On t-Motifs
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Chapter 8

The Period Functor

8.1 A Tannakian Category of Periods

8.1.1. Let $L$ be a commutative ring, $A$ a commutative $L$-algebra and $B$ an $A$-algebra such that $A \to B$ is injective. Consider the category whose objects are triples $(V, W, \alpha)$ consisting of

- a projective and finitely generated $L$-module $V$;
- a projective $A$-module $W$;
- an isomorphism of $B$-modules $\alpha : V_B \to W_B$.

Morphisms in this category are pairs $(f, g)$ of an $L$-linear homomorphism $f : V_1 \to V_2$ and an $A$-linear homomorphism $g : W_1 \to W_2$ such that the obvious square commutes:

$$g_B \circ \alpha_1 = \alpha_2 \circ f_B.$$

Thus, $f$ determines $g_B$ and even $g$, since $A \to B$ is injective. It follows that $\text{Hom}((V_1, W_1, \alpha_1), (V_2, W_2, \alpha_2))$ is a sub-$L$-module of $\text{Hom}(V_1, V_2)$, and that the category we are dealing with is $L$-linear.

This category has a natural tensor product

$$(V_1, W_1, \alpha_1) \otimes (V_2, W_2, \alpha_2) \overset{\text{def}}{=} (V_1 \otimes_L V_2, W_1 \otimes_A W_2, \alpha_1 \otimes_B \alpha_2)$$

and the unit for tensor product is $(L, A, \text{id})$. The dual of a triple takes the contragredient $\alpha$, thus $(V, W, \alpha)^\vee \overset{\text{def}}{=} (V^\vee, W^\vee, (\alpha^\vee)^{-1})$. There is a
trace map from \((V, W, \alpha) \otimes (V, W, \alpha)^V\) to the unit \((L, A, \text{id})\):
\[
\text{tr} : (v \otimes \lambda, w \otimes \mu) \mapsto (\lambda(v), \mu(w)).
\]
This category of triples is an \(L\)-linear rigid tensor category and we shall denote it by \(\mathcal{P}(L, A, B)\).

8.1.2. Such a category arises naturally in the context of periods of algebraic varieties.\(^{(1)}\)

Let \(X\) be a smooth and projective variety defined over \(F\). On the one hand, the algebraic De Rham complex yields cohomology with \(F\)-coefficients \(H^\bullet_{\text{dR}}(X, F)\). The singular cohomology \(H^\bullet(X(C), \mathbb{Q})\), on the other hand, has rational coefficients. They are related by the De Rham Theorem,\(^{(2)}\) which furnishes in every degree \(i\) an isomorphism
\[
\omega_X : H^i(X(C), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H^i_{\text{dR}}(X, F) \otimes_{F} \mathbb{C},
\]
functorial in \(X\). The resulting functor
\[
X \rightsquigarrow (H^\bullet(X(C), \mathbb{Q}), H^\bullet_{\text{dR}}(X, F), \omega_X)
\]
to \(\mathcal{P}(\mathbb{Q}, F, \mathbb{C})\) is conjectured to factor through the motif of \(X\) (here loosely defined as an element of the universal Tannakian category for the collection of \(\ell\)-adic cohomology theories):
\[
X \rightsquigarrow h(X, \mathbb{Q}) \rightsquigarrow (H^\bullet(X(C), \mathbb{Q}), H^\bullet_{\text{dR}}(X, F), \omega_X).
\]
Consider as an example the cohomology in degree 2\(d\) of \(d\)-dimensional projective space. There is an isomorphism
\[
(H^{2d}(\mathbb{P}^d(C), \mathbb{Q}), H^{2d}_{\text{dR}}(\mathbb{P}^d, F), \omega) \approx (\mathbb{Q}, F, \text{multiplication by } (2\pi i)^{-d}).
\]

8.2 Construction of the Period Functor

8.2.1. Let \(k[t] \rightarrow K\) be injective and let \(K^\dagger\) be any complete and algebraically closed field containing \(K\) such that \(\|\theta\| > 1\). Denote by

\(^{(1)}\)See [André 2004], in particular section 7.1.6.

