpolating configurations. Secondly, in some cases, there exists a precise correspondence between instantons and solitons. In the next chapter, we will actually see an explicit example of this. Because solitons are not the main focus of this text, I will only briefly introduce them and will refer the interested reader to Coleman 's book [14] and Rajaraman's book [15] for a careful introduction, and Zee's book [17] for a short but very clear exposition of the topic.

2.3.1 Solitons: Definition and examples

Definition: A soliton is a time-independent extremum of the *Mikowskian* action with finite non-zero energy.[7]

Note that we are now back to Minkowski spacetime. Time-independent means that the field configuration has no non-trivial time-dependence that could for instance be obtained by boosting a static solution.

Let us take a look at the simplest soliton, the *kink* solution. We define the following field theory in (1 + 1)-dimensions:

$$\mathcal{L} = -\tfrac{1}{2}\,(\partial\phi)^2 - V(\phi)\,, \tag{2.65}$$

with

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2\,, \tag{2.66}$$

where ϕ is the field, and λ and v are parameters. This is a double-well potential. Note that we are working in the *mostly plus* convention, which is why the kinetic term has a minus sign. We instinctively know that this Lagrangian has two very simple solutions, namely the two vacua $\phi(t, x) = \pm v$. They both have energy zero. In standard perturbation theory we are instructed to pick one of the two vacua and study the fluctuations around it. In practice, this means rewriting the scalar field as $\phi \to v + \chi$, and treating the fluctuation χ as the fundamental field. Plugging this back into (2.65) we will find that χ is a scalar particle with mass $\mu = (\lambda\, v^2)^{1/2}$.

One can, however, also look for a solution with non-trivial conditions, namely a configuration that interpolates between those two vacua, i.e. $\phi \to \pm v$ for $x \to \pm\infty$. Such a solution will look qualitatively like the kink we saw in section 2.1, see figure 2.6. In fact, this solution is also known as the kink solution. Because it is time-independent, we can write its energy density as follows:

$$\mathcal{E} = \tfrac{1}{2}\,\phi'^2 + V\,, \tag{2.67}$$

where the prime denotes differentiation w.r.t. the spatial coordinate x. Because $\phi \to \pm v$ for $x \to \pm\infty$, the energy density is non-zero only within a localized region. This means that the

[7] This is not the only possible definition. A stricter one, stated in [15], also requires that a soliton's shape be left unaffected by scattering against another soliton, but we will not be exploring this property here.

total energy will be finite. Since this energy density is positive, we can rewrite it as a square plus a positive term:

$$\tfrac{1}{2}\left(\phi' \pm \sqrt{2\,V}\right)^2 \mp \phi' \sqrt{2\,V}\,. \tag{2.68}$$

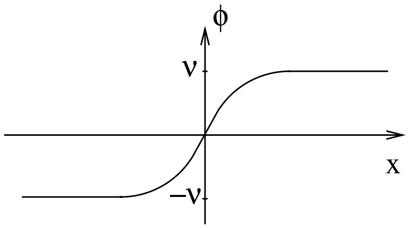

Figure 2.6: *The kink solution: a classical field configuration trajectory that interpolates between the two classical vacua of the double-well potential.*

This means that the energy of any solution to this system satisfies a bound:

$$E \geq \left| \int dx\, \phi' \sqrt{2\,V} \right| = \left| \int_{\phi(x=-\infty)}^{\phi(x=+\infty)} d\phi \sqrt{2\,V} \right|\,. \tag{2.69}$$

