On t-Motifs
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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2007

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Chapter 8

The Period Functor

8.1 A Tannakian Category of Periods

8.1.1 Let $L$ be a commutative ring, $A$ a commutative $L$-algebra and $B$ an $A$-algebra such that $A \rightarrow B$ is injective. Consider the category whose objects are triples $(V, W, \alpha)$ consisting of

- a projective and finitely generated $L$-module $V$;
- a projective $A$-module $W$;
- an isomorphism of $B$-modules $\alpha : V_B \rightarrow W_B$.

Morphisms in this category are pairs $(f, g)$ of an $L$-linear homomorphism $f : V_1 \rightarrow V_2$ and an $A$-linear homomorphism $g : W_1 \rightarrow W_2$ such that the obvious square commutes:

$$g_B \circ \alpha_1 = \alpha_2 \circ f_B.$$

Thus, $f$ determines $g_B$ and even $g$, since $A \rightarrow B$ is injective. It follows that $\text{Hom}((V_1, W_1, \alpha_1), (V_2, W_2, \alpha_2))$ is a sub-$L$-module of $\text{Hom}(V_1, V_2)$, and that the category we are dealing with is $L$-linear.

This category has a natural tensor product

$$(V_1, W_1, \alpha_1) \otimes (V_2, W_2, \alpha_2) \overset{\text{def}}{=} (V_1 \otimes_L V_2, W_1 \otimes_A W_2, \alpha_1 \otimes_B \alpha_2)$$

and the unit for tensor product is $(L, A, \text{id})$. The dual of a triple takes the contragredient $\alpha$, thus $(V, W, \alpha)^\vee \overset{\text{def}}{=} (V^\vee, W^\vee, (\alpha^\vee)^{-1})$. There is a
trace map from $(V, W, \alpha) \otimes (V, W, \alpha)^V$ to the unit $(L, A, \text{id})$:

$$\text{tr} : (v \otimes \lambda, w \otimes \mu) \mapsto (\lambda(v), \mu(w)).$$

This category of triples is an $L$-linear rigid tensor category and we shall denote it by $\mathcal{P}(L, A, B)$.

8.1.2. Such a category arises naturally in the context of periods of algebraic varieties.\(^{(1)}\) Fix a subfield $F$ of $\mathbb{C}$.

Let $X$ be a smooth and projective variety defined over $F$. On the one hand, the algebraic De Rham complex yields cohomology with $F$-coefficients $H^i_{\text{dR}}(X, F)$. The singular cohomology $H^\bullet(X(\mathbb{C}), \mathbb{Q})$, on the other hand, has rational coefficients. They are related by the De Rham Theorem,\(^{(2)}\) which furnishes in every degree $i$ an isomorphism

$$\omega_X : H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q} \mathbb{C} \rightarrow H^i_{\text{dR}}(X, F) \otimes_F \mathbb{C},$$

functorial in $X$. The resulting functor

$$X \rightsquigarrow (H^\bullet(X(\mathbb{C}), \mathbb{Q}), H^\bullet_{\text{dR}}(X, F), \omega_X)$$

to $\mathcal{P}(\mathbb{Q}, F, \mathbb{C})$ is conjectured to factor through the motif of $X$ (here loosely defined as an element of the universal Tannakian category for the collection of $\ell$-adic cohomology theories):

$$X \rightsquigarrow h(X, \mathbb{Q}) \rightsquigarrow (H^\bullet(X(\mathbb{C}), \mathbb{Q}), H^\bullet_{\text{dR}}(X, F), \omega_X).$$

Consider as an example the cohomology in degree $2d$ of $d$-dimensional projective space. There is an isomorphism

$$(H^{2d}(\mathbb{P}^d(\mathbb{C}), \mathbb{Q}), H^{2d}_{\text{dR}}(\mathbb{P}^d, F), \omega) \approx (\mathbb{Q}, F, \text{multiplication by } (2\pi i)^{-d}).$$

8.2 Construction of the Period Functor

8.2.1. Let $k[t] \rightarrow K$ be injective and let $K^\dagger$ be any complete and algebraically closed field containing $K$ such that $\|\theta\| > 1$. Denote by