\(^{(2)}\)Theorem 1’ of [Grothendieck 1966].
$R \subset K^\dagger \{t\}$ the subring consisting of those power series that have an infinite radius of convergence. Thus $R$ is the ring of entire functions on the affine line over $K^\dagger$. Let $M$ be an effective $t$-motif over $K$. Recall that $H_{an}(M, k[t])$ is the set of $\sigma$-invariant vectors in $M\{t\} \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger \{t\}$.

**Lemma.** $H_{an}(M, k[t]) \subset M \otimes_{K[t]} R$.

**Proof.** This is essentially Proposition 3.1.3 of [Anderson et al. 2004]—to see this it suffices to choose a basis of $M$ and to express everything in terms of matrices.

8.2.2. By the Lemma, there is a natural map

$$H_{an}(M, k[t]) \rightarrow M[[t - \theta]] \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger[[t - \theta]]$$

obtained through Taylor expansion in $t = \theta$. It induces a map

$$H_{an}(M, k[t]) \otimes_{k[t]} K^\dagger((t - \theta)) \overset{\omega_M}{\rightarrow} M[[t - \theta]] \otimes_{K^\dagger[[t - \theta]]} K^\dagger((t - \theta))$$

and this map is an isomorphism if and only if $M$ is analytically trivial.

With $M = \mathbb{C}$, we have that on the natural bases $\omega_C$ equals multiplication with $\Omega$.

8.2.3. The construction

$$M \rightsquigarrow (H_{an}(M, k[t]), M[[t - \theta]], \omega_M) \quad (8.1)$$

defines a $\otimes$-functor from $t\mathcal{M}_{\text{eff,a.t.}}$ to $\mathcal{P}(k[t], K^\dagger[[t - \theta]], K((t - \theta)))$. Extending scalars from $k[t]$ to $k(t)$ yields a functor

$$M \rightsquigarrow (H_{an}(M, k(t)), M[[t - \theta]], \omega_M) \quad (8.2)$$

from $t\mathcal{M}_{\text{eff,a.t.}}$ to $\mathcal{P}(k(t), K^\dagger[[t - \theta]], K((t - \theta)))$. Note that $K^\dagger[[t - \theta]]$ is indeed a $k(t)$-algebra, since $\theta$ is transcendental over $k$.

As the target categories $\mathcal{P}$ are rigid, both functors extend to non-effective $t$-motifs in a purely formal way.
8.3  Fully Faithfulness on Pure t-Motifs

8.3.1. Denote by the letter $P$ the functor
\[ P : tM_{a.t.} \rightarrow \mathcal{P} \left( k[t], K^\dagger[[t - \theta]], K^\dagger((t - \theta)) \right) \]
described in the preceding section.

8.3.2. For the sake of brevity, we write $\mathcal{P}$ for the target category of the functors $P$. Assume that $K = K^\dagger$. Here is the main Theorem of this chapter:

**Theorem (first form).** If $M_1$ and $M_2$ are t-motifs that are analytically trivial and pure of the same weight, then the natural map
\[ \text{Hom}_{t \mathcal{M}}(M_1, M_2) \rightarrow \text{Hom}_{\mathcal{P}}(P(M_1), P(M_2)) \]
is an isomorphism of $k[t]$-modules.

Note that we have already shown that the map is injective (3.2.6.)

8.3.3. The Theorem is equivalent with the apparently weaker claim:

**Theorem (second form).** Let $n$ be a non-negative integer and $M$ an analytically trivial effective t-motif that is pure of weight $n$, then
\[ \text{Hom}_{t \mathcal{M}}(C^n, M) \rightarrow \text{Hom}_{\mathcal{P}}(P(C^n), P(M)) \]
is an isomorphism of $k[t]$-modules.

