University of Groningen

# Theory and history of geometric models

Polo-Blanco, Irene

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

*Document Version*
Publisher's PDF, also known as Version of record

*Publication date:*
2007

Link to publication in University of Groningen/UMCG research database

*Citation for published version (APA):*
Polo-Blanco, I. (2007). *Theory and history of geometric models*. [s.n.].

**Copyright**
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

**Take-down policy**
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*

Download date: 26-09-2022

# Chapter 2

# Smooth cubic surfaces

In this chapter, the classical theory concerning smooth cubic surfaces is reviewed. Such a surface can be defined as the set of zeroes of a homogeneous polynomial of degree three in $\mathbb{P}^3$. A cubic surface can also be seen as the blow up of the projective plane in a set of six points, as was originally proved by A. Clebsch in 1871.

In Section 2.2, Schläfli's classification of real cubic surfaces (1858) in terms of the number of real lines and of real tritangent planes is compared to the topological one, described by H. Knoerrer and T. Miller (1987). Using this, explicit examples are given. A paper based on this chapter and written together with Jaap Top is accepted for publication in the *Canadian Mathematical Bulletin.*

In Section 2.3, an algorithm is presented that computes the blow down morphism of a cubic surface. This algorithm provides a new proof of the result of A. Clebsch (1871). The explicit morphisms for the cases of the Clebsch diagonal surface and the Fermat cubic surface are calculated.

In the last section, we define twists of a surface, and calculate explicit twists of the Clebsch surface and the Fermat surface over $\mathbb{Q}$.

## 2.1 An introduction to smooth cubic surfaces

The origin of the study of cubic surfaces takes place in the beginning of the 19th century. Mathematicians were studying the structure of algebraic surfaces in the projective space, in particular, they studied cubic surfaces in $\mathbb{P}^3$. Such a cubic surface $S$ is defined as the set of zeroes of a homogeneous polynomial $f$ of degree three in $\mathbb{P}^3$, i.e.,

$$S = \{(x : y : z : t) \in \mathbb{P}^3 \mid f(x : y : z : t) = 0\}.$$

In this section, we present some chronological facts of importance in the history of cubic surfaces. We begin with Cayley and Salmon's discovery of the existence of 27 lines on a smooth cubic surface and proceed with Schläfli's main results: the first one on the configuration of the lines, and the second one, on the real classification of cubic surfaces. At the end, Clebsch's notion of blow-up of a cubic surface is presented. The rest of the section concerns well known facts on smooth cubic surface as blow-ups.

### 2.1.1 The 27 lines

The interest in cubic surfaces grew enormously after 1849, when the English mathematicians Arthur Cayley and George Salmon made the following discovery:

**Theorem 1.** *Any smooth cubic surface over $\mathbb{C}$ contains precisely 27 lines.*

Cayley and Salmon started a correspondence about the number of lines on a smooth cubic surface. Cayley discovered that the number of lines must be finite, and communicated this to Salmon, who quickly replied with a proof that the number of lines is in fact 27. We sketch Salmon's proof:

*Proof.*

- $S$ contains a line: an elementary and self-contained proof of this can be found in [Reid, pp. 102-106] and [Top]. The classical proof (compare [Reid, (8.15)] and [Top, pp. 42]) runs as follows. Write $G = G(1,3)$ for the Grassmannian of lines in $\mathbb{P}^3$; this is a variety of dimension 4. Define
$$V := \{(\ell, F) \in G \times \mathbb{P}^{19} \mid F \text{ vanishes on } \ell\}.$$
It is easy to verify that the projection $\pi_1 : V \to G$ is surjective and its fibers have dimension 15. Hence $V$ has dimension 19. The existence of cubic surfaces for which the set of lines on them is finite and non-empty implies that the projection $\pi_2 : V \to \mathbb{P}^{19}$ is surjective. Hence every cubic surface contains a line.

- Now suppose $\ell_1$ is a line on the cubic surface $S$. We claim that there are exactly 5 planes $H$ passing through $\ell_1$ such that the intersection $H \cap S = \ell_1 \cup \ell_2 \cup \ell_3$ with $\ell_i$ all distinct.

  In general, if $H$ is a plane passing through $\ell_1$, the intersection $H \cap S = \ell_1 \cup Q$ where $Q$ is a conic. Since $S$ is smooth, $Q$ can either be irreducible or consist of two different lines. We may choose coordinates on $\mathbb{P}^3$ such that $\ell_1$ is given by $x_2 = x_3 = 0$ and $H$ by $x_2 = \mu x_3$. The intersection $H \cap S$ is given by: $f = x_3 Q(x_0, x_1, x_3)$ where

$$\begin{aligned} Q(x_0, x_1, x_3) &= x_0^2 A(\mu, 1) + x_0 x_1 B(\mu, 1) + x_1^2 C(\mu, 1) \\ &\quad + x_0 x_3 D(\mu, 1) + x_1 x_3 E(\mu, 1) + x_3^2 F(\mu, 1). \end{aligned}$$

  Consider the last equation as a conic in $x_0$, $x_1$ and $x_3$. One has that $Q$ is singular if and only if the determinant:

$$\Delta = \begin{vmatrix} 2A & B & D \\ B & 2C & E \\ D & E & 2F \end{vmatrix} = 0.$$

  This gives us

$$4ACF + BDE - CD^2 - AE^2 - B^2F = 0, \tag{2.1}$$

  which is an equation in $\mu$ of degree 5 (except when $x_3 = 0$ is one of the planes $H$, which may be excluded after a change of coordinates). If we prove that this equation has only simple roots, we are done. Changing coordinates again if necessary, we may assume that $x_2 = 0$ is a root of $\Delta$, and prove that $x_2^2$ does not divide $\Delta$. Let $H = \{x_2 = 0\}$ be the corresponding plane. Then, $H \cap S = \ell_1 \cup \ell_2 \cup \ell_3$. One can assume that the lines contained in $H = \{x_2 = 0\}$ are given by the following equations, taking into consideration whether or not the three lines are concurrent.

  1. $\ell_1 = \{x_3 = 0\}, \ell_2 = \{x_0 = 0\}, \ell_3 = \{x_1 = 0\}$ or
  2. $\ell_1 = \{x_3 = 0\}, \ell_2 = \{x_0 = 0\}, \ell_3 = \{x_0 = x_3\}$.

  In the first case, $f$ can be written as: $f = x_0 x_1 x_3 + x_2 \cdot g$ with $g$ a quadratic form. Comparing this with the expression of $Q(x_0, x_1, x_3)$, we conclude that $B(x_2, x_3) = x_3 + \text{const} \cdot x_2$ and $x_2 \,| A, C, D, E, F$. Placing the last relations in (2.1) and reducing it modulo $x_2^2$ one obtains:

$$\Delta = -x_3^2 F \mod x_2^2.$$

Let us see that $x_2^2$ does not divide $F$. For that, consider the nonsingular point $(0, 0, 0, 1) \in S$. Since $x_2 | F$ we have that $F = a x_3^2 x_2 + b x_3 x_2^2 + x_2^3$, and since $(0, 0, 0, 1)$ is not singular, we conclude that the coefficient $a \neq 0$, that is, $x_2^2$ does not divide $\Delta$.

In a very similar manner, case 2 can be proved. We have therefore proved that $\Delta$ has only simple roots. This means that there are 5 values of $\mu$ for which the corresponding plane $H$ intersects $S$ in three lines, or in other words, the line $\ell_1$ lies in 5 such planes.

- There are at least 27 lines on $S$: Fix a plane $H$ containing $\ell_1$ and intersecting $S$ in $\ell_1$ and two other lines. Then $H \cap S = \ell_1 \cup \ell_2 \cup \ell_3$. Apart from the plane $H$, there are 4 other planes containing $\ell_1$ where two more lines lie. The same occurs for the lines $\ell_2$ and $\ell_3$, that is, the total number of lines is: $3 + 4 \times 2 + 4 \times 2 + 4 \times 2 = 27$.

- There are no more than 27: Let $\ell$ be a line on $S$. Fix a plane $H$ such that $H \cap S = \ell_1 \cup \ell_2 \cup \ell_3$. The line $\ell$ intersects the plane $H$ in a point $p$. On the one hand, it holds that $H \cap S \cap \ell \subseteq H \cap S = \ell_1 \cup \ell_2 \cup \ell_3$. On the other hand, $H \cap S \cap \ell = H \cap \ell = \{p\}$. Hence, $p \in \ell_1 \cup \ell_2 \cup \ell_3$, in particular, $p$ is in one of the lines $\ell_i$ with $i = 1, 2, 3$. The plane generated by the two lines $\ell$ and $\ell_i$ is a plane passing through $\ell_i$ and intersecting $S$ in two other lines. Since we have already counted all such planes, we conclude that $\ell$ is one of the lines counted in the previous paragraph.

□

### 2.1.2 Schläfli double six

The 27 lines on a cubic surface form a very special configuration. To be able to describe the intersection behaviour of the 27 lines, the Swiss mathematician Ludwig Schläfli introduced in 1858 the concept of *double six* (see [Sch]). A double six is a set of 12 of the 27 lines on a cubic surface $S$, represented with Schläfli's notation as:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \end{pmatrix},$$

with the following intersection behaviour: $a_{ij}$ intersects $a_{kl}$ if and only if $i \neq k$ and $j \neq l$. A smooth cubic surface has 36 double sixes.

With this new definition, Schläfli described in [Sch] the intersection of the 27 lines as follows:

**Theorem 2.** *The* 27 *lines on a smooth cubic surface can be given by* 12 *lines* $a_{i,j}$ *(with* $i \in \{1,2\}$ *and* $j \in \{1,\dots,6\}$*) and* 15 *lines* $c_{n,m}$ *(with* $n < m$ *and* $m, n \in \{1,\dots,6\}$*). The lines* $a_{i,j}$ *form a double six. The line* $c_{n,m}$ *intersects the lines* $a_{i,n}, a_{i,m}$ *and all* $c_{i,j}$ *for* $i, j \notin \{n,m\}$.

### 2.1.3 The real case

Schläfli was the first to study cubic surfaces defined over $\mathbb{R}$. A plane intersecting a smooth cubic surface in three lines is called a *tritangent plane.* In [Sch], Schläfli classified the real cubic surfaces according to their number of real lines and real tritangents. The same result was also obtained by Cremona in [Cr].

**Theorem 3 (Schläfli, 1858 and Cremona, 1868).** *The number of real lines and of real tritangent planes on any smooth, real cubic surface is one of the following, in which each pair really occurs:* $(27,15)$, $(15,15)$, $(7,5)$, $(3,7)$, $(3,13)$.

### 2.1.4 Clebsch's result

Alfred Clebsch proved in [Cl] that any smooth cubic surface over the complex numbers can be obtained by mapping $\mathbb{P}^2$ to $\mathbb{P}^3$ by a space of cubics passing through six points in general position (i.e., no three on a line and no six on a conic).

**Theorem 4 (Clebsch, 1871)).** *Let* $\{p_1,\dots,p_6\} \subseteq \mathbb{P}^2$ *be a set of six points in general position. Consider the space of cubic curves in* $\mathbb{P}^2$ *passing through the six points. This space has dimension* 4. *Let* $\{f_1, f_2, f_3, f_4\}$ *be a basis of the space, and define*

$$\begin{array}{rcl} \Phi : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ p & \mapsto & (f_1(p) : f_2(p) : f_3(p) : f_4(p)). \end{array}$$

*Then,*

*i) The closure of* $\Phi(\mathbb{P}^2 - \{p_1, \ldots, p_6\})$ *is a smooth cubic surface.*

*ii) Every smooth cubic surface can be obtained in this way.*

### 2.1.5 Cubic surfaces as blow ups

Clebsch's result in Theorem 4 can be rephrased in a modern way using the notion of blow-up, which we briefly recall here:

**Theorem 5.** *Let* $X$ *be a smooth surface and* $p \in X$*. Then, there exists a surface* $Bl_pX$*, and a morphism* $\epsilon : Bl_pX \longrightarrow X$*, unique up to isomorphism, such that:*

1. *The restriction of* $\epsilon$ *to* $\epsilon^{-1}(X - p)$ *is an isomorphism onto* $X - p$*,*
2. $\epsilon^{-1}(p)$ *is isomorphic to* $\mathbb{P}^1$*.*

*Proof.* Compare [Be], [Ha]. □

We say that $Bl_pX$ is the *blow-up* of $X$ at $p$, and $\epsilon^{-1}(p)$ is called the *exceptional divisor* of the blow-up. Let $C$ denote an irreducible curve contained in a surface $X$ that passes through $p$. Then, the closure of $\epsilon(C - p)$ in $Bl_pX$ is an irreducible curve called the *strict transform* of $C$.

The definition of blow-up of a surface $X$ at a point can be naturally extended to a notion of blow-up of a surface $X$ at a finite set of points $\{p_1, \ldots, p_n\} \subseteq X$. For instance, the blow-up at $\{p_1, p_2\} \subseteq X$ is defined as $Bl_{\epsilon^{-1}(p_2)}Bl_{p_1}X$.

In terms of this, the second part of the result of Clebsch can be stated as:

**Theorem 6.** *Every smooth cubic surface* $S \subset \mathbb{P}^3(\mathbb{C})$ *can be obtained by embedding the blow-up of* $\mathbb{P}^2$ *at some set of six points in general position in* $\mathbb{P}^3$*, using the anti-canonical map (compare [Ha, V, 4]).*

*Proof.* A proof of this result can be found in [Be], Theorem IV.1 and Proposition IV.9. It also follows from Section 2.3. □

In other words, $S$ admits a 'blow-down morphism' $\varphi : S \to \mathbb{P}^2$, characterized by the properties that $\varphi$ is an isomorphism from $S \setminus \cup_{i=1}^6 \ell_i$ to $\mathbb{P}^2 \setminus \{p_1, \dots, p_6\}$, in which the $\ell_i \subset S$ are six pairwise disjoint lines and the $p_i \in \mathbb{P}^2$ are six pairwise different points. Furthermore, $\varphi(\ell_i) = p_i$. (See Section 2.2 for explicit examples of blow-down morphisms).

The 27 lines on a smooth cubic surface can be described in terms of blow-ups:

**Theorem 7.** *Let $S$ be a smooth cubic surface obtained by blowing up $\mathbb{P}^2$ at the points $p_1, \dots, p_6$. Then, $S$ contains 27 lines which are:*

1. *The six exceptional divisors $E_i$ for $i = 1, \dots, 6$.*
2. *The strict transform $F_{ij}$ of the line passing through $p_i$ and $p_j$, for all $1 \le i < j \le 6$. There are $\binom{6}{2} = 15$ such lines.*
3. *The strict transforms $G_j$ of the six conics in $\mathbb{P}^2$ passing through $\{p_1, \dots, p_6\} \backslash \{p_j\}$ for $j = 1, \dots, 6$.*

*Proof.* See [Be], Proposition IV.12. □

Schläfli's notation for the 27 lines is:

1. The exceptional divisors $E_i$ are denoted by $i$.
2. The strict transforms $F_{ij}$ are denoted by $ij$.
3. The strict transforms $G_j$ are denoted by $j'$.

Note that this explains his results mentioned in Section 2.1.2. The twelve lines from the first and third group form a double six:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' & 5' & 6' \end{pmatrix}.$$

We end by giving a small lemma which will be needed in the course of the chapter.

**Lemma 1.** *Let $S$ be a smooth cubic surface. Given two skew lines $\ell_1$ and $\ell_2$ on $S$, there exist precisely five lines $m_1, \dots, m_5$ on $S$ such that $\ell_1 \cap m_i \neq \emptyset \neq \ell_2 \cap m_i$ for $i = 1, \dots, 5$.*

*Proof.* Suppose that the surface is obtained by blowing up the six points $p_1, \dots, p_6$. It is known that we can suppose that the two skew lines are, using Schläfli's notation, 1 and $1'$. Then, the only lines that intersect both 1 and $1'$ are: 12,13,14,15,16. □

## 2.2 Real cubic surfaces

In this section, the topological classification of smooth real cubic surfaces is recalled, and compared to Schläfli's classification. In the last part, explicit examples of surfaces of every possible type are given. Given such a smooth, cubic surface $S$ over $\mathbb{R}$, we recall Schläfli's classification:

**Theorem 8.** *The number of real lines and of real tritangent planes on any smooth, real cubic surface is one of the following, in which each pair really occurs:* $(27, 15)$, $(15, 15)$, $(7, 5)$, $(3, 7)$, $(3, 13)$.

An alternative way to classify, is by the topological structure on the space of real points $S(\mathbb{R}) \subset \mathbb{P}^3(\mathbb{R})$. This was done by several authors including H. Knörrer and T. Miller (1987), R. Silhol (1989) and J. Kollár (1997). The result is as follows [KM], [Si], [K].

**Theorem 9.** *A smooth, real cubic surface is isomorphic over $\mathbb{R}$ to a surface of one of the following types.*

1. *A surface $S$ obtained by blowing up $\mathbb{P}^2$ in $6$ real points (no $3$ on a line, not all $6$ on a conic). In this case, $S(\mathbb{R})$ is the non-orientable compact connected surface of Euler characteristic $-5$.*

2. *A surface $S$ obtained by blowing up $\mathbb{P}^2$ in $4$ real points and a pair of complex conjugate points (again, no $3$ on a line, not all $6$ on a conic). In this case, $S(\mathbb{R})$ is the non-orientable compact connected surface of euler characteristic $-3$.*

3. *A surface $S$ obtained by blowing up $\mathbb{P}^2$ in $2$ real points and $2$ pairs of complex conjugate points (again, no $3$ on a line, not all $6$ on a conic). In this case, $S(\mathbb{R})$ is the non-orientable compact connected surface of euler characteristic $-1$.*

4. *A surface $S$ obtained by blowing up $\mathbb{P}^2$ in $3$ pairs of complex conjugate points (again, no $3$ on a line, not all $6$ on a conic). In this case, $S(\mathbb{R})$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$, which has euler characteristic $1$.*

5. *A surface $S$ constructed as follows. Take a smooth, real conic given as $F = 0$ and $5$ real points $P_1, \ldots, P_5$ on it. Take a $6$th real point $P_6 \neq P_5$ on the tangent line to the conic at $P_5$, such that no $3$ of the $6$ points are on a line. Write the conic through $P_1, \ldots, P_4, P_6$ as*

*$G = 0$. The rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^2$ defined by $P \mapsto \tau(P)$, in which $\{P, \tau(P)\}$ is the intersection of the conic given by $G(P)F - F(P)G = 0$ and the line through $P$ and $P_6$, then extends to an involution $\tau$ on the blow-up $B$ of $\mathbb{P}^2$ in the points $P_1, \ldots, P_6$. Define $X$ as the quotient $B \times \mathrm{Spec}(\mathbb{C})/(\tau \times c)$, in which $c$ is complex conjugation. Then $S(\mathbb{R})$ consists of two connected components, one homeomorphic to the 2-sphere $S^2$ and the other to $\mathbb{P}^2(\mathbb{R})$.*

A straightforward and amusing consequence is

**Corollary 1.** *A smooth, cubic surface $S$ defined over $\mathbb{R}$ does not admit any real blow-down morphism to $\mathbb{P}^2$ if, and only if $S(\mathbb{R})$ is not connected.*

Note that the 'if' part of this corollary is obvious for a purely topological reason: any surface admitting a real blow-down morphism to a variety whose real points form a connected space, is itself connected. The corollary yields a 'visual' way to recognize real, smooth cubic surfaces not admitting any real blow-down morphism to $\mathbb{P}^2$.

