University of Groningen

Theory and history of geometric models

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Chapter 3

Models of curves

In the present chapter, we will treat various models representing different types of curves. The first section deals with models of cubic curves over $\mathbb{P}^2(\mathbb{R})$ as classified by Möbius in [M]. The second section of the chapter studies two different groups of models illustrating space curves.

3.1 Models of cubic curves over $\mathbb{P}^2(\mathbb{R})$

The models treated in this section belong to Martin Schilling's collection. Namely, we study the 7 string models from Series XXV and 2 plaster models from Series XVII. These models have in common that they are all representing particular cubic curves. The string models from Series XXV were made in 1899 by Hermann Wiener, and are linked to his text [H-W1] of 1901. The complete series is shown in Figure 3.1.
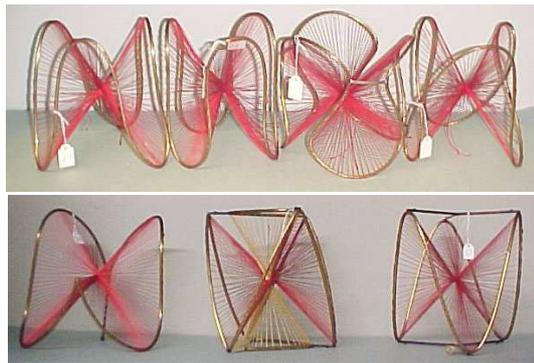

Figure 3.1: Seven string models, Series XXV, University of Amsterdam.

The two plaster models from Series XVII correspond to the numbers 2a and 2b of Schilling's catalogue [Sch]. These models are shown in Figure 3.2, and consist of two balls made of plaster with curves of degree three drawn on them. They are discussed in more detail in Section 3.1.3.

Figure 3.2: Plaster models, Series XVII nr. 2a and 2b, University of Groningen.

The nine models that have been mentioned all represent real curves of degree three in the real projective plane $\mathbb{P}^2(\mathbb{R})$. The models illustrate the seven different types of real cubic curves as classified by Möbius (see [M]). These curves are represented in thread or in plaster depending on the ways of visualizing the real projective space $\mathbb{P}^2(\mathbb{R})$. One can consider the following:

Setup:

1. The real projective plane $\mathbb{P}^2(\mathbb{R})$ can be considered as the space of all lines in $\mathbb{R}^3$ that pass through the origin.

2. Alternatively, the real projective plane $\mathbb{P}^2(\mathbb{R})$ can be considered as the set of pairs of antipodal points on the sphere S^2. I.e., the sphere as the usual $2 : 1$ covering of $\mathbb{P}^2(\mathbb{R})$

$$\Phi : S^2 \longrightarrow \mathbb{P}^2(\mathbb{R}).$$

One can now look at curves $\Gamma \subseteq \mathbb{P}^2(\mathbb{R})$ in two different ways:

1. As families of lines in $\mathbb{R}^3$ through the origin.

2. As their preimage under Φ:

$$\Phi^{-1}(\Gamma) \in S^2.$$

The seven thread models represent cubic curves described as families of lines. The seven curves shown on the two plaster models represent cubic curves as preimages under Φ.

3.1.1 Results on cubic curves

In order to understand Möbius' classification, we need some background knowledge on the theory of cubic curves. We begin by giving the following definition:

Definition 4. Let C be a plane curve given by a homogeneous polynomial F. An *inflection point* or a *flex point* p of C is a non-singular point of C such that the intersection of the tangent line at p with the curve C has multiplicity ≥ 3.

The tangent line to C at a flex point p is called a *flex line* of C.

In order to compute the flex points on a curve C, one can use the following result:

Theorem 16. *Let C be a curve given by $F = 0$. The flex points of C are the smooth points of C satisfying*

$$H(x) = det\left(\frac{\partial^2 F}{\partial x_i x_j}\right) = 0.$$

Proof. See [W, Thm. 6.3]. □

The polynomial H in this theorem is called the *Hessian of C* (or of F), and $H = 0$ defines the so-called Hessian curve of C.

Remark: Suppose that C is an irreducible cubic curve in $\mathbb{P}^2$. Then its Hessian curve is also a cubic. Since not every smooth point of C is a flex, it follows that C has at most nine flex points.

The following theorem is well known and simplifies the study of many properties of the cubic curves in $\mathbb{P}^2(\mathbb{R})$.

Theorem 17. *Let C denote an irreducible cubic curve defined over $\mathbb{R}$. Then C has a real flex point and after a linear change of variables defined over $\mathbb{R}$, C is given by $y^2z - f(x, z)$, where f is a homogeneous polynomial of degree 3 in x. In this form, the real flex point is given as $(0 : 1 : 0)$ and the corresponding flex line is $z = 0$.*

Proof. See [W, Thm. 6.4]. □

3.1.2   Newton's classification of cubic curves

In 1704 Isaac Newton classified the irreducible curves of degree 3 in $\mathbb{P}^2(\mathbb{R})$. This classification appears in the appendix *Enumeratio Linearum Tertii Ordinis* of his book *Opticks* (see [N]). In this appendix, Newton considers the irreducible cubic curves given in its Weierstrass form:

$$y^2z = x^3 + ax^2z + bxz^2 + cz^3 = f(x, z)$$

and classifies them in terms of $f(x, 1) = x^3 + ax^2 + bx + c$ into the following five types :

1. *Parabola pura*: $f(x, 1)$ has precisely one real root which is simple. The curve is non-singular and its real locus has one component. (Fig. 9^a and 9^b by Möbius [M]).

