Chapter 7

A Few Notes on Extensions

7.1 Extensions of Effective \( t \)-Motifs

7.1.1. Let \( 0 \rightarrow M_2 \rightarrow M \rightarrow M_1 \rightarrow 0 \) be an exact sequence of \( K[t, \sigma] \)-modules. If \( M_1 \) and \( M_2 \) are effective \( t \)-motifs, then so is \( M \). We say that \( M \) is an extension of the effective \( t \)-motif \( M_1 \) by \( M_2 \). The set of isomorphism classes of extensions forms an abelian group \( \text{Ext}_{t,M_{\text{eff}}}(M_1, M_2) \) under the Baer sum\(^{(1)}\) and this group is naturally a \( k[t] \)-module.

7.1.2. This module can be given a more explicit description. Put \( H \overset{\text{def}}{=} \text{Hom}_{K[t]}(M_1, M_2) \), the space of linear maps of \( M_1 \) to \( M_2 \), and write \( H' \) for the space of semi-linear maps of \( M_1 \) to \( M_2 \). The \( K[t] \)-modules \( H \) and \( H' \) are isomorphic, since both are free of rank \( \text{rk}(M_1) \text{rk}(M_2) \), but there is no natural isomorphism (unless \( K = k \)). Consider the \( k[t] \)-linear map

\[
\delta : H \rightarrow H' : f \mapsto \sigma_2 \circ f - f \circ \sigma_1.
\]

Note that \( \ker(\delta) = \text{Hom}(M_1, M_2) \). We contend that

\[
\text{coker}(\delta) = \text{Ext}_{t,M_{\text{eff}}}(M_1, M_2).
\]

(7.1)

In fact, if \( M \) is an extension of \( M_1 \) by \( M_2 \) then as \( K[t] \)-modules \( M \approx M_1 \oplus M_2 \) and \( \sigma(m_1, m_2) = (\sigma_1(m_1), \sigma_2(m_2) + \gamma(m_1)) \) with \( \gamma \in H' \). This

\(^{(1)}\)See [BAER 1934] or Chapter XIV of [CARTAN AND EILENBERG 1956].
extension splits if and only if there exists a linear \( f : M_1 \to M_2 \) such that \( \gamma = \delta(f) \).

### 7.1.3. The \( k[t] \)-module \( \text{Ext} \) is almost a divisible module:

**Proposition.** If \( K = K^2 \) and \( \lambda \in k[t] \) has an invertible image under \( k[t] \to K \) then multiplication with \( \lambda \) is surjective on \( \text{Ext}_{tM_{\text{eff}}}(M_1, M_2) \).

**Proof of the proposition.** Consider the commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H & \overset{\lambda}{\longrightarrow} & H & \longrightarrow & H/\lambda H & \longrightarrow & 0 \\
\downarrow{\delta} & & \downarrow{\delta} & & \downarrow{\delta} & & \\
0 & \longrightarrow & H' & \overset{\lambda}{\longrightarrow} & H' & \longrightarrow & H'/\lambda H' & \longrightarrow & 0
\end{array}
\]

with exact rows. This gives an exact sequence of cokernels

\[
\cdots \to \text{Ext}(M_1, M_2) \overset{\lambda}{\longrightarrow} \text{Ext}(M_1, M_2) \to \text{coker}(\bar{\delta}) \to 0.
\]

It remains to show that \( \text{coker}(\bar{\delta}) = 0 \), that is, that \( \bar{\delta} \) is surjective. The vector space \( H/\lambda H \) is of finite dimension over \( K \). Since \( t - \theta \) is invertible in \( K[t]/\lambda K[t] \) (here the condition on \( \lambda \) is needed), the map \( \bar{\delta} \) is the sum of a non-degenerate semi-linear map \( (\sigma_2 \circ f) \) and a linear isomorphism \( (f \circ \sigma_1) \). It follows from Corollary b.2.2 that \( \bar{\delta} \) is surjective.

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### 7.2. Ext\((1,1)\)

**7.2.1.** As was already indicated in §5.2, knowledge about the group of extensions \( \text{Ext}(1,1) \) in some Tannakian category leads to information on the underlying fundamental group. We have calculated \( \text{Ext}(1,1) \) in the category of constant \( t \)-motifs (essentially 5.2.2) and deduced from it the Artin-Schreier Theorem on degree \( p \) extensions in characteristic \( p \). Similarly we calculated \( \text{Ext}(1,1) \) in the category of interior \( t \)-motifs (5.2.3), to find that the abelianisation of the affine group scheme involved is of multiplicative type. In this section we will repeat the exercise and calculate \( \text{Ext}(1,1) \) in the categories of \( t \)-motifs.
7.2.2. One should realise that a $t$-motif that is an extension of two effective $t$-motifs need not be effective. In fact, it follows from the definition of the category $t\mathcal{M}$ that for every pair $M_1, M_2$ of $t$-motifs

$$\text{Ext}_{t\mathcal{M}}(M_1, M_2) = \lim_{\rightarrow n} \text{Ext}_{t\mathcal{M}_{\text{eff}}}(M_1 \otimes C^n, M_2 \otimes C^n)$$

where the limit is for increasing $n$, starting at a sufficiently large value for the right-hand-side to make sense.

