Instantons and cosmologies in string theory

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Chapter 1

String theory in a nutshell

1.1 Introduction

In this chapter, the basic definitions and foundations of string theory will be laid. We will start by reviewing the relativistic point-particle in the formalism of the variational principle. Then, we will repeat this for the relativistic bosonic string. After a brief introduction into the canonical quantization of the string and the resulting spectrum, we will study the string in the path integral formalism. The notions of vertex operators, and the genus expansion of string Feynman diagrams will be introduced. This will allow us to understand how non-trivial spacetime backgrounds, on which the string can propagate, can be interpreted as coherent states of strings. Then, we will briefly see that requiring classical symmetries to hold quantum mechanically imposes constraints on spacetime backgrounds by means of β-functions. In the low energy approximation, these constraints can be interpreted as spacetime field theories. Field theories obtained as low energy approximations to string theory will be the main framework of this thesis. Finally, a brief summary of supersymmetric string theories and their low energy limits will be provided.

In the following, I will be borrowing heavily (and sometimes verbatim) from Polchinski's textbooks [2, 3]. However, the philosophy behind this chapter is *not* to provide the reader with yet another carbon copy of the standard textbooks, and certainly not to improve on the latter. The main goal of this chapter is to show a minimal selection from the standard textbooks in order to schematically explain how the low energy limit of the quantized theory of relativistic strings (which is a QFT in the two world-sheet dimensions) is a classical field theory in spacetime.

1.1.1 The relativistic point-particle

Before we begin our journey into the theory of strings, let us review our knowledge of relativistic point-particles through the action principle.

To describe the motion of a particle moving in a D-dimensional Minkowski spacetime we can define $D-1$ functions of time $X^1(X^0), ..., X^{D-1}(X^0)$, which give the particle's position in space at any given time X^0. We can also make this description covariant by parametrizing the particle's *world-line* with a variable τ, such that we now have D functions $X^0(\tau), ..., X^{D-1}(\tau)$ on equal footing. One can derive the equations of motion from the variational principle through the following action:

$$S = -m \int d\tau \, (-\dot{X}^\mu(\tau)\, \dot{X}_\mu(\tau))^{1/2}, \tag{1.1}$$

where m is the particle's mass, and the dot represents a τ-derivative. This action measures the relativistically invariant arc-length (or proper time) of the world-line, and the classical particle will move along the trajectory that extremizes this quantity. The Euler-Lagrange equations for the X^μ's are then,

$$\partial_\tau \left(\frac{m\, \dot{X}^\mu}{(-\dot{X}^\mu\, \dot{X}_\mu)^{1/2}} \right) = 0\,. \tag{1.2}$$

The conjugate momenta to the particle's spacetime coordinates are the following:

$$P^\mu = \frac{m\, \dot{X}^\mu}{(-\dot{X}^\mu\, \dot{X}_\mu)^{1/2}}\,, \tag{1.3}$$

from which we easily derive the on-mass-shell constraint

$$P^2 + m^2 = 0. \tag{1.4}$$

Although this action allows for an easy derivation of the classical equations of motion and on-shell condition, it does not accommodate the case of the massless particle. Moreover, the square root of the integrand makes this action awkward to work with in a path integral calculation. Fortunately, there is a more convenient form which eliminates these two features by introducing an auxiliary field:

$$S' = \tfrac{1}{2} \int d\tau \, (e^{-1}(\tau)\, \dot{X}^\mu(\tau)\, \dot{X}_\mu(\tau) - e(\tau)\, m^2)\,. \tag{1.5}$$

The auxiliary field $e(\tau)$ is the world-line *einbein*. In other words, it is the square root of (minus) the metric $g_{\tau\tau}(\tau) = -e(\tau)^2$ that lives on the one-dimensional τ-space. This metric is an independent field and is therefore *not* induced by the spacetime Minkowski metric $g_{\mu\nu}$. The first property we should establish about this action is that it is equivalent to the previous one (1.1) (except for the massless case). To show this we compute the equations of motion of the *einbein*:

$$m^2\, e^2 + \dot{X}^\mu\, \dot{X}_\mu = 0\,. \tag{1.6}$$

Substituting this back into (1.5) we find that $S = S'$. Notice also that the conjugate momenta are now given by

$$P^\mu = \frac{\dot{X}^\mu}{e}\,, \tag{1.7}$$

which, combined with (1.6) gives the on-mass-shell constraint as an equation of motion.

Let us list the symmetries of the action (1.5):

- D-dimensional spacetime Poincaré transformations:

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu\, X^\nu + A^\mu\,, \tag{1.8}$$

where $\Lambda^\mu{}_\nu$ is an $SO(1, D-1)$ matrix and A^μ is an arbitrary D-vector.

- World-line reparametrizations:

$$\begin{aligned}\tau &\to \tau' \\ e(\tau) &\to e'(\tau') = e(\tau)\,\frac{d\tau}{d\tau'} \\ X^\mu(\tau) &\to X^\mu(\tau')\,.\end{aligned} \tag{1.9}$$

This action is by construction Poincaré invariant. The second symmetry merely confirms the fact that the physics of a particle should be independent of how one chooses to parametrize its world-line. Notice that we could make a paradigm shift and regard this system (1.5) as a one-dimensional theory of D scalar fields $X^\mu(\tau)$ and a metric $g_{\tau\tau}(\tau) = -e(\tau)^2$. In that case, the D-dimensional Poincaré symmetry would be interpreted as an internal symmetry of the scalar fields, and world-line reparametrization invariance would be seen as invariance under general coordinate transformations in one dimension. Although this interpretation may appear strange in this case, this point of view will prove to be a very powerful tool in string theory.