In this section we show (Proposition 1) how the description given in Theorem 9 implies, and in fact yields the same classes of real cubic surfaces, as the classical classification of Schläfli. We also present explicit examples, in two ways: directly working from cubic equations, and also starting from the description given in Theorem 9.

The work reported on in this section started when Remke Kloosterman showed us an abstract proof of the existence of a so-called real Del Pezzo surface of degree two which does not admit a real blow-down morphism to $\mathbb{P}^2$; see also the survey paper of Kollár [K]. Explicit equations for all cases described in Theorem 9, are also presented in a recent paper by Holzer and Labs [HL, Table 2].

### 2.2.1 Real lines and real tritangent planes

In this section we show that Schläfli's classification is in fact the same as the topological one.

**Proposition 1.** *The enumerative classification of Schläfli presented in Theorem 3 yields the same five classes as the topological one given in Theorem 9. More precisely, in the notation of the latter theorem: the number of real lines and of real tritangent planes for each type is as follows.*

| *type (1)* | *type (2)* | *type (3)* | *type (4)* | *type (5)* |
|---|---|---|---|---|
| $(27, 45)$ | $(15, 15)$ | $(7, 5)$ | $(3, 7)$ | $(3, 13)$ |

This is achieved by directly describing the real lines and the real tritangent planes for each of the five types given in Theorem 9.

We start by recalling some standard notation for the lines and tritangent planes, originally used by Schäfli.

**Notation.** Let $S$ be the smooth cubic surface obtained by blowing up $\mathbb{P}^2$ at the six points $\{p_1, \ldots, p_6\}$. Denote the set of 27 lines on $S$ as follows. The image in $S \subset \mathbb{P}^3$ of the exceptional line corresponding to $p_i$, is denoted as $i$, for $i = 1, \ldots 6$. Next, $ij$ is the image in $\mathbb{P}^3$ of the strict transforms of the line passing through $p_i$ and $p_j$. Finally, $j'$ will be the image in $\mathbb{P}^3$ of the strict transform of the conic passing through all $p_i$, for $i \neq j$.

A tritangent plane $P$ is completely determined by the three lines $l_i$ such that $P \cap S = l_1 \cup l_2 \cup l_3$. Denote $P = \langle l_1, l_2, l_3 \rangle$.

**Lemma 2.** *Let $S$ be the smooth cubic surface obtained by blowing up $\mathbb{P}^2$ at the six points $\{p_1, \ldots, p_6\}$. Then, the 45 tritangent planes of $S$ are:*

1. *The 30 planes $\langle i, j', ij \rangle$ with $i, j \in \{1, \ldots, 6\}, i \neq j$ and*
2. *The 15 planes $\langle ij, kl, mn \rangle$, with $\{i, j, k, l, m, n\} = \{1, 2, 3, 4, 5, 6\}$.*

*Proof.* In fact, the lemma provides the description of the tritangent planes in terms of a 'Schläfli double six', as was already given by L. Schläfli [Sch, p. 116, 117]. We sketch the (quite obvious) proof for convenience.

Suppose the plane $P$ contains the line $i$. Then it does not contain any line $j$ for $j \neq i$ since such lines $i, j$ do not intersect. If $P$ contains $j'$, then for the same reason $j \neq i$. Under this condition, $i$ and $j'$ indeed intersect, and both intersect the line $ij$. Hence $P = \langle i, j', ij \rangle$. It is easily seen that this is the only type of tritangent plane containing a line $i$ or a line $j'$.

Two different lines $ij$ and $kl$ intersect precisely, when the numbers $\{i, j\}$ and $\{k, l\}$ are disjoint. Is this the case, then $mn$ (with $\{m, n\}$ the remaining two numbers in $\{1, \ldots, 6\}$) is the unique third line intersecting both $ij$ and $kl$. This yields the tritangent planes $\langle ij, kl, mn \rangle$.

□

We now describe the number of real lines and real tritangent planes for each of the cases described in Theorem 9. This is quite simple, and somewhat similar to arguments as can be found in 19th century papers by L. Schläfli, L. Cremona and A. Clebsch. However, surprisingly, we did not find the result in the classical literature, nor in modern texts on cubic surfaces such as the books by B. Segre [Se], by Yu.I. Manin [M] or by R. Silhol [Si].

**Notation** Let '$^-$' denote complex conjugation. We will write $\bar{j}$ (and in the same manner $\bar{j}'$... et cetera) to denote the exceptional line corresponding to the point $\bar{p}_i$. Note that this equals the conjugate line of the exceptional line corresponding to $p_i$.

We discuss the various types described in Theorem 9 case by case.

**Type (1).**

Here, the cubic surface $S$ is obtained by blowing up six real points in $\mathbb{P}^2$.

In this case, all 27 lines and all 45 tritangent planes are real.

**Type (2).**

Now $S$ is obtained by blowing up four real points and one pair of complex conjugate points. I.e., the set of six points is given by: $\{a_1, \bar{a_1}, a_2, a_3, a_4, a_5\}$.

i) The real lines are:

of type $i$: 2, 3, 4 and 5;

of type $j'$: $2'$, $3'$, $4'$ and $5'$;

of type $(ij)$: $\{i, j\} = \{\bar{i}, \bar{j}\} \Leftrightarrow ij \in \{1\bar{1}, 23, 24, 25, 34, 35, 45\}$.

In total, there are $4 + 4 + 7 = 15$ real lines.

ii) The real tritangent planes are:

of type $\langle i, j', ij\rangle$: all 12 planes with $i, j \in \{2, 3, 4, 5\}$;

of type $\langle ij, kl, mn\rangle$: the planes $\langle 1\bar{1}, 23, 45\rangle$, $\langle 1\bar{1}, 24, 35\rangle$, $\langle 1\bar{1}, 25, 34\rangle$.

In total, there are $12 + 3 = 15$ real tritangent planes.

**Type (3).**

Here $S$ is obtained by blowing up two real points and two pairs of complex conjugate points. I.e., the set of six points is given by: $\{a_1, \bar{a_1}, a_2, \bar{a_2}, a_3, a_4\}$.

i) The real lines are:
of the kind $i$: 3 and 4;
of the kind $j'$: $3'$ and $4'$;
of the kind $(ij)$: $1\bar{1}, 2\bar{2}, 34$.
In total, there are 7 real lines.

ii) The real tritangent planes are:
of type $\langle i, j', ij\rangle$: $\langle 3, 4', 34\rangle$ and $\langle 4, 3', 43\rangle$;
of type $\langle ij, kl, mn\rangle$: $\langle 1\bar{1}, 2\bar{2}, 34\rangle$, $\langle 34, 1\bar{2}, \bar{1}2\rangle$ and $\langle 34, 12, \bar{1}\bar{2}\rangle$.
In total, there are 5 real tritangent planes.

**Type (4).**

In this case, the cubic surface $S$ is obtained by blowing up three pairs of complex conjugate points. Write the points as $\{a_1, \bar{a_1}, a_2, \bar{a_2}, a_3, \bar{a_3}\}$.

i) The only real lines in this case are $1\bar{1}, 2\bar{2}, 3\bar{3}$. Hence there are precisely three real lines in $S$.

ii) We now calculate the number of real tritangent planes of $S$. Clearly, no tritangent plane of the form $\langle i, j', ij\rangle$ is fixed by complex conjugation. Furthermore, every real tritangent plane must contain a real line, i.e., one of the three lines $\{1\bar{1}, 2\bar{2}, 3\bar{3}\}$.

The planes $\langle i, \bar{i}', i\bar{i}\rangle$ are not fixed by complex conjugation. The real ones are $\langle 1\bar{1}, 2\bar{2}, 3\bar{3}\rangle$, $\langle 1\bar{1}, 23, \bar{2}\bar{3}\rangle$, $\langle 1\bar{1}, 2\bar{3}, \bar{2}3\rangle$,
$\langle 2\bar{2}, 13, \bar{1}\bar{3}\rangle$, $\langle 2\bar{2}, 1\bar{3}, \bar{1}3\rangle$, $\langle 3\bar{3}, 12, \bar{1}\bar{2}\rangle$ and $\langle 3\bar{3}, 1\bar{2}, \bar{1}2\rangle$.

There are 7 real tritangent planes.

**Type(5).**

This case is obtained by changing the real structure on a special kind of surface as described in type (1).

Let $a_1, \ldots, a_6$ be the six real points in $\mathbb{P}^2$ which will be blown up. By construction, the line connecting $a_5$ and $a_6$ is tangent in $a_5$ to the conic passing through $a_1, \ldots, a_5$. Let $B$ be the real, cubic surface in $\mathbb{P}^3$ obtained by the usual embedding of the blow-up of $\mathbb{P}^2$ in the points $a_i$. We first describe the action of the involution $\tau$ on the 27 lines in $B$.

Firstly, let $\{1,2,3,4\} = \{i,j\} \cup \{k,l\}$. Then

$$\tau : \qquad ij \longleftrightarrow kl.$$

Indeed, take a general point $u$ on the line $L_{ij}$ containing $a_i$ and $a_j$. The (singular) conic through $a_1, \ldots, a_4, u$ equals the union $L_{ij} \cup L_{kl}$, and since $\{u, \tau(u)\} = (L_{ij} \cup L_{kl}) \cap L$ where $L$ is the line connecting $u$ and $a_6$, the image $\tau(u)$ is a general point on $L_{kl}$. In a similar way, using the conic through $a_1, \ldots, a_5$ and observing that $L_{56}$ is tangent to this conic, it follows that

$$\tau : \qquad 5 \longleftrightarrow 5$$

and

$$\tau : \qquad 6' \longleftrightarrow 6'.$$

Now take a general point $u$ on the line $L_{56}$. Clearly, its image under $\tau$ is again a general point on $L_{56}$, hence

$$\tau : \qquad 56 \longleftrightarrow 56.$$

Next, take $j \in \{1,2,3,4\}$ and consider a general point $u \in L_{j6}$. The conic $C_u$ passing through $a_1, \ldots, a_4, u$ then obviously intersects $L_{j6}$ in the points $u$ and $a_j$. This means $\tau(u) = a_j$, so in particular, the rational involution map $\tau$ on $\mathbb{P}^2$ restricted to $L_{j6}$ is not even bijective. However, changing the point $u$ also changes the direction of the tangent line to $C_u$ at $a_j$. On the blow up $B$, this shows

$$\tau : \qquad j6 \longleftrightarrow j.$$

A similar argument yields that $\tau$ maps the general point on the conic through $a_1, \ldots, a_4, a_6$, to $a_6$, and

$$\tau : \qquad 5' \longleftrightarrow 6.$$

It is now a straightforward computation to determine the action of $\tau$ on the remaining lines of $B$: one may use the observation that for any pair of lines $\ell_1, \ell_2 \subset B$, the intersection number $\ell_1 \cdot \ell_2$ equals $\tau(\ell_1) \cdot \tau(\ell_2)$. Using this, it follows that for each $j \in \{1,2,3,4\}$ one has

$$\tau : \qquad j' \longleftrightarrow j5.$$

With this information one can also describe the action of complex conjugation $c$ on the real cubic surface $S = B \times \mathrm{Spec}(\mathbb{C})/(\tau \times c)$. Namely, over $\mathbb{C}$

this surface is isomorphic to $B$, giving an identification of the lines and tritangent planes of $B$ with those of $S$. Then by construction a line/tritangent plane of $S$ is real if, and only if the corresponding line/tritangent plane of $B$ is fixed by $\tau \times c$. Since $c$ fixes all lines and tritangent planes of $B$, this means we have to count the ones fixed by $\tau$.

i) As is explained above, $5, 6'$ and $56$ are the only lines in $B$ fixed by $\tau$. Hence $S$ contains precisely 3 real lines.

ii) To determine the tritangent planes fixed by $\tau$, observe that any such plane contains at least 1 line in $B$ fixed by $\tau$, so at least one of the lines $5, 6'$ or $56$.

The tritangent planes fixed by $\tau$ then are:

of type $\langle i, j', ij\rangle$: the 4 planes $\langle 5, j', j5\rangle$ with $j \in \{1, 2, 3, 4\}$ and the 4 planes $\langle i, 6', i6\rangle$ with $i \in \{1, 2, 3, 4\}$ and the 2 planes $\langle 5, 6', 56\rangle$ and $\langle 6, 5', 56\rangle$;

of type $\langle ij, kl, mn\rangle$: the 3 planes $\langle ij, kl, 56\rangle$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

This gives 13 planes, hence $S$ contains precisely 13 real tritangent planes. Note that in exactly one of them, namely in the one corresponding to $\langle 5, 6', 56\rangle$, the 3 lines in $S$ contained in it are all real as well.

**Remark:** In particular, the above description shows that every smooth, real cubic surface has at least one real tritangent plane that contains 3 real lines of the surface. In the case that the total number of real lines equals 3, this means that these three lines pairwise intersect and that in fact this real tritangent plane is unique. This is an observation already made by Schläfli [Sch, p. 118 cases D, E].

### 2.2.2 Examples

In this section, we present explicit examples of surfaces of every possible type in the classification. More kinds are also obtained in Section 2.4.2.

**The family $S_\lambda$**

Let $S_\lambda$ be the cubic surface in $\mathbb{P}^4$ given by the equations:

$$S_\lambda = \begin{cases} \lambda x^3 + y^3 + z^3 + w^3 + t^3 & = & 0 \\ x + y + z + w + t & = & 0. \end{cases}$$

An equation for $S_\lambda$ in $\mathbb{P}^3$ is

$$\lambda x^3 + y^3 + z^3 + w^3 + (-x - y - z - w)^3 = 0.$$

A straightforward calculation shows that $S_\lambda$ is a smooth cubic surface for all $\lambda \in \mathbb{C}$, except $\lambda = 1/4$ and $\lambda = 1/16$. In case $\lambda = 1/16$, the unique singular point is $(x : y : z : w : t) = (-4 : 1 : 1 : 1 : 1)$. For $\lambda = 1/4$, there are exactly 4 singular points, given by taking $x = -2$, three of the remaining coordinates $+1$, and the last one $-1$. In fact, the number of isolated singularities on a cubic surface cannot exceed 4, and the surface with 4 such singular points is unique. We give a proof of this in the following theorem:

**Theorem 10.** *If a cubic surface has finitely many singular points, then the number of singular points is less than or equal to four. Moreover, the cubic surface containing precisely* 4 *singular points is unique up to linear isomorphism.*

*Proof.* Let $F(x_1, x_2, x_3, x_4) = 0$ define a cubic surface $S$ with finitely many singular points. Suppose that the number of singular points is at least 4. Put four singular points in general position (the four points are not contained in a plane, since otherwise there would be infinitely many singular points)

$$(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1).$$

Consider a plane through 3 of those points. It cuts $S$ in a curve of degree 3 with 3 singular points. We conclude that the intersection consists of 3 lines (counted with multiplicity).

We conclude that $F(0, 0, x_3, x_4)$ is identical 0, because the line through $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ lies on $S$. Therefore, $F$ does not contain the terms: $x_3^3, x_3^2 x_4, x_3 x_4^2, x_4^3$. The same holds for every pair of coordinates. Therefore,

$$F = c_1 x_2 x_3 x_4 + c_2 x_1 x_3 x_4 + c_3 x_1 x_2 x_4 + c_4 x_1 x_2 x_3.$$

None of the $c_i$ is 0 since otherwise $F$ would be reducible. The transformation $x_i \mapsto c_i x_i$ for $i = 1, \ldots, 4$, and division by $c_1 \cdots c_4$ brings $F$ into the form:

$$F = x_2 x_3 x_4 + x_1 x_3 x_4 + x_1 x_2 x_4 + x_1 x_2 x_3.$$

A computation shows that the only singular points on $F = 0$ are $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$. We conclude that 4 is the maximum number of singular points and that the surface with four singular points is unique up to coordinate changes.

□

This surface was first studied by A. Cayley and it is called the 'Cayley cubic surface'; compare [Hu, pp. 115–122]. Some 19th century plaster models of it appear in the celebrated 'Rodenberg series'; see [F, pp. 16-17], [Po-Za, Serie VII, nr. 2-6].

We now continue the discussion on the family $S_\lambda$. For real $\lambda > 1/4$, all $S_\lambda(\mathbb{R})$ are topologically the same. The special one with $\lambda = 1$ is the Clebsch diagonal surface. Hence for $\lambda > 1/4$, the surface $S_\lambda$ is of type (1). An explicit blow-down morphism, defined over $\mathbb{Q}$, in the case $\lambda = 1$ is constructed in section 2.3.1.

Similarly, for all real $\lambda < 1/16$ the $S_\lambda(\mathbb{R})$ are homeomorphic. The case $\lambda = 0$ yields the Fermat cubic surface. It is a well-know exercise (e.g., [Ha, Exc. V 4.16]), to show that it contains exactly 3 real lines and 7 real tritangent planes. Hence for $\lambda < 1/16$, the surface $S_\lambda$ is of type (4) (see [Se] for an intuitive proof of this). An explicit blow-down morphism, defined over $\mathbb{Q}$, in the case $\lambda = 0$ was first presented by N. Elkies [El]. In the next section, we present another blow-down morphism for the Fermat.

In the singular surface $S = S_{1/16}$, the point $(-4 : 1 : 1 : 1)$ is an isolated point of the real locus $S(\mathbb{R})$. To see this, use the equation

$$x^3/16 + y^3 + z^3 + 1 - (x + y + z + 1)^3 = 0$$

for an open affine part of $S$; on this part, we consider the point $(-4, 1, 1)$. The change of variables

$$r = x + 4, \qquad s = y - 1, \qquad t = z - 1$$

yields the new equation

$$6t^2 + 6s^2 + \frac{9}{4}r^2 + 6rs + 6st + 6rt + F(r, s, t)$$

in which $F$ is a homogeneous cubic. Clearly, the quadratic part of this defines a positive definite form, hence $(0, 0, 0)$ is an isolated point in the real locus defined by the equation.

