Chapter 7

A Few Notes on Extensions

7.1 Extensions of Effective $t$-Motifs

7.1.1. Let $0 \rightarrow M_2 \rightarrow M \rightarrow M_1 \rightarrow 0$ be an exact sequence of $K[t, \sigma]$-modules. If $M_1$ and $M_2$ are effective $t$-motifs, then so is $M$. We say that $M$ is an extension of the effective $t$-motif $M_1$ by $M_2$. The set of isomorphism classes of extensions forms an abelian group $\text{Ext}_{tM_{\text{eff}}}(M_1, M_2)$ under the Baer sum\(^{(1)}\) and this group is naturally a $k[t]$-module.

7.1.2. This module can be given a more explicit description. Put $H \overset{\text{def}}{=} \text{Hom}_{K[t]}(M_1, M_2)$, the space of linear maps of $M_1$ to $M_2$, and write $H'$ for the space of semi-linear maps of $M_1$ to $M_2$. The $K[t]$-modules $H$ and $H'$ are isomorphic, since both are free of rank $\text{rk}(M_1) \text{rk}(M_2)$, but there is no natural isomorphism (unless $K = k$). Consider the $k[t]$-linear map

$$\delta : H \rightarrow H' : f \mapsto \sigma_2 \circ f - f \circ \sigma_1.$$ 

Note that $\ker(\delta) = \text{Hom}(M_1, M_2)$. We contend that

$$\text{coker}(\delta) = \text{Ext}_{tM_{\text{eff}}}(M_1, M_2). \quad (7.1)$$

In fact, if $M$ is an extension of $M_1$ by $M_2$ then as $K[t]$-modules $M \approx M_1 \oplus M_2$ and $\sigma(m_1, m_2) = (\sigma_1(m_1), \sigma_2(m_2) + \gamma(m_1))$ with $\gamma \in H'$. This

\(^{(1)}\)See [BAER 1934] or Chapter XIV of [CARTAN AND EILENBERG 1956].
extension splits if and only if there exists a linear $f : M_1 \to M_2$ such that $\gamma = \delta(f)$.

### 7.1.3. The $k[t]$-module Ext is almost a divisible module:

**Proposition.** If $K = K^2$ and $\lambda \in k[t]$ has an invertible image under $k[t] \to K$ then multiplication with $\lambda$ is surjective on $\text{Ext}_{M_{\text{eff}}}(M_1, M_2)$.

**Proof of the proposition.** Consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H & \xrightarrow{\lambda} & H & \xrightarrow{\delta} & H/\lambda H & \longrightarrow & 0 \\
& & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\bar{\delta}} \\
0 & \longrightarrow & H' & \xrightarrow{\lambda} & H' & \xrightarrow{\delta} & H'/\lambda H' & \longrightarrow & 0
\end{array}
$$

with exact rows. This gives an exact sequence of cokernels

$$
\cdots \to \text{Ext}(M_1, M_2) \xrightarrow{\lambda} \text{Ext}(M_1, M_2) \to \text{coker}(\bar{\delta}) \to 0.
$$

It remains to show that $\text{coker}(\bar{\delta}) = 0$, that is, that $\bar{\delta}$ is surjective. The vector space $H/\lambda H$ is of finite dimension over $K$. Since $t - \theta$ is invertible in $K[t]/\lambda K[t]$ (here the condition on $\lambda$ is needed), the map $\bar{\delta}$ is the sum of a non-degenerate semi-linear map (‘$\sigma_2 \circ f$’) and a linear isomorphism (‘$f \circ \sigma_1$’). It follows from Corollary b.2.2 that $\bar{\delta}$ is surjective. □

### 7.2 $\text{Ext}(1,1)$

#### 7.2.1. As was already indicated in §5.2, knowledge about the group of extensions $\text{Ext}(1,1)$ in some Tannakian category leads to information on the underlying fundamental group. We have calculated $\text{Ext}(1,1)$ in the category of constant $t$-motifs (essentially 5.2.2) and deduced from it the Artin-Schreier Theorem on degree $p$ extensions in characteristic $p$. Similarly we calculated $\text{Ext}(1,1)$ in the category of interior $t$-motifs (5.2.3), to find that the abelianisation of the affine group scheme involved is of multiplicative type. In this section we will repeat the exercise and calculate $\text{Ext}(1,1)$ in the categories of $t$-motifs.
7.2.2. One should realise that a $t$-motif that is an extension of two effective $t$-motifs need not be effective. In fact, it follows from the definition of the category $t\mathcal{M}$ that for every pair $M_1, M_2$ of $t$-motifs

$$\text{Ext}_{t\mathcal{M}}(M_1, M_2) = \lim_{\rightarrow n} \text{Ext}_{t\mathcal{M}_{\text{eff}}}(M_1 \otimes C^n, M_2 \otimes C^n)$$

where the limit is for increasing $n$, starting at a sufficiently large value for the right-hand-side to make sense.

