On t-Motifs
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Chapter 7

A Few Notes on Extensions

7.1 Extensions of Effective $t$-Motifs

7.1.1. Let $0 \to M_2 \to M \to M_1 \to 0$ be an exact sequence of $K[t, \sigma]$-modules. If $M_1$ and $M_2$ are effective $t$-motifs, then so is $M$. We say that $M$ is an extension of the effective $t$-motif $M_1$ by $M_2$. The set of isomorphism classes of extensions forms an abelian group $\text{Ext}_{t,M_{\text{eff}}}(M_1, M_2)$ under the Baer sum\(^{(1)}\) and this group is naturally a $k[t]$-module.

7.1.2. This module can be given a more explicit description. Put $H \overset{\text{def}}{=} \text{Hom}_{K[t]}(M_1, M_2)$, the space of linear maps of $M_1$ to $M_2$, and write $H'$ for the space of semi-linear maps of $M_1$ to $M_2$. The $K[t]$-modules $H$ and $H'$ are isomorphic, since both are free of rank $\text{rk}(M_1) \text{rk}(M_2)$, but there is no natural isomorphism (unless $K = k$). Consider the $k[t]$-linear map

$$\delta : H \to H' : f \mapsto \sigma_2 \circ f - f \circ \sigma_1.$$ 

Note that $\text{ker}(\delta) = \text{Hom}(M_1, M_2)$. We contend that

$$\text{coker}(\delta) = \text{Ext}_{t,M_{\text{eff}}}(M_1, M_2). \quad (7.1)$$

In fact, if $M$ is an extension of $M_1$ by $M_2$ then as $K[t]$-modules $M \cong M_1 \oplus M_2$ and $\sigma(m_1, m_2) = (\sigma_1(m_1), \sigma_2(m_2) + \gamma(m_1))$ with $\gamma \in H'$. This

\(^{(1)}\text{See [Baer 1934] or Chapter XIV of [Cartan and Eilenberg 1956].}\)
extension splits if and only if there exists a linear \( f : M_1 \to M_2 \) such that \( \gamma = \delta(f) \).

7.1.3. The \( k[t] \)-module \( \text{Ext} \) is almost a divisible module:

**Proposition.** If \( K = \mathbb{K}^2 \) and \( \lambda \in k[t] \) has an invertible image under \( k[t] \to K \) then multiplication with \( \lambda \) is surjective on \( \text{Ext}_{\text{eff}}(M_1, M_2) \).

**Proof of the proposition.** Consider the commutative diagram

\[
\begin{array}{c}
0 \to H \xrightarrow{\lambda} H \to H/\lambda H \to 0 \\
\downarrow \delta \downarrow \delta \downarrow \delta \\
0 \to H' \xrightarrow{\lambda} H' \to H'/\lambda H' \to 0
\end{array}
\]

with exact rows. This gives an exact sequence of cokernels

\[
\cdots \to \text{Ext}(M_1, M_2) \xrightarrow{\lambda} \text{Ext}(M_1, M_2) \to \text{coker}(\delta) \to 0.
\]

It remains to show that \( \text{coker}(\delta) = 0 \), that is, that \( \delta \) is surjective. The vector space \( H/\lambda H \) is of finite dimension over \( K \). Since \( t - \theta \) is invertible in \( K[t]/\lambda K[t] \) (here the condition on \( \lambda \) is needed), the map \( \tilde{\delta} \) is the sum of a non-degenerate semi-linear map (\( \sigma_2 \circ f' \)) and a linear isomorphism (\( f \circ \sigma_1' \)). It follows from Corollary b.2.2 that \( \tilde{\delta} \) is surjective.

\[\Box\]

7.2 \( \text{Ext}(1,1) \)

7.2.1. As was already indicated in §5.2, knowledge about the group of extensions \( \text{Ext}(1,1) \) in some Tannakian category leads to information on the underlying fundamental group. We have calculated \( \text{Ext}(1,1) \) in the category of constant \( t \)-motifs (essentially 5.2.2) and deduced from it the Artin-Schreier Theorem on degree \( p \) extensions in characteristic \( p \). Similarly we calculated \( \text{Ext}(1,1) \) in the category of interior \( t \)-motifs (5.2.3), to find that the abelianisation of the affine group scheme involved is of multiplicative type. In this section we will repeat the exercise and calculate \( \text{Ext}(1,1) \) in the categories of \( t \)-motifs.
7.2.2. One should realise that a $t$-motif that is an extension of two effective $t$-motifs need not be effective. In fact, it follows from the definition of the category $t\mathcal{M}$ that for every pair $M_1, M_2$ of $t$-motifs

$$\text{Ext}_{t\mathcal{M}}(M_1, M_2) = \lim_{\rightarrow n} \text{Ext}_{t\mathcal{M}_{\text{eff}}}(M_1 \otimes C^n, M_2 \otimes C^n)$$

where the limit is for increasing $n$, starting at a sufficiently large value for the right-hand-side to make sense.