1.1.2 The relativistic bosonic string

Now we are ready to deal with the bosonic string. We will proceed by analogy with the case of the particle. A particle sweeps out a world-*line* in spacetime, which means that we can describe it as an embedding of a one-dimensional manifold into a D-dimensional Minkowski spacetime. A string sweeps out a two dimensional world-*sheet*, this requires an embedding of a two dimensional manifold into D-dimensional Minkowski spacetime. The string coordinates will then be functions of two parameters $X^\mu(\tau,\sigma)$. We can derive equations of motion for the string by requiring that the world-sheet extremize its invariant surface. In order to measure that surface we define the *induced* metric on the world-sheet h_{ab}, where a, b run over the world-sheet indices:

$$h_{ab} = \partial_a X^\mu\, \partial_b X^\nu\, \eta_{\mu\nu}\,. \tag{1.10}$$

Then, the string will extremize the so-called Nambu-Goto action:

$$S_{NG} = -\frac{1}{2\pi\,\alpha'} \int d\tau\, d\sigma\, (-\det h_{ab})^{1/2}\,. \tag{1.11}$$

In the case of the point particle we needed a constant with units of energy to make the action dimensionless (i.e. the mass), in this case we need energy per unit length. Hence, the constant $1/(2\pi\,\alpha')$ will play the role of the string tension.

Once again, we can derive equations of motion from this action; however, if we expect to use it in a path integral formalism we should find an action without a square root. In order to achieve this we must again introduce an auxiliary world-sheet metric γ_{ab}. The action we are after is called the Brink-Di Vecchia-Howe-Deser-Zumino action [4, 5] or Polyakov action [6, 7]:

$$S_P = -\frac{1}{4\pi\,\alpha'} \int d\tau\, d\sigma\, (-\gamma)^{1/2}\, \gamma^{ab}\, \partial_a X^\mu\, \partial_b X_\mu\,, \tag{1.12}$$

where $\gamma = \det \gamma_{ab}$. This action has a more familiar kinetic term for the X^μ, which makes the path integral easy to evaluate. If we eliminate the auxiliary metric γ_{ab} from this action by using its equations of motion, we will find that the Polyakov action is equivalent to the Nambu-Goto action (1.11).

Let us list the symmetries of the Polyakov action:

- Poincaré transformations in D-dimensional spacetime:

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu \,, \tag{1.13}$$

where $\Lambda^\mu{}_\nu$ is an SO$(1, D-1)$ matrix and A^μ is an arbitrary D-vector.

- World-sheet reparametrizations:
Defining a generalized world-sheet coordinate $\sigma^a = (\tau, \sigma)$ for $a = 0, 1$ we have,

$$\begin{aligned} \sigma^a &\to \sigma'^a(\tau,\sigma), \\ X^\mu(\tau,\sigma) &\to X^\mu(\tau',\sigma'), \\ \gamma_{ab} &\to \gamma'_{cd}(\tau',\sigma')\,\frac{\partial\sigma^c}{\partial\sigma'^a}\,\frac{\partial\sigma^d}{\partial\sigma'^b}\,. \end{aligned} \tag{1.14}$$

- World-sheet Weyl rescalings:

$$\gamma_{ab} \to \gamma'_{ab} = e^{2\,\omega(\tau,\sigma)}\,\gamma_{ab}\,. \tag{1.15}$$

The first two symmetries, (1.13) and (1.14) are analogous to the point-particle symmetries, (1.8) and (1.9). The last one (1.15), however, is specifically due to the fact that we are dealing with a two dimensional extended object. This symmetry tells us that we should regard all Weyl-equivalent metrics on the world-sheet as physically equivalent. From the two dimensional point of view, we have a scalar field theory with an internal D-dimensional Poincaré invariance, Weyl-rescaling invariance, and invariance under two-dimensional general coordinate transformation. This field theory falls under the category of *conformal field theories*. Two dimensional CFT's are a very special kind of CFT when it comes to doing both classical *and* quantum computations, due to techniques that exist only for two dimensions. This is analogous to the fact that there are much more powerful techniques to do analysis on the complex plane than there are for higher dimensional complex spaces.

Let us now write down and solve the equations of motion for the Polyakov action (1.12). Varying the string coordinates X^μ, we get the following equation:

$$\begin{aligned} \delta S_P = \frac{1}{2\pi\,\alpha'}\int d\tau\, d\sigma\, \partial_a \{ (-\gamma)^{1/2}\,\gamma^{ab}\,\partial_b X^\mu \}\,\delta X^\mu \\ - \frac{1}{2\pi\,\alpha'}\int d\tau\,(-\gamma)^{1/2}\,\partial_\sigma X_\mu\,\delta X^\mu\Big|_{\sigma=0}^{\sigma=l}\,. \end{aligned} \tag{1.16}$$

To make this variation zero both terms must vanish independently. The first term requires the two-dimensional Laplacian of the X^μ's to vanish. The second term requires a choice of boundary conditions, for which there are three possibilities:

- Open string Neumann b.c.s:

$$\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,l) = 0\,. \tag{1.17}$$

These conditions imply that no momentum flows in or out through the string endpoints, and, hence, that these move freely.