2. *Parabola campaniformis cum ovali*: $f(x, 1)$ has three distinct real roots. The curve is non-singular and its real locus has two components. (See Fig. 7 by Möbius [M]).

3. *Parabola nodata*: $f(x, 1)$ has a real double root α and a real simple root β with $\alpha > \beta$. The curve has an ordinary double point with real tangents. (Fig. 10 by Möbius [M]).

4. *Parabola punctata*: $f(x, 1)$ has a real double root α and a real simple root β with $\alpha < \beta$. The curve has an isolated double point with conjugate tangents. (Fig. 8 by Möbius [M]).

5. *Parabola cuspidata*: $f(x, 1)$ has one triple real root (Fig. 11 by Möbius [M]).

Obviously, Newton's classification of irreducible cubic curves over $\mathbb{R}$ uses Theorem 17 above. Although he stated this theorem, he did not provide a proof. Plücker [P] refined Newton's classification and gave a complete detailed proof (see [At]).

3.1.3   Möbius' classification

As we have seen in Section 3.1.1, an irreducible cubic curve has at most 9 flex points over $\mathbb{C}$. In fact, a real irreducible cubic curve can either have 1 real flex point or 3. Möbius studies which types of Newton's classification contain 1 or 3 real flex points. His results are collected in the following theorem:
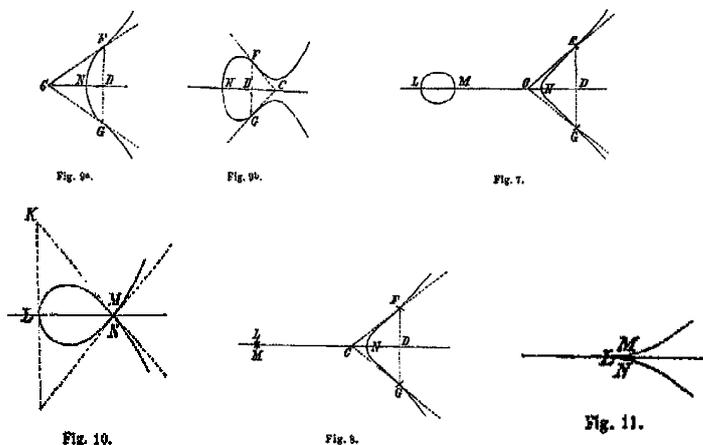
Figure 3.3: Möbius' drawings on Newton's types of cubic curves ([M]).

Theorem 18. *Let C be an irreducible cubic curve over $\mathbb{R}$. The number of real flex points on the curve is as follows, depending on its type in Newton's classification:*

1. *The Parabola pura, Parabola campaniformis cum ovali and Parabola punctata have three real flex points.*

2. *The Parabola nodata and Parabola cuspidata have one real flex point.*

In case 1., the three real flex points are collinear.

Proof. We present a modern proof of this, based on the fact that a group structure can be defined on a smooth irreducible plane cubic curve. Let C_s be the set of smooth points on C. It is well known ([W, Thm. 6.4]) that C_s contains a real flex point. After a real, linear change of coordinates we may assume that $O = (0 : 1 : 0)$ is a flex point and $z = 0$ is the associated flex line. One can consider the group structure on the set C_s defined as: choose the neutral element of the group to be $O = (0 : 1 : 0)$. One can define the sum of two points $P + Q$ to be the third point of intersection of the line PQ with the curve C.

It follows that P is a flex point of C if and only if $3P = 0$ in the group C_s. In order to find the real flex points, one can study the real locus of C_s for the five types of irreducible cubic curves:

1. The *Parabola pura* is smooth and its real locus has one component. Hence $C_s(\mathbb{R})$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

2. The *Parabola campaniformis cum ovali* is smooth and its real locus has two components. Then, $C_s(\mathbb{R}) \simeq \mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

3. For the *Parabola nodata*, $C_s(\mathbb{R}) \simeq \mathbb{R}^* \simeq \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$.

4. For the *Parabola punctata*, $C_s(\mathbb{R}) \simeq \{z \in \mathbb{C}^* \mid |z| = 1\} \simeq \mathbb{R}/\mathbb{Z}$.

5. For the *Parabola cuspidata*, $C_s(\mathbb{R}) \simeq \mathbb{R}$.

It follows immediately that for the cases *iii*) and *v*) the only element of order 3 in C_s is the point $O = (0 : 1 : 0)$. For the remaining cases, C_s has three elements of order 3, that is, C has 3 real flex points.

For the last assertion, observe that if $P \neq Q$ are real flex points, then $3(-P - Q) = -3P - 3Q = 0$ hence $-(P + Q)$ is also a real flex point. Since P, Q and $-(P + Q)$ add up to zero, they are on a line, and these are all the real flex points. An elementary proof, relying on the fact that C may be assumed to be given by an equation $y^2 z = f(x, z)$, is presented in [T]. □

Cubic curves on S^2

As we mentioned in the beginning of the chapter, Möbius uses for his classification the antipodal map

$$\Phi : S^2 \longrightarrow \mathbb{P}^2(\mathbb{R}).$$

In this way, lines in $\mathbb{P}^2(\mathbb{R})$ correspond to large circles in S^2.