In fact, what happens in $S_\lambda(\mathbb{R})$ when $\lambda$ decreases from $1/4$ to $1/16$, is that for $\lambda = 1/4$ we have a 2-sphere which is connected to the rest of the surface in the 4 singular points. When $\lambda$ decreases, the contact of this 2-sphere with the rest of the surface disappears. The 2-sphere becomes smaller until at $\lambda = 1/16$ it shrinks to a single point (the singular point), and for $\lambda < 1/16$ this point has disappeared. In particular, what we claim here, is that for $1/16 < \lambda < 1/4$, the real locus $S_\lambda(\mathbb{R})$ is not connected and hence $S_\lambda$ is of type (5). This can be shown topologically; instead, we briefly sketch below how one finds the 3 real lines and 13 real tritangent planes in the case $1/16 < \lambda < 1/4$.

Observe that the three real lines given in parametric equations as

$$l_1 = (0 : -t : t : 1),\ l_2 = (0 : -1 : t : 1),\ l_3 = (0 : t : -1 : 1)$$

are contained in $S_\lambda$.

Let $H$ be a plane containing $l_1$. One can write $H = \{ax + b(y + z) = 0\}$. Suppose first that $a = 0$. This describes the real tritangent plane $H = \{y + z = 0\}$, meeting $S$ in $l_1$ and in a pair of complex conjugate lines. In case $a \neq 0$, the plane can be given as $H_b = \{x + by + bz = 0\}$. We count the number of tritangent planes of this type.

One has

$$H_b \cap S_\lambda = l_1 \cup C_b$$

where $C_b$ is a (possibly reducible) conic. In order for $H_b$ to be a tritangent plane, we need $C_b$ to be singular. Since $1/16 < \lambda < 1/4$, this happens for exactly four different real values of $b$, which gives us 4 real tritangent planes (in particular, the case $b = 0$ yields $H = \{x = 0\}$, which is the plane that contains the three real lines $l_1, l_2$ and $l_3$). We have then counted 5 real tritangent planes. Again using $1/16 < \lambda < 1/4$, it follows that each of them, apart from $\{x = 0\}$, contains $l_1$ and a pair of complex conjugate lines. By proceeding in the same way with the tritangent planes containing $l_2$ and $l_3$ respectively, we find 4 real tritangent planes for each case, all containing $l_2$ resp. $l_3$ as the only real line (apart from the plane $H = \{x = 0\}$ that contains all of $l_1, l_2$ and $l_3$). In total, we have now counted 12 pairs of complex conjugate lines and 3 real lines, hence these are all the lines of $S_\lambda$. Since every real tritangent plane contains a real line of $S_\lambda$, the above calculation in fact yields all real tritangent planes of $S_\lambda$. Therefore, the surface contains precisely $5 + 4 + 4 = 13$ real tritangent planes.

This finishes the calculation.

### Shioda's construction

Instead of starting from cubic equations directly, one can of course derive such equations starting from 6 points in the plane. A very convenient way to do this, is described by T. Shioda [Sh, § 6]. His observation is, that although the 6 points are required not to be contained in any conic, they will certainly be smooth points on some irreducible, cuspidal cubic curve. After a linear transformation, one may assume that this cubic curve $C$ is given by the equation $y^2z - x^3 = 0$. Then

$$t \mapsto a_t := (t^{-2} : t^{-3} : 1)$$

defines an isomorphism of algebraic groups over $\mathbb{Q}$ from the additive group $\mathbb{G}_a$ to the group of smooth points on $C$ (in which $t = 0$ maps to the point $(0 : 1 : 0)$ which is taken as the neutral element). The latter group has the property that 3 points $a_{t_1}, a_{t_2}, a_{t_3}$ in it, are on a line precisely when $t_1 + t_2 + t_3 = 0$ and similarly, 6 smooth points $a_{t_j}$ are on a conic if, and only if $\sum t_j = 0$.

Now, given 6 values $t_j \in \mathbb{G}_a$ which are pairwise different, such that no 3 of them add to 0 and $\sum t_j \neq 0$, Shioda [Sh, Thm. 14] explicitly gives an equation for the corresponding smooth, cubic surface. His result is the following:

Let $c$, $a_i$, $a'_i$ and $a'_{ij}$ be the following linear forms in $t_1, \ldots, t_6$:

$$c := -\sum_{i=1}^{6} t_i, \quad a_i := \frac{c}{3} - t_i, \quad a'_i := -\frac{2c}{3} - t_i, \quad a_{ij} := \frac{c}{3} + t_i + t_j,$$

and $f$ the polynomial

$$f(x) := \prod_{i=1}^{6}(x + a_i)(x + a'_i) \prod_{1 \leq i < j \leq 6} (x + a_{ij}).$$

Let $e_i$ denote the coefficient of $f$ in $x^{27-i}$. Shioda's result is the following:

**Theorem 11.** *Let $Q_i = (t_i^{-2} : t_i^{-3} : 1) \in \mathbb{P}^2 (1 \leq i \leq 6)$ be six points on the cuspidal cubic $X^3 - Y^2Z = 0$ and assume the condition*

$$(\#) \quad t_i \neq t_j \quad (i \neq j), \quad t_i + t_j + t_k \neq 0 \quad (i, j, k \quad \textit{distinct}), \quad \sum_{i=1}^{6} t_i \neq 0.$$

*Then the cubic surface $V \subseteq \mathbb{P}^3$ obtained by blowing up these points has the following defining equation*

$$Y^2W + 2YZ^2 = X^3 + X(p_0W^2 + p_1ZW + p_2Z^2) + q_0W^3 + q_1ZW^2 + q_2Z^2W$$

*where $p_i, q_j$ are given by:*

$$\begin{aligned}
p_2 &= \epsilon_2/12\\
p_1 &= \epsilon_5/48\\
q_2 &= (\epsilon_6 - 168p_2^3)/96\\
p_0 &= (\epsilon_8 - 294p_2^4 - 528p_2q_2)/480\\
q_1 &= (\epsilon_9 - 1008p_1p_2^2)/1344\\
q_0 &= (\epsilon_{12} - 608p_1^2p_2 - 4768p_0p_2^2 - 252p_2^6 - 1200p_2^3q_2 + 1248q_2^2)/17280.
\end{aligned}$$

*Moreover, the above cubic surface $V$ is smooth if and only if the assumption (#) is satisfied.*

We give some examples to illustrate Shioda's result. Take 4 values $t_1,\ldots,t_4$ and put $t_5 := -(t_1 + t_2 + t_3 + t_4)/2$ and $t_6 := t_1 + t_2 + t_3 + t_4$. Assume that the $t_j$ are pairwise different, no 3 of them add to 0 and $\sum t_j \neq 0$. Take $(t_1, \ldots, t_6) := (0, 1, 2, 5, -4, 8)$. With this choice, the conic through $a_{t_1}, \ldots, a_{t_5}$ meets $C$ in $a_{t_5}$ with multiplicity 2 (since $t_1+t_2+t_3+t_4+2\cdot t_5 = 0$). The tangent line to this conic at $a_{t_5}$ is by construction tangent to $C$ as well, and because $t_6 + t_5 + t_5 = 0$, the third point of intersection with $C$ is $a_{t_6}$. Hence this is a configuration as is used in the construction of cubic surfaces of type (5). As an explicit example, take $(t_1, \ldots, t_6) := (0, 1, 2, 5, -4, 8)$. The corresponding cubic surface of type (1) has equation

$$y^2w + 2yz^2 = x^3 - \frac{964825}{768}xw^2 - \frac{79}{2}xz^2 + \frac{433748125}{55296}w^3 + \frac{141859}{96}z^2w.$$

The involution $\tau$ is in this case given as

$$\tau(x : y : z : w) := (x : y : -z : w).$$

This implies that after changing the sign of the coefficients of $yz^2$ and $xz^2$ and $z^2w$, a cubic surface of type (5) is obtained.

Starting from the set $\{\pm\sqrt{-1}, 1 \pm \sqrt{-1}, \pm 1\}$, the method yields the cubic surface of type (3), given by $y^2w + 2yz^2 =$

$$x^3 - \frac{3025}{8748}xw^2 + \frac{55}{81}xzw - \frac{1}{3}xz^2 - \frac{5525}{354294}w^3 + \frac{8345}{39366}zw^2 + \frac{67}{243}z^2w.$$

To obtain an example of type (2), one takes 4 real values and one pair of complex conjugates. For example, $\{\pm 1, \pm\sqrt{-1}, 2, 3\}$ yields the equation $y^2w + 2yz^2 =$

$$x^3 - \frac{138025}{34992}xw^2 - \frac{245}{81}xzw - \frac{16}{3}xz^2 + \frac{48982975}{5668704}w^3 + \frac{600565}{78732}zw^2 + \frac{1439}{486}z^2w.$$

## 2.3 Algorithm for the blow down morphism

Let $S$ be a smooth cubic surface over $\mathbb{C}$. Recall that $S$ admits a blow-down morphism $\Psi : S \longrightarrow \mathbb{P}^2$, characterized by the properties that:

1. $\Phi$ is an isomorphism from $S \setminus \cup_{i=1}^6 \ell_i$ to $\mathbb{P}^2 \setminus \cup_{i=1}^6 p_i$, with $\ell_i \subseteq S$ six pairwise disjoint lines and $p_i \in \mathbb{P}^2$ six points in general position for $i = \{1, \ldots, 6\}$,
2. $\Phi(\ell_i) = p_i$.

In this section we provide a new, algorithmic proof of Theorem 4, (ii) of Clebsch.

**Remark:** In [B-P] there is a completely different approach, using syzygies, for computing the cubic equation from the 6 points in $\mathbb{P}^2$. This method also includes the computation of the equation of the cubic surface obtained by blowing up 5 points of $\mathbb{P}^1 \times \mathbb{P}^1$.

**Theorem 12.** *The following algorithm, which takes as input a smooth cubic surface $S$ over $\mathbb{C}$ and a line $\ell_1$ on it, realizes a nonsingular cubic surface as a blow up of 6 points.*

Let $S$ be a smooth cubic surface in $\mathbb{P}^3(\mathbb{C})$ given by a cubic homogeneous polynomial $f$ in $\mathbb{C}[X_0, X_1, X_2, X_3]$.

**Sketch of the Algorithm**

- Find a line $\ell_2 \subseteq S$ such that $\ell_1 \cap \ell_2 = \emptyset$.
- Define the rational map $\Phi_1 : S \subseteq \mathbb{P}^3 \dashrightarrow \ell_1 \times \ell_2$ as follows. Let $p \in S \setminus (\ell_1 \cup \ell_2)$ be a point on the surface. Calculate the unique line $\ell \in \mathbb{P}^3$, such that $p \in \ell$, $\ell \cap \ell_1 = \{q_1\}$ and $\ell \cap \ell_2 = \{q_2\}$. Define $\Phi_1(p) = (q_1, q_2) \in \ell_1 \times \ell_2$.

- Consider $V$ the space of homogeneous polynomials of degree 1 in $\mathbb{C}[x_1, \dots, x_4]$, and $W \subseteq V$ the subspace given by $\{h \in V \mid h(\ell_1) = 0\}$. The space $V/W$ has dimension 2. Let $\{\beta_1, \beta_2\}$ be a basis of $V/W$. Then, the map

$$\begin{array}{ccc} \ell_1 & \longrightarrow & \mathbb{P}(V/W) \simeq \mathbb{P}^1 \\ x & \mapsto & (\beta_1(x) : \beta_2(x)) \end{array}$$

  is an isomorphism.

  Actually one can take for $\beta_1, \beta_2$ two distinct elements of $\{x_1, \dots, x_4\}$, say $x_j, x_k$.

  The same can be done for the line $\ell_2$, and we have a basis $\{\tilde{\beta}_1, \tilde{\beta}_2\} = \{x_l, x_m\}$. We have the isomorphism given by:

$$\begin{array}{rcl} f : \ell_1 \times \ell_2 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ (x, x') & \mapsto & ((\beta_1(x) : \beta_2(x)), (\tilde{\beta}_1(x') : \tilde{\beta}_2(x'))) \end{array}$$

  Note that $f$ is well defined up to isomorphisms of the two $\mathbb{P}^1$'s.

- Find the five lines $m_1, \dots, m_5$ on $S$ that intersect both $\ell_1$ and $\ell_2$ (see lemma 1).

- The map $f \circ \Phi_1 : S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ blows down the five lines $m_i$ to five points $p_i = ((x_j(m_i) : x_k(m_i)), (x'_l(m_i) : x'_m(m_i)) \in \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \dots, 5$. Choose one such point, say $p_1$, and compute its coordinates.

- Define $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ as follows: Consider the space of forms of bidegree $(1, 1)$ in $\mathbb{C}[X_j, X_k, X'_l, X'_m]$ (i.e., polynomials that are homogeneous of degree 1 in $(X_j, X_k)$ and in $(X'_l, X'_m)$). Any such a form is given by:

$$\alpha X_j X'_l + \beta X_j X'_m + \gamma X_k X'_l + \delta X_k X'_m,$$

  with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

  Consider the subspace of the forms that vanish at $p_1$. This space has dimension 3. Find a basis $\{f_1, f_2, f_3\}$ of this space and define $\Phi_2$ as:

$$\begin{array}{rccc} \Phi_2 : & \mathbb{P}^1 \times \mathbb{P}^1 & \dashrightarrow & \mathbb{P}^2 \\ & p & \mapsto & (f_1(p) : f_2(p) : f_3(p)). \end{array}$$

- The complete blow down morphism is $\Phi = \Phi_2 \circ f \circ \Phi_1$.

In the next proposition it will be verified that the algorithm has the property of Theorem 12.

**Proposition 2.** *The map $\Phi$ defines a birational morphism $S \to \mathbb{P}^2$ which blows down 6 lines and is an isomorphism outside these lines.*

*Proof.* One can assume after a linear change of variables that the smooth cubic surface $S$ contains the two skew lines

$$\ell_1 = \begin{cases} x_0 = 0 \\ x_1 = 0 \end{cases} \text{ and } \ell_2 = \begin{cases} x_2 = 0 \\ x_3 = 0. \end{cases}$$

Since $\ell_i \subseteq S$, the equation defining $S$ is given by:

$$\alpha x_0^2 x_3 + \beta x_1^2 x_2 + x_0 x_1 h_1(x_2, x_3) + x_3 x_0 f_1(x_2, x_3) + x_1 x_2 g_1(x_2, x_3) = 0$$

where $h_1, f_1$ and $g_1$ are linear forms in the variables $x_2$ and $x_3$.

In order to define $f$, we will suppose that the bases considered are: $\{\beta_1, \beta_2\} = \{x_2, x_3\}$ and $\{\tilde{\beta}_1, \tilde{\beta}_2\} = \{x_0, x_1\}$. We can therefore assume that the map $f \circ \Phi_1$ is in this case given by:

$$((a : b : c : d)) \overset{f \circ \Phi_1}{\mapsto} ((c : d), (a : b)),$$

defined on $S$ outside $\ell_1 \cup \ell_2$. After a possible change of coordinates $\binom{x_0}{x_1} = A\binom{x'_0}{x'_1}$ and $\binom{x_2}{x_3} = A'\binom{x'_2}{x'_3}$ , one can assume $m_1$ to be given by

$$m_1 = \begin{cases} x_0 = 0 \\ x_2 = 0. \end{cases}$$

The point $p_1 = f \circ \Phi_1(m_1)$ is $p_1 = ((0 : 1), (0 : 1))$. A basis for the space of forms of bidegree $(1, 1)$ vanishing at $((0 : 1), (0 : 1))$ is $\{ac, ad, bc\}$. Hence, the total map $\Phi : S \dashrightarrow \mathbb{P}^2$ is given by:

$$((a : b : c : d)) \overset{\Phi}{\mapsto} (ac : ad : bc).$$

Let us now see that $\Phi$ is everywhere defined on $S$. The map is not defined if and only if:

$$ac = ad = bc = 0$$

which happens if and only if

$$a = b = 0 \text{ or } c = d = 0 \text{ or } a = c = 0,$$

i.e., the lines $\ell_1$, $\ell_2$ and $m_1$.

We begin by considering a point of the form $p = (0 : 0 : p_3 : p_4) \in \ell_1 \subset S$. We will show that $\Phi$ is well defined at $p$. In order to do that, we consider a point $q = (q_1 : q_2 : q_3 : q_4)$ in the tangent space $T_pS$ of $S$ at $p$, with $q_1 \neq 0$ and $q_2 \neq 0$. Then, $p + \epsilon q \notin \ell_1$ and $\Phi(p + \epsilon q)$ with $\epsilon^2 = 0$ should not depend on $q$.

On the one hand, one has that the image

$$\Phi(p + \epsilon q) = \Phi(\epsilon q_1 : \epsilon q_2 : p_3 + \epsilon q_3 : p_4 + \epsilon q_4) = (q_1 p_3 : q_1 p_4 : q_2 p_3).$$

If $p_3 = 0$, i.e., for the point $(0 : 0 : 0 : 1)$ on $\ell_1$, one has that $\Phi(p + \epsilon q) = (0 : 1 : 0)$, independent of $q$. If $p_4 = 0$, then $q_2 = 0$ and $\Phi(p+\epsilon q) = (1 : 0 : 0)$ is also independent of $q$.

On the other hand, $q \in T_pS$ implies $p_4 f_1(p_3, p_4) q_1 + p_3 g_1(p_3, p_4) q_2 = 0$. It follows that $\Phi(p + \epsilon q)$ is independent of $q$, and in fact

$$\Phi(p) = (-p_3 g_1(p_3, p_4) : -p_4 g_1(p_3, p_4) : p_4 f_1(p_3, p_4)).$$

The same holds for the points on $\ell_2$.

For a point $p = (0 : p_2 : 0 : p_4)$, the tangent space $T_pS$ is given by $(p_2 h_1 + p_4 f_1) x_0 + (\beta p_2^2 + p_2 g_1) x_2 = 0$. It follows that $q \in T_pS$ implies $(p_2 h_1 + p_4 f_1) q_1 + (\beta p_2^2 + p_2 g_1) q_3 = 0$. We conclude that $\Phi(p + \epsilon q) = (0 : q_1 p_4 : p_2 q_3)$ is also independent of $q$.