This is known as the *Bogomol'nyi bound*. Because we are choosing a time-independent Ansatz, we can easily see that the Lagrangian density of this system (2.65) is equal to minus the energy density (2.67), i.e. $\mathcal{L} = -\mathcal{E}$. This is more than a mere curiosity, this is at the heart of the instanton-soliton correspondence. Therefore, solving the equations of motion with this Ansatz means extremizing not only the action, but also the energy. This means that the soliton actually saturates the Bogomol'nyi bound (2.69). In other words, a soliton is the configuration of least energy within its class of boundary conditions or topological class. To saturate the bound, the field has to satisfy:

$$\phi' = \pm\sqrt{2\,V}\,. \tag{2.70}$$

This is often referred to as the BPS condition. Note that if a field satisfies this equation, it automatically satisfies the equations of motions. However, we have now simplified the task of solving a second order differential equation into solving a first order equation. In supergravity, p-branes are solutions, which satisfy an analogous form of the BPS condition. The latter implies that the solution preserves a certain amount of the supersymmetry of the theory it lives in. Using (2.68) and (2.70) we find that the energy is given by:

$$E = \left| \int_{\phi=-\nu}^{\phi=\nu} d\phi \sqrt{2\,V} \right|\,. \tag{2.71}$$

This depends only on the potential and the boundary conditions, and not on any parameters of the solution. In our case, $E \sim \mu^3/\lambda$. So the kink is very massive (energetic) for small coupling constant. This means that object is non-perturbative, i.e. it cannot be found by doing some sort of pertubation theory around the vacuum. The kink is at least pertubatively a stable configuration. Its non-trivial boundary conditions prevent it from simply decaying into an object with lower energy. It is not a simple ripple in the field. Mathematically this translates into the statement that the kink has a conserved *topological current*[8]

$$J^\mu = \frac{1}{2}\, \epsilon^{\mu\nu}\, \partial_\nu \phi \,, \tag{2.72}$$

yielding a conserved *topological charge*

$$Q = \int_{+\infty}^{-\infty} dx\, J^0 = \frac{1}{2\,v} \Big(\phi(+\infty) - \phi(-\infty) \Big) . \tag{2.73}$$

Solitons are also present in more complicated field theories, such as gauge theories. Magnetic monopoles are an example of solitons. Depending on the dimensionality of the soliton it may be called, *monopole, string or vortex, membrane*, or *texture*, if it 'stretches' over $0, 1, 2$ and 3 spatial directions respectively. If it only has one transverse spatial direction, such as the kink in $1 + 1$ dimensions, it is called a *domain wall*. All of these objects are characterized by some topological charge. In gauge theories this charge will be a Pontryagin index.

In gravitational theories, there are objects analogous to solitons. The simplest one is the Schwarzschild black hole. Its metric is the following:

$$ds^2 = -\left(1 - \frac{2\,G\,M}{r}\right) dt^2 + \left(1 - \frac{2\,G\,M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2_{S^2} \,, \tag{2.74}$$

where G is the Newton constant and M is a parameter of the solution. For an introduction to black holes, the reader is referred to the pedagogical lecture notes by S. Carroll [18] (or his book [19]), and to Townsend's extensive lecture notes [20]. The spacetime geometry of the Schwarzschild black hole is non-trivial in that it interpolates between flat Minkowski spacetime at spatial infinity, and $AdS_2 \times S^2$ near its horizon at $r = 2\,G\,M$. Although energy is a tricky subject in General Relativity, it can be defined via the ADM mass formula, which can be found in [20]. Once it is calculated, one finds that it is equal to the parameter M in the solution for the Schwarzschild metric (2.74). From the solution, we see that this object is also non-perturbative. No matter how 'small' we make the mass, its effect will be very dramatic near the horizon. In supergravity, p-branes play the role of the soliton. They are the higher-dimensional generalization of the charged Reissner-Nordström black hole. A p-brane has a $p+1$-dimensional world-volume and is charged under a $p+2$-form field-strength. For an introduction into p-brane solutions, the reader is referred to "String Solitons" [21], and to "Gravity and Strings" [22].

2.3.2 The correspondence

Now that we have seen the definition of solitons and have seen some examples of them, let us study their correspondence with instantons. We will first look at the simplest example of this correspondence, and then explain it in a more general context.