7.2.3. For $\text{Ext}(1, 1)$ in $t\mathcal{M}^\circ$ we have:

**Proposition.** If $K = K^\circ$ then

$$\text{Ext}_{t\mathcal{M}^\circ}(1, 1) = \begin{cases} \bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t] & \text{if } k[t] \rightarrow K \text{ is injective} \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** By 7.1.2, the group of effective extensions of $C^n$ by $C^n$ is the cokernel of the map

$$\delta : K[t] \rightarrow K[t] : f \mapsto (t - \theta)^n (\sigma(f) - f)).$$

By the Artin-Schreier Theorem ($K$ is separably closed), the image of $\delta$ is $(t - \theta)^n K[t]$, hence in the category $t\mathcal{M}$ we have

$$\text{Ext}_{t\mathcal{M}}(1, 1) = \bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t].$$

From this the Ext in the category $t\mathcal{M}^\circ$ of $t$-motifs up to isogeny can be calculated: use that

$$\text{Ext}_{t\mathcal{M}^\circ}(-,-) = \text{Ext}_{t\mathcal{M}}(-,-) \otimes_{k[t]} k(t)$$

to obtain the modules as stated in the Proposition.

7.2.4. If $k[t] \rightarrow K$ is injective then the above calculation also yields the $\text{Ext}(1, 1)$ in the Tannakian category $t\mathcal{M}^\circ_{\text{a.a.}}$. In fact,

**Lemma.** Every extension of an analytically trivial $t$-motif by an analytically trivial $t$-motif is itself analytically trivial.
and therefore
\[ \text{Hom}(\Gamma, G_{a,k(t)}) = \text{Ext}_{t,M^\circ} (\mathbf{1}, \mathbf{1}) \]
\[ = \text{Ext}_{t,M^\circ} (\mathbf{1}, \mathbf{1}). \]

This gives a description of the ‘additive part’ of the abelianisation of the affine group scheme \( \Gamma \).

**Proof of the Lemma.** Let \( E \) be an extension of \( M_2 \) by \( M_1 \), where both \( M_i \) are analytically trivial. This yields an exact sequence
\[ 0 \to M_1\{t\} \to E\{t\} \to M_2\{t\} \to 0. \]

Both \( M_1\{t\} \) and \( M_2\{t\} \) have a \( \sigma \)-invariant basis, and these define a basis for \( E\{t\} \) on which \( \sigma \) acts by an upper triangular block matrix. To show that \( E\{t\} \) is analytically trivial (has an invariant basis) it hence suffices to show that the map
\[ K^\dagger\{t\} \to K^\dagger\{t\} : \sum_i a_i t_i \mapsto \sum_i (a^q_i - a_i) t^i \]
is surjective. This is immediate from the observation that if \( b \in K^\dagger \) with \( \|b\| \leq 1 \) then the equation
\[ x^q - x = b \]
has a (in fact, unique) solution with \( \|x\| = \|b\|. \)

### 7.3 \( t \)-Motifs over Finite Fields

**7.3.1.** The situation is quite a bit simpler when \( K \) is a finite field.

**Proposition.** Let \( K \) be a finite field and \( M_1 \) and \( M_2 \) be two effective \( t \)-motifs over \( K \). Then \( \text{Ext}_{t,M^\text{eff}} (M_1, M_2) \) is a finitely generated \( k[t] \)-module and
\[ \text{rk}_{k[t]} \text{Hom}(M_1, M_2) = \text{rk}_{k[t]} \text{Ext}_{t,M^\text{eff}} (M_1, M_2). \]

Note that while in \( tM^\circ \) the \( k(t) \)-space \( \text{Ext}(\mathbf{1}, \mathbf{1}) \) is one-dimensional when \( K \) is a finite field, it vanishes when \( K \) is the algebraic closure of a finite field (7.2.3).
Proof of the Proposition. Take $H$, $H'$ and $\delta$ as in 7.1.2. Then

$$0 \to \text{Hom}(M_1, M_2) \to H \xrightarrow{\delta} H' \to \text{Ext}(M_1, M_2) \to 0$$

is an exact sequence of $k[t]$-modules. Since $K$ is finite over $k$, the modules $H$ and $H'$ are free and of the same finite rank over $k[t]$, whence the claims of the Proposition.