The map $\Phi$ is a birational morphism: one can find its inverse as $\Phi_1^{-1} \circ f^{-1} \circ \Phi_2^{-1}$: A point $(x_0 : x_1 : 1) \in \mathbb{P}^2$ is sent to $((x_0/x_1 : 1), (x_0 : 1))$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The two points on the lines $\ell_1$ and $\ell_2$ are given by:

$$(p_1, p_2) = ((0 : 0 : x_0 : x_1), (x_0 : 1 : 0 : 0)) \subseteq \ell_1 \times \ell_2.$$

One can calculate the line $\ell_{p_1 p_2}$ passing through $p_1$ and $p_2$:

$$\ell_{p_1 p_2} = \{(\lambda x_0 : \lambda : x_0 : x_1)\} \subseteq \mathbb{P}^3$$

and intersect it with $S$. We recall that the equation of $S$ can be given by:

$$\begin{aligned} &\alpha x_0^2 x_3 + \beta x_1^2 x_2 + x_0 x_1 (a_1 x_2 + b_1 x_3) + x_3 x_0 (a_2 x_2 + b_2 x_3) \\ &+ x_1 x_2 (a_3 x_2 + b_3 x_3) = 0. \end{aligned}$$

The intersection $\ell \cap S$ consists of the points $p_1$,$p_2$ and a third point $p \in S$ that defines the rational map $\Phi^{-1}$:

$$\begin{array}{rcl} \Phi^{-1} : \mathbb{P}^2 \setminus \cup_{i=1}^6 p_i & \longrightarrow & \mathbb{P}^3 \\ (x:y:z) & \mapsto & \big(x(-a_3xz - b_3yz - a_2xy - b_2y^2) : \\ & & z(-a_3xz - b_3yz - a_2xy - b_2y^2) : \\ & & x(\beta z^2 + a_1xz + b_1yz + \alpha xy) : \\ & & y(\beta z^2 + a_1xz + b_1yz + \alpha xy)\big) . \end{array}$$

□

**Remark:** Note that the map $\Phi_2$ blows up the point $p_1 = f \circ \Phi_1(m_1) = ((0 : 1), (0 : 1))$, and blows down the following two lines that intersect on the point $p_1$:

$$r_1 : \{(0:1)\} \times \mathbb{P}^1 \quad \text{and} \quad r_2 : \mathbb{P}^1 \times \{(0:1)\}.$$

The six points $\{p_1, \dots, p_6\}$ are obtained as follows: Four of them are $\Phi(m_2), \Phi(m_3), \Phi(m_4), \Phi(m_5)$. The remaining two are $\Phi_2(r_1)$ and $\Phi_2(r_2)$.

Note that the above map $\Phi^{-1}$ is given by $p \mapsto (f_1(p) : \dots : f_4(p))$, where $\{f_1, \dots, f_4\}$ is a basis of the space of cubic curves passing through the points $p_1, \dots, p_6$. In particular, it is a map given by cubic polynomials.

In the next section we present some examples of real surfaces for which the explicit blow down morphism is calculated.

**Remark:** The algorithm really requires a given line $\ell_1 \subseteq S$ as input. Given the equation of a surface, we do not know an efficient way of finding a line on it.

### Algorithm for real surfaces

Now we assume that the cubic surface $S$ is given over $\mathbb{R}$. Then $S \subseteq \mathbb{P}^3_{\mathbb{R}}$ and $S$ is given by a cubic form in $\mathbb{R}[X_1, \dots, X_4]$ (where $\mathbb{R}[X_1, \dots, X_4]$ denotes the homogeneous coordinate ring of $\mathbb{P}^3_{\mathbb{R}}$). As we saw in Section 2.2, a real blow down morphism exists in this case precisely when the surface is of type 1), 2), 3) or 4) given in Theorem 9. We assume that $S$ is of type 1), 2), 3) or 4) and distinguish the following two cases:

1. $S$ is of type 1), 2) or 3): then, $S$ contains a skew pair of real lines $\{\ell_1, \ell_2\}$. In this case, the algorithm is exactly the same as in the complex case, where one should replace $\mathbb{C}$ by $\mathbb{R}$ everywhere. The morphism $\Phi : S \longrightarrow \mathbb{P}^2_{\mathbb{R}}$ blows down a real configuration of six lines. This configuration depends on the type 1), 2) or 3).

2. $S$ is of type 4): then $S$ does not contain a skew pair of real lines. In this case, there exists a skew pair of complex lines $\{\ell, \bar{\ell}\}$, where $\bar{\ell}$ is the complex conjugate of the line $\ell$.

   We briefly sketch the steps to follow in such case:

   - Choose $\{\ell, \bar{\ell}\}$, a pair of skew conjugate lines on $S$.
   - Build a map $\Phi_1 : S \subseteq \mathbb{P}^3 \dashrightarrow \ell \times \bar{\ell}$ as in the complex case.
   - Let $V_{\mathbb{R}} = \mathbb{R}x_1 + \cdots + \mathbb{R}x_4$ and $V = \mathbb{C}x_1 + \cdots + \mathbb{C}x_4$. Complex conjugation on $V$ is defined by: $\overline{\sum a_i x_i} = \sum \overline{a_i} x_i$. Let $\ell$ define a 2-dimensional subspace $W \subseteq V$ and $\bar{\ell}$ define the subspace $\overline{W} \subseteq V$. Choose $\beta_1, \beta_2 \in V$ such that their image in $V/W$, denoted by the same symbols, form a basis.

     For $V/\overline{W}$ we choose $\overline{\beta_1}, \overline{\beta_2}$ as representatives of a basis. This defines the map $f : \ell \times \bar{\ell} \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The real structure considered on $\mathbb{P}^1 \times \mathbb{P}^1$ is the one induced by $f$. In particular, for $p = (a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$ one has $\overline{p} = (\overline{b}, \overline{a})$.
   - Find the five lines $m_1, \ldots, m_5$ on $S$ that intersect both $\ell$ and $\bar{\ell}$. Since the set $\{\ell, \bar{\ell}\}$ is defined over $\mathbb{R}$, so is $\{m_1, \ldots, m_5\}$. In particular, this means that at least one of the lines $m_i$, say $m_1$, is real.
   - The map $f \circ \Phi_1$ blows down the five lines $m_i$ to five points $p_i \in \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \ldots, 5$. Compute the coordinates of the real point $p_1$, i.e, $p_1 = (a, \overline{a}) \in \mathbb{P}^1 \times \mathbb{P}^1$.
   - Let $X_1, X_2$ and $X'_1, X'_2$ denote the homogeneous coordinates of $\mathbb{P}^1$ and $\mathbb{P}^1$. Complex conjugation on the forms of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (with the above real structure) is given by: $\overline{\sum \alpha_{ij} X_i X'_j} = \sum \overline{\alpha_{ij}} X'_i X_j$. A basis $\{f_1, f_2, f_3\}$ for the 3-dimensional real space of the forms of bidegree $(1, 1)$ vanishing at $p_1$, and invariant under the complex conjugation as defined here, defines the rational map $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ by $p \mapsto (f_1(p) : \ldots : f_3(p))$.

- $\Phi = \Phi_2 \circ f \circ \Phi_1 : S \longrightarrow \mathbb{P}^2_{\mathbb{R}}$, is a real morphism blowing down 6 lines consisting of 3 pairs of conjugated complex lines.

**Remark:** As we know from section 2.2, a real morphism to $\mathbb{P}^2_{\mathbb{R}}$ blowing down 6 lines does not exist for surfaces of type 5). In fact, the above algorithm fails already in the very first step: there exists no pair of skew conjugate lines on the surface.

### 2.3.1 Examples: The Clebsch and the Fermat

Next, we calculate the explicit blow down morphism for two surfaces: the Clebsch diagonal surface and the Fermat cubic surface.

#### The Clebsch diagonal surface

Recall that the Clebsch diagonal surface is a smooth cubic surface $S$ given by the following equations in $\mathbb{P}^4$:

$$S = \begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

An equation for $S$ in $\mathbb{P}^3$ is:

$$S = x_0^3 + x_1^3 + x_2^3 + x_3^3 + (-x_0 - x_1 - x_2 - x_3)^3 = 0.$$

Consider the line

$$\ell_1 = \begin{cases} x_0 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

on $S$.

- The Clebsch surface has the property that all its 27 lines are defined over $\mathbb{R}$. The 27 lines can be given in two groups as:

  i) 15 lines defined over $\mathbb{Q}$ of the form:

  $$l_{i,jk} = \begin{cases} x_i = 0 \\ x_j + x_k = 0 \end{cases}$$

  where $i, j, k \in \{0, 1, 2, 3, 4\}$.

ii) 12 lines defined over $\mathbb{Q}(\sqrt{5})$ and not over $\mathbb{Q}$. According to Clebsch's construction in [Cl], these lines can be given in the following form. Let $\zeta$ be a fifth root of unity $\zeta := e^{2\pi i/5}$. Write $\{1,2,3,4\} = \{j,k,l,m\}$ and consider the point

$$p_{jkmn} = (\zeta^j, \zeta^k, \zeta^m, \zeta^n, 1) \in \mathbb{P}^4$$

and its complex conjugate $\overline{p_{jkmn}}$. The line connecting $p_{jkmn}$ and $\overline{p_{jkmn}}$ is real and it is contained in $S$. There are 12 pairs of lines $\{p_{jkmn}, \overline{p_{jkmn}}\}$ (since we may assume that $j = 1$ or 2. For each of the possibilities, there are 3 possible choices for $k$, 2 for $l$ and 1 for $m$. I.e., one has $2 \times 3 \times 2 = 12$ pairs). The set of these 12 lines can be rewritten in the following manner:

$$\Delta_{(j,k,m,n)} = \begin{cases} x_j + \tau x_k + x_m = 0 \\ x_k + \tau x_j + x_n = 0 \\ \tau x_j + \tau x_k - x_4 = 0 \end{cases}$$

with $\{j,k,m,n\} = \{0,1,2,3\}$, $j < k$ and $\tau = \frac{1+\sqrt{5}}{2}$.

Let us consider the line

$$\ell_2 = \begin{cases} x_4 = 0 \\ x_0 + x_2 = 0 \end{cases}$$

which is skew to $\ell_1$, and apply the algorithm to the two lines $\{\ell_1, \ell_2\}$.

- As it is explained in the algorithm, the blow-down morphism is built in two steps:

$$S \subseteq \mathbb{P}^3 \xrightarrow{\Phi_1} \ell_1 \times \ell_2 \;\simeq\; \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\Phi_2} \mathbb{P}^2.$$

Recall that $\Phi_1(p) = (p_1, p_2) \in \ell_1 \times \ell_2$, where $p_1$ and $p_2$ are the intersections points of $\ell$ with $\ell_1$ and $\ell_2$ respectively, and $\ell$ is the unique line passing through $p$ and intersecting both the lines. In order to find $p_1$ and $p_2$, let us consider the plane $H_i$ passing through $\ell_i$ and $p = (a:b:c:d)$ for $i = 1, 2$:

$H_1 = \{x_0(-b-c) + ax_1 + ax_2 = 0\}$ and
$H_2 = \{x_0(d+b) + x_1(-a-c) + x_2(d+b) - a - c = 0\}$.

The point $p_1$ is the intersection point $H_2 \cap \ell_1$:

$$p_1 = (0 : -a - c : a + c : a + b + c + d),$$

and $p_2$ is the intersection point $H_1 \cap \ell_2$:

$$p_2 = (-a : -a - b - c : a : a + b + c).$$

- Consider $V$ the space of homogeneous polynomials of degree 1 in $\mathbb{C}[x_1, \ldots, x_4]$, and $W \subseteq V$ the subspace given by $\{h \in V \mid h(\ell_1) = 0\}$ and $W'$ the subspace $\{h \in V \mid h(\ell_2) = 0\}$. Consider the bases $\{\beta_1, \beta_2\} = \{x_2, x_3\}$ and $\{\tilde{\beta_1}, \tilde{\beta_2}\} = \{x_2', x_3'\}$ of the spaces $V/W$ and $V/W'$ respectively. Define $f$ as:

$$\begin{array}{rcl} f : \ell_1 \times \ell_2 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ ((x_0 : x_1 : x_2 : x_3), (x_0' : x_1' : x_2' : x_3')) & \mapsto & ((x_2 : x_3), (x_2' : x_3')). \end{array}$$

The map $f \circ \Phi_1$ is given by:

$$\begin{array}{rcl} S \subseteq \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ (a : b : c : d) & \mapsto & ((a + c : a + b + c + d), (a : a + b + c)). \end{array}$$

- By lemma 1, we know that there exist precisely 5 lines on $S$ that intersect both $\ell_1$ and $\ell_2$. These five lines are:

$$m_1 = \left\{ \begin{array}{l} x_4 = 0 \\ x_0 + x_3 = 0 \end{array} \right. , \ m_2 = \left\{ \begin{array}{l} x_0 = 0 \\ x_1 + x_3 = 0 \end{array} \right. , \ m_3 = \left\{ \begin{array}{l} x_1 = 0 \\ x_0 + x_2 = 0 \end{array} \right. ,$$

$$m_4 = \left\{ \begin{array}{l} x_0 + \tau x_2 + x_3 = 0 \\ x_2 + \tau x_0 + x_1 = 0 \end{array} \right. \quad \text{and} \quad m_5 = \overline{m_4} = \left\{ \begin{array}{l} x_0 + \overline{\tau} x_2 + x_3 = 0 \\ x_2 + \overline{\tau} x_0 + x_1 = 0 \end{array} \right.$$

where $\overline{\tau} = \frac{1-\sqrt{5}}{2}$.

- By the definition of $\Phi_1$, the five lines $m_i$ are blown down to five points $p_i$:

  i) $f \circ \Phi_1(m_1) = ((\infty : 1), (-1 : 1))$,
  ii) $f \circ \Phi_1(m_2) = ((1 : 1), (0 : 1))$,

iii) $f \circ \Phi_1(m_3) = ((0:1),(\infty:1))$,

iv) $f \circ \Phi_1(m_4) = ((\overline{\tau}:1),(-\tau:1))$,

v) $f \circ \Phi_1(m_5) = ((\tau:1),(-\overline{\tau}:1))$.

Consider any of the points $p_i$ (since they are all real), for instance, the point $p_2 = \Phi_1(m_2) = ((1:1),(0:1))$.

- Consider a basis for the subspace of the forms of bidegree $(1,1)$ in $\mathbb{C}[X_2, X_3, X_2', X_3']$ that vanish in the point $((1:1),(0:1))$, for example the basis
$$\mathcal{B} = \{X_2X_3' - X_3X_3', X_3X_2', X_2X_2'\}.$$
This basis defines the map $\Phi_2$:
$$\begin{array}{rcl} \Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ ((x_2:x_3),(x_2':x_3')) & \mapsto & (x_2x_3' - x_3x_3' : x_3x_2' : x_2x_2') \end{array}$$

- The complete blow down morphism is given by $\Phi = \Phi_2 \circ f \circ \Phi_1$:
$$\begin{array}{rcl} \Phi : S \subseteq \mathbb{P}^3 & \longrightarrow & \mathbb{P}^2 \\ (a:b:c:d) & \mapsto & ((-b-d)(a+b+c) : \\ & & a(a+b+c+d) : a(a+c)). \end{array}$$

In order to find the six points in $\mathbb{P}^2$, note that four of them are the images $\Phi(m_1)$, $\Phi(m_3)$, $\Phi(m_4)$, $\Phi(m_5)$. The remaining two are the images of the lines $r_1 = \mathbb{P}^1 \times \{(1:1)\}$ and $r_2 = \{(0:1)\} \times \mathbb{P}^1$ under $\Phi_2$. That is, the six points are:
$$(1:0:1),(0:1:0),(-\tau:-\tau:1),(-\overline{\tau},-\overline{\tau}:1),(1:0:0),(0:1:1).$$

The six lines that are blown down to the six points are the four lines $m_1, m_3, m_4, m_5$ and the two lines:
$$\Phi_1^{-1}(r_1) = \begin{cases} x_1 = 0 \\ x_2 + x_4 = 0 \end{cases} \qquad \text{and} \qquad \Phi_1^{-1}(r_2) = \begin{cases} x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

The blow-up map $\Psi = \Phi^{-1}$ is:

$$\begin{array}{rcl} \Psi : \mathbb{P}^2 & \dashrightarrow & S \subseteq \mathbb{P}^3 \\ (x : y : z) & \mapsto & ((-z+y)(-z^2+xz+xy) : -z^3+xz^2+yz^2-x^2y : \\ & & x(z^2-xz-y^2) : z(-xz+x^2+yz-y^2)). \end{array}$$

### An application to Coding Theory

In this section, we give an application of some of the results obtained in the previous section to coding theory. In particular, we calculate the number of words of minimum weight in a BCH-code. This number is relevant, since it is related to the maximal number of errors that the code can detect.

We begin by defining some basic notions of coding theory.

**Definition 1.**

1. A *linear code* $\mathcal{C}$ over $\mathbb{F}_q$ of length $n$ is a linear subspace of the vector space $\mathbb{F}_q^n$ consisting on $n$-tuples $c = (a_0, \ldots, a_{n-1})$ with $a_i \in \mathbb{F}_q$. The elements $c = (a_0, \ldots, a_{n-1}) \in \mathcal{C}$ are called *words*.

2. A linear code $\mathcal{C}$ is a *cyclic* code if for every word $c = (a_0, \ldots, a_{n-1}) \in \mathcal{C}$, the word $c = (a_{n-1}, a_0, \ldots, a_{n-2})$ is also in $\mathcal{C}$.

3. The *Hamming distance* $d(c_1, c_2)$ between two words $c_1, c_2 \in \mathcal{C}$ is defined as the number of coefficients where $c_1$ and $c_2$ differ. The Hamming distance defines a metric on $\mathbb{F}_q^n$.

4. The *weight* $w(c)$ of a word $c \in \mathcal{C}$ is the number of non-zero coordinates of $c$. In other words, $w(c) = d(c, 0)$.

