Theory and history of geometric models

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Introduction

If you visit the second floor of the department of mathematics at the University of Groningen, you may appreciate the beauty of some plaster and string models kept in a glass showcase at the coffee corner. A closer look at these models shows that they are accompanied with short mathematical explanations, which describe them as different types of algebraic surfaces, surfaces from differential geometry, curves etc. And there are more models. Stepping into the individual offices and taking a look in the library, one discovers a collection of more than 100 mathematical models in the department.

Model building started in the second half of the 19th century. Many new algebraic curves and surfaces were discovered, along with other objects in mathematics and physics. In order to be able to visualize such objects and their properties, the mathematicians started building models to make them concrete. An important example of building of models takes place in Germany. Mathematicians such as Felix Klein and Alexander von Brill were responsible for the design of many mathematical models. Between 1880 and 1935 the German companies L. Brill and M. Schilling distributed massively copies of these models to the universities in Europe and in America. The 1911 catalogue of M. Schilling [Sch] describes 40 series consisting of almost 400 models. A survey of this collection can be found in [F]. This two volume book includes photographs of many of these models, as well as a mathematical treatment for many of them.

In Chapter 1 we begin with an account of these German companies and, later on, we focus on the collection of models in The Netherlands. The last part of this chapter is dedicated to the collection of Groningen. For this collection we have tried to answer questions of the following kind: Who was responsible for buying the models? What was their purpose? Were the models used in lectures? Were the models appreciated for their artistic value? And, were they on display?

We found that the Groningen professor of geometry Pieter Hendrik Schoute was the one in charge of buying the models and that he probably used them during his lectures. As to the later use of the models, we have tried to find information by interviewing several alumni of the University of Groningen who studied mathematics between 1930 and 1960. In Section 1.4.4 we draw the conclusion that, although the artistic value of the models might still have been appreciated, their pedagogical purpose was almost forgotten already in the 30's. In view of the large purchase of models in 1908 by Schoute, we conjecture at the end of Chapter 1 that a major part of the collection of mathematical models was destroyed in 1906 in the fire of the Academy Building. The large purchase from 1908 was maybe intended to replace the previous collection.

In Chapter 1 we also describe how the inventory of the collection was carried out, and how the restoration of many of the models was done. A summary of this inventory can be found on the webpage [Po-Za], where a list of all the models appear along with photographs and short mathematical explanations. With regard to the restoration, we give an account of the work of the sculptor Cayetano Ramírez López, who restored in 2005 all plaster models in the Groningen collection. Many of the string models have been restored by Marius van der Put (professor of algebra and geometry).

From a scientific point of view, one soon realizes the complexity, diversity and depth of the mathematics behind the models. In Chapters 2, 3 and 4 we focus on the study of the mathematics of the curves and surfaces present in some of the series in Schilling's collection. In Chapter 2 we study cubic surfaces. George Salmon and Arthur Cayley proved in 1849 that a smooth cubic surface contains precisely 27 lines over the complex numbers. In 1872 Alfred Clebsch discovered the *diagonal surface* (also known as the Clebsch surface), which is given by the equations

$$\begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0. \end{cases}$$

This is a smooth cubic surface with the property that all the 27 lines are real. This surface is represented by the first model from Series VII in Schilling's collection. It is remarkable that this German model of the diagonal surface is not in the Groningen collection. For this reason, the sculptor Ramírez López was asked, in 2005, to construct for the Groningen collection a plaster reproduction of the original model of the diagonal surface.

In Section 2.2 we consider smooth cubic surfaces over the real numbers. Schläfli's classification of real cubic surfaces [Sch] in terms of the number of real lines and real tritangent planes is compared to the topological classification, described by H. Knoerrer and T. Miller [KM]. Explicit examples for this comparison are given.

In 1871 Alfred Clebsch proved that a smooth cubic surface can be obtained as the space of cubic curves in $\mathbb{P}^2$ that pass through six points in general position. In more modern terminology, Clebsch's result can be rephrased as follows: every smooth cubic surface can be obtained by blowing up $\mathbb{P}^2$ at six points in general position. In Section 2.3, an algorithm is presented that computes the blow down morphism of a cubic surface. This algorithm provides a new proof of the result of Clebsch. The explicit morphisms for the cases of the Clebsch diagonal surface and the Fermat cubic surface are calculated (the latter is defined by the equation $x_0^3+x_1^3+x_2^3+x_3^3=0$ in $\mathbb{P}^3$). In the last section of Chapter 2, we treat twists of surfaces and calculate explicit twists of the Clebsch surface and the Fermat surface over $\mathbb{Q}$.

In Chapter 3 we treat various models representing different types of curves. The first part of the chapter deals with models of cubic curves over $\mathbb{P}^2(\mathbb{R})$ as classified by Möbius in [M]. Newton had classified in 1704 the irreducible curves of degree 3 into five types. Finally, we study twists of curves and observe that Möbius' classification does not *behave well* with respect to twists, in other words, his classification is not consistent with twisting the curves.

The second part of the chapter studies two different groups of models of space curves. Both groups illustrate a classification of singular space curves into 8 types. A model in the first group of 8 objects consists of a wire representing the curve itself. A model in the second group of 16 objects consists of strings. These strings represent tangent lines to the space curve. Each of the 8 types of space curves appears in two different ways, depending on choices for the osculation plane of the space curve and its projective position.

Ruled surfaces in $\mathbb{P}^3$, defined as union of straight lines, were studied since the second half of the 19th century. Series XII in Schilling's catalogue [Sch] consists of 10 string models representing ruled surfaces of degree 4. The mathematical text explaining these models was written by Karl Rohn in 1904 [Ro], and presents an exhaustive and precise study of these surfaces. The real case gives rise to several surfaces, 10 of which are represented in the models from Series XII. In Chapter 4 we give a modern interpretation of the method used by Rohn to find the equations of the surfaces. Ruled surfaces of order four have earlier been classified by Ludwig Cremona [Cr] and Arthur Cayley [Ca] in 1868 and 1869 respectively.

Finally, Chapter 5 is devoted to the study of some cardboard models of polyhedra present in the Groningen collection. These models were constructed by the amateur mathematician Alicia Boole Stott at the end of the 19th century. Boole Stott rediscovered the six regular polytopes, commonly known by the names: hypertetrahedron, hypercube, hyperoctahedron, 24-cell, 120-cell and 600-cell. These objects are the four-dimensional analogous to the Platonic Solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron. The set of models made by Boole Stott are series of polyhedra, first increasing and then decreasing in size, that illustrate the three-dimensional sections of the 120-cell and 600-cell. These sections, obtained by intersecting the polytopes with a three-dimensional space, are treated in detail by Boole Stott in her publication [B-S]. The presence of these models in the University of Groningen is explained by the collaboration between Boole Stott and Pieter Hendrik Schoute. At the celebration of the 300th birthday of the University of Groningen, Boole Stott was awarded a honorary doctorate. In this chapter we give an account of Boole Stott's life and describe her main results on four-dimensional geometry.

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