7.2.3. For $\text{Ext}(1, 1)$ in $t\mathcal{M}^\circ$ we have:

**Proposition.** If $K = K^\circ$ then

$$\text{Ext}_{t\mathcal{M}^\circ}(1, 1) = \begin{cases} \bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t] & \text{if } k[t] \to K \text{ is injective} \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** By 7.1.2, the group of effective extensions of $C^n$ by $C^n$ is the cokernel of the map

$$\delta : K[t] \to K[t] : f \mapsto (t - \theta)^n (\sigma(f) - f)).$$

By the Artin-Schreier Theorem ($K$ is separably closed), the image of $\delta$ is $(t - \theta)^n K[t]$, hence in the category $t\mathcal{M}$ we have

$$\text{Ext}_{t\mathcal{M}}(1, 1) = \bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t].$$

From this the $\text{Ext}$ in the category $t\mathcal{M}^\circ$ of $t$-motifs up to isogeny can be calculated: use that

$$\text{Ext}_{t\mathcal{M}^\circ}(-, -) = \text{Ext}_{t\mathcal{M}}(-, -) \otimes_{k[t]} k(t)$$

to obtain the modules as stated in the Proposition. \qed

7.2.4. If $k[t] \to K$ is injective then the above calculation also yields the $\text{Ext}(1, 1)$ in the Tannakian category $t\mathcal{M}_{\text{a.t.}}^\circ$. In fact,

**Lemma.** Every extension of an analytically trivial $t$-motif by an analytically trivial $t$-motif is itself analytically trivial.
and therefore
\[ \text{Hom}(\Gamma, G_{a,k(t)}) = \text{Ext}_{tM^0}^1(1, 1) = \text{Ext}_{tM^0}^1(1, 1). \]

This gives a description of the ‘additive part’ of the abelianisation of the affine group scheme \( \Gamma \).

**Proof of the Lemma.** Let \( E \) be an extension of \( M_2 \) by \( M_1 \), where both \( M_i \) are analytically trivial. This yields an exact sequence
\[ 0 \rightarrow M_1\{t\} \rightarrow E\{t\} \rightarrow M_2\{t\} \rightarrow 0. \]
Both \( M_1\{t\} \) and \( M_2\{t\} \) have a \( \sigma \)-invariant basis, and these define a basis for \( E\{t\} \) on which \( \sigma \) acts by an upper triangular block matrix. To show that \( E\{t\} \) is analytically trivial (has an invariant basis) it hence suffices to show that the map
\[ K^\dagger\{t\} \rightarrow K^\dagger\{t\} : \sum_i a_i t^i \mapsto \sum_i (a_i^q - a_i) t^i \]
is surjective. This is immediate from the observation that if \( b \in K^\dagger \) with \( \|b\| \leq 1 \) then the equation
\[ x^q - x = b \]
has a (in fact, unique) solution with \( \|x\| = \|b\| \).

**7.3 \( t \)-Motifs over Finite Fields**

**7.3.1.** The situation is quite a bit simpler when \( K \) is a finite field.

**Proposition.** Let \( K \) be a finite field and \( M_1 \) and \( M_2 \) be two effective \( t \)-motifs over \( K \). Then \( \text{Ext}_{tM_{\text{eff}}}^1(M_1, M_2) \) is a finitely generated \( k[t] \)-module and
\[ \text{rk}_{k[t]} \text{Hom}(M_1, M_2) = \text{rk}_{k[t]} \text{Ext}_{tM_{\text{eff}}}^1(M_1, M_2). \]

Note that while in \( tM^0 \) the \( k(t) \)-space \( \text{Ext}(1, 1) \) is one-dimensional when \( K \) is a finite field, it vanishes when \( K \) is the algebraic closure of a finite field (7.2.3).
Proof of the Proposition. Take $H$, $H'$ and $\delta$ as in 7.1.2. Then

$$0 \rightarrow \text{Hom}(M_1, M_2) \rightarrow H \xrightarrow{\delta} H' \rightarrow \text{Ext}(M_1, M_2) \rightarrow 0$$

is an exact sequence of $k[t]$-modules. Since $K$ is finite over $k$, the modules $H$ and $H'$ are free and of the same finite rank over $k[t]$, whence the claims of the Proposition. \hfill $\Box$