5. The *minimum weight* of a linear code $\mathcal{C}$ is given by

$$d_{min}(\mathcal{C}) = min\{w(c) : c \in \mathcal{C}, c \neq 0\}.$$

BCH codes are named after Bose, Chaudhuri and Hocquenghem. To define them, consider the finite field $\mathbb{F}_{2^m}$ and a generator $\alpha$ of the multiplicative group $\mathbb{F}_{2^m}^*$. Put $n = 2^m - 1$ and define the binary linear cyclic code of length $n$ as:

$$\mathcal{C} = \{(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_2^n \mid \sum_{i=0}^{n-1} a_i\alpha_i = 0 \text{ and } \sum_{i=0}^{n-1} a_i\alpha_i^3 = 0\}.$$

One can calculate the minimal weight of a given code $\mathcal{C}$ by finding the solutions to the system

$$\sum_{i=0}^{n-1} a_i \alpha_i = 0 \text{ and } \sum_{i=0}^{n-1} a_i \alpha_i^3 = 0.$$

It can easily be proven that the system has no solutions with distinct $\alpha_i$ for $n \leq 4$. Hence, $d_{min}(\mathcal{C}) \geq 5$. If $c = (a_0, \ldots, a_{n-1}) \in \mathcal{C}$ has weight 5, this means there exist

$$0 \leq i_1 < i_2 < \ldots < i_5 \leq n-1$$

such that $a_j = 0$ if $j \notin \{i_1, \ldots, i_5\}$ and $a_j = 1$ otherwise. Since $c \in \mathcal{C}$, the point $(\alpha_{i_1} : \cdots : \alpha_{i_5})$ is a point on the Clebsch with coordinates in $\mathbb{F}_{2^m}$, that is

$$\begin{cases} \alpha_{i_1}^3 + \alpha_{i_2}^3 + \alpha_{i_3}^3 + \alpha_{i_4}^3 + \alpha_{i_5}^3 = 0 \\ \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} + \alpha_{i_4} + \alpha_{i_5} = 0. \end{cases}$$

Calculating the solutions to this system is equivalent to counting the number $\#S(\mathbb{F}_{2^m})$ of points on the Clebsch with coordinates in $\mathbb{F}_{2^m}$.

**Lemma 3.** *The number $\#S(\mathbb{F}_{2^m})$ of points of the Clebsch with coordinates in $\mathbb{F}_{2^m}$ is given by:*

$$\#S(\mathbb{F}_{2^m}) = \begin{cases} 2^{2m} + 7 \cdot 2^m + 1 & \textit{if } m \textit{ is even} \\ 2^{2m} + 5 \cdot 2^m + 1 & \textit{if } m \textit{ is odd.} \end{cases}$$

*Proof.* Recall that the Clebsch diagonal surface is obtained by blowing up the six points:

$$(1:0:1), (0:1:0), (1:0:0), (0:1:1), (-\tau : -\tau : 1), (-\overline{\tau}, -\overline{\tau} : 1)$$

where $\tau = \frac{1+\sqrt{5}}{2}$ and $\overline{\tau} = \frac{1-\sqrt{5}}{2}$. The six points are in $\mathbb{P}^2(\mathbb{Z}[\tau])$.

One can define $\mathbb{F}_4$ as $\mathbb{Z}[\tau]/(2) = \mathbb{F}_2[\tau + (2)]$, with $\tau + (2)$ a root of $x^2 + x + 1 = 0$. After reducing the six points modulo 2, one sees that they are all in $\mathbb{P}^2(\mathbb{F}_4)$ and that they are still in general position. The four points

$$(1:0:1), (0:1:0), (1:0:0), (0:1:1)$$

are in $\mathbb{P}^2(\mathbb{F}_2)$, and the remaining two are in $\mathbb{P}^2(\mathbb{F}_4) \setminus \mathbb{P}^2(\mathbb{F}_2)$.

Hence, the number $a$ of points in $\mathbb{P}^2(\mathbb{F}_{2^m})$ is

$$\begin{cases} a = 4 & \text{if } m \text{ even} \\ a = 6 & \text{if } m \text{ odd.} \end{cases}$$

We recall that the Clebsch can be obtained by blowing up the $\mathbb{P}^2$ at six point ($a$ of them in $\mathbb{P}^2(\mathbb{F}_{2^m})$). Since these points are replaced by $\mathbb{P}^1$'s, one has that the total number of points in $S$ in $\mathbb{F}_{2^m}$ can be calculated as:

$$\#\mathbb{P}^2(\mathbb{F}_{2^m}) + a - a \cdot \#\mathbb{P}^1(\mathbb{F}_{2^m}).$$

One can write $\mathbb{P}^2(\mathbb{F}_{2^m})$ as $\{(0:0:1), (0:1:x), (1:y:z) \mid x, y, z \in \mathbb{F}_{2^m}\}$. Hence, $\#\mathbb{P}^2(\mathbb{F}_{2^m}) = 1 + 2^m + 2^{2m}$. The number of points of $\mathbb{P}^1(\mathbb{F}_{2^m})$ is $2^m + 1$, that is,

$$\#S(\mathbb{F}_{2^m}) = \begin{cases} 2^{2m} + 7 \cdot 2^m + 1 & \text{if } m \text{ is even} \\ 2^{2m} + 5 \cdot 2^m + 1 & \text{if } m \text{ is odd.} \end{cases}$$

□

We can now prove the result:

**Theorem 13.** *The number of words of weight* 5 *in* $\mathcal{C}$ *is given by:*

$$\begin{cases} (q-1)(q-4)^2/5! & \text{for} \quad q = 2^m \quad \text{if } m \text{ is even} \\ (q-1)(q-2)(q-8)/5! & \text{for} \quad q = 2^m \quad \text{if } m \text{ is odd.} \end{cases}$$

*Proof.* Among all points in $\#S(\mathbb{F}_{2^m})$, we only need those whose coordinates are all distinct and differ from zero. After a short computation (see [Vlugt]), one sees that we must leave aside a total of $15(2^m) - 15$ points, namely: $\binom{5}{3} = 10$ points in the orbit of $(1:1:0:0:0)$ under the automorphism group $S_5$ of $S$, $\binom{5}{1} = 5$ points in the orbit of $(1:1:1:1:0)$ and $15(2^m - 2)$ points in the orbit of $(1:1:a:a:0)$ with $a \in \mathbb{F}_{2^m} \backslash \mathbb{F}_2$.

In the case when $m$ is even, the number of words of weight 5 are counted as follows: By lemma 3, the number of points of $S$ that are in $\mathbb{F}_{2^m}$ whose homogeneous coordinates are distinct and differ from zero is:

$$(2^m)^2 + 7(2^m) + 1 - (15(2^m) - 15) = (2^m - 4)^2.$$

Every projective solution gives $2^m - 1$ affine ones. Since the solution are up to permuting the five coordinates, one has that the total number or words of weight five for $m$ even is:

$$(2^m - 1)(2^m - 4)^2/5!.$$

In a similar way, the formula for $m$ odd is obtained:

$$(2^m - 1)(2^m - 2)(2^m - 8)/5!.$$

$\square$

Alternative proofs of this theorem can be found in [Vlugt] and [McW-Sl].

### The Fermat cubic surface

The Fermat cubic is a surface of type 4) in Schläfli's classification. It does not contain a skew pair of real lines. Hence, we will consider a pair of skew complex conjugate lines.

We recall that the Fermat cubic surface $S$ is the smooth cubic surface in $\mathbb{P}^3$ defined by the polynomial

$$x_0^3 + x_1^3 + x_2^3 + x_3^3.$$

The 27 lines on the Fermat can be given in three groups as:

i) 9 lines of the form:

$$\begin{cases} x_0 + \omega x_1 = 0 \\ x_2 + \omega' x_3 = 0, \end{cases}$$

ii) 9 lines of the form:

$$\begin{cases} x_0 + \omega x_2 = 0 \\ x_1 + \omega' x_3 = 0, \end{cases}$$

iii) 9 lines of the form:

$$\begin{cases} x_0 + \omega x_3 = 0 \\ x_1 + \omega' x_2 = 0, \end{cases}$$

where $\omega$ and $\omega'$ are cubic roots of unity (not necessarily primitive). Now fix a primitive third root of unity $\omega$ and consider the line on $S$:

$$\ell = \begin{cases} x_0 + \omega x_1 = 0 \\ x_2 + \omega x_3 = 0, \end{cases}$$

and the conjugate line $\overline{\ell}$

$$\overline{\ell} = \begin{cases} x_0 + \omega^2 x_1 = 0 \\ x_2 + \omega^2 x_3 = 0 \end{cases}$$

which is skew to $\ell$.

- We proceed to build the map $\Phi_1(p) = (p_1, p_2) \in \ell \times \overline{\ell}$. We find $p_1$ and $p_2$ as follows. Let $p = (a : b : c : d) \in \mathbb{P}^3$. Let $H_2$ be the plane passing through $\overline{\ell}$ and $p$. Such a plane can be given by:

$$H_2 = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \mid \\ (c + d\omega^2)x_0 + (c\omega^2 + d\omega)x_1 - (a + b\omega^2)x_2 - (a\omega^2 + b\omega)x_3 = 0\}.$$

Intersect the plane $H_2$ with the line $\ell$ and obtain $p_1$:

$$p_1 = (-a\omega - b : a + b\omega^2 : -\omega(c + d\omega^2) : c + d\omega^2).$$

The second coordinate $p_2 \in \overline{\ell}$ is:

$$p_2 = (-a\omega^2 - b : a + b\omega : -\omega^2(c + d\omega) : c + d\omega).$$

Define $\Phi_1$ as $\Phi_1(p) = (p_1, p_2)$.

- Let $V_{\mathbb{R}} = \mathbb{R}x_1 + \cdots + \mathbb{R}x_4$ and $V = \mathbb{C}x_1 + \cdots + \mathbb{C}x_4$. Let $\ell$ define a 2-dimensional subspace $W \subseteq V$ and $\overline{\ell}$ define the subspace $\overline{W} \subseteq V$. Consider the bases $\{x_1, x_3\}$ and $\{x_1', x_3'\}$ of the spaces $V/W$ and $V/\overline{W}$ respectively. Define $f$ as:

$$\begin{array}{ccc} \ell \times \overline{\ell} & \xrightarrow{f} & \mathbb{P}^1 \times \mathbb{P}^1 \\ ((x_0 : x_1 : x_2 : x_3), (x_0' : x_1' : x_2' : x_3')) & \mapsto & ((x_1 : x_3), (x_1' : x_3')) \end{array}$$

- In total, the map $f \circ \Phi_1$ is given by:

$$\begin{array}{ccc} S \subseteq \mathbb{A}^3 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ (a : b : c : d) & \mapsto & ((a + b\omega^2 : c + d\omega^2), (a + b\omega : c + d\omega)). \end{array}$$

- By applying lemma 1 to the two lines $\ell$ and $\overline{\ell}$, we have that there exist 5 lines on $S$ intersecting both $\ell$ and $\overline{\ell}$. The five lines are given by:

$$m_1 = \begin{cases} x_0 + \omega x_1 = 0 \\ x_2 + \omega^2 x_3 = 0, \end{cases} \quad m_2 = \begin{cases} x_0 + \omega^2 x_1 = 0 \\ x_2 + \omega x_3 = 0, \end{cases}$$

$$m_3 = \begin{cases} x_0 + \omega x_2 = 0 \\ x_1 + \omega x_3 = 0, \end{cases} \quad m_4 = \begin{cases} x_0 + \omega^2 x_2 = 0 \\ x_1 + \omega^2 x_3 = 0 \end{cases}$$

and the real line

$$m_5 = \begin{cases} x_0 + x_2 = 0 \\ x_1 + x_3 = 0. \end{cases}$$

- By the definition of $\Phi_1$, the five lines $m_1 \dots m_5$ are sent to five points:

  i) $f \circ \Phi_1(m_1) = ((\infty : 1), (0 : 1))$,
  ii) $f \circ \Phi_1(m_2) = ((0 : 1), (\infty : 1))$,
  iii) $f \circ \Phi_1(m_3) = ((-\omega : 1), (-\omega : 1))$,
  iv) $f \circ \Phi_1(m_4) = ((-\omega^2 : 1), (-\omega^2 : 1))$,
  v) $f \circ \Phi_1(m_5) = ((-1 : 1), (-1 : 1))$.

  Consider the real point $p_5 = ((-1 : 1), (-1 : 1))$.

- We look for forms of bidegree $(1, 1)$

$$\alpha X_1 X_1' + \beta X_1 X_3' + \gamma X_3 X_1' + \delta X_3 X_3',$$

  with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, that are invariant under complex conjugation $\sum \alpha_{ij} X_i X_j' \mapsto \sum \overline{\alpha_{ij}} X_i' X_j$.
  A basis for the subspace of forms that vanish in $((-1 : 1), (-1 : 1))$ is:

$$\{f_1, f_2, f_3\} = \{2X_1X_1' + X_1X_3' + X_3X_1', \frac{X_1X_3' - X_3X_1'}{2\omega + 1}, -X_1X_1' + X_3X_3'\}.$$

  Define $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ that maps $((x_1 : x_3), (x_1' : x_3'))$ to

$$(2x_1x_1' + x_1x_3' + x_1'x_3 : \frac{x_1x_3' - x_1'x_3}{2\omega + 1} : -x_1x_1' + x_3x_3').$$

- The complete blow down morphism is given by $\Phi_2 \circ f \circ \Phi_1$:

$$\begin{array}{rcl} S \subseteq \mathbb{P}^3 & \longrightarrow & \mathbb{P}^2 \\ (a : b : c : d) & \mapsto & (2a^2 - a(2b - 2c + d) + b(2b - c + 2d) : \\ & & ad - bc : -a^2 - b^2 + c^2 + ab - cd + d^2). \end{array}$$

The map $\Phi_2$ blows up the point $((-1 : 1), (-1, 1))$ and blows down the following two lines that intersect in the point $((-1 : 1), (-1, 1))$:

$$r_1 = \{(-1 : 1)\} \times \mathbb{P}^1 \text{ and } r_2 = \mathbb{P}^1 \times \{(-1 : 1)\}.$$

Next, we present the six points in $\mathbb{P}^2$ for the Fermat cubic that correspond to this blow down morphism: Four of the points are the images of the four lines $m_1, m_2, m_3, m_4$ under the blow down morphism. The remaining two can be obtained by applying $\Phi_2$ to the lines $r_1 = \{(-1:1)\} \times \mathbb{P}^1$ and $r_2 = \mathbb{P}^1 \times \{(-1:1)\}$.

$$(\omega : 1 : 0), (\omega^2 : 1 : 0), (2 : 0 : \omega), (2 : 0 : \omega^2), (\omega : 1 : -\omega), (\omega^2 : 1 : -\omega^2).$$

The six lines that are blown down to these six points are the four lines $m_1, m_2, m_3, m_4$ and the lines

$$\Phi_1^{-1}(r_1) = \begin{cases} x_0 + \omega^2 x_3 = 0 \\ x_2 + \omega^2 x_1 = 0 \end{cases} \text{ and } \Phi_1^{-1}(r_2) = \begin{cases} x_0 + \omega x_3 = 0 \\ x_2 + \omega x_1 = 0. \end{cases}$$

The blow-up map $\Psi = \Phi^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ maps $(x : y : z)$ to:

$$\begin{gathered}(-x^3 - 2x^2z + 3x^2y + 12yxz - 3xy^2 - 4xz^2 + 6y^2z + 12yz^2 + 9y^3 : \\ x^3 + 2x^2z + 3x^2y + 12yxz + 3xy^2 + 4xz^2 - 6y^2z + 12yz^2 + 9y^3 : \\ -8z^3 - 8xz^2 - 9y^3 - x^3 - 3x^2y - 3xy^2 - 4x^2z - 12y^2z : \\ 8z^3 + 8xz^2 - 9y^3 + x^3 - 3x^2y + 3xy^2 + 4x^2z + 12y^2z)\,.\end{gathered}$$

**Remark 1:** This differs from the blow-up map given by Elkies [El]. We do not know whether or not his blow-up map is related to ours.

**Remark 2:** Euler was the first one to give all rational solutions of $\sum_{i=0}^{3} x_i^3 = 0$. His parameterization is given by the following polynomials of degree 4:

$$\begin{aligned} p &= 3(bc - ad)(c^2 + 3d^2) \\ q &= (a^2 + 3b^2)^2 - (ac + 3bd)(c^2 + 3d^2) \\ r &= 3(bc - ad)(a^2 + 3b^2) \\ s &= -(c^2 + 3d^2)^2 + (ac + 3bd)(a^2 + 3b^2). \end{aligned}$$

The details of his method can be found in [D, pp. 552-554]. A general method that yields parameterizations of cubic surfaces by polynomials of degree 4, is described in [Ba-Ho-Ne]. The algorithms described in this section may be regarded as an improvement of their method.

## 2.4 Twists of surfaces over $\mathbb{Q}$

In this section we compute twists over $\mathbb{Q}$ of the Clebsch diagonal surface and the Fermat cubic surface. First, we define $\mathrm{H}^1$ and give some background knowledge on the theory of twists.

### 2.4.1 $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{G})$

Let $X$ be an algebraic surface defined over $\mathbb{Q}$ and $G = \mathrm{Aut}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ the group of the $\overline{\mathbb{Q}}$-automorphisms of $X \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $G$ as follows:

Let $g \in G$, and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Define $\sigma(g)$ as the unique automorphism in $G$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times_{\mathbb{Q}} \overline{\mathbb{Q}} & \xrightarrow{g} & X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \\ id_X \times \sigma \downarrow & & \downarrow id_X \times \sigma \\ X \times_{\mathbb{Q}} \overline{\mathbb{Q}} & \xrightarrow{\sigma(g)} & X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \end{array} ,$$

It is immediate to verify that: (To ease notation we will write composition of maps as $hg$ instead of $h \circ g$.):

1. $\sigma(g) = (id_X \times \sigma)g(id_X \times \sigma)^{-1} \in G$.
2. $\sigma(gh) = \sigma(g)\sigma(h)$
3. $(\sigma\tau)(g) = \sigma(\tau(g))$.

In particular, $\sigma(g)$ defines an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $G$.

**Definition 2.** Let $\sigma(g)$ be the action described before. A 1-*cocycle* is a map $c : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow G$ such that $c(\sigma\tau) = c(\sigma)\sigma(c(\tau))$.

Two 1-cocycles $c, c'$ are equivalent if there is an $h \in G$ such that for all $\sigma$, $c(\sigma) = h^{-1}c'(\sigma)\sigma(h)$. This defines an equivalence relation. The set of equivalence classes is denoted by $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{G})$.

### 2.4.2 Explicit twists of surfaces

**Definition 3.** Let $X$ be an algebraic surface over $\mathbb{Q}$. A *form* or a *twist* of $X$ is another algebraic surface $Y$ defined over $\mathbb{Q}$ such that there exists a $\overline{\mathbb{Q}}$-isomorphism

$$f : X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \longrightarrow Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

The set of twists of $X$ modulo $\mathbb{Q}$-isomorphism is denoted by $Twist(X)$.