7.3.2. In particular, when $M_1$ and $M_2$ are pure of different weights, the module of homomorphisms is trivial (6.2.2) and therefore $\text{Ext}(M_1, M_2)$ is torsion. This is in line with Algebraic Geometry, where it is expected that every mixed motif over a finite field and with $\mathbb{Q}$-coefficients decomposes as a direct sum of pure motifs.\(^{(2)}\) Only, here we have to deal with the pathology that there exist $t$-motifs that do not have a filtration with pure quotients. We shall proceed immediately to exhibit an example.

7.3.3. Let $\theta = 0$, that is, $K$ has ‘characteristic $t$’. Consider the effective $t$-motif

$$M = K[t]e_1 \oplus K[t]e_2 \text{ with } \begin{cases} \sigma(e_1) = te_1 + e_2 \\ \sigma(e_2) = te_1 \end{cases}$$

Proposition. $M$ has weights 0 and 1, yet $M_K$ has no proper pure sub-$t$-motifs.

Sketch of proof. On the given basis, the characteristic polynomial of $\sigma$ is $f(X) = X^2 - tX - t$. Using the Newton polygon, one verifies that the valuations of zeroes of $f$ are 0 and $-1$, whence the weights are 0 and 1.

If $M$ contains pure sub-$t$-motif then it is either isomorphic to 1 or to $C$. In other words, $M$ must contain a vector $v = ae_1 + be_2$ such that either $\sigma(v) = v$ or $\sigma(v) = tv$. The former can be excluded by an argument on the degrees of $a$ and $b$, the latter by a calculation modulo $t$.

\(\square\)

7.4 Higher Ext

7.4.1. If an abelian category has sufficient injectives or projectives then functors $\text{Ext}^i(-, -)$ can be defined using resolutions. By [YONEDA 1954]

\(^{(2)}\)See Theorem 2.49 in [MILNE 1994], where this is credited to GROTHENDIECK.
these functors have a definition independent of the existence of resolutions, valid on any abelian category. This generalises the identification of $\text{Ext}^1$ with the group of extensions under Baer sum.

7.4.2. On the abelian category $t\mathcal{M}^\circ$ they vanish:

**Theorem.** $\text{Ext}^i_{t\mathcal{M}^\circ}(-,-) = 0$, for all $i > 1$.

**Corollary.** $\text{Ext}^i_{t\mathcal{M}^\circ_{\text{eff}}}(−,−) = 0$, for all $i > 1$.  

*Proof of the Theorem.* Clearly it suffices to show that the higher Ext are trivial on $t\mathcal{M}^\circ_{\text{eff}}$. Denote by $\mathcal{C}$ the category of left modules over the ring $K[t,\sigma] \otimes k[t]$.

The functor

$$M \mapsto M \otimes_{k[t]} k(t)$$

defines a fully faithful embedding of $t\mathcal{M}^\circ_{\text{eff}}$ into $\mathcal{C}$. This induces for every pair $M_1, M_2$ of effective $t$-motifs natural maps

$$\phi^i : \text{Ext}^i_{t\mathcal{M}^\circ_{\text{eff}}}(M_1, M_2) \to \text{Ext}^i_{\mathcal{C}}(M_1, M_2).$$

The category $\mathcal{C}$ has sufficient projectives and hence the target groups can be calculated using resolutions. In fact, using a $K[t]$-basis of an effective $t$-motif $M$ as a set of generators in $\mathcal{C}$ one sees that every effective $t$-motif $M$ has a free resolution of length at most 1 in $\mathcal{C}$: the free set of relations expresses the action of $\sigma$ on the basis. Thus for $i > 1$ the target groups of $\phi^i$ vanish.

Proposition 3.3 of [OORT 1964] asserts that if for some $i$ and all $M_1$ and $M_2$ the map $\phi^i$ is bijective, then (for all $M_1$ and $M_2$) the map $\phi^{i+1}$ is injective. Thus the Theorem will be shown as soon as $\phi^1$ is bijective.

This is indeed the case:

$$\text{Ext}^1_{t\mathcal{M}^\circ_{\text{eff}}}(M_1, M_2) = \text{coker}(H \xrightarrow{\delta} H') \otimes_{k[t]} k(t)$$

$$= \text{coker}(H \otimes k(t) \xrightarrow{\delta \otimes k(t)} H' \otimes k(t))$$

$$= \text{Ext}^1_{\mathcal{C}}(M_1 \otimes k(t), M_2 \otimes k(t)),$$

using the flatness of the $k[t]$-module $k(t)$.  

\[\square\]