- Open string Dirichlet b.c.s:

$$\delta X^\mu(\tau, 0) = \delta X^\mu(\tau, l) = 0\,. \tag{1.18}$$

These conditions mean that we are fixing the string endpoints and no longer consider them as dynamical.

- Closed string (periodic b.c.s):

$$X^\mu(\tau, 0) = X^\mu(\tau, l)\,. \tag{1.19}$$

This is the requirement that the string be closed, i.e. that it have no endpoints.

For open strings, the Neumann boundary conditions (1.17) are the only conditions that are consistent with spacetime Poincaré invariance, whereas the Dirichlet b.c.'s (1.18) explicitly break it. For instance, if one imposes Neumann b.c.s on $D-p-1$ string coordinates and Dirichlet b.c.s on $p+1$ of the coordinates, this means that the string endpoints are stuck to a $p+1$-dimensional hypersurface of spacetime called Dp-brane, where the 'D' stands for Dirichlet. That's why the latter were discarded for a long time as unphysical until Polchinski discovered in 1995 [8] that D-branes are an integral part of string theory.

Let us now focus on the open string with Neumann b.c.s and solve the equations of motion from the first term in (1.16):

$$\partial_a((-\gamma)^{1/2}\,\gamma^{ab}\,\partial_b X^\mu) = 0\,. \tag{1.20}$$

For general γ_{ab} this can be a non-trivial equation to solve. However, we are in luck. In two dimensions there are enough symmetries to make this equation trivial. The first symmetry we make use of is invariance under general coordinate transformations (1.14). One can show that, in two dimensions, it is *locally* possible to bring *any* metric to a *conformally flat* form through an appropriate coordinate transformation:

$$\sigma^a \to \sigma'^a \tag{1.21}$$

$$\gamma^{ab} \to \gamma'^{ab} = e^\phi\,\eta^{ab} = e^\phi \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.22}$$

where ϕ is some function of τ and σ. Now we are only a Weyl transformation (1.15) away from a flat metric. However, by inspecting of (1.20), we see that the conformal factor simply drops out. The solution for X in (1.20) is the following:

$$X^\mu(\tau,\sigma) = x^\mu + 2\,\alpha'\,p^\mu\,\tau + i\,(2\,\alpha')^{1/2}\sum_{n\neq 0}\frac{1}{n}\,\alpha^\mu_n\,e^{-in\tau}\,\cos n\sigma\,, \tag{1.23}$$

where we require $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$ to ensure reality. The parameter x^μ can be thought of as the string's initial center-of-mass position, p^μ as its center-of-mass momentum, and the α^μ_n as the oscillation modes of the string. Note that in the string action we did not fix the mass but rather the tension or energy per unit length of the string. Since length of the string depends on its oscillatory state, so will its mass. This makes sense relativistically, exciting the string's oscillatory modes means putting energy into it, and energy is the same as mass. In fact, by using Hamiltonian dynamics, one can show that the string's mass is given by the following relation:

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{1.24}$$

For the closed string one follows an analogous procedure to that for the open string case. One discovers, however, that the closed string has two sets of oscillators α^μ and $\tilde{\alpha}^\mu$, the so-called *right-* and *left-movers*, which can be viewed as non-stationary waves on the world-sheet traveling to the right and to the left respectively.

1.1.3 The bosonic string spectrum

We will now schematically study the quantum spectrum of the bosonic string. For a detailed account of what we are about to do, the reader is referred to any standard textbook on String theory such as [2] and [9].

Let us begin with the canonical quantization of the open string. Just as in the case of the point particle, the string is quantized by replacing Poisson bracket into commutators:

$$\begin{aligned} \{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} &\to [X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = i\,\eta^{\mu\nu}\,\delta(\sigma-\sigma')\,, \\ \text{and} \qquad \{x^\mu, p^\nu\} &\to [x^\mu, p^\nu] = i\,\eta^{\mu\nu}\,, \end{aligned} \tag{1.25}$$

where $\Pi^\mu = (1/2\pi\,\alpha')\,\dot{X}^\mu$. Promoting the string coordinates to operators implies that the string oscillators α_n^μ are themselves promoted to operators. In fact, they acquire the following commutation relations:

$$[\alpha_m^\mu, \alpha_n^\nu] = i\,m\,\delta_{m+n}\eta^{\mu\nu}\,, \tag{1.26}$$

which we recognize as the commutation relations of the harmonic oscillator, where the α_{-n} and α_n are the creation and annihilation operators, respectively. So the string can be thought of as an eigenstate of the momentum operator p^μ with an infinite number of harmonic oscillators, each at a different excitation level. To create a state, define a "vacuum" state with some definite momentum $|\,p; 0, 0, ...\,\rangle$, and then act on it with α_{-n}^μ operators. This will generate a string with definite momentum and oscillatory modes. The mass of the string will be given by a modified version of the classical formula (1.24). The quantum formula will count the number of harmonic oscillators and add a zero-point energy:

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - 1\right). \tag{1.27}$$

Note that every oscillator α^μ carries a spacetime Lorentz index. It can be shown that the state created by acting with an oscillator on the vacuum, $\alpha_{-n}^\mu\,|\,p;0\,\rangle$, behaves as a vector boson. More generally, it can be shown that any string state will behave as a particle with mass and spin determined by the number of its oscillators and their indices. The closed string spectrum is also generated by harmonic oscillators. Its spectrum, however, is different from that of the open string. One key difference is that the closed string spectrum contains a massless spin-two particle which behaves like a *graviton*, whereas the open string does not.