Let $X$ be an algebraic surface. For each twist $Y \in Twist(X)$ choose a $\overline{\mathbb{Q}}$-isomorphism $f : X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \longrightarrow Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Define $\sigma(f) : X \times_{\mathbb{Q}} \overline{\mathbb{Q}} \longrightarrow Y \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that the following diagram is commutative.

$$\begin{array}{ccc} X \times_{\mathbb{Q}} \overline{\mathbb{Q}} & \xrightarrow{f} & Y \times_{\mathbb{Q}} \overline{\mathbb{Q}} \\ id_X \times \sigma \downarrow & & \downarrow id_Y \times \sigma \\ X \times_{\mathbb{Q}} \overline{\mathbb{Q}} & \xrightarrow{\sigma(f)} & Y \times_{\mathbb{Q}} \overline{\mathbb{Q}} \end{array} ,$$

Define the map $c : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow G$ attached to $f$ by $c(\sigma) := f^{-1}\sigma(f) \in \mathrm{Aut}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$. Then, the following statements hold:

**Theorem 14.** *Using the previous notation,*

1. *$c$ is a cocycle (i.e., for all $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $c(\sigma\tau) = (c(\sigma))\sigma(c(\tau))$).*
2. *A different choice of $f$ leads to an equivalent cocycle.*
3. *The natural map*

$$Twist(X) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{G})$$

*is a bijection, that is, the twists of $X$ are in one-to-one correspondence with the elements of $\mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{G})$.*

*Proof.* See [Sil]. □

### 2.4.3 Twists of the Clebsch

Recall that the Clebsch diagonal surface is the smooth cubic surface $S$ given by the following homogeneous equations in $\mathbb{P}^4$:

$$S = \begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

The description of all lines on $S$ as given in 2.3.1 shows:

**Proposition 3.** *Let $S$ be the Clebsch diagonal surface.*

1. *The number of rational lines on $S$ is 15.*

*2. The number of rational tritangents on $S$ is* 15.

*Proof.* Recall that $S$ is obtained by blowing up 6 points: 4 defined over $\mathbb{Q}$ and one $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$ conjugate pair. $\square$

We now calculate twists of this surface. It is known that $\mathrm{Aut}(S \times_{\mathbb{Q}} \overline{\mathbb{Q}}) = S_5$ (see [Hu]). The automorphisms group $S_5$ acts on $S$ by permuting the set $\{x_0, x_1, x_2, x_3, x_4\}$.

Note that, in this case, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S_5$ is trivial. Hence, it holds that $c(\sigma_1\sigma_2) = c(\sigma_1)\sigma_1(c(\sigma_2)) = c(\sigma_1)c(\sigma_2)$, i.e., $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), S_5) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), S_5)/\mathrm{conj}$, where "conj" denotes conjugation by elements in $S_5$.

The Clebsch surface can be represented by its homogeneous coordinate ring $A$:

$$A = \frac{\mathbb{Q}[x_0, x_1, x_2, x_3, x_4]}{(\sum_{j=0}^{4} x_j, \sum_{j=0}^{4} x_j^3)}$$

Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and let $h : \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q}) \longrightarrow S_5$ be a homomorphism. By the general theory as described in Theorem 14, $h$ defines a twists $S^{twist}$ of $S$. We will now describe this twist.

We define a new action $\rho_h$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} A$ as follows:

$$\rho_h(\sigma)(\lambda \otimes a) = \sigma(\lambda) \otimes h(\sigma)(a) \text{ for all } \lambda \otimes a \in \overline{\mathbb{Q}} \otimes A.$$

More precisely, $\rho_h$ acts on $A$ as:

$$\rho_h(\sigma)(x_i) = x_{h(\sigma)i},$$

for all $i = 0, \ldots, 4$.

The elements of the homogeneous coordinate ring of the twisted surface $S^{twist}$ are the invariants of $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} A = \overline{\mathbb{Q}}[x_0, x_1, x_2, x_3, x_4]/(\sum_{j=0}^{4} x_j, \sum_{j=0}^{4} x_j^3)$ under $\rho_h$. In fact

$$(\overline{\mathbb{Q}}x_0 + \overline{\mathbb{Q}}x_1 + \overline{\mathbb{Q}}x_2 + \overline{\mathbb{Q}}x_3 + \overline{\mathbb{Q}}x_4)^{\rho_h} = \mathbb{Q}y_0 + \mathbb{Q}y_1 + \mathbb{Q}y_2 + \mathbb{Q}y_3 + \mathbb{Q}y_4$$

for certain linear forms $y_i$. These $y_i$ generate the coordinate ring of $S^{twist}$.

In the following, we illustrate this for the case of homomorphisms $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow S_5$ which have as image a subgroup of order 2, and for one example in which the image is a subgroup of order 5.

**Notation**: We denote by $\{0, 1, 2, 3, 4\}$ the set of five elements that $S_5$ permutes. Then, $(i, j) \in S_5$ denotes the two-cycle that permutes the elements $i$ and $j$, and so on.

**Twist by $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ and a 2-cycle.**

All 2-cycles give the same class. Then, one can consider the 2-cycle given by $\langle(0,1)\rangle$, with $(0,1) \in S_5$. Let $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) = \{1,\sigma\}$. Define $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \langle(0,1)\rangle$ by the diagram:

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr, "h"] \arrow[dr, "res"'] & & \langle(0,1)\rangle \leq S_5 \\ & \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \arrow[ur, "c"'] & \end{tikzcd}$$

where $c$ is an isomorphism such that $c(\sigma) = (0,1)$.

**Proposition 4.** *Let $S^{twist}$ be the twisted surface of the Clebsch $S$ by the homomorphism $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \langle(0,1)\rangle$. Then,*

1. *The surface $S^{twist}$ is given by the equations*

$$S^{twist} = \begin{cases} \dfrac{y_0^3}{4} + \dfrac{3y_0y_1^2}{8d} + y_2^3 + y_3^3 + y_4^3 = 0 \\ y_0 + y_2 + y_3 + y_4 = 0 \end{cases}$$

2. *The number of rational lines and rational tritangents on $S^{twist}$ is $(3,13)$.*

*Proof.*

1. **Notation:** In the following, $s$ will denote the 2-cycle $(0,1)$. The action of $\rho_s(\sigma)$ of $\sigma$ on $\mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} A$ is given by: $\rho_s(\lambda \otimes x_i) = \sigma(\lambda) \otimes x_{s(i)}$ for all $i = 0,\dots,4$. The invariants in $\mathbb{Q}(\sqrt{d})[x_0,x_1,x_2,x_3,x_4]$ under this action are generated by $y_0 = x_0+x_1$, $y_1 = \sqrt{d}(x_0-x_1)$, $y_2 = x_2$, $y_3 = x_3$ and $y_4 = x_4$. We now write the $x_i$'s in terms of the $y_i$'s:

$$x_0 = \frac{y_0}{2} + \frac{y_1}{2\sqrt{d}},\ x_1 = \frac{y_0}{2} - \frac{y_1}{2\sqrt{d}},\ x_2 = y_2,\ x_3 = y_3,\ x_4 = y_4.$$

   and substitute them in the equations $\sum_{j=0}^4 x_j = 0$ and $\sum_{j=0}^4 x_j^3 = 0$. The new equations in terms of the $y_i$'s that define the twisted surface are:

$$S^{twist} = \begin{cases} \dfrac{y_0^3}{4} + 3\dfrac{y_0y_1^2}{8d} + y_2^3 + y_3^3 + y_4^3 = 0 \\ y_0 + y_2 + y_3 + y_4 = 0. \end{cases}$$

2. Let us now calculate the number of lines of $S^{twist}$ that are defined over $\mathbb{Q}$.

   In general, a line $l \subseteq S$, where $S$ is the original surface, is defined over $\mathbb{Q} \Leftrightarrow \sigma(l) = l$. We now have an isomorphism: $S \times_{\mathbb{Q}} \overline{\mathbb{Q}} \simeq S^{twist} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$. A line $\tilde{l} \subseteq S^{twist}$ is defined over $\mathbb{Q} \Leftrightarrow \rho_s(\sigma)(l) = l$.

   We recall (see Section 2.3.1) that the 27 lines on the Clebsch can be given in the following two groups:

   (a) 15 lines defined over $\mathbb{Q}$ of the form:

   $$\ell_{i,jk} = \begin{cases} x_i = 0 \\ x_j + x_k = 0 \end{cases}$$

   where $i, j, k \in \{0, 1, 2, 3, 4\}$. Since all lines of this type are defined over $\mathbb{Q}$, the map $\rho_s(\sigma)$ acts on $\ell_{i,jk}$ by: $\rho_s(\sigma)(\ell_{i,jk}) = \ell_{s(i),s(j)s(k)}$. It is easy to see that the only rational lines in this group are $\ell_{2,01}$, $\ell_{3,01}$ and $\ell_{4,01}$.

   (b) 12 lines defined over $\mathbb{Q}(\sqrt{5})$ as follows:

   $$\Delta_{(j,k,m,n)} = \begin{cases} x_j + \tau x_k + x_m = 0 \\ x_k + \tau x_j + x_n = 0 \\ \tau x_j + \tau x_k - x_4 = 0 \end{cases}$$

   with $\{j, k, m, n\} = \{0, 1, 2, 3\}$, $j < k$ and $\tau = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. These twelve lines form the following double six $D$:

   $$\begin{pmatrix} \Delta_{(0,1,2,3)} & \Delta_{(0,2,3,1)} & \Delta_{(0,3,1,2)} & \Delta_{(1,2,0,3)} & \Delta_{(1,3,2,0)} & \Delta_{(2,3,0,1)} \\ \Delta_{(0,1,3,2)} & \Delta_{(0,2,1,3)} & \Delta_{(0,3,2,1)} & \Delta_{(1,2,3,0)} & \Delta_{(1,3,0,2)} & \Delta_{(2,3,1,0)} \end{pmatrix}.$$

   To calculate the action of $\rho_s(\sigma)$ on the lines, we distinguish two cases:

   (a) Suppose that $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{5})$. We look for lines $\ell \subseteq S$ such that $\rho_s(\tilde{\sigma})(\ell) = \ell$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One must distinguish the following cases:

      i. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} \neq id$. In this case, the action sends $\sqrt{5}$ to $-\sqrt{5}$ and interchanges $x_0 \leftrightarrow x_1$. The action on the rational lines is the following: $\rho_s(\tilde{\sigma})(\ell_{i,jk}) = \ell_{s(i),s(j)s(k)}$. Hence, the rational lines fixed by this action are: $\ell_{2,01}, \ell_{3,01}, \ell_{4,01}$.

Concerning the 12 lines defined over $\mathbb{Q}(\sqrt{5})$ that form the double six $D$, one has that $\rho_s(\tilde{\sigma})(D)$ equals:

$$\begin{array}{cccccc} \Delta_{(0,1,3,2)} & \Delta_{(0,2,1,3)} & \Delta_{(0,3,2,1)} & \Delta_{(1,2,3,0)} & \Delta_{(1,3,0,2)} & \Delta_{(2,3,1,0)} \\ \Delta_{(0,1,2,3)} & \Delta_{(0,2,3,1)} & \Delta_{(0,3,1,2)} & \Delta_{(1,2,0,3)} & \Delta_{(1,3,2,0)} & \Delta_{(2,3,0,1)} \end{array}$$

That is, there are no lines of this type fixed by $\rho_s(\tilde{\sigma})$.
Therefore, in the remaining cases we will only look at the action of $\rho_s(\tilde{\sigma})$ on the rational lines.

ii. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} = id$. In this case, the action on the rational lines is the same as in previous case. That is, the action fixes the lines $\ell_{2,01}, \ell_{3,01}, \ell_{4,01}$.

iii. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} \neq id$. The action sends $\sqrt{5}$ to $-\sqrt{5}$ and leaves the coordinates $x_i$'s invariant. Hence, $\rho_s(\tilde{\sigma})$ fixes all 15 rational lines $\ell_{i,jk}$.

iv. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} = id$. Again, $\rho_s(\tilde{\sigma})$ fixes all rational lines.

The lines on $S$ that are fixed by $\rho_s(\tilde{\sigma})$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are the three lines $\ell_{2,01}, \ell_{3,01}, \ell_{4,01}$. The same can be done to calculate the number of tritangents of $S^{twist}$ that are defined over $\mathbb{Q}$ (i.e., the pairs of intersecting lines $\{\ell_1, \ell_2\}$ on $S$ such that $\{\rho_s(\tilde{\sigma})(\ell_1), \rho_s(\tilde{\sigma})(\ell_2)\} = \{\ell_1, \ell_2\}$). This leads to 13 tritangent planes, corresponding to the pairs: $\{\ell_{0,12}, \ell_{1,02}\}$, $\{\ell_{0,13}, \ell_{1,03}\}$, $\{\ell_{0,14}, \ell_{1,04}\}$, $\{\ell_{0,23}, \ell_{1,23}\}$, $\{\ell_{0,24}, \ell_{1,24}\}$, $\{\ell_{0,34}, \ell_{1,34}\}$, $\{\ell_{2,34}, \ell_{3,42}\}$, $\{\ell_{2,03}, \ell_{2,13}\}$, $\{\ell_{2,04}, \ell_{2,14}\}$, $\{\ell_{3,02}, \ell_{3,12}\}$, $\{\ell_{3,04}, \ell_{3,14}\}$, $\{\ell_{4,02}, \ell_{4,12}\}$, $\{\ell_{4,03}, \ell_{4,13}\}$.

(b) Suppose that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{5})$. We repeat the reasoning and distinguish two cases:

i. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} \neq id$. In this case, the action sends $\sqrt{5}$ to $-\sqrt{5}$ and interchanges $x_0 \leftrightarrow x_1$. As before, $\rho_s(\tilde{\sigma})$ fixes the three rational lines $\ell_{2,01}, \ell_{3,01}, \ell_{4,01}$ and no line of type $\Delta_{(j,k,m,n)}$. The action fixes the 13 tritangents given above.

ii. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} = id$. The action fixes all the lines and tritangents on $S$.

Again, one has that the twisted surface has 3 rational lines and 13 rational tritangents.

$\square$

**Twist by $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ and the product of two 2-cycles.**

Consider the homomorphism $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow S_5$ given by $h(\sigma) = (0,1)(2,3)$. In this case, $h$ factors as $h = cf$ in the following manner:

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[rr, "h"] \arrow[dr, "res"'] & & \langle (0,1)(2,3) \rangle \leq S_5 \\ & \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) \arrow[ur, "c"'] & \end{tikzcd}$$

where $c$ is an isomorphism such that $c(\sigma) = (0,1)(2,3)$.

**Notation:** In the following we will denote the permutation $(0,1)(2,3)$ by $s$. We have the following result:

**Proposition 5.** *Let $S^{twist}$ be the twisted surface of the Clebsch $S$ by the homomorphism $h$. Then,*

1. *$S^{twist}$ is given by the equations:*

$$S^{twist} = \begin{cases} \dfrac{y_0^3}{4} + \dfrac{3y_0y_1^2}{8d} + \dfrac{y_2^3}{4} + \dfrac{3y_2y_3^2}{8d} + y_4^3 = 0 \\ y_0 + y_2 + y_4 = 0. \end{cases}$$

2. *The number of rational lines and rational tritangents on $S^{twist}$ is* $(3,7)$.

*Proof.* 1. Denote by $\rho_s(\sigma)$ the action on $\mathbb{Q}(\sqrt{d}) \otimes_{\mathbb{Q}} A$ given by: $\rho_s(\sigma)(\lambda \otimes x_i) = \sigma(\lambda) \otimes x_{s(i)}$ for all $i = 0, \ldots, 4$. The invariants in $\mathbb{Q}(\sqrt{d})[x_0, x_1, x_2, x_3, x_4]$ under this action are generated by $y_0 = x_0 + x_1$, $y_1 = \sqrt{d}(x_0 - x_1)$, $y_2 = x_2$, $y_3 = \sqrt{d}(x_2 - x_3)$ and $y_4 = x_4$. One can write the $x_i$'s in terms of the $y_i$'s:

$$x_0 = \frac{y_0}{2} + \frac{y_1}{2\sqrt{d}}, x_1 = \frac{y_0}{2} - \frac{y_1}{2\sqrt{d}}, x_2 = \frac{y_2}{2} + \frac{y_3}{2\sqrt{d}}, x_3 = \frac{y_2}{2} - \frac{y_3}{2\sqrt{d}}, x_4 = y_4.$$

and substitute them in the equations $\sum_{j=0}^{4} x_j = 0$ and $\sum_{j=0}^{4} x_j^3 = 0$, obtaining the new equations in terms of the $y_i s$ of the twisted surface:

$$\begin{cases} \dfrac{y_0^3}{4} + \dfrac{3y_0y_1^2}{8d} + \dfrac{y_2^3}{4} + \dfrac{3y_2y_3^2}{8d} + y_4^3 = 0 \\ y_0 + y_2 + y_4 = 0. \end{cases}$$

2. In the same manner as in the previous case, we now calculate the number of lines of $S^{twist}$ that are defined over $\mathbb{Q}$, i.e., the lines $l \in S$ such that $\rho_s(\tilde{\sigma})(l) = l$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Again, we distinguish the following cases:

   (a) Suppose that $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{5})$. One must again distinguish the following cases:

      i. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} \neq id$. In this case, the action sends $\sqrt{5}$ to $-\sqrt{5}$ and interchanges $x_0 \leftrightarrow x_1$ and $x_2 \leftrightarrow x_3$. The action on the rational lines is the following: $\rho_s(\tilde{\sigma})(\ell_{i,jk}) = \ell_{s(i),s(j)s(k)}$. Hence, the rational lines fixed by this action are: $\ell_{4,01}, \ell_{4,02}, \ell_{4,03}$.
      One can easily see that there are no lines of type $\Delta_{(j,k,m,n)}$ that remain fixed under the action of $\rho_s(\tilde{\sigma})$.
      Hence, in the remaining cases we will only look at the action of $\rho_s(\tilde{\sigma})$ on the rational lines.

      ii. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} = id$. In this case, the action on the rational lines is the same as in previous case. That is, the action fixes the lines $\ell_{4,01}, \ell_{4,02}, \ell_{4,03}$.

      iii. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} \neq id$. The action sends $\sqrt{5}$ to $-\sqrt{5}$ and leaves the coordinates $x_i$ invariant. Hence, $\rho_s(\tilde{\sigma})$ fixes all 15 rational lines $\ell_{i,jk}$.

      iv. $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{5})} = id$. Again, $\rho_s(\tilde{\sigma})$ fixes all rational lines.

   The lines on $S$ that are fixed by $\rho_s(\tilde{\sigma})$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are the three lines $\ell_{4,01}, \ell_{4,02}, \ell_{4,03}$. There are 7 tritangents that are fixed by the action, namely the ones corresponding to the pairs: $\{\Delta_{(0,1,2,3)}, \Delta_{(2,3,1,0)}\}$, $\{\Delta_{(0,2,3,1)}, \Delta_{(0,2,1,3)}\}$, $\{\Delta_{(0,2,1,3)}, \Delta_{(0,3,2,1)}\}$, $\{\Delta_{(1,2,0,3)}, \Delta_{(1,2,3,0)}\}$, $\{\Delta_{(1,3,2,0)}, \Delta_{(1,3,0,2)}\}$, $\{\Delta_{(2,3,0,1)}, \Delta_{(0,1,3,2)}\}$, $\{\ell_{4,01}, \ell_{4,02}\}$.