Suppose that Γ is a cubic curve in $\mathbb{P}^2(\mathbb{R})$ with only one real flex point. Then, Möbius' classification and Newton's classification coincide. The *Parabola nodata* is called type 6 by Möbius and the *Parabola cuspidata* type 7.

Suppose that Γ is a cubic curve in $\mathbb{P}^2(\mathbb{R})$ that has 3 real flex points. Möbius considers the following four lines in $\mathbb{P}^2(\mathbb{R})$.

1. The flex lines ℓ_1, ℓ_2, ℓ_3 at the three real flex points.

2. The real line ℓ that contains the three real flex points (see Theorem 18).

Via the antipodal map, these four lines correspond to four large circles on S^2. These circles give a tiling of S^2. Möbius obtains the following result:

Theorem 19. *Let* $\Gamma \subseteq \mathbb{P}^2(\mathbb{R})$ *be a cubic curve with three real flex points. Let* $\ell_1, \ell_2, \ell_3, \ell$ *be the lines described before, and consider the tiling of the sphere given by the large circles* $\{\Phi^{-1}(\ell_1), \Phi^{-1}(\ell_2), \Phi^{-1}(\ell_3), \Phi^{-1}(\ell)\}$. *Then, the subdivision can only consist of triangles and quadrangles. Furthermore, one of the following holds.*

1. *The subdivision contains only triangles (this happens if and only if the three flex lines* ℓ_1, ℓ_2, ℓ_3 *are concurrent in* $\mathbb{P}^2(\mathbb{R})$). *This cannot occur with a Parabola Campaniformis cum ovali or a Parabola punctata. Möbius calls these curves* neutral curves *and they are of type 5 in his classification.*

2. *The subdivision contains triangles and quadrangles and the curve lies in the area of the quadrangles. This happens for the Parabola Campaniformis cum ovali (called type 3 by Möbius), the Parabola punctata (called type 2 by Möbius) and the Parabola pura (called type 1 in his classification). Möbius calls these curves* 3-curves.

3. *The subdivision contains triangles and quadrangles and the curve lies in the area of the quadrangles. As in case 1, this cannot happen for the Parabola Campaniformis cum ovali and the Parabola punctata. Möbius calls these curves* 4-curves *and they are of type 4 in his classification.*

Proof. To verify this, suppose that ℓ_1 is the line at infinity $(z = 0)$ in $\mathbb{P}^2$. The tiling of $\mathbb{R}^2 = \mathbb{P}^2(\mathbb{R}) \backslash \ell_1$ is then given by the affine parts of the lines ℓ_2, ℓ_3 and ℓ. Since ℓ contains the unique point at infinity on C and ℓ_2 does not, ℓ and ℓ_2 are not parallel. Similarly, ℓ and ℓ_3 are not parallel. One has the following two possibilities:

1. ℓ_2 and ℓ_3 are parallel. The tiling of $\mathbb{R}^2$ obtained here then yields case 1. We will prove below that this can only occur for certain *Parabola pura*.

2. ℓ_2 and ℓ_3 are not parallel. Then, ℓ, ℓ_2 and ℓ_3 cut out a triangle in $\mathbb{R}^2$ with vertices the two affine flex points and the point of intersection of ℓ_2 and ℓ_3. This gives 8 triangles in S^2, bordered by part of three of the four circles $\Phi^{-1}(\ell)$, $\Phi^{-1}(\ell_i)$, and 6 quadrangles.

   - If the curve C intersects the triangle in $\mathbb{R}^2$ bordered by ℓ_1, ℓ_2 and ℓ_3, then it crosses the boundary only at the flex points $\ell \cap \ell_2$ and

$\ell \cap \ell_3$. These points being flexes and the curve being cubic means, that at $\ell \cap \ell_2$ the curve passes from one side of ℓ_2 to the other, and from one side of ℓ to the other (see Figure 3.4).
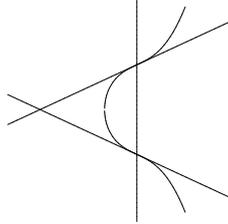

Figure 3.4: 3-curve.

Similarly at $\ell \cap \ell_3$. This implies that the curve passes from a triangle to a triangle. This yields case 2.

This settles the proof in the case that the real locus consists of only one component. For the *Parabola campaniformis cum ovali* and *Parabola punctata* we may assume that C is given by $y^2 z = f(x, z)$. The component not meeting the line $\ell_1 : z = 0$ is then fixed by the map $(x : y : z) \mapsto (x : -y : z)$. This implies the curve is in the triangle.

- Similarly, case 3. is obtained.

$\square$

Möbius also proved that for the *Parabola pura*, all three cases of Theorem 19 can occur.

In this way, Möbius obtains his classification of the cubic curves in $\mathbb{P}^2(\mathbb{R})$ into 7 species. We give examples of this in the next section.

Note that this subdivision is consistent, that is, it remains invariant under real projective transformations.