Note also that the mass of a state is inversely proportional to α'. This means that in the low energy approximation to string theory (low α'), the massive states will become very massive and will be difficult to excite. That's why one can focus on the massless states when doing this approximation.

The goal of these first three subsections was to introduce the classical string and its quantum mechanical spectrum in a fair amount of detail. In the next two subsections, I will explain the Feynmann path integral quantization of the string and show that, in the low energy approximation (i.e. $\alpha' \to 0$), string theory can be effectively described by a spacetime field theory containing gravity, an antisymmetric tensor, and a scalar. This is a rather ambitious goal and a detailed treatment of this subject would require a lot of formalism and space, and would divert us from the main topic of this thesis: to study particular field theory configurations with gravity and scalar fields. I will, therefore, not show any detailed calculations; however, I will try to give an overview that is self-contained in that it does not require any new concepts beyond those of basic quantum field theory and path integrals. For an account that really does justice to the subject of the path integral quantization of the string, the reader is referred to [2, 9, 10].

1.1.4 The string path integral

Now that we know how the string spectrum comes about, let us turn to the path integral formalism to see how string amplitudes are defined.

When we want to compute quantum mechanical amplitudes for a point particle, the Feynmann path integral procedure instructs us to sum over all possible histories (world-lines) $x(t)$ that the particle can take, given some initial and final positions x_i and x_f, and to weight each with the phase $\exp(iS/\hbar)$, where $S = S[x(t)]$ is the action evaluated on the path. The partition function is then the following:

$$Z = \int d[x]\, e^{-iS[x]} \,. \tag{1.28}$$

This is sometimes referred to as first quantization in old fashioned language. It is nothing other than *quantum mechanics*. When we want to compute a quantum field theory amplitude using path integrals, we have to sum over all possible configurations a field ϕ can take given some spacetime boundary conditions, each weighted again by a phase. This yields the following partition function:

$$Z = \int d[\phi]\, e^{-iS[\phi]} \,, \tag{1.29}$$

where Z stands for *Zustandssumme* (sum of states). In the old fashioned language, this is second quantization. However, many physicists regard this as a misnomer because the procedure quantizes the field only once. This should just be called *quantum field theory*

In string theory we will be summing over all trajectories the string can take, i.e. over all possible world-sheets, and weight each with the Polyakov action (1.12):

$$Z = \int d[X]\, d[\gamma]\, e^{-iS[X,g]} \,, \tag{1.30}$$

where X represents the spacetime coordinates of the string and γ the world-sheet metric. This is the analog of (1.28). In other words, this path integral describes the quantum mechanics of the string.

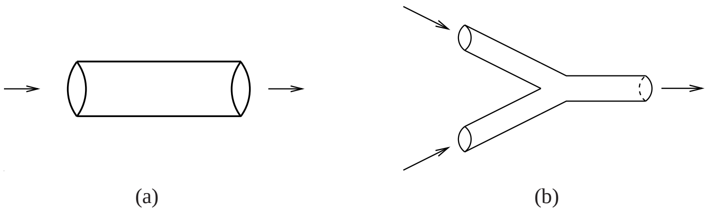

Figure 1.1: *Feynman diagram of a closed string: (a) propagator; (b) three-point function.*

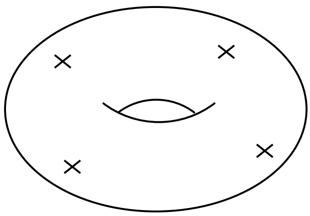

Figure 1.2: *Feynman diagram of the one-loop four-point diagram.*

We can take, however, a radically different point of view. If we view the Polyakov action as a two-dimensional action of fields, the path integral (1.30) becomes the analog of (1.29), summing over all configurations the fields $X^\mu(\sigma,\tau)$ and $\gamma_{ab}(\sigma,\tau)$ can take: This means that we have to sum over all scalar field configurations and all world-sheet geometries with given boundary conditions. For instance, the open and closed string propagators and string 3-point functions will contain diagrams[1] such as those in figures 1.1(a) and (b). By working in the Euclidean (Wick rotated) path integral formalism, and thus summing over Euclidean[2] two-dimensional metrics, one can use the conformal symmetry of the theory to map all world-sheets to compact *Riemann* surfaces. All external legs, which are infinitely long, are brought to a finite distance from each other. For instance, the closed string propagator diagram in figure 1.1(a), which was a cylinder, gets mapped to a sphere and the external legs get mapped to two points on the sphere. The "one-loop" four-point function diagram gets mapped to a torus with four points as external legs, see figure 1.2. The general rule is that all diagrams are mapped to compact closed or open surfaces and their external legs are mapped to points on the surfaces. However, it seems strange to map the external legs to points. After all, these external legs are supposed to represent initial and final states of strings, so mapping these to points seems to lose all the stringy information of these states. It turns out that the proper way to do this is to include what are called *vertex operators* on the compact surfaces. These operators $V(\sigma,\tau)$, which are inserted in the path integral, will supplement the latter with all the stringy information about initial and final states. For example, the state with no oscillators excited (the tachyon), but with some momentum p is

[1] It is also possible to draw diagrams representing the process where two open strings join at their endpoints, thereby forming a closed string. This implies that open string theory must include closed string modes.