**Proof of equivalence.** Take $M_1$ and $M_2$ as in the first formulation. Then for a sufficiently large $n$ the t-motif $M \overset{\text{def}}{=} \text{Hom}(M_1, M_2) \otimes C^n$ is effective and by adjunction (2.2.5) it follows that
\[ \text{Hom}_{t \mathcal{M}}(M_1, M_2) = \text{Hom}_{t \mathcal{M}}(C^n, M) \]
and similarly that
\[ \text{Hom}_{\mathcal{P}}(P(M_1), P(M_2)) = \text{Hom}_{\mathcal{P}}(P(C^n), P(M)) \]
8.34. Proof of the Theorem in its second form. Let $f : P(C^n) \to P(M)$ be given. Pick generators to identify

$$P(C^n) = (k[t], K[[t - \theta]], \Omega^n)$$

Denote the image of $1 \in k[t]$ under $f$ by $v \in M\{t\}^\sigma$. The fact that $f$ is a morphism in $P$ implies that

$$\omega_{\Omega}(v) \in \Omega^n M[[t - \theta]] = (t - \theta)^n M[[t - \theta]]. \quad (8.3)$$

Claim: $v \in \Omega^n M \otimes_{K[t]} R$.

By Lemma 8.2.1 $v$ lies in $M \otimes_{K[t]} R$. The claim states that $v$ has zeroes of order at least $n$ at $t = \theta^i$ for all $i \geq 0$. For $i = 0$ this is precisely what is asserted in equation (8.3). The presence of the other zeroes follows by induction on $i$ since $\sigma(v) = v$ and the action of $\sigma$ is defined over $K[t]$.

The claim shows that

$$C^n \otimes R \to M \otimes R : g e \mapsto \sigma(g) \frac{v}{\Omega^n}$$

is a well-defined $\sigma$-equivariant $R$-homomorphism. By the GAGA-style Proposition 4.4 of [GARDEYN 2003], this ‘analytic’ map descends to an ‘algebraic’ morphism $C^n \to M$ of $t$-motifs which by construction gets mapped to $f$ under the functor $P$. \hfill \Box

8.3.5. Remarks. The Theorem can also be deduced from ANDERSON’s results on scattering matrices\(^{(3)}\)

A more general result, accounting also for non-pure $t$-motifs, has been announced by PINK\(^{(4)}\) but unfortunately no precise statement, nor a proof, has been published so far.

8.4 Corollary: $\Gamma_K$ is Connected

8.4.1. Using the above fully faithfulness we obtain the following important Theorem.

\(^{(3)}\)See §3 of [ANDERSON 1986].
\(^{(4)}\)See the pre-print [PINK 1997].
**Theorem.** An effective and analytically trivial \( t \)-motif \( M \) that is pure of weight zero is constant.

**Proof.** It is sufficient to prove the Theorem for \( K = K^\dagger \).

Denote by \( r \) the rank of \( M \), so that \( \wedge^r M \) is isomorphic to the unique rank one effective \( t \)-motif that is pure of weight zero: \( \wedge^r M \approx 1 \). By functoriality this translates into an isomorphism

\[
P(\wedge^r M) \approx P(1)
\]

which implies that the matrix expressing \( \omega_M \) lies in \( \text{GL}(r, K[[t - \theta]]) \).

From this it follows that there is an isomorphism

\[
P(M) \approx P(r1)
\]

whence by Theorem 8.3.2 the effective \( t \)-motif \( M \) is constant. \( \square \)

8.4.2. This allows us to finish the proof of the connectedness of \( \Gamma_K \) (4.2.3). Recall that assuming the non-connectedness of \( \Gamma \) we have obtained an effective \( t \)-motif \( M \) that is analytically trivial and such that \( M \otimes M \) is a subquotient of \( nM \) for some \( n \). We were left with the task of showing that \( M \) is constant.

*End of the proof of Theorem 4.2.3.* Let \( \lambda \) be the maximal weight of \( M \). Then \( 2\lambda \) is the maximal weight of \( M \otimes M \) and hence \( 2\lambda \) is a weight of \( nM \), hence \( 2\lambda \leq \lambda \), hence \( \lambda = 0 \), hence \( M \) is pure of weight 0, hence, by the above Theorem, \( M \) is constant. \( \square \)