Weierstrass form

In order to prove Möbius' classification of cubic curves, we first fix $\mathbb{R}^2 = \mathbb{P}^2(\mathbb{R}) \backslash \ell_1$, that is, $\mathbb{R}^2$ with coordinates $\frac{x}{z}$ and $\frac{y}{z}$. Consider such a curve given by $y^2 z = f(x, z) = x^3 + ax^2 z + bxz^2 + cz^3$. In this form, one of the three real flex points is the point at infinity $(0 : 1 : 0)$. Note also that if $(\alpha : \beta : \gamma)$

is a real flex point of Γ different from $(0 : 1 : 0)$, then $(\alpha : -\beta : \gamma)$ is also a real flex point of the curve. Moreover, $\beta \neq 0$ since otherwise the tangent line $x = \alpha z$ at $(\alpha : 0 : 1)$ also meets the curve at $(0 : 1 : 0)$, so $(\alpha : 0 : 1)$ is not a flex point. This provides an alternative proof of the fact that the real flex points are collinear (see Theorem 18).

One can consider the four lines $\ell_1, \ell_2, \ell_3, \ell$ as before. These lines are now given by:

1. The flex line ℓ_1 of the flex point $(0 : 1 : 0)$ (given by the line at infinity $z = 0$).

2. The flex line ℓ_2 of the flex point $(\alpha : \beta : 1)$, with $\beta > 0$.

3. The flex line ℓ_3 of the flex point $(\alpha : -\beta : 1)$, with $\beta > 0$.

4. The line ℓ that contains the three real flex points (given by the equation $x - \alpha z = 0$).

Observe that the two flex lines ℓ_2 and ℓ_3 considered in the affine plane are the mirror image of each other under reflection int he line $y = 0$. One can look at the affine line ℓ_2 and consider the *slope* of ℓ_2. We have the following correspondence with Möbius' result:

Theorem 20. *Let* $\Gamma \subseteq \mathbb{P}^2(\mathbb{R})$ *be an irreducible cubic curve given by* $y^2 z = f(x, z) = x^3 + ax^2 z + bxz^2 + cz^3$, *and consider the lines* ℓ_1, ℓ_2, ℓ_3 *and* ℓ *as before. Consider the tiling of* S^2 *given by the large circles*

$$\{\Phi^{-1}(\ell_1), \Phi^{-1}(\ell_2), \Phi^{-1}(\ell_3), \Phi^{-1}(\ell)\}$$

. *Let* ℓ_{2*} *be the affine flex line of the flex point* (α, β). *Then,*

1. *The slope of* ℓ_{2*} *equals* $0 \Leftrightarrow$ *the subdivision of* S^2 *contains only triangles.*

2. *The slope of* ℓ_{2*} *is* $> 0 \Leftrightarrow$ *the subdivision of* S^2 *contains triangles and quadrangles and the curve lies in the area of the triangles (i.e.,* Γ *is a 3-curve).*

3. *The slope of* ℓ_{2*} *is* $< 0 \Leftrightarrow$ *the subdivision of* S^2 *contains triangles and quadrangles and the curve lies in the area of the quadrangles (i.e.,* Γ *is a 4-curve).*

Proof. For the first case, note that the slope of ℓ_{2*} equals 0 if and only if ℓ_{2*} and ℓ_{3*} are parallel. The last holds if and only if ℓ_1, ℓ_2 and ℓ_3 are concurrent in $\mathbb{P}^2(\mathbb{R})$. This implies, by Theorem 19, that the tiling of S^2 contains only triangles.

Let $(\alpha : \beta : 1)$ be the real flex point with $\beta > 0$. Suppose that the largest zero of $f(x,1) = x^3 + ax^2 + bx + c$ occurs at $x = \zeta$. Then, $\zeta < \alpha$ and the graph of $\sqrt{f}$ is convex in an interval $(\alpha - \epsilon, \alpha)$ and lies below ℓ_{2*}, and concave in an interval $(\alpha, \alpha + \epsilon)$ and lies above ℓ_{2*}. The triangle bounded by ℓ, ℓ_2 and ℓ_3 being to the left or to the right of ℓ determines that the curve lies inside or outside the triangle. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Figure 3.1.3 shows the 7 types of cubic curves on the sphere by Möbius:
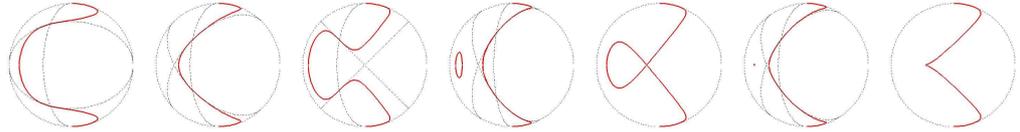
Figure 3.5: From left to right: P. pura 1, 2 and 3, P. campaniformis cum ovali, P. nodata, P. punctata, P. cuspidata.