[2] It is not always possible to perform the Wick rotation. When dealing with cosmological models, i.e. time-dependent spacetimes, Wick rotations can make the metric complex, see [11]

translated into the following vertex operator:

$$|\,0;p\,\rangle \qquad \Rightarrow \qquad \int d^2z\ :e^{i\,p\cdot X}:\,, \tag{1.31}$$

where z is a complex coordinate representing τ and σ, and : : represents normal ordering. Then, the two-point function for a tachyon with momentum p is computed as follows:

$$\langle\,0;p\,|\,e^{iHT}\,|\,0;p\,\rangle = \langle\,0\,|\left(\int d^2z\ :e^{i\,p\cdot X}:\right)^{\dagger}\left(\int d^2z'\ :e^{i\,p\cdot X}:\right)|\,0\,\rangle \tag{1.32}$$

$$= \int d[X]\,d[\gamma]\left(\int d^2z\ :e^{i\,p\cdot X}:\right)^{\dagger}\left(\int d^2z'\ :e^{i\,p\cdot X}:\right)e^{-iS[X,g]}\,. \tag{1.33}$$

The state that corresponds to the closed string graviton looks as follows:

$$\zeta_{\mu\nu}\,\alpha^{\mu}_{-1}\,\alpha^{\nu}_{-1}\,|\,0;p\,\rangle \qquad \Rightarrow \qquad \int d^2z\ :\zeta_{\mu\nu}\,\partial_z X^{\mu}\,\partial_{\bar z}X^{\nu}\,e^{i\,p\cdot X}:\,, \tag{1.34}$$

where $\zeta_{\mu\nu}$ is a symmetric tensor. This is actually not all that strange and new. In ordinary QFT one must also use operator insertions in the path integral in order to "prepare" the initial and final states of an amplitude. For instance:

$$\langle\,x_{i_1}\cdots x_{i_n}\,|\,e^{iHT}\,|\,x_{f_1}\cdots x_{f_n}\,\rangle = \langle\,0\,|\,\phi(x_{i_1})\cdots\phi(x_{i_n})\,\phi(x_{f_1})\cdots\phi(x_{f_n})\,|\,0\,\rangle \tag{1.35}$$

$$= \frac{1}{Z}\int d[\phi]\,\phi(x_{i_1})\cdots\phi(x_{i_n})\,\phi(x_{f_1})\cdots\phi(x_{f_n})\,e^{-S}\,, \tag{1.36}$$

where $|\,0\,\rangle$ is the Fock space vacuum.

It now seems like we have a rule for computing amplitudes, represent all external legs with operator insertions in the path integral, and sum over all two-dimensional compact surfaces. Summing over all surfaces means summing over all metrics and topologies of surfaces. The topology of a two dimensional compact surface is completely specified by the number of its boundaries, crosscaps, and handles (genus). But this procedure, being so similar to what we usually do in QFT, raises a very important question. The genus of a diagram is pictorially very reminiscent of the number of loops of a quantum field theory diagram. For instance, take the torus diagram with four vertex operators, shrink the string to a point particle and you will recover a one-loop diagram for a 4-point function in ϕ^4 theory. In a weakly coupled field theory, loop diagrams are usually suppressed by the coupling constant. So the big question is: where is the analog of this in string theory? Is there such a thing as a string coupling constant that keeps track of the loop order? Well, it turns out that when we wrote down the Polyakov action (1.12), we didn't write the most general action consistent with all the symmetries we found so far ((1.13), (1.14), (1.15)). There is one more piece we could have added, the two-dimensional gravity action [3]:

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d\tau\,d\sigma(\gamma)^{1/2}\,R + \frac{1}{2\pi}\int_{\partial M} ds\,K\,, \tag{1.37}$$

[3] Note that we are now working in the Euclidean formalism, so there is no minus sign under the square root in $(\gamma)^{1/2}$

where the first term is the Ricci scalar and the second term is the extrinsic curvature for a manifold with a boundary (an open string world-sheet). Although very geometric in nature, this action is a topological invariant for two-dimensional manifolds. It basically counts the genus and the number of boundaries and crosscaps of a surface.

$$\chi = 2 - 2g - b - c\,, \tag{1.38}$$

where g is the genus, b the number of boundaries, and c the number of crosscaps. Therefore, if we write the string action as follows:

$$S = S_P + \lambda\chi\,, \tag{1.39}$$

then diagrams will be weighted by a factor $e^{-\lambda\chi}$. If λ is small, we will say that string theory is *weakly coupled* and hence defined *perturbatively*. In this case, diagrams will be suppressed as their genus grows, just like QFT diagrams are suppressed as their loop number grows. If it is large, then we are in the *strongly coupled regime* of string theory, where most of the known techniques from field theory break down and very little is known. In the next section, we will see where this string coupling constant λ comes from; the answer will be quite surprising.