\(^{(1)}\) See [André 2004], in particular section 7.1.6.  
\(^{(2)}\) Theorem 1’ of [Grothendieck 1966].
\( R \subset K^\dagger \{ t \} \) the subring consisting of those power series that have an infinite radius of convergence. Thus \( R \) is the ring of entire functions on the affine line over \( K^\dagger \). Let \( M \) be an effective \( t \)-motif over \( K \). Recall that \( H_{an}(M, k[t]) \) is the set of \( \sigma \)-invariant vectors in \( M \{ t \} \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger \{ t \} \).

**Lemma.** \( H_{an}(M, k[t]) \subset M \otimes_{K[t]} R \).

**Proof.** This is essentially Proposition 3.1.3 of [Anderson et al. 2004]—to see this it suffices to choose a basis of \( M \) and to express everything in terms of matrices. \( \square \)

**8.2.2.** By the Lemma, there is a natural map

\[
H_{an}(M, k[t]) \to M[[t - \theta]] \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger[[t - \theta]]
\]

obtained through Taylor expansion in \( t = \theta \). It induces a map

\[
H_{an}(M, k[t]) \otimes_{k[t]} K^\dagger((t - \theta)) \xrightarrow{\omega_M} M[[t - \theta]] \otimes_{K^\dagger[[t - \theta]]} K^\dagger((t - \theta))
\]

and this map is an isomorphism if and only if \( M \) is analytically trivial.

With \( M = C \), we have that on the natural bases \( \omega_C \) equals multiplication with \( \Omega \).

**8.2.3.** The construction

\[
\forall M \rightsquigarrow (H_{an}(M, k[t]), M[[t - \theta]], \omega_M) \quad (8.1)
\]

defines a \( \otimes \)-functor from \( tM_{\text{eff,a.t.}} \) to \( \mathcal{P}(k[t], K^\dagger[[t - \theta]], K^\dagger((t - \theta))) \). Extending scalars from \( k[t] \) to \( k(t) \) yields a functor

\[
\forall M \rightsquigarrow (H_{an}(M, k(t)), M[[t - \theta]], \omega_M) \quad (8.2)
\]

from \( tM_{\text{eff,a.t.}} \) to \( \mathcal{P}(k(t), K^\dagger[[t - \theta]], K((t - \theta))) \). Note that \( K^\dagger[[t - \theta]] \) is indeed a \( k(t) \)-algebra, since \( \theta \) is transcendental over \( k \).

As the target categories \( \mathcal{P} \) are rigid, both functors extend to non-effective \( t \)-motifs in a purely formal way.
8.3 Fully Faithfulness on Pure $t$-Motifs

8.3.1. Denote by the letter $P$ the functor

$$P : tM_{a.t.} \rightarrow \mathcal{P} \left( k[t], K^+[[t - \theta]], K^+((t - \theta)) \right)$$

described in the preceding section.

8.3.2. For the sake of brevity, we write $\mathcal{P}$ for the target category of the functors $P$. Assume that $K = K^+$. Here is the main Theorem of this chapter:

**Theorem (first form).** If $M_1$ and $M_2$ are $t$-motifs that are analytically trivial and pure of the same weight, then the natural map

$$\text{Hom}_{tM}(M_1, M_2) \rightarrow \text{Hom}_{\mathcal{P}}(P(M_1), P(M_2))$$

is an isomorphism of $k[t]$-modules.

Note that we have already shown that the map is injective (3.2.6.)

8.3.3. The Theorem is equivalent with the apparently weaker claim:

**Theorem (second form).** Let $n$ be a non-negative integer and $M$ an analytically trivial effective $t$-motif that is pure of weight $n$, then

$$\text{Hom}_{tM}(C^n, M) \rightarrow \text{Hom}_{\mathcal{P}}(P(C^n), P(M))$$

is an isomorphism of $k[t]$-modules.