Cubic curves on S^2: plaster models

The two models from Series XVII nr. 2a and 2b represent two balls made of plaster, and are shown in Figure 3.2. They contain the seven types of cubic curves in Möbius' classification. We recall from [Do] the equations of the curves that are represented in the two balls:

1. The model on the left corresponds to Series XVII, nr. 2a. It contains 4 curves given by the following types and equations:

   - *Parabola campaniformis cum ovali, Möbius type 3*:

   $$y^2 z = \frac{1}{4}(x + 12z)(x + 5z)(x + 2z)$$

   - *Parabola nodata, Möbius type 6*:

   $$y^2 x = -9(x + z)^2(x + 6z)$$

- *Parabola punctata, Möbius type 2*:

$$y^2 z = \frac{1}{4}(x + \frac{9}{2}z)^2(x + z)$$

- *Parabola cuspidata, Möbius type 7*:

$$y^2 x = -(x + \frac{9}{2}z)^3$$

2. The model on the right corresponds to Series XVVII, nr. 2b. It contains 3 curves given by the following types and equations:

- *Parabola pura, Möbius type 1*:

$$y^2 = \frac{4}{9}\{(x + 3z)^2 - \frac{20x^3}{x - 4z}\}$$

- *Parabola pura, Möbius type 4*:

$$y^2 = \frac{4}{9}\{(x + 3z)^2 - \frac{25x^3}{x + 6z}\}$$

- *Parabola pura, Möbius type 5*:

$$y^2 = \frac{4}{9}\{(x + 3z)^2 - \frac{(4x)^3}{x + 3z}\}$$

It is clear, by looking at the zeroes of the righthand side of the equations, that the curves on the first ball are of the type given above.

Let us now study the three examples on the second ball. The three curves are given in such a way that two of the flex points have coordinates $(0 : 2 : 1)$ and $(0 : -2 : 1)$ with associate flex lines $y = \frac{2}{3}(x + 3z)$ and $y = -\frac{2}{3}(x + 3z)$ respectively. The third flex point is the point $(0 : 1 : 0)$. We study the three cases in detail:

(a) The curve is given by $y^2 = \frac{4}{9}\{(x + 3z)^2 - \frac{20x^3}{x - 4z}\}$: The third flex line is given by $x = 4z$. Since the three flex lines are not concurrent, the tiling of the sphere will contain triangles (bounded by the three lines $y = \pm\frac{2}{3}(x + 3z)$ and $x = 0$) and quadrangles (bounded by the four lines $y = \pm\frac{2}{3}(x + 3z)$, $x = 0$ and the line $x = 4z$). We claim that the curve is a 3-*curve* (i.e., it lies in the area of the triangles): observe that the curve goes to infinity when x approaches 4 from the left. In particular, this means that the curve has no points inside the quadrangles. See Fig 3.6 (a).
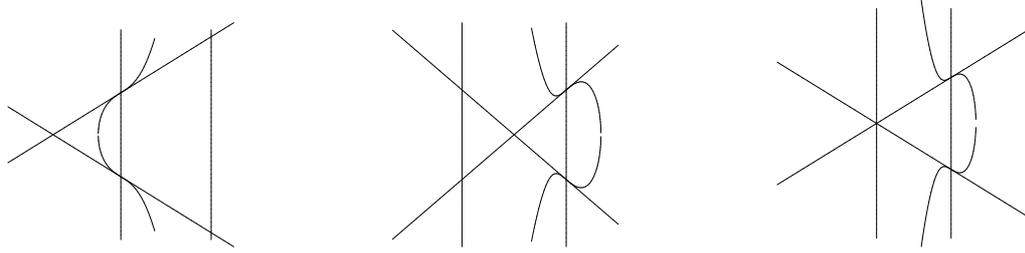
Figure 3.6: From left to right: 3-*curve* from (a), 4-*curve* from (b) and *neutral curve* from (c).

(b) The curve is given by $y^2 = \frac{4}{9}\{(x + 3z)^2 - \frac{25x^3}{x+6z}\}$. The third flex line is given by $x = -6z$. Since the three flex lines lines are not concurrent, the tiling of the sphere will contain triangles and quadrangles. The triangles are bounded by the three lines $y = \pm\frac{2}{3}(x + 3z)$ and $x = 0$. The quadrangles are bounded by these three lines and the line $x = -6z$.

The curve goes to infinity when x approaches -6 from the right, so the curve remains inside the quadrangles, i.e., it is a 4-curve. See Figure 3.6 (b).

(c) In the last case, the three flex lines are $y = \frac{2}{3}(x + 3z)$, $y = -\frac{2}{3}(x + 3z)$ and $x = -3z$ which intersect in the point $(-3 : 0 : 1)$. Therefore, the tiling of the sphere contains only triangles, i.e., the curve is of type 5 in Möbius classification. See Figure 3.6 (c).

3.1.4   Twists of curves

Consider the irreducible cubic curve C over $\mathbb{R}$ given in the form $y^2z = x^3 + ax^2z+bxz^2+cz^3$. We study the twists of the curve over $\mathbb{R}$. Let $\text{Aut}(C)$ be the group of automorphisms of C over $\mathbb{C}$ that send $(0 : 1 : 0)$ to $(0 : 1 : 0)$. Recall that the twists of C are given by the elements of $\text{H}^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(\text{C}))$. It is known (see [Sil, Cor. 5.4.1]) that $\text{Aut}(C)$ is isomorphic to the group μ_i for $n = 2$, 4 and 6, where μ_i the group of i-th roots of unity. Moreover, the isomorphism $\text{Aut}(\text{C}) \longrightarrow \mu_i$ commutes with the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Let $\text{Gal}(\mathbb{C}/\mathbb{R}) = \langle\sigma\rangle$. Then, the elements of $\text{H}^1(\langle\sigma\rangle, \mu_i)$ are given by the equivalence classes of cocycles $c : \text{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow \mu_i$. We study the twists of C for the three cases μ_2, μ_4 and μ_6 in the following lemma:

Lemma 5. $\mathrm{H}^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mu_i)$ *is in all cases, a group of order* 2 *generated by the class of the cocycle* $c : \sigma \mapsto a$ *with* $c(1) = 1$ *and* $c(\sigma) = a$, *where* a *is a generator of* μ_i *(note that* $a\sigma(a) = 1$).