1.1.5 Strings in background fields

So far, we have been studying the theory of a string propagating in a D-dimensional flat spacetime. An obvious generalization at this point would be to start all over again with a Polyakov-like action that has a curved spacetime metric:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, (\gamma)^{1/2}\,\gamma^{ab}\,G_{\mu\nu}(X)\,\partial_a X^\mu\,\partial_b X^\nu\,. \tag{1.40}$$

This action is called a *non-linear sigma model*. From the two-dimensional perspective, this non-trivial spacetime metric $G_{\mu\nu}(X)$ plays the role of a *field-dependent coupling*, where the fields in question are the D scalar fields X^μ.

The attentive reader should be skeptical about this operation. Although it seems natural to replace the flat spacetime metric with a curved one, we should ask ourselves the following question: if the string is supposed to be the fundamental object which generates all particles and forces including gravity, are we allowed to simply put in by hand a curved metric in the action from which we will derive the string spectrum? In other words, if the graviton is a state of the string, how can we include gravity into the action that we must quantize in order to *find* the graviton? This seems like a vicious circle. However, there is a way out of it. The following explanation is borrowed from Polchinski's textbook [2].

Let us first consider a background spacetime metric that is nearly flat:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)\,, \tag{1.41}$$

where $h_{\mu\nu}(X)$ is small. If we expand the integrand of the path integral we obtain the following:

$$e^{-S_\sigma} = e^{-S_P}\left(1 - \frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\,\gamma^{1/2}\,\gamma^{ab}\,h_{\mu\nu}\,\partial_a X^\mu\,\partial_b X^\nu + \cdots\right), \tag{1.42}$$

where S_σ is the sigma model action (1.40), and S_P is the Polyakov action (1.12). The second term in the parenthesis is of the form of a vertex operator for a closed string graviton state (1.34), with $h_{\mu\nu} \propto \zeta_{\mu\nu}\, e^{i\,p\,X}$. So this perturbation of the background metric (1.41) can be viewed as the emission or absorption of a graviton state. Furthermore, if we have a full $G_{\mu\nu}$ background metric, we can view it as an exponentiation of a graviton vertex operator; i.e. a coherent state of gravitons. This validates our naive replacement of the Minkowski spacetime metric for a general curved metric in the non-linear sigma model (1.40).

Let us look back on what we have done so far. We wrote down an action for a string that propagates in a flat spacetime. By quantizing it we found that the string generates particles of different spin, including the graviton. Then, we included gravity into our starting action and discovered that this operation was merely an insertion of a coherent state of gravitons. A natural question at this point would be: can we include other fields in our action that can be viewed as coherent superpositions of other string states? The answer is yes.

Focusing on the massless closed string modes we can write the following action:

$$S_\sigma = \frac{1}{4\pi\,\alpha'} \int \gamma^{1/2} \left[(\gamma^{ab}\, G_{\mu\nu}(X) + i\,\epsilon^{ab}\, B_{\mu\nu}(X))\, \partial_a X^\mu\, \partial_b X^\nu + \alpha'\, R\, \Phi(X) \right], \tag{1.43}$$

where $B_{\mu\nu}$ is the background antisymmetric tensor, Φ is the background scalar (called dilaton), and R is the two-dimensional Ricci scalar. This is the most general action consistent with Poincaré invariance, two-dimensional g.c.t. invariance, and Weyl invariance, containing all massless closed string modes as background fields. In order for this theory to be consistent from the two-dimensional point of view, one needs to make sure that the classical Weyl symmetry is also a symmetry of the quantum theory. This is accomplished by requiring that the expectation value of the trace of the stress-energy tensor of the CFT vanish. This is just the requirement that a current that is classically conserved also be quantum mechanically conserved. This calculation, which we will not contemplate here, is called *anomaly cancelation*. Canceling the Weyl anomaly implies requiring that certain functions called *beta*-functions vanish. Up to first order in α', they look as follows:

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' \left(R_{\mu\nu} + 2\,\nabla_\mu\,\nabla_\nu\Phi - \frac{1}{4}\, H_{\mu\kappa\sigma}\, H_\nu{}^{\kappa\sigma} \right) + O(\alpha'^2)\,, \\
\beta^B_{\mu\nu} &= \alpha' \left(-\frac{1}{2}\, \nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\Phi\, H_{\kappa\mu\nu} \right) + O(\alpha'^2)\,, \\
\beta^\Phi &= \alpha' \left(\frac{D-26}{6\,\alpha'} - \frac{1}{2}\, \nabla^2\Phi + \nabla_\kappa\Phi\, \nabla^\kappa\Phi - \frac{1}{24}\, H_{\kappa\mu\nu}\, H^{\kappa\mu\nu} \right) + O(\alpha'^2)\,,
\end{aligned} \tag{1.44}$$

where $H_{\mu\nu\kappa} \equiv \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$. These three β-functions must vanish independently. By taking proper linear combinations of these equations we are left with something very peculiar. We are left with equations that look like equations of motion for the spacetime background fields. For instance, the equation for the background spacetime metric $G_{\mu\nu}(X)$ turns out to be the Einstein equation. So the quantum string imposes constraints on its field-dependent couplings that look like spacetime field equations! Another peculiarity about these equations is that they require[4] that $D = 26$. The quantum mechanical string is thus only consistent in 26 spacetime

[4] They only require $D = 26$ if the background dilaton Φ is constant. Solutions such as the so-called *linear dilaton theory* with $D < 26$ do exist, however, they are not phenomenologically attractive.

dimensions! There is one more peculiar thing we should notice. The string coupling constant, which we called λ in (1.39) is actually the background value of the dilaton Φ, as can be seen from (1.43). So the string coupling is not a free parameter of the theory, it is determined by a background field of the string itself!