   (b) Suppose that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{5})$. Proceeding in the same way as in the case of a 2-cycle, we see that we obtain the same result as above, namely, the twisted surface has 3 rational lines and 7 rational tritangents.

$\square$

**Twists of the Clebsch diagonal surface over $\mathbb{R}$**

If we consider $S$ as a cubic surface over $\mathbb{R}$, its twist over $\mathbb{R}$ are, completely analogously, in bijection with conjugate classes of homomorphisms $h : \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow S_5$. For the case when $S$ is the Clebsch diagonal surface, one obtains that:

**Proposition 6.** *There are precisely three quadratic twists over $\mathbb{R}$ for the Clebsch diagonal surface. They are the following, namely*

1. *The trivial twist $S$. It has $(27, 45)$ lines and tritangents over $\mathbb{R}$.*
2. *The twisted surface*

$$S^{(01)} = \begin{cases} \dfrac{y_0^3}{4} - 3\dfrac{y_0 y_1^2}{8} + y_2^3 + y_3^3 + y_4^3 = 0 \\ y_0 + y_2 + y_3 + y_4 = 0. \end{cases}$$

   *This surface has $(3, 13)$ lines and tritangents over $\mathbb{R}$.*

3. *The twisted surface*

$$S^{(01)(23)} = \begin{cases} \dfrac{y_0^3}{4} - \dfrac{3y_0 y_1^2}{8} + \dfrac{y_2^3}{4} - \dfrac{3y_2 y_3^2}{8} + y_4^3 = 0 \\ y_0 + y_2 + y_4 = 0. \end{cases}$$

   *This surface has $(3, 7)$ lines and tritangents over $\mathbb{R}$.*

**Cyclic extension of degree 5. Case $\langle(0, 1, 2, 3, 4)\rangle$.**

Consider the homomorphism $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \langle(0, 1, 2, 3, 4)\rangle \leq S_5$. We look for fields $K$ such that $h$ factors as $h = cf$ in the following manner:

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{h} & \langle(0,1,2,3,4)\rangle \leq S_5 \\ res \searrow & & \nearrow c \\ & \mathrm{Gal}(K/\mathbb{Q}) & \end{array}$$

where $c$ is an isomorphism. Since $\langle(0, 1, 2, 3, 4)\rangle$ is cyclic of order 5, the field $K$ satisfies $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(e^{\frac{2\pi i}{n}})$ with $n$ such that $5 \mid \varphi(n)$, and $\mathbb{Q} \subseteq K$ a Galois extension of degree 5. We choose $n = 11$ so $K = \mathbb{Q}(\cos(\frac{2\pi}{11})) = \mathbb{Q}(\frac{\zeta+\zeta^{-1}}{2}) = \mathbb{Q}(\zeta + \zeta^{-1})$, with $\zeta$ a primitive 11-th root of unity.

An automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ of order 5 is given by $\sigma(\zeta) = \zeta^2$. The restriction of $\sigma$ to $\mathrm{Gal}(\mathbb{Q}(\frac{\zeta+\zeta^{-1}}{2})/\mathbb{Q})$ generates this Galois group. We put $c(\sigma) = (0, 1, 2, 3, 4)$.

In the next proposition, we give the equations for the twist of $S$ and the number of rational lines and rational tritangents. The example shows that it is possible to obtain complicated examples from simple ones.

**Proposition 7.** *Let $S^{twist}$ be the twisted surface of the Clebsch $S$ by $h$ described above. Then,*

1. *$S^{twist}$ is given by the equations:*

$$S^{twist} = \begin{cases} y_0 + y_1 + y_2 + y_3 + y_4 = 0 \\ -4y_0^3 - y_1^3 - y_2^3 - y_3^3 - y_4^3 + 7y_0^2y_1 - 4y_0^2y_2 - 4y_0^2y_3 - 4y_0^2y_4 \\ -2y_0y_1^2 - y_1^2y_2 - y_1^2y_3 - y_1^2y_4 - y_1y_2^2 - y_2^2y_3 - y_2^2y_4 - 2y_0y_3^2 \\ -y_1y_3^2 - y_2y_3^2 - y_4y_3^2 - 2y_0y_4^2 - y_1y_4^2 - y_2y_4^2 - y_3y_4^2 \\ -2y_0y_1y_2 - 2y_0y_1y_3 + 9y_0y_1y_4 + 9y_0y_2y_3 - 2y_0y_2y_4 \\ -2y_0y_3y_4 - y_1y_2y_3 - y_1y_2y_4 - y_1y_3y_4 - y_2y_3y_4 = 0. \end{cases}$$

2. *The number of rational lines and rational tritangents on $S^{twist}$ is* $(0, 0)$.

*Proof.* 1. We have a new action $\rho_h(\sigma)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} A$ as follows:

$$\rho_h(\sigma)(\zeta \otimes x_i) = \sigma(\zeta) \otimes c(\sigma)(x_i) = \zeta^2 \otimes x_{c(\sigma)(i)},$$

with $\gamma = \zeta+\zeta^{-1}$. We now look for the invariants in $\mathbb{Q}(\zeta+\zeta^{-1})[x_0, x_1, x_2, x_3, x_4]$ under the action of $\rho_h(\sigma)$. These invariants will be generated over $\mathbb{Q}$ by forms $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4$ satisfying:

$$\begin{aligned} a_0x_0 + \cdots + a_4x_4 = \rho_h(\sigma)(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4) = \\ \sigma(a_0)x_1 + \sigma(a_1)x_2 + \sigma(a_2)x_3 + \sigma(a_3)x_4 + \sigma(a_4)x_0. \end{aligned}$$

I.e., we look for $a_0, a_1, a_2, a_3, a_4$ such that $\sigma(a_0) = a_1$, $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, $\sigma(a_3) = a_4$, $\sigma(a_4) = a_0$.

One can verify that the relation between the $x_i$'s and suitable invariants $y_i$'s is the following:

$$\begin{aligned} x_0 &= \gamma y_0 + \sigma(\gamma)y_1 + \sigma^2(\gamma)y_2 + \sigma^3(\gamma)y_3 + \sigma^4(\gamma)y_4 \\ x_1 &= \sigma(\gamma)y_0 + \sigma^2(\gamma)y_1 + \sigma^3(\gamma)y_2 + \sigma^4(\gamma)y_3 + \gamma y_4 \\ x_2 &= \sigma^2(\gamma)y_0 + \sigma^3(\gamma)y_1 + \sigma^4(\gamma)y_2 + \gamma y_3 + \sigma(\gamma)y_4 \\ x_3 &= \sigma^3(\gamma)y_0 + \sigma^4(\gamma)y_1 + \gamma y_2 + \sigma(\gamma)y_3 + \sigma^2(\gamma)y_4 \\ x_4 &= \sigma^4(\gamma)y_0 + \gamma y_1 + \sigma(\gamma)y_2 + \sigma^2(\gamma)y_3 + \sigma^3(\gamma)y_4 \end{aligned}$$

where $\gamma = \zeta + \zeta^{-1}$, $\sigma(\zeta) = \zeta^2$ and the determinant of the matrix of the coefficients is non-zero.

The equation $\sum_{i=0}^4 x_i = 0$ gives $-y_0 - y_1 - y_2 - y_3 - y_4 = 0$. Let us see what happens when we calculate $\sum_{i=0}^4 x_i^3 = 0$. One has for instance that $x_0^3 = \sum_{0\le i,j,k\le 4} \sigma^i(c)\sigma^j(c)\sigma^k(c) y_i y_j y_k$. Hence,

$$\sum_{i=0}^4 x_i^3 = \sum_{0\le i,j,k\le 4} trace(\sigma^i(c)\sigma^j(c)\sigma^k(c)) y_i y_j y_k.$$

To obtain the explicit new equation, one must calculate every coefficient of $y_i y_j y_k$. We recall that $trace_{\mathbb{Q}(\zeta+\zeta^{-1})/\mathbb{Q}}(x) = \frac{1}{2} trace_{\mathbb{Q}(\zeta)/\mathbb{Q}}(x)$ and that $trace_{\mathbb{Q}(\zeta)/\mathbb{Q}}(c) = \sum_{\sigma\in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \sigma(c)$.

For instance, the coefficient of $y_0^3$ equals $trace(c^3) = trace(\zeta^3 + 3\zeta + 3\zeta^{-1} + \zeta^{-3}) = \frac{1}{2}(-1-3-3-1) = -4$. In a similar manner, the rest of the coefficients can be calculated. The final equation is:

$$\begin{array}{l}
-4y_0^3 - y_1^3 - y_2^3 - y_3^3 - y_4^3 + 7y_0^2y_1 - 4y_0^2y_2 - 4y_0^2y_3 - 4y_0^2y_4 \\
-2y_0y_1^2 - y_1^2y_2 - y_1^2y_3 - y_1^2y_4 - y_1y_2^2 - y_2^2y_3 - y_2^2y_4 - 2y_0y_3^2 \\
-y_1y_3^2 - y_2y_3^2 - y_4y_3^2 - 2y_0y_4^2 - y_1y_4^2 - y_2y_4^2 - y_3y_4^2 - 2y_0y_1y_2 \\
-2y_0y_1y_3 + 9y_0y_1y_4 + 9y_0y_2y_3 - 2y_0y_2y_4 - 2y_0y_3y_4 - y_1y_2y_3 \\
-y_1y_2y_4 - y_1y_3y_4 - y_2y_3y_4 = 0.
\end{array}$$

2. The rational lines of the new surface will be those lines $\ell \in S$ such that $\rho_h(\tilde\sigma)(\ell) = \ell$ for all $\tilde\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The first group of 15 lines are of the form:

$$\ell_{i,jk} = \left\{ \begin{array}{l} x_i = 0 \\ x_j + x_k = 0. \end{array} \right.$$

In this case,

$$\rho_h(\tilde\sigma)(\ell_{i,jk}) = \left\{ \begin{array}{l} x_{c(\sigma)(i)} = 0 \\ x_{c(\sigma)(j)} + x_{c(\sigma)(k)} = 0 \end{array} \right.$$

with $\tilde\sigma$ such that $\tilde\sigma|_{\mathbb{Q}(\zeta)}$ is $\sigma$. Since $c(\sigma) = (0,1,2,3,4)$, we have that $\rho_h(\ell_{i,jk}) \neq \ell_{i,jk}$ for all $i,j,k \in \{0,1,2,3,4\}$. Thus, there are no rational lines on the new surface coming from this group.

$\rho_h(\tilde{\sigma})$ acts on the double six formed by the 12 other lines of the Clebsch. One can easily see that $\rho_h(\tilde{\sigma})(D)$ equals

$$\begin{pmatrix} \Delta_{(2,3,0,1)} & \Delta_{(1,2,0,3)} & \Delta_{(0,3,1,2)} & \Delta_{(1,3,2,0)} & \Delta_{(0,1,2,3)} & \Delta_{(0,2,3,1)} \\ \Delta_{(2,3,1,0)} & \Delta_{(1,2,3,0)} & \Delta_{(0,3,2,1)} & \Delta_{(1,3,0,2)} & \Delta_{(0,1,3,2)} & \Delta_{(0,2,1,3)} \end{pmatrix},$$

i.e., there are no rational lines of this type either.

Hence, the twisted surface contains no rational lines.

Since there are no rational lines there are also no rational tritangents: indeed, if there was one, then the Galois group of $\overline{\mathbb{Q}}/\mathbb{Q}$ would cyclically permute the three lines on it; hence so would $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta))$. This is impossible, since over $\mathbb{Q}(\zeta)$, $S^{twist}$ and $S$ become isomorphic.

□

We complete our treatment of the Clebsch diagonal surface by showing:

**Theorem 15.** *There is no form of the Clebsch with all the* 27 *lines defined over* $\mathbb{Q}$.

*Proof.* Let $S$ be the Clebsch cubic surface. Consider $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}\,(S) = S_5$ a homomorphism that defines $S^{twist}$ with all lines defined over $\mathbb{Q}$. We recall that a line $\ell \subseteq S$ is rational in $S^{twist}$ if and only if $\rho_h(\sigma)(\ell) = \ell$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Therefore $h(\sigma)$ fixes all lines on $S$ defined over $\mathbb{Q}$. In particular, $h(\sigma)$ fixes the five rational lines given by: $\ell_1 : (0 : \lambda : -\lambda : \mu : -\mu)$, $\ell_2 : (\lambda : 0 : -\lambda : \mu : -\mu)$, $\ell_3 : (\lambda : -\lambda : 0 : \mu : -\mu)$, $\ell_4 : (\lambda : -\lambda : \mu : 0 : -\mu)$, $\ell_5 : (\lambda : -\lambda : \mu : -\mu : 0)$. Then, $c(\sigma)$ is the identity and $S = S^{twist}$ has not all lines defined over $\mathbb{Q}$, contradicting the assumption on $S^{twist}$.

□

**Remark:** In [Swin], Swinnerton-Dyer formulates a necessary and sufficient criterion for a cubic surface $S$ defined over a number field $K$ to be rational over $K$ to $\mathbb{P}^2$: $S$ should contain a $\mathrm{Gal}(\overline{K}/K)$-stable set of $n$ skew lines for $n \in \{2, 3, 6\}$. Using the methods of this section, this criterion can be checked for forms of the Clebsch diagonal surface over $\mathbb{Q}$. We have not done this yet.

### 2.4.4 Twists of the Fermat

We recall that the Fermat cubic is the smooth cubic surface given by the following polynomial in $\mathbb{P}^3$:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3.$$

Consider the vector space $V = \mathbb{C}e_0 + \cdots + \mathbb{C}e_3$. The group $S_4$ acts on $V$ as follows: $\pi \in S_4$ acts by $\pi e_i = e_{\pi(i)}$ for $i = 0, \ldots, 3$. The group

$$N = \{\zeta = (\zeta_0, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^4 \mid \zeta_i^3 = 1 \text{ for all } i \text{ and } \zeta_0 \ldots \zeta_3 = 1\}$$

acts on $V$ by $\zeta e_i = \zeta_i e_i$ for $i = 0, \ldots, 3$. The group $G$ of the automorphisms is the semidirect product $N \rtimes S_4$. The group structure is given by

$$\pi(\zeta_0, \ldots, \zeta_3) = (\zeta_{\pi^{-1}(0)}, \ldots, \zeta_{\pi^{-1}(3)})\pi.$$

Any element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $G$ by

$$\sigma((\zeta_0, \ldots, \zeta_3), \pi) = ((\sigma(\zeta_0), \ldots, \sigma(\zeta_3)), \pi).$$

One has the following result:

**Lemma 4.** *Fix a quadratic field $\mathbb{Q}(\sqrt{d})$. There are precisely three quadratic twists over $\mathbb{Q}$ for the Fermat cubic $S$ which become isomorphic to $S$ over $\mathbb{Q}(\sqrt{d})$. They are given by the homomorphisms: $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow S_4$ with images the subgroups: $< (0,1) >$, $< (0,1)(2,3) >$ and $< (1) >$.*

*Proof.* A 1-cocycle $c : \mathrm{Gal}(\mathbb{Q}(\sqrt{d}/\mathbb{Q}) = \{1, \sigma\} \longrightarrow G$ is given by $c(\sigma) = \zeta\pi \in G$ satisfying $c(\sigma)\sigma(c(\sigma)) = 1$. In particular, $\pi$ has order 1 or 2 and we may suppose that $\pi \in \{1, (01), (01)(23)\}$. Now, we have to distinguish some cases:

1. $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)$, where $\omega$ is a primitive third root of unity.

    - Case $\pi = 1$. Now $c(\sigma) = (\zeta_0, \ldots, \zeta_3)$ is any element of $N$. An equivalent cocycle $c'$ has the form $c'(\sigma) = h^{-1}c(\sigma)\sigma(h)$. One can choose $h \in N$ such that $c'(\sigma) = 1$. Indeed, this follows from $(\omega^2)^{-1}\omega\sigma(\omega^2) = 1$ and $(\omega)^{-1}\omega^2\sigma(\omega) = 1$. In other words, this subcase produces as twist $S$ itself.

- Case $\pi = (01)$. In this case $c(\sigma) = \zeta\pi = (\zeta_0, \ldots, \zeta_3)\pi$ and $(\zeta_0, \ldots, \zeta_3)\pi(\sigma(\zeta_0), \ldots, \sigma(\zeta_3))\pi^{-1} = 1$. This implies that $(\zeta_0\sigma(\zeta_1), \zeta_1\sigma(\zeta_0), \zeta_2^2, \zeta_3^2) = 1$. Hence $\zeta_2 = \zeta_3 = 1$ and $\zeta_0\zeta_1 = 1$ and $\sigma(\zeta_0) = \zeta_0$, $\sigma(\zeta_1) = \zeta_1$. Hence, $(\zeta_0, \ldots, \zeta_3) = 1$.
- Case $\pi = (01)(23)$. Now $c(\sigma) = \zeta\pi = (\zeta_0, \ldots, \zeta_3)\pi$ and $(\zeta_0\sigma(\zeta_1), \zeta_1\sigma(\zeta_0), \zeta_2\sigma(\zeta_3), \zeta_3\sigma(\zeta_2)) = 1$. The possibilities for $\zeta$ are $1$,$(\omega, \omega, \omega^2, \omega^2)$,$(\omega^2, \omega^2, \omega, \omega)$. Consider the equivalent 1-cocycle $c'$ given by $c'(\sigma) = h^{-1}c(\sigma)\sigma(h)$ for some $h = (h_0, \ldots, h_3) \in N$. Now $c'(\sigma)$=$(h_0, h_1, h_2, h_3)^{-1}\zeta\pi(\sigma h_0, \sigma h_1, \sigma h_2, \sigma h_3)\pi^{-1}\pi$ and we would like $(h_0, h_1, h_2, h_3)^{-1}\zeta(\sigma h_1, \sigma h_0, \sigma h_3, \sigma h_2) = 1$. For $\zeta = 1$ we take $h = 1$. For $\zeta = (\omega, \omega, \omega^2, \omega^2)$ we take $h = (\omega, 1, \omega^2, 1)$. For $\zeta = (\omega^2, \omega^2, \omega, \omega)$ we take $h = (\omega^2, 1, \omega, 1)$.

  We have therefore seen that there are three twists for $S$ over the field $\mathbb{Q}(\sqrt{-3})$.