Proof. In all cases, we have a surjective homomorphism:

$$\Phi : \quad \mu_i \longrightarrow \quad \mathrm{H}^1(\langle\sigma\rangle, \mu_i)$$
$$\zeta \mapsto \quad [c \text{ with } c(1) = 1, \ c(\sigma) = \zeta]$$

whose kernel are the coboundaries $\{\frac{\sigma(\zeta)}{\zeta} \mid \zeta \in \mu_i\}$. Hence,

$$\mathrm{H}^1(\langle\sigma\rangle, \mu_i) \simeq \frac{\mu_i}{\mathrm{Ker}(\Phi)}.$$

We study $\mathrm{H}^1(\langle\sigma\rangle, \mu_i)$ for the three cases $i = 2$, 4 and 6:

1. $\mathrm{Aut}(C) \simeq \mu_2$. In this case, the kernel of $\Phi : \mu_2 \longrightarrow \mathrm{H}^1(\langle\sigma\rangle, \mu_2)$ is trivial, so one has that the twists are given by $\mathrm{H}^1(\langle\sigma\rangle, \mu_2) \simeq \mu_2$. This gives us only one nontrivial twist, namely the one given by the cocycle of order 2: $c(\sigma) = (x : y : z) \mapsto (x : -y : z)$. To calculate the equation of the twisted curve of the curve given by $y^2z = x^3 + ax^2z + bxz^2 + cz^3$, we calculate the invariants in $\mathbb{C}[x, y, z]$ under the action:

$$x, y, z \quad \mapsto x, -y, z$$
$$i \quad \mapsto -i.$$

   Three invariants are given by the elements $\mu = x$ and $\psi = iy$ and $\zeta = z$ and an equation for the twisted curve is:

$$\psi^2\zeta = -\mu^3 - a\mu^2\zeta - b\mu\zeta^2 - c\zeta^3.$$

2. $\mathrm{Aut}(C) \simeq \mu_4$, which holds for the curves $y^2z = x^3 + axz^2$. In this case, $\mathrm{Ker}(\Phi) \simeq \langle\pm 1\rangle \subseteq \mu_4$ and there is again only one non-trivial twist given by the cocycle of order 4: $c(\sigma) = (x : y : z) \mapsto (-x : iy : z)$. The invariants in $\mathbb{C}[x, y, z]$ of the action

$$x, y, z \quad \mapsto -x, iy, z$$
$$i \quad \mapsto -i$$

   are generated by $\mu = -ix$ and $\psi = \frac{1+i}{\sqrt{2}}y$ and $\zeta = z$ and the equation for the twisted curve is: $\psi^2\zeta = \mu^3 - a\mu\zeta^2$.

3. $\mathrm{Aut}(C) \simeq \mu_6$, which holds for the curves $y^2 z = x^3 + a z^3$. In this case, the kernel of Φ is isomorphic to $\langle \omega \rangle \subseteq \mu_6$, with ω a cube root of unity. Therefore, there is again only one non-trivial twist that can be given by the cocycle of order 2: $c(\sigma) = (x : y : z) \mapsto (x : -y : z)$. This twist is identical with the quadratic twist. As in case 1., the equation of the twisted curve is $\psi^2 \zeta = \mu^3 - a\zeta^3$.

$\square$

Let us study the non-trivial twists of the curves in Newton's and Möbius' classifications.

1. *Quadratic twist.* We first study what happens to the curves in Newton's classification:

   (a) C **is a *Parabola pura*** : C is of the form $y^2 z = (x - az)(x^2 + bz^2)$, with $a, b \in \mathbb{R}$, $b > 0$. An equation for the twisted curve is: $\psi^2 \zeta = (-\mu + a\zeta)(\mu^2 + b\zeta^2)$ which defines again a *Parabola pura*.

   (b) C **is a *Parabola campaniformis cum ovali*** : C is of the form $y^2 z = (x - az)(x - bz)(x - cz)$, with $a, b, c \in \mathbb{R}$. An equation for the twisted curve is: $\psi^2 \zeta = (\mu + a\zeta)(\mu + b\zeta)(\mu + c\zeta)$ which defines again a *Parabola campaniformis cum ovali*.

   (c) C **is a *Parabola nodata*** : C is of the form $y^2 z = (x - az)^2 (x - bz)$, with $a > b$. An equation for the twisted curve is: $\psi^2 \zeta = (\mu + a\zeta)^2 (\mu + b\zeta)$ with $-a < -b$ which defines a *Parabola punctata*.

   (d) C **is a *Parabola punctata*** : As studied before, one sees that the twisted curve defines a *Parabola nodata*.