The constraints for the background fields not only look like equations of motion for spacetime fields, they can also be derived from a spacetime action:

$$S = \frac{1}{2\kappa_0^2} \int d^D X \, (-G)^{1/2} \, e^{-2\Phi} \Big[R + 4 \, \nabla_\mu \Phi \, \nabla^\mu \Phi - \frac{1}{12} \, H_{\mu\nu\lambda} \, H^{\mu\nu\lambda} - \frac{2\,(D-26)}{3\,\alpha'} + O(\alpha') \Big], \tag{1.45}$$

where κ_0 is some physically meaningless constant. Whenever we are working in the low energy approximation of string perturbation theory, we can simply regard string theory as a spacetime field theory defined by the action above. We focus on the massless modes of the string because, for small α', the massive modes become very heavy and they decouple from the theory.

The search for solutions to (1.45) is also a search for a string theory background to quantize the string on. Such a background is often called a string theory *vacuum* because, after quantizing the string around it, it acquires the interpretation of a local minimum of some ‘potential’ for the string to oscillate about. The question however remains: what potential, and of what theory? The β-functions provide us with consistency conditions that dictate in what spacetime backgrounds strings are allowed to propagate. However, because string theory must ultimately be a theory of spacetime, and not just a two-dimensional CFT, one would like to be able to treat these backgrounds as vacuum states of some quantum theory. Such a theory does not yet exist. Because of that, there are a myriad of backgrounds to choose from and no principle that allows us to distinguish them. There is, in some sense, a vacuum *degeneracy*, because we do not have something like a potential that can help us distinguish the different ‘states’ of spacetime. One of the great challenges in string theory is finding what is called a *vacuum selection principle* that will actually pinpoint what background is *the* background for strings.

In recent years, however, the debate has shifted from the question: "what is the vacuum selection principle?", to the question: "should there be a vacuum selection principle?" L. Susskind has proposed a scenario, in which all possible allowed vacua actually exist [12]. This *landscape* scenario consists of stating that our universe is just one constituent of a *megaverse*, in which all kinds of universes (corresponding to all kinds of string theory backgrounds) exist, but are causally disconnected. In this approach, there is no room for a vacuum selection principle.

1.2 Superstrings and supergravities

1.2.1 Superstring theories

So far we have been studying the bosonic string, which is a fine toy model, but not a realistic description of particle physics for two reasons: first, the spectrum of the bosonic string contains a tachyon (i.e. a particle with negative mass), which indicates an instability of the string background. Second, it doesn’t contain any fermions since the oscillators only generate integer spin particles. To overcome these problems, we need to generalize the string to a supersymmetric string. By upgrading the two dimensional CFT to a supersymmetric conformal field theory, or *superconformal* field theory, and imposing consistency conditions on the quantized theory, the string will turn out to have spacetime supersymmetry, the tachyon will be projected out of the spectrum, and, as a bonus, the number of required spacetime dimensions will be reduced from 26 to 10. I will now give an intuitive overview of how the superstring is developed, the theories it leads to, and what its low energy approximations are (i.e. supergravities).

The basic form of the supersymmetric world-sheet action is as follows:

$$S = \frac{1}{4\pi} \int d\tau \, d\sigma \left(\frac{2}{\alpha'} \, \partial X^\mu \, \bar{\partial} X_\mu + \psi^\mu \, \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \, \partial \tilde{\psi}_\mu \right), \tag{1.46}$$

where the ψ^μ are D two-dimensional fermions. This theory is a superconformal field theory. By analyzing its spectrum in analogy with the bosonic string, one will find that the world-sheet fermions also have oscillators, which act as raising and lowering operators on the vacuum. This will give rise to not only spacetime bosonic states, but also spacetime fermionic states. In fact, by properly counting the bosonic and fermionic states that are generated, one finds that this theory has spacetime supersymmetry. This means that the number of bosonic degrees of freedom matches the number of fermionic degrees of freedom. This theory turns to be anomaly-free only in ten spacetime dimensions.

A more detailed study of the superstring will show that it is actually possible to define *five* different consistent supersymmetric string theories:

- **Type I**: This is a theory of unoriented open strings.
- **Type II**: There are two theories in this category, **Type IIA** and **Type IIB**. These are theories of closed strings, and they differ in the boundary conditions applied on the world-sheet fermions.
- **Heterotic**: There are two heterotic string theories. These theories are constructed in very peculiar ways, and they naturally have non-Abelian spacetime gauge symmetries. Their groups are indicated by their names: **Het** $E_8 \times E_8$, and **Het** SO(32).

Type I and the Type II theories are *a priori* not free of tachyons. However, a certain projection must be performed on the spectrum for consistency conditions, after which all tachyons are gone. This projection is called the *GSO* projection after Gliozzi, Scherk and Olive [13]. All five of these string theories live in ten spacetime dimensions.

1.2.2 Supergravities

We will now write down the low energy effective actions for these five supersymmetric string theories. But before we do so, let us look back to the case of the bosonic string for some moral guidance. When we quantized the string, we required that the classical symmetries (spacetime Poincaré, and 2-D Weyl invariance) be respected at the quantum level. This lead to the vanishing of the β-functions of the field-dependent couplings $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$, $\Phi(X)$, which carved out for us a procedure to write down a unique spacetime field theory that describes the massless modes of the string at low energy.