2. $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$. The Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}) = \{1, \sigma\}$ acts trivially on $G$. A 1-cocycle is now determined by $c(\sigma) = \zeta\pi$ and has to satisfy $\zeta\pi\zeta\pi^{-1}\pi^2 = 1$.

   - Case $\pi = 1$. Then, $1 = \zeta\pi\zeta\pi^{-1} = \zeta^2$ implies that $\zeta = 1$. In this case, the twist is just $S$ itself.
   - Case $\pi = (01)$. Then, $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)(\zeta_1, \zeta_0, \zeta_2, \zeta_3) = 1$ is equivalent to $\zeta_0\zeta_1 = \zeta_2 = \zeta_3 = 1$. This leads the possibilities $\zeta = 1, (\omega, \omega^2, 1, 1), (\omega^2, \omega, 1, 1)$. We will produce $h \in N$ such that the equivalent 1-cocycle $c'$ given by $c'(\sigma) = h^{-1}c(\sigma)h$ satisfies $c'(\sigma) = \pi$. In this first case $h = 1$ works. In case $\zeta = (\omega, \omega^2, 1, 1)$, $h = (\omega, 1, \omega^2, 1)$ works. In case $\zeta = (\omega^2, \omega, 1, 1)$, $h = (\omega^2, 1, \omega, 1)$ works.
   - Case $\pi = (01)(23)$. Then $c(\sigma) = \zeta\pi$ and $\zeta = (\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ satisfies $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)(\zeta_1, \zeta_0, \zeta_3, \zeta_2) = 1$. As before one can find $h \in N$ such that $c'(\sigma) = h^{-1}c(\sigma)h = \pi$. Consider for example $\zeta = (\omega, \omega^2, \omega, \omega^2)$. Then $h = (1, \omega^2, \omega, 1)$ works. The other cases are similar. We conclude that there are three twists of $S$ over the field $\mathbb{Q}(\sqrt{d})$.

$\square$

We proceed to calculate the quadratic twists of the Fermat cubic surface. First, we describe the 27 lines and the 45 tritangents on the Fermat cubic as follows:

1. The 27 lines on the Fermat can be given in three groups as:

$$\begin{cases} L_{0,m,n} = \{(-\omega^m : -\omega^n z : z : 1) \mid z \in \mathbb{C})\} \\ L_{1,m,n} = \{(-\omega^n z : -\omega^m : z : 1) \mid z \in \mathbb{C})\} \\ L_{2,m,n} = \{(-\omega^n z : z : -\omega^m : 1) \mid z \in \mathbb{C})\}, \end{cases}$$

   with $n, m \in \mathbb{Z}/3\mathbb{Z}$.

2. The 45 tritangents are:

   - 27 planes given by the polynomials:

$$P_{a,b,c} : x_0 + \omega^a x_1 + \omega^b x_2 + \omega^c x_3 \ (a, b, c \in \mathbb{Z}/3\mathbb{Z}),$$

   - 18 planes given by:

$$Q_{n,m,a} : x_n + \omega^a x_m = 0, \text{ with } n < m \text{ and } a \in \mathbb{Z}/3\mathbb{Z}.$$

We deduce from this that the number of rational lines and rational tritangents on the Fermat cubic surface is $(3, 7)$.

**Proposition 8.** *Let $S^{twist}$ be the twist surface of the Fermat surface $S$ by $c(\sigma) = (0, 1)$. Then, an equation for $S^{twist}$ is:*

$$2(\frac{y_0}{2})^3 + 3\frac{y_0 y_1^2}{4d} + y_2^3 + y_3^3 = 0.$$

*The number of rational lines and rational tritangents on $S^{twist}$ is*

1. $(1, 3)$ *if* $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$.
2. $(3, 7)$ *if* $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$.

*Proof.* One can calculate the invariants of $\mathbb{Q}(\sqrt{d})x_0 + \mathbb{Q}(\sqrt{d})x_1 + \mathbb{Q}(\sqrt{d})x_2 + \mathbb{Q}(\sqrt{d})x_3$ under the action that sends $\sqrt{d}$ to $-\sqrt{d}$ and interchanges $x_0$ and $x_1$. They are generated by $y_0 = x_0 + x_1$, $y_1 = \sqrt{d}(x_0 - x_1)$, $y_2 = x_2$ and

$y_3 = x_3$. By substituting the invariants in the equation $\sum_{j=0}^{3} x_j^3 = 0$ we obtain the new equation:

$$2(\frac{y_0}{2})^3 + 3\frac{y_0 y_1^2}{4d} + y_2^3 + y_3^3 = 0.$$

We proceed to calculate the number or rational lines and rational tritangents on the twisted surface.

1. Suppose that $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$. We look for lines $\ell \subseteq S$ such that $\rho_c(\tilde{\sigma})(\ell) = \ell$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One must distinguish the following cases:

   (a) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} \neq id$. In this case, the action sends $\omega$ to $\omega^2$ and interchanges $x_0 \leftrightarrow x_1$.

   - The action on the lines is the following: $\rho_c(\tilde{\sigma})$ interchanges $L_{0,m,n} \leftrightarrow L_{1,-m,-n}$ and $L_{2,m,n} \leftrightarrow L_{2,-m,n}$. The lines fixed by this action are: $L_{2,0,0}, L_{2,0,1}, L_{2,0,2}$.
   - The action fixes the 3 planes $P_{0,0,0}$, $P_{2,1,1}$ and $P_{1,2,2}$ and the 6 planes $Q_{0,1,a}$ and $Q_{2,3,a}$ for $a, b \in \mathbb{Z}/3\mathbb{Z}$.

   (b) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = \sigma$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} = id$. In this case, the action leaves $\omega$ invariant and interchanges $x_0 \leftrightarrow x_1$.

   - The action on the lines is: $\rho_c(\tilde{\sigma})$ interchanges $L_{0,m,n} \leftrightarrow L_{1,m,n}$ and $L_{2,m,n} \leftrightarrow L_{2,m,-n}$. The lines fixed by this action are: $L_{2,0,0}, L_{2,1,0}, L_{2,2,0}$.
   - The action fixes the 18 planes $P_{0,b,c}$ and the 4 planes $Q_{0,1,0}$ and $Q_{2,3,a}$.

   (c) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} \neq id$. In this case, the action sends $\omega$ to $\omega^2$ and leaves the coordinates $x_i$'s invariant.

   - The action on the lines is: $\rho_c(\tilde{\sigma})$ interchanges $L_{0,m,n} \leftrightarrow L_{0,-m,-n}$, $L_{1,m,n} \leftrightarrow L_{1,-m,-n}$ and $L_{2,m,n} \leftrightarrow L_{2,-m,-n}$. The lines fixed by this action are: $L_{0,0,0}, L_{1,0,0}, L_{2,0,0}$.
   - The action fixes the plane $P_{0,0,0}$ and the 6 planes $Q_{n,m,0}$.

   (d) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{d})} = id$ and $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} = id$. In this case, $\rho_c(\tilde{\sigma})$ fixes all the lines and tritangents.

The only line fixed by $\rho_c(\tilde{\sigma})$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the line $L_{2,0,0}$. There are 3 tritangents fixed by the action, namely the planes: $P_{0,0,0}$, $Q_{0,1,0}$ and $Q_{2,3,0}$.

**Remark:** Although it is not necessary, one can check that these are indeed rational tritangents in the twisted surface by interchanging the $x_i$'s with the $y_i$'s in the equations of the tritangents and observing that the result is a rational tritangent. Indeed, in this way one obtains the tritangents: $y_0 + y_2 + y_3 = 0$, $y_0 = 0$, $y_2 + y_3 = 0$ and the line $(0 : z : -1 : 1)$.

2. Suppose that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$. We repeat the reasoning and distinguish two cases:

   (a) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} \neq id$. In this case, the action sends $\omega$ to $\omega^2$ and interchanges $x_0 \leftrightarrow x_1$.

   - The action on the lines is: $\rho_c(\tilde{\sigma})$ interchanges $L_{0,m,n} \leftrightarrow L_{1,-m,-n}$ and $L_{2,m,n} \leftrightarrow L_{2,-m,n}$. The lines fixed by this action are: $L_{2,0,0}, L_{2,0,1}, L_{2,0,2}$.
   - The action fixes the 3 planes $P_{0,0,0}$, $P_{2,1,1}$ and $P_{1,2,2}$ and the 4 planes $Q_{0,1,a}$ and $Q_{2,3,0}$.

   (b) $\tilde{\sigma}|_{\mathbb{Q}(\sqrt{-3})} = id$. The action fixes all the lines and tritangents on $S$.

   In total, one has that $\rho_c(\tilde{\sigma})$ fixes the 3 lines $L_{2,0,0}, L_{2,0,1}, L_{2,0,2}$ and the 7 tritangents $P_{0,0,0}$, $P_{2,1,1}$, $P_{1,2,2}$, $Q_{0,1,0}$, $Q_{0,1,1}$, $Q_{0,1,2}$, $Q_{2,3,0}$. .

□

**Proposition 9.** *Let $S^{twist}$ be the twist surface of the Fermat surface $S$ by $c(\sigma) = (0,1)(2,3)$. Then, an equation for $S^{twist}$ is:*

$$2(\frac{y_0}{2})^3 + 3\frac{y_0 y_1^2}{4d} + 2(\frac{y_2}{2})^3 + 3\frac{y_2 y_3^2}{4d} = 0.$$

*The number of rational lines and rational tritangents on $S^{twist}$ is*

1. $(3,3)$ *if* $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$.
2. $(15,15)$ *if* $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$.

*Proof.* One can calculate the invariants of $\mathbb{Q}(\sqrt{d})x_0+\mathbb{Q}(\sqrt{d})x_1+\mathbb{Q}(\sqrt{d})x_2+\mathbb{Q}(\sqrt{d})x_3$ under the action that sends $\sqrt{d}$ to $-\sqrt{d}$ and interchanges $x_0 \leftrightarrow x_1$ and $x_2 \leftrightarrow x_3$. They are generated by $y_0 = x_0 + x_1$, $y_1 = \sqrt{d}(x_0 - x_1)$, $y_2 = x_2 + x_3$ and $y_3 = \sqrt{d}(x_2 - x_3)$. By substituting the invariants in the equation $\sum_{j=0}^{3} x_j^3 = 0$ we obtain the new equation:

$$2(\frac{y_0}{2})^3 + 3\frac{y_0 y_1^2}{4d} + 2(\frac{y_2}{2})^3 + 3\frac{y_2 y_3^2}{4d} = 0.$$

1. Suppose that $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{-3})$. As in the previous case, one must calculate the action of $\rho_c(\tilde{\sigma})$ on the lines on $S$. Again, we distinguish the same cases as above:

   (a) In this case, $\rho_c(\tilde{\sigma})$ interchanges $L_{0mn} \leftrightarrow L_{0,-n,-m}$, $L_{1,m,n} \leftrightarrow L_{1,-n,-m}$ and $L_{2,m,n} \leftrightarrow L_{2,m,n}$. The lines fixed by this action are: $L_{0,0,0}$, $L_{0,1,2}$, $L_{0,2,1}$, $L_{1,0,0}$, $L_{1,1,2}$, $L_{1,2,1}$ and all the lines of type $L_{2,m,n}$. The action fixes the 9 planes $P_{a,a-b,b}$ and the two planes $Q_{0,1,0}$ and $Q_{2,3,0}$.

   (b) The action $\rho_c(\tilde{\sigma})$ interchanges $L_{0,m,n} \leftrightarrow L_{0,n,m}$, $L_{1,m,n} \leftrightarrow L_{1,n,m}$ and $L_{2,m,n} \leftrightarrow L_{2,-m,-n}$. The lines fixed by this action are: $L_{0,0,0}$, $L_{0,1,1}$, $L_{0,2,2}$, $L_{1,0,0}$, $L_{1,1,1}$, $L_{1,2,2}$ and $L_{2,0,0}$. The fixed tritangents are: $P_{0,b,b}$, $Q_{0,1,0}$ and $Q_{2,3,0}$.

   (c) The action interchanges $L_{0mn} \leftrightarrow L_{0,-m,-n}$, $L_{1,m,n} \leftrightarrow L_{1,-m,-n}$ and $L_{2,m,n} \leftrightarrow L_{2,-m,-n}$. The lines fixed by this action are: $L_{0,0,0}$, $L_{1,0,0}$ and $L_{2,0,0}$. The planes fixed are: $P_{0,0,0}$ and $Q_{n,m,0}$.

   (d) $\rho_c(\tilde{\sigma})$ fixes all lines and tritangents.

   The lines fixed by $\rho_c(\tilde{\sigma})$ for all $\tilde{\sigma} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ are the lines $L_{0,0,0}$,$L_{1,0,0}$ and $L_{2,0,0}$. The 3 tritangents fixed are $P_{0,0,0}$, $Q_{0,1,0}$ and $Q_{2,3,0}$.

2. Suppose that $\mathbb{Q}(\sqrt{d}) = \mathbb{Q}(\sqrt{-3})$. By the same reasoning as above, one has that the action fixes the 15 lines: $L_{0,0,0}$,$L_{0,1,2}$,$L_{0,2,1}$,$L_{1,0,0}$,$L_{1,1,2}$,$L_{1,2,1}$ and all the lines of type $L_{2,m,n}$ and the 15 planes: $P_{1,0,1}$, $P_{2,0,2}$, $P_{0,0,0}$, $P_{1,1,0}$, $P_{2,1,1}$, $P_{0,1,2}$, $P_{2,2,0}$, $P_{0,2,1}$, $P_{1,2,2}$, $Q_{0,1,0}$, $Q_{0,1,1}$, $Q_{0,1,2}$, $Q_{2,3,0}$, $Q_{2,3,1}$, $Q_{2,3,2}$.

□

**Twists of the Fermat cubic surface over $\mathbb{R}$**

Viewing the Fermat cubic $S$ as a surface over $\mathbb{R}$, one obtains in the same way twists $S^{(01)}$ and $S^{(01)(23)}$ over $\mathbb{R}$ corresponding to $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to G$ with image generated by $(1,(01))$ resp. $(1,(01)(23))$.

**Proposition 10.** *The twists of the Fermat cubic $S$ over $\mathbb{R}$ are described as follows.*

1. *The trivial twist $S$.*
2. *The twisted surface*
$$S^{(01)} = 2(\frac{y_0}{2})^3 - 3\frac{y_0 y_1^2}{4} + y_2^3 + y_3^3 = 0.$$
3. *The twisted surface*
$$S^{(01)(23)} = 2(\frac{y_0}{2})^3 - 3\frac{y_0 y_1^2}{4} + 2(\frac{y_2}{2})^3 - 3\frac{y_2 y_3^2}{4} = 0.$$

*In all three cases, the surface has $(3,7)$ lines and tritangents over $\mathbb{R}$.*

# Bibliography


[AKP] Artebani, A. and Kloosterman, R. and Pacini, M. A new model for the theta divisor of the cubic threefold, *Le Matematiche*, LVIII (2003), 201–236.

[Ba-Ho-Ne] Bajaj, C.L., Holt, R.L., Netravali, A.N. Rational parametrizations of nonsingular real cubic surfaces, *Transactions on Graphics*, vol. 17, nr. 1 (1998), 1–31.

[Be] Beauville, A. *Complex Algebraic Surfaces*, Cambridge University Press, 1983.

[B-P] Berry, T. G. and Patterson, R. R. Implicitization and parametrization of nonsingular cubic surfaces, *Computer Aided Geometric Design*, vol. 18, nr. 8, (2001), 723–738.

[Cl] Clebsch, A. Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5ten Grades und die geometrische Theorie des ebenen Fünfseits, *Math. Ann.*, vol. 4 (1871), 284–345.

[Cr] Cremona, L. Mémoire de géométrie pure sur les surfaces du troisiéme ordre, *Journ. des Mathématiques pures et appliquées*, vol. 68 (1868), 1–133.

[D] Dickson, L., *History of the Theory of Numbers*, Vol 2, Carnegie Institute of Washington, 1919.

[El] Elkies, N. www.math.harvard.edu/∼elkies/4cubes.html

[F] Fischer, G. *Mathematical Models*, Vieweg, 1986.

[Ha] Hartshorne, R. *Algebraic Geometry*, Springer-Verlag, 1977.

[HL] Holzer, S. and Labs, O. Illustrating the classification of real cubic surfaces, preprint, University of Mainz, 2005.

[Hu] Hunt, B. *The Geometry of some special Arithmetic Quotients*, Springer-Verlag, Lecture Notes in Math., vol. 1637, 1996.

[KM] Knörrer, H. and Miller, T. Topologische Typen reeller kubischer Flächen, *Math. Z.*, vol. 195 (1987), 51–67.

[K] Kollár, J. Real algebraic surfaces, preprint, arXiv:alg-geom/9712003, 1997.

[M] Manin, Yu.I. *Cubic Forms*, North-Holland, 1974, second expanded edition, 1986.

[McW-Sl] MacWilliams, F. J. and Sloane, N. J. A. *The theory of error correcting codes*, North-Holland, Amsterdam, 1977.

[Po-Za] Polo-Blanco, I. and Zalm, L. van der, *Inventory of the collection of mathematical models at the University of Groningen*, 2004. `http://www.math.rug.nl/models`

[Reid] Reid, M. *Undergraduate Algebraic Geometry*, Cambridge University Press, 1988.

[Sch] Schläfli, L. An attempt to determine the twenty-seven lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface, *Quart. J. pure appl. math.*, vol. 2 (1858), 110–120.

[Se] Segre, B. *The non-singular cubic surfaces: A new method of investigation with special reference to questions of reality*, Oxford, 1942.

[Sh] Shioda, T. Weierstrass Transformations and Cubic Surfaces, *Comment. Math. univ. St. Pauli*, vol. 44 (1995), 109–128.

[Si] Silhol, R. *Real Algebraic Surfaces*, Springer-Verlag, Lecture Notes in Math., vol. 1392, 1989.

[Sil] Silverman, J.H. *The Arithmetic of Elliptic Curves*, Springer-Verlag, Graduate Texts in Math., vol. 106, 1986.

[Swin] Swinnerton-Dyer, H.P.F. Applications of algebraic geometry to number theory, *Proc. 1969 summer inst. number theory*, *Proc. Symp. Pure Math*, *AMS* vol. 20 (1971), 1–52.

[Top] Top, J. Kubische gipsmodellen, *Structuur in schoonheid*, Vakantiecursus 2004, CWI, Amsterdam.

[Vlugt] van der Vlugt, M. An application of Elementary Algebraic Geometry in Coding Theory, *Nieuw Archief voor Wiskunde*, vol. 14 (1996), 159–165.