   (e) C **is a *Parabola cuspidata*** : C is of the form $y^2 z = (x - az)^3$. The twisted curve can be given by: $(\mu + az)^3$ which defines again a *Parabola cuspidata*.

   We note that cases (c), (d), (e) concern singular curves for which Lemma 5 also holds. Let us now look at Möbius' classification into three types of the *Parabola pura*. Suppose that the *Parabola pura* is given by the equation $y^2 z = x^3 + a z^3$. For $a > 0$ the curve has a real flex point at $(0, \sqrt{a})$. The flex line associated to $(0, \sqrt{a})$ is $y = \sqrt{a}$ which has slope 0. Hence, the curve is a *neutral curve*. For $a < 0$, the curve has a real flex point at $(\sqrt[3]{-4a}, \sqrt{-3a})$ and the slope of the flex

line associated to the flex point is $\frac{3\sqrt[3]{(-4a)^2}}{2\sqrt{-3a}}$ which is positive. Hence, the equation defines a 3-*curve*.

The curve becomes $\psi^2\zeta = \mu^3 - az^3$ after twisting, which defines a 3-*curve* when $a > 0$ and a *neutral-curve* when $a < 0$.

We study another example of *Parabola pura*, namely the one given by the equation $y^2z = (x+z)^3 + x^2z$. This curve has a real flex point at $(-1, 1)$, and the associated flex line is $y = -x$. Hence it is a 4-curve. An equation for its twist is $\psi^2\zeta = (\mu-\zeta)^3 - \mu^2\zeta$, which defines a 3-curve (one can again calculate the real flex point with positive y-coordinate, and see that the slope of its associated flex line is positive).

On the other hand, the 3-curve given by $y^2z = (x-z)^3 + x^2z$ has as twist a curve given by $\psi^2\zeta = (\mu+\zeta)^3 - \mu^2\zeta$ which is again a 3-curve. We have seen that one obtains different results depending on the equation of the curve, which shows that Möbius' classification is not consistent with respect of twisting of the curves.

2. *Twist of degree* 4. The curve $y^2z = x^3 + axz^2$ defines a *Parabola pura* of type 3-curve when $a > 0$, a *Parabola cuspidata* when $a = 0$ and a *Parabola campaniformis cum ovali* when $a < 0$. After twisting, the *Parabola pura* of type 3-curve becomes a *campaniformis cum ovali* and vice versa. The *cuspidata* remains a *cuspidata* after twisting.

3. *Twist of degree* 6. The twists of the curve $y^2z = x^3 + az^3$ are the same as its quadratic twists, hence have been studied in the first case already.

3.2 Models of space curves

In this section we study two families of models of space curves. The first family consists of 8 wire models that form Series XI in [Sch]. The second family belongs to Wiener's catalogue [H-W2] and consists of 16 models made of string. Both families represent types of space curves.

3.2.1 Wire models of space curves

In this section, we study the models of Series XI in Schilling's collection. The Series consists of 8 models representing space curves. In Figure 3.7 one of them is displayed.

Figure 3.7: Wire model, Series XI nr. 2, University of Groningen.

Most of the contents of this section can be found in [F]. These 8 models were designed by C. Wiener to illustrate the results on singularities of space curves of his text [C-W]. In order to understand the models, we recall some facts concerning differential geometry:

Consider the *space curve* given by the map

$$\Phi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \mapsto (a_1(t), a_2(t), a_3(t)),$$

where I is an interval containing 0 and $a_i(t) = \sum_{j=1}^{\infty} b_{ij} t^j$ are convergent power series in a neighbourhood of $t = 0$, for $i = 1, 2, 3$ (that is, $a_i(t) \in t\mathbb{R}\{t\}$ where $\mathbb{R}\{t\}$ is the ring of convergent power series in the variable t).

We say that the curve is *singular* at $t = 0$ if $a_i(t) \in t^2\mathbb{R}\{t\}$ for all $i = 1, 2, 3$, and there exists $i_0 \in \{1, 2, 3\}$ such that $a_{i_0}(t)$ is not in $\mathbb{R}\{t^n\}$ for $n \geq 2$. Suppose that the curve $\Phi(t)$ does not lie on a plane. Then one can define the *tangent line* of the curve at $t = 0$ as the limit where $t \to 0$ of the line generated by $\Phi'(t)$. Analogously, the *osculation plane* of the curve is defined as the limit where $t \to 0$ of the plane generated by the vectors $\Phi'(t)$ and $\Phi''(t)$. The normal vector of the osculation plane is the limit of $\Phi'(t) \wedge \Phi''(t)$. Any other point of the curve is non-singular and one defines the tangent line and osculation plane in the usual way.