**Proof of equivalence.** Take $M_1$ and $M_2$ as in the first formulation. Then for a sufficiently large $n$ the $t$-motif $M \overset{\text{def}}{=} \text{Hom}(M_1, M_2) \otimes C^n$ is effective and by adjunction (2.2.5) it follows that

$$\text{Hom}_{tM}(M_1, M_2) = \text{Hom}_{tM}(C^n, M)$$

and similarly that

$$\text{Hom}_{\mathcal{P}}(P(M_1), P(M_2)) = \text{Hom}_{\mathcal{P}}(P(C^n), P(M)).$$
8.34. Proof of the Theorem in its second form. Let \( f : P(C^n) \to P(M) \) be given. Pick generators to identify

\[
P(C^n) = (k[t], K[[t - \theta]], \Omega^n)
\]

Denote the image of \( 1 \in k[t] \) under \( f \) by \( v \in M\{t\}^\sigma \). The fact that \( f \) is a morphism in \( P \) implies that

\[
\omega_M(v) \in \Omega^n M[[t - \theta]] = (t - \theta)^n M[[t - \theta]]. \tag{8.3}
\]

**Claim:** \( v \in \Omega^n M \otimes_{K[t]} R \).

By Lemma 8.2.1 \( v \) lies in \( M \otimes_{K[t]} R \). The claim states that \( v \) has zeroes of order at least \( n \) at \( t = \theta^i \) for all \( i \geq 0 \). For \( i = 0 \) this is precisely what is asserted in equation (8.3). The presence of the other zeroes follows by induction on \( i \) since \( \sigma(v) = v \) and the action of \( \sigma \) is defined over \( K[t] \).

The claim shows that

\[
\Gamma_K \otimes R \to M \otimes R : g e \mapsto \sigma(g) \frac{v}{\Omega^n}
\]

is a well-defined \( \sigma \)-equivariant \( R \)-homomorphism. By the GAGA-style Proposition 4.4 of [Gardeyn 2003], this ‘analytic’ map descends to an ‘algebraic’ morphism \( C^n \to M \) of \( t \)-motifs which by construction gets mapped to \( f \) under the functor \( P \).

8.35. Remarks. The Theorem can also be deduced from Anderson’s results on scattering matrices\(^{(3)}\).

A more general result, accounting also for non-pure \( t \)-motifs, has been announced by Pink\(^{(4)}\) but unfortunately no precise statement, nor a proof, has been published so far.

8.4 Corollary: \( \Gamma_K \) is Connected

8.4.1. Using the above fully faithfulness we obtain the following important Theorem.

\(^{(3)}\) See §3 of [Anderson 1986].

\(^{(4)}\) See the pre-print [Pink 1997].
Theorem. An effective and analytically trivial $t$-motif $M$ that is pure of weight zero is constant.

Proof. It is sufficient to prove the Theorem for $K = K^\dagger$.

Denote by $r$ the rank of $M$, so that $\wedge^r M$ is isomorphic to the unique rank one effective $t$-motif that is pure of weight zero: $\wedge^r M \approx 1$. By functoriality this translates into an isomorphism

$$P(\wedge^r M) \approx P(1)$$

which implies that the matrix expressing $\omega_M$ lies in $GL(r, K[[t - \theta]])$. From this it follows that there is an isomorphism

$$P(M) \approx P(r 1)$$

whence by Theorem 8.3.2 the effective $t$-motif $M$ is constant. □

8.4.2. This allows us to finish the proof of the connectedness of $\Gamma_{K^\dagger}$ (4.2.3). Recall that assuming the non-connectedness of $\Gamma$ we have obtained an effective $t$-motif $M$ that is analytically trivial and such that $M \otimes M$ is a subquotient of $nM$ for some $n$. We were left with the task of showing that $M$ is constant.

End of the proof of Theorem 4.2.3. Let $\lambda$ be the maximal weight of $M$. Then $2\lambda$ is the maximal weight of $M \otimes M$ and hence $2\lambda$ is a weight of $nM$, hence $2\lambda \leq \lambda$, hence $\lambda = 0$, hence $M$ is pure of weight 0, hence, by the above Theorem, $M$ is constant. □