The five supersymmetric string theories not only have the symmetries of the bosonic string, but they each come with a different form of spacetime supersymmetry. It turns out that supersymmetry is a stringent enough constraint that, given the dimensionality and field content of a theory, there is only one possible spacetime action one can write down. This means that all we need to know to construct the low energy effective actions for the massless modes for these five string theories is their spectrum and the kind of supersymmetry they have. The resulting actions are called *supergravities*. Their symmetries naturally combine general coordinate transformation invariance and local supersymmetry as the name suggests. The bosonic parts of the actions for the five supergravities are the following:

- **Type IIA**

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{2\kappa_0^2} \int d^{10} (-G)^{1/2} \Big(e^{-2\Phi} \Big[R + 4 (\nabla \Phi)^2 - \frac{1}{12} (H^{(3)})^2 \Big] \\ - \frac{1}{4} (G^{(2)})^2 - \frac{1}{48} (G^{(4)})^2 \Big) - \frac{1}{4\kappa_0} \int B^{(2)} \, dC^{(3)} \, dC^{(3)} , \end{aligned} \tag{1.47}$$

  where G is the 10-dimensional metric, Φ the dilaton, $H^{(3)} = dB^{(2)}$ the field strength of a two-form, $G^{(2)} = dC^{(1)}$ the field strength of a one-form, and $G^{(4)} = dC^{(3)} + H^{(3)} \wedge C^{(1)}$ can be seen as the modified field strength of a three-form.

- **Type IIB**

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa_0^2} \int d^{10} (-G)^{1/2} \Big(e^{-2\Phi} \Big[R + 4 (\nabla \Phi)^2 - \frac{1}{12} (H^{(3)})^2 \Big] \\ - \frac{1}{12} (G^{(3)} + C^{(0)} H^{(3)})^2 - \frac{1}{2} (dC^{(0)})^2 - \frac{1}{480} (G^{(5)})^2 \Big) \\ + \frac{1}{4\kappa_0^2} \int \Big(C^{(4)} + \frac{1}{2} B^{(2)} C^{(2)} \Big) G^{(3)} H^{(3)} , \end{aligned} \tag{1.48}$$

  where $G^{(3)} = dC^{(2)}$, $G^{(5)} = dC^{(4)} + H^{(3)} \wedge C^{(2)}$, and $C^{(0)}$ is a scalar. To get the right number of degrees of freedom, one must impose that the field strength of the four-form $F^{(5)} = dC^{(4)}$ be self-dual: $F^{(5)} = *F^{(5)}$. However, this constraint can only be imposed at the level of the equations of motion.

- **Type I**

$$\begin{aligned} S_{\mathrm{I}} = \frac{1}{2\kappa_0^2} \int d^{10} (-G)^{1/2} \Big(e^{-2\Phi} \Big[R + 4 (\nabla \Phi)^2 \Big] \\ - \frac{1}{12} \Big(\tilde{G}^{(3)} \Big)^2 - \frac{\alpha'}{8} e^{-\Phi} \, Tr (F^{(2)})^2 \Big) , \end{aligned} \tag{1.49}$$

  where $\tilde{G}^{(3)} = dC^{(2)} - \frac{\alpha'}{4} \left[\frac{1}{30} Tr \left(A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right) \right]$. The trace '$Tr$' runs over Yang-Mills group indices.

- **Heterotic**

$$S_{\text{Het}} = \frac{1}{2\kappa_0^2} \int d^{10}\,(-G)^{1/2}\, e^{-2\,\Phi} \Big(R + 4\,(\nabla\,\Phi)^2 - \frac{1}{12}\left(\tilde{H}^{(3)}\right)^2 - \frac{\alpha'}{8}\, e^{-\Phi}\, Tr(F^{(2)})^2 \Big), \tag{1.50}$$

where $\tilde{H}^{(3)} = dB^{(2)} - \frac{\alpha'}{4}\left[\frac{1}{30}\, Tr\left(A \wedge dA + \frac{1}{3}\, A \wedge A \wedge A\right)\right]$.

This concludes the introduction to string theory. The main goal of this chapter was to explain how a quantum theory of relativistic strings can, in a certain approximation, lead to a spacetime gravitational field theory. Actually, two approximations were made. The first one is the assumption that strings interact weakly, i.e. that the string coupling constant given by the constant part of the dilaton is small. This allows us to define a CFT on the world-sheet perturbatively. The second assumption is the low energy approximation. At low energies only the massless states of the string are excited. In the β-functions this is manifested by a truncation of α' corrections. This is what allows us to write down a spacetime classical field theory, such as a supergravity, as an effective description of string theory.

Throughout this thesis we will be working with these approximations. In the next part, which consists of two chapters, we will study instantons and their role in string theory. In the second part, chapters 4, 5 and 6, we will study cosmology in the context of scalar-gravity theories. These theories are often supergravity Lagrangians that have been dimensionally reduced and truncated to contain only the metric and scalar fields. In the final part of this thesis, chapter 7, we will see how the first two parts come together in two different ways: first, we will see how Wick rotations can relate supergravity instantons to cosmological solutions. Then, we will make a paradigm shift and treat those two kinds of solutions on equal footing, by regarding them as trajectories of a particle in a fictitious *target space* parametrized by the scalar fields of the theory.