Let $t \in (-\epsilon, \epsilon)$ and $\Phi(t) = (a_1(t), a_2(t), a_3(t))$. The curve $\Phi(t)$ can be put in its *standard form*, i.e.,

$$\Phi(t) = (t^\ell, c_1 t^m + \text{ higher order terms}, c_2 t^n + \text{ higher order terms})$$

where $\ell < m < n$. Then, we can normalize $\Phi'(t) = (\ell t^{\ell-1}, c_1 m t^{m-1} + \ldots, c_2 n t^{n-1} + \ldots)$ to

$$(1, \frac{c_1 m}{\ell} t^{m-\ell} + \ldots, \frac{c_2 n}{\ell} t^{n-\ell} + \ldots).$$

Analogously, $\Phi''(t) = (\ell(\ell-1)t^{\ell-2}, c_1 m(m-1)t^{m-2} + \ldots, c_2 n(n-1)t^{n-2} + \ldots)$ is normalized to

$$(1, \frac{c_1 m(m-1)}{\ell(\ell-1)} t^{m-\ell} + \ldots, \frac{c_2 n(n-1)}{\ell(\ell-1)} t^{n-\ell} + \ldots).$$

Then, one obtains the eight cases in Wiener's classification by assigning a set of three signs $(\pm, \pm, \pm)$ to each of the curves following the criteria given below (this classification can be found in [F]).

1. A '+' sign is placed in the first place $(+, \ , \ )$ when the curve $\Phi(t)$ considered in a small neighborhood of t proceeds after $t = 0$ in the same direction. This happens when ℓ is odd.

2. A '+' sign is placed in the second place $(\ , +, \ )$ when the tangent line considered in a small neighbourhood of t proceeds to rotate after $t = 0$ in the same direction. This means that in the normalization of $\Phi'(t)$, the order of $\frac{c_1 m}{\ell} t^{m-\ell} + \ldots$ is odd. That is, $m - \ell$ is odd.

3. A '+' sign is placed in the third place $(\ , \ , +)$ when the osculation plane considered in a small neighborhood of t proceeds to revolve after $t = 0$ in the same direction. This means that the smallest non-zero exponent of t in $\Phi'(t) \wedge \Phi''(t) = (d_1 t^{n-\ell} + \ldots, d_2 t^{n-m} + \ldots, d_3 + \ldots)$. is odd. That is, $n - m$ is odd.

The equations of the curves $\Phi : I \longrightarrow \mathbb{R}^3$ that the eight models from Series XI are representing have been simplified from the ones in [F] and are given below, together with their type.

1. $\Phi(t) = (t, t^2, t^3)$ of type $(+, +, +)$.

2. $\Phi(t) = (t, t^2, t^4)$ of type $(+, +, -)$.

3. $\Phi(t) = (t, t^3, t^6)$ of type $(+, -, +)$.

4. $\Phi(t) = (t, t^3, t^9)$ of type $(+, -, -)$.

5. $\Phi(t) = (t^2, t^3, t^6)$ of type $(-, +, +)$.

6. $\Phi(t) = (t^2, t^3, t^9)$ of type $(-, +, -)$.

7. $\Phi(t) = (t^2, t^4, t^7)$ of type $(-, -, +)$.

8. $\Phi(t) = (t^2, t^4, t^8)$ of type $(-, -, -)$.

Remark: The classification of space curves into 8 types were interpreted by Felix Klein [K] as follows: Let the curve $\Phi(t) = (a_1(t), a_2(t), a_3(t))$ be in its *standard form*. In this form, the curve lies in the positive octant (i.e., $a_i(t) > 0$ for $i = 1, 2, 3$) for $t > 0$. For $t < 0$ the curve can lie in 8 possible octants, which gives the 8 possible cases given above.

3.2.2   String models of space curves

H. Wiener constructed a series of 16 string models to exhibit the 8 cases of space curves described in previous section (see [H-W2]). The strings on the models are representing the tangent lines at points of the space curve. This results in a string model contained in a cube. H. Wiener uses the opposite sign convention. The first 8 models from Reihe VII in [H-W2] are representing the following space curves:

1. $\Phi(t) = (12t, 12t^2, 12t^3)$ of type $(-, -, -)$.

2. $\Phi(t) = (12t, 12t^2, 12t^4)$ of type $(-, -, +)$.

3. $\Phi(t) = (12t, 12t^3, 12t^4)$ of type $(-, +, -)$.

4. $\Phi(t) = (12t^2, 12t^3, 12t^4)$ of type $(+, -, -)$.

5. $\Phi(t) = (12t, 12t^3, 12t^5)$ of type $(-, +, +)$.

6. $\Phi(t) = (12t^2, 12t^3, 12t^5)$ of type $(+, -, +)$.

7. $\Phi(t) = (12t^2, 12t^4, 12t^5)$ of type $(+, +, -)$.

8. $\Phi(t) = (12t^2, 8t^4 + 2t^5 + 2t^6, 9t^6 + 3/2t^7 + 3/2t^8)$ of type $(+, +, +)$.

The additional 8 models in this series are obtained by placing the osculation plane in another projective position. The University of Groningen keeps three models that look like the ones in H. Wiener's collection. The model displayed in Figure 3.8 shows the tangent lines at points of the space curve given by $\Phi(t) = (12t^2, 12t^3, 12t^4)$. However, we have been unable to identify the remaining two models of the collection of Groningen. This could mean that the three string models in Groningen do not belong to Wiener's collection, but were made by somebody else. This is also suggested by the fact that the size of these models is $21.5 \times 21.5 \times 21.5$ cm, while the catalogue of Wiener mentions the size $24 \times 24 \times 24$ cm.

Figure 3.8: String model representing tangent lines to the curve $\Phi(t) = (12t^2, 12t^3, 12t^4)$, University of Groningen.

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