Chapter 8

The Period Functor

8.1 A Tannakian Category of Periods

8.1.1. Let $L$ be a commutative ring, $A$ a commutative $L$-algebra and $B$ an $A$-algebra such that $A \rightarrow B$ is injective. Consider the category whose objects are triples $(V, W, \alpha)$ consisting of

- a projective and finitely generated $L$-module $V$;
- a projective $A$-module $W$;
- an isomorphism of $B$-modules $\alpha : V_B \rightarrow W_B$.

Morphisms in this category are pairs $(f, g)$ of an $L$-linear homomorphism $f : V_1 \rightarrow V_2$ and an $A$-linear homomorphism $g : W_1 \rightarrow W_2$ such that the obvious square commutes:

$$g_B \circ \alpha_1 = \alpha_2 \circ f_B.$$

Thus, $f$ determines $g_B$ and even $g$, since $A \rightarrow B$ is injective. It follows that $\text{Hom}((V_1, W_1, \alpha_1), (V_2, W_2, \alpha_2))$ is a sub-$L$-module of $\text{Hom}(V_1, V_2)$, and that the category we are dealing with is $L$-linear.

This category has a natural tensor product

$$(V_1, W_1, \alpha_1) \otimes (V_2, W_2, \alpha_2) \overset{\text{def}}{=} (V_1 \otimes_L V_2, W_1 \otimes_A W_2, \alpha_1 \otimes_B \alpha_2)$$

and the unit for tensor product is $(L, A, \text{id})$. The dual of a triple takes the contragredient $\alpha$, thus $(V, W, \alpha)^\vee \overset{\text{def}}{=} (V^\vee, W^\vee, (\alpha^\vee)^{-1})$. There is a
trace map from \((V, W, \alpha) \otimes (V, W, \alpha)^V\) to the unit \((L, A, \text{id})\):

\[
\text{tr} : (v \otimes \lambda, w \otimes \mu) \mapsto (\lambda(v), \mu(w)).
\]

This category of triples is an \(L\)-linear rigid tensor category and we shall denote it by \(\mathcal{P}(L, A, B)\).

8.1.2. Such a category arises naturally in the context of periods of algebraic varieties\(^{(1)}\). Fix a subfield \(F\) of \(\mathbb{C}\).

Let \(X\) be a smooth and projective variety defined over \(F\). On the one hand, the algebraic De Rham complex yields cohomology with \(F\)-coefficients \(H^i_{dR}(X, F)\). The singular cohomology \(H^\bullet(X(\mathbb{C}), \mathbb{Q})\), on the other hand, has rational coefficients. They are related by the De Rham Theorem\(^{(2)}\) which furnishes in every degree \(i\) an isomorphism

\[
\omega_X : H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_{dR}(X, F) \otimes_F \mathbb{C},
\]

functorial in \(X\). The resulting functor

\[
X \rightsquigarrow (H^\bullet(X(\mathbb{C}), \mathbb{Q}), H^\bullet_{dR}(X, F), \omega_X)
\]

to \(\mathcal{P}(\mathbb{Q}, F, \mathbb{C})\) is conjectured to factor through the motif of \(X\) (here loosely defined as an element of the universal Tannakian category for the collection of \(\ell\)-adic cohomology theories):

\[
X \rightsquigarrow h(X, \mathbb{Q}) \rightsquigarrow (H^\bullet(X(\mathbb{C}), \mathbb{Q}), H^\bullet_{dR}(X, F), \omega_X).
\]

Consider as an example the cohomology in degree \(2d\) of \(d\)-dimensional projective space. There is an isomorphism

\[
(H^{2d}(\mathbb{P}^d(\mathbb{C}), \mathbb{Q}), H^{2d}_{dR}(\mathbb{P}^d, F), \omega) \approx (\mathbb{Q}, F, \text{multiplication by } (2\pi i)^{-d}).
\]

8.2 Construction of the Period Functor

8.2.1. Let \(k[t] \to K\) be injective and let \(K^\dagger\) be any complete and algebraically closed field containing \(K\) such that \(|\theta| > 1\). Denote by

\(^{(1)}\)See [André 2004], in particular section 7.1.6.
\(^{(2)}\)Theorem 1' of [Grothendieck 1966].
$R \subseteq K^\dagger \{t\}$ the subring consisting of those power series that have an infinite radius of convergence. Thus $R$ is the ring of entire functions on the affine line over $K^\dagger$. Let $M$ be an effective $t$-motif over $K$. Recall that $H_{\text{an}}(M, k[t])$ is the set of $\sigma$-invariant vectors in $M\{t\} \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger \{t\}$.