7.3.2. In particular, when $M_1$ and $M_2$ are pure of different weights, the module of homomorphisms is trivial (6.2.2) and therefore $\text{Ext}(M_1, M_2)$ is torsion. This is in line with Algebraic Geometry, where it is expected that every mixed motif over a finite field and with $\mathbb{Q}$-coefficients decomposes as a direct sum of pure motifs.$^{(2)}$ Only, here we have to deal with the pathology that there exist $t$-motifs that do not have a filtration with pure quotients. We shall proceed immediately to exhibit an example.

7.3.3. Let $\theta = 0$, that is, $K$ has ‘characteristic $t$’. Consider the effective $t$-motif

$$M = K[t]e_1 \oplus K[t]e_2 \quad \text{with} \quad \begin{cases} \sigma(e_1) = te_1 + e_2 \\ \sigma(e_2) = te_1 \end{cases}$$

Proposition. $M$ has weights 0 and 1, yet $M_K$ has no proper pure sub-$t$-motifs.

Sketch of proof. On the given basis, the characteristic polynomial of $\sigma$ is $f(X) = X^2 - tX - t$. Using the Newton polygon, one verifies that the valuations of zeroes of $f$ are 0 and $-1$, whence the weights are 0 and 1.

If $M$ contains pure sub-$t$-motif then it is either isomorphic to 1 or to $C$. In other words, $M$ must contain a vector $v = ae_1 + be_2$ such that either $\sigma(v) = v$ or $\sigma(v) = tv$. The former can be excluded by an argument on the degrees of $a$ and $b$, the latter by a calculation modulo $t$. \hfill $\Box$

7.4 Higher Ext

7.4.1. If an abelian category has sufficient injectives or projectives then functors $\text{Ext}^i(-, -)$ can be defined using resolutions. By [YONEDA 1954]

$^{(2)}$See Theorem 2.49 in [MILNE 1994], where this is credited to GROTHENDIECK.
these functors have a definition independent of the existence of resolutions, valid on any abelian category. This generalises the identification of $\text{Ext}^1$ with the group of extensions under Baer sum.

**7.4.2.** On the abelian category $tM^o$ they vanish:

**Theorem.** $\text{Ext}^i_{tM^o}(-,-) = 0$, for all $i > 1$.

**Corollary.** $\text{Ext}^i_{tM^o_{\mathit{eff}}}(-,-) = 0$, for all $i > 1$. □

**Proof of the Theorem.** Clearly it suffices to show that the higher Ext are trivial on $tM^o_{\mathit{eff}}$. Denote by $C$ the category of left modules over the ring $K[t,\sigma] \otimes k[t] k(t)$.

The functor $M \rightsquigarrow M \otimes_{k[t]} k(t)$ defines a fully faithful embedding of $tM^o_{\mathit{eff}}$ into $C$. This induces for every pair $M_1, M_2$ of effective $t$-motifs natural maps $\phi^i : \text{Ext}^i_{tM^o_{\mathit{eff}}}(M_1, M_2) \to \text{Ext}^i_C(M_1, M_2)$.

The category $C$ has sufficient projectives and hence the target groups can be calculated using resolutions. In fact, using a $K[t]$-basis of an effective $t$-motif $M$ as a set of generators in $C$ one sees that every effective $t$-motif $M$ has a free resolution of length at most 1 in $C$: the free set of relations expresses the action of $\sigma$ on the basis. Thus for $i > 1$ the target groups of $\phi^i$ vanish.

Proposition 3.3 of [Oort 1964] asserts that if for some $i$ and all $M_1$ and $M_2$ the map $\phi^i$ is bijective, then (for all $M_1$ and $M_2$) the map $\phi^{i+1}$ is injective. Thus the Theorem will be shown as soon as $\phi^1$ is bijective.

This is indeed the case:

$$\text{Ext}^1_{tM^o_{\mathit{eff}}}(M_1, M_2) = \text{coker}(H \xrightarrow{\delta} H') \otimes_{k[t]} k(t)$$

$$= \text{coker}(H \otimes k(t) \xrightarrow{\delta \otimes k(t)} H' \otimes k(t))$$

$$= \text{Ext}^1_C(M_1 \otimes k(t), M_2 \otimes k(t)),$$

using the flatness of the $k[t]$-module $k(t)$. □