Taking the example of the scalar field in $1 + 1$ dimensions from the previous subsection, the reader will recall that if we make take the time-independent Ansatz, which is what we do

[8]Note that this current is not a Noether current, as it does not follow from a continous symmetry.

when looking for solitons, and substitute it into the Lagrangian density (2.65), the latter takes the following form:

$$\mathcal{L} = -\tfrac{1}{2}\,\partial_x\phi^2 - V = -\mathcal{E}\,, \tag{2.75}$$

where $\phi = \phi(x)$, and $\mathcal{E}$ is the energy density of the system. A soliton is defined as being an extremum of the action defined by this Lagrangian density *and* as having finite energy. Note that this Lagrangian density is, up to a minus sign, equivalent to that of a scalar field in *one* Euclidean dimension if we define Euclidean time τ as $\tau \equiv x$. Hence, the equations of motion for a soliton in $1+1$ dimensions are the same as the equations for an instanton in one Euclidean dimension, and the requirement that the soliton have finite *energy*

$$E = \int dx\,\mathcal{E}\,, \tag{2.76}$$

is equivalent to the requirement that the instanton in one dimension have finite *action*. So the kink-soliton in $1+1$ dimensions corresponds to the instanton in one dimension[9]. The relation is simply $\phi_{sol}(x) = \phi_{inst}(\tau)$.

This is not specific to the kink model, one can show a more general correspondence. Let us define a system in $d+1$ spacetime dimensions with general degrees of freedom, which we denote by ϕ_I, where the I can stand for a collection Lorentz indices, or internal indices, and a Lagrangian density

$$\mathcal{L} = \mathcal{L}(\phi_I, \partial\phi_I)\,, \tag{2.77}$$

where both temporal and spatial derivatives are implied by the symbol '∂'. The conjugate momenta of the system are defined as follows:

$$\pi_I^\mu \equiv \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi_I\right)}\,. \tag{2.78}$$

Using the time-dependent Ansatz we can write the energy as follows:

$$E = \int d^d x\,\left[\pi_I^0\,\dot\phi_I(t,\vec{x}) - \mathcal{L}\left(\phi_I(t,\vec{x})\right)\right)] = -\int d^d x \mathcal{L}\left(\phi(\vec{x})\right)\,, \tag{2.79}$$

where the dot, as usual, represents a time derivative, and the first term on the LHS vanishes due to the time-independence of the solution. A soliton solution will be an extremum of this energy (since S=-E), and will have finite energy. Since all degrees of freedom depend only on the spatial directions, we can view this Lagrangian density as that of a d-dimensional Euclidean system (up to a minus sign), and this energy can be viewed as its action. The soliton can then be called an instanton in d dimensions. In practice, all one has to do is a Kaluza-Klein reduction over time, but without the interpretation that time is compactified. One is simply truncating time.

To summarize all this, the statement is the following: *A soliton in $d+1$ dimensions is equivalent to an instanton in d dimensions*. In the next chapter, we will see that charged black holes can be viewed as a certain kind of supergravity instantons called *D-instantons*. An interesting

[9]The kink instanton solution in $(1+1)$-dimensional quantum mechanics can be viewed as an instanton in $(0+1)$-dimensional quantum field theory.

question that comes to mind based on the statement we have made, is whether its converse is true. In other words: *When is an instanton in d dimensions equivalent to a soliton in $d + 1$ dimensions*? The answer depends on the Lagrangian. If a Euclidean Lagrangian can be obtained as the time truncation of a $d + 1$-dimensional Lagrangian, in other words, if it can be *uplifted* to $d + 1$ dimensions, then the instanton will give rise to a soliton. In the next chapter we will establish the necessary condition for a D-instanton to give rise to a black hole.

In this chapter, we studied the basics about instantons in quantum mechanics and quantum field theory. We learned that instantons provide us with non-perturbative information, by telling us that a naïve perturbative vacuum is not really the vacuum state of a theory, because the system can tunnel out of it. This requires that one rewrite a path integral with a new topological term that properly takes this fact into account.

In the next chapter, we will be looking at instantons in gravitational field theories, such as supergravities. Although defining a path integral for a gravitational theory is tricky business and requires unnatural adjustments in order to be well-defined, it is possible to talk about instantons and non-perturbative tunneling effects in gravity.