**Lemma.** $H_{\text{an}}(M, k[t]) \subseteq M \otimes_{K[t]} R$.

**Proof.** This is essentially Proposition 3.1.3 of [Anderson et al. 2004]—to see this it suffices to choose a basis of $M$ and to express everything in terms of matrices. $\square$

**8.2.2.** By the Lemma, there is a natural map

$$H_{\text{an}}(M, k[t]) \to M[[t - \theta]] \overset{\text{def}}{=} M \otimes_{K[t]} K^\dagger [[t - \theta]]$$

obtained through Taylor expansion in $t = \theta$. It induces a map

$$H_{\text{an}}(M, k[t]) \otimes_{k[t]} K^\dagger ((t - \theta)) \overset{\omega_M}{\longrightarrow} M[[t - \theta]] \otimes_{K^\dagger [[t-\theta]]} K^\dagger ((t - \theta))$$

and this map is an isomorphism if and only if $M$ is analytically trivial. With $M = C$, we have that on the natural bases $\omega_C$ equals multiplication with $\Omega$.

**8.2.3.** The construction

$$M \leadsto (H_{\text{an}}(M, k[t]), M[[t - \theta]], \omega_M) \quad (8.1)$$

defines a $\otimes$-functor from $tM_{\text{eff,a.t.}}$ to $\mathcal{P}(k[t], K^\dagger [[t - \theta]], K^\dagger ((t - \theta)))$. Extending scalars from $k[t]$ to $k(t)$ yields a functor

$$M \leadsto (H_{\text{an}}(M, k(t)), M[[t - \theta]], \omega_M) \quad (8.2)$$

from $tM_{\text{eff,a.t.}}^\circ$ to $\mathcal{P}(k(t), K^\dagger [[t - \theta]], K((t - \theta)))$. Note that $K^\dagger [[t - \theta]]$ is indeed a $k(t)$-algebra, since $\theta$ is transcendental over $k$.

As the target categories $\mathcal{P}$ are rigid, both functors extend to non-effective $t$-motifs in a purely formal way.
8.3 Fully Faithfulness on Pure $t$-Motifs

8.3.1. Denote by the letter $P$ the functor

$$P : t\text{M}_{a.t.} \to \mathcal{P} \left( k[t], K^\dagger[[t - \theta]], K^\dagger((t - \theta)) \right)$$
described in the preceding section.

8.3.2. For the sake of brevity, we write $\mathcal{P}$ for the target category of the functors $P$. Assume that $K = K^\dagger$. Here is the main Theorem of this chapter:

**Theorem (first form).** If $M_1$ and $M_2$ are $t$-motifs that are analytically trivial and pure of the same weight, then the natural map

$$\text{Hom}_{t\mathcal{M}}(M_1, M_2) \to \text{Hom}_\mathcal{P}(P(M_1), P(M_2))$$

is an isomorphism of $k[t]$-modules.

Note that we have already shown that the map is injective (3.2.6.)

8.3.3. The Theorem is equivalent with the apparently weaker claim:

**Theorem (second form).** Let $n$ be a non-negative integer and $M$ an analytically trivial effective $t$-motif that is pure of weight $n$, then

$$\text{Hom}_{t\mathcal{M}}(C^n, M) \to \text{Hom}_\mathcal{P}(P(C^n), P(M))$$

is an isomorphism of $k[t]$-modules.

**Proof of equivalence.** Take $M_1$ and $M_2$ as in the first formulation. Then for a sufficiently large $n$ the $t$-motif $M \overset{\text{def}}{=} \text{Hom}(M_1, M_2) \otimes C^n$ is effective and by adjunction (2.2.5) it follows that

$$\text{Hom}_{t\mathcal{M}}(M_1, M_2) = \text{Hom}_{t\mathcal{M}}(C^n, M)$$

and similarly that

$$\text{Hom}_\mathcal{P}(P(M_1), P(M_2)) = \text{Hom}_\mathcal{P}(P(C^n), P(M)).$$
8.34. Proof of the Theorem in its second form. Let \( f : P(C^n) \to P(M) \) be given. Pick generators to identify

\[
P(C^n) = (k[t], K[[t - \theta]], \Omega^n)
\]

Denote the image of 1 \( \in k[t] \) under \( f \) by \( v \in M\{t\}^\sigma \). The fact that \( f \) is a morphism in \( P \) implies that

\[
\omega_M(v) \in \Omega^n M[[t - \theta]] = (t - \theta)^n M[[t - \theta]]. \tag{8.3}
\]

Claim: \( v \in \Omega^n M \otimes_{K[t]} R \).

By Lemma 8.2.1 \( v \) lies in \( M \otimes_{K[t]} R \). The claim states that \( v \) has zeroes of order at least \( n \) at \( t = \theta^i \) for all \( i \geq 0 \). For \( i = 0 \) this is precisely what is asserted in equation (8.3). The presence of the other zeroes follows by induction on \( i \) since \( \sigma(v) = v \) and the action of \( \sigma \) is defined over \( K[t] \).

The claim shows that \( C^n \otimes R \to M \otimes R : g e \mapsto \sigma(g) \frac{v}{\Omega^n} \)

is a well-defined \( \sigma \)-equivariant \( R \)-homomorphism. By the GAGA-style Proposition 4.4 of [Gardeyn 2003], this ‘analytic’ map descends to an ‘algebraic’ morphism \( C^n \to M \) of \( t \)-motifs which by construction gets mapped to \( f \) under the functor \( P \).

8.35. Remarks. The Theorem can also be deduced from Anderson’s results on scattering matrices\(^{(3)}\)

A more general result, accounting also for non-pure \( t \)-motifs, has been announced by Pink\(^{(4)}\) but unfortunately no precise statement, nor a proof, has been published so far.

8.4 Corollary: \( \Gamma_{K^s} \) is Connected

8.4.1. Using the above fully faithfulness we obtain the following important Theorem.

\(^{(3)}\)See §3 of [Anderson 1986].

\(^{(4)}\)See the pre-print [Pink 1997].
**Theorem.** An effective and analytically trivial $t$-motif $M$ that is pure of weight zero is constant.

**Proof.** It is sufficient to prove the Theorem for $K = K^\dagger$.

Denote by $r$ the rank of $M$, so that $\wedge^r M$ is isomorphic to the unique rank one effective $t$-motif that is pure of weight zero: $\wedge^r M \approx 1$. By functoriality this translates into an isomorphism

$$P(\wedge^r M) \approx P(1)$$

which implies that the matrix expressing $\omega_M$ lies in $GL(r, K[[t - \theta]])$. From this it follows that there is an isomorphism

$$P(M) \approx P(r1)$$

whence by Theorem 8.3.2 the effective $t$-motif $M$ is constant.

8.4.2. This allows us to finish the proof of the connectedness of $\Gamma_{K'}$ (4.2.3). Recall that assuming the non-connectedness of $\Gamma$ we have obtained an effective $t$-motif $M$ that is analytically trivial and such that $M \otimes M$ is a subquotient of $nM$ for some $n$. We were left with the task of showing that $M$ is constant.

*End of the proof of Theorem 4.2.3.* Let $\lambda$ be the maximal weight of $M$. Then $2\lambda$ is the maximal weight of $M \otimes M$ and hence $2\lambda$ is a weight of $nM$, hence $2\lambda \leq \lambda$, hence $\lambda = 0$, hence $M$ is pure of weight 0, hence, by the above Theorem, $M$ is constant.