7.2.3. For $\text{Ext}(1, 1)$ in $t\mathcal{M}^\circ$ we have:

**Proposition.** If $K = K^\circ$ then

$$\text{Ext}_{t\mathcal{M}^\circ}(1, 1) = \begin{cases} \bigcup_{n \geq 0} (t - \theta)^{-n}K[t]/K[t] & \text{if } k[t] \to K \text{ is injective} \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** By 7.1.2, the group of effective extensions of $C^n$ by $C^n$ is the cokernel of the map

$$\delta : K[t] \to K[t] : f \mapsto (t - \theta)^n(\sigma(f) - f)).$$

By the Artin-Schreier Theorem ($K$ is separably closed), the image of $\delta$ is $(t - \theta)^nK[t]$, hence in the category $t\mathcal{M}$ we have

$$\text{Ext}_{t\mathcal{M}}(1, 1) = \bigcup_{n \geq 0} (t - \theta)^{-n}K[t]/K[t].$$

From this the Ext in the category $t\mathcal{M}^\circ$ of $t$-motifs up to isogeny can be calculated: use that

$$\text{Ext}_{t\mathcal{M}^\circ}(-, -) = \text{Ext}_{t\mathcal{M}}(-, -) \otimes_{k[t]} k(t)$$

to obtain the modules as stated in the Proposition. \hfill \square

7.2.4. If $k[t] \to K$ is injective then the above calculation also yields the $\text{Ext}(1, 1)$ in the Tannakian category $t\mathcal{M}_{a.a}^\circ$. In fact,

**Lemma.** Every extension of an analytically trivial $t$-motif by an analytically trivial $t$-motif is itself analytically trivial.
and therefore
\[
\text{Hom}(\Gamma, \mathbb{G}_{a,k(t)}) = \text{Ext}_{t,M^\circ}^1(1,1) \\
= \text{Ext}_{t,M^\circ}^1(1,1).
\]
This gives a description of the ‘additive part’ of the abelianisation of the affine group scheme \( \Gamma \).

**Proof of the Lemma.** Let \( E \) be an extension of \( M_2 \) by \( M_1 \), where both \( M_i \) are analytically trivial. This yields an exact sequence

\[
0 \rightarrow M_1 \{t\} \rightarrow E \{t\} \rightarrow M_2 \{t\} \rightarrow 0.
\]

Both \( M_1 \{t\} \) and \( M_2 \{t\} \) have a \( \sigma \)-invariant basis, and these define a basis for \( E \{t\} \) on which \( \sigma \) acts by an upper triangular block matrix. To show that \( E \{t\} \) is analytically trivial (has an invariant basis) it hence suffices to show that the map

\[
K^\dagger \{t\} \rightarrow K^\dagger \{t\} : \sum a_i t^i \mapsto \sum (a_i^q - a_i) t^i
\]

is surjective. This is immediate from the observation that if \( b \in K^\dagger \) with \( \|b\| \leq 1 \) then the equation

\[
x^q - x = b
\]

has a (in fact, unique) solution with \( \|x\| = \|b\| \).

### 7.3 \( t \)-Motifs over Finite Fields

**7.3.1.** The situation is quite a bit simpler when \( K \) is a finite field.

**Proposition.** Let \( K \) be a finite field and \( M_1 \) and \( M_2 \) be two effective \( t \)-motifs over \( K \). Then \( \text{Ext}_{t,M^\circ}^1(M_1, M_2) \) is a finitely generated \( k[t] \)-module and

\[
\text{rk}_{k[t]} \text{Hom}(M_1, M_2) = \text{rk}_{k[t]} \text{Ext}_{t,M^\circ}^1(M_1, M_2).
\]

Note that while in \( t,M^\circ \) the \( k(t) \)-space \( \text{Ext}(1,1) \) is one-dimensional when \( K \) is a finite field, it vanishes when \( K \) is the algebraic closure of a finite field (7.2.3).
Proof of the Proposition. Take \( H, H' \) and \( \delta \) as in 7.1.2. Then

\[
0 \rightarrow \text{Hom}(M_1, M_2) \rightarrow H \overset{\delta}{\rightarrow} H' \rightarrow \text{Ext}(M_1, M_2) \rightarrow 0
\]

is an exact sequence of \( k[t] \)-modules. Since \( K \) is finite over \( k \), the modules \( H \) and \( H' \) are free and of the same finite rank over \( k[t] \), whence the claims of the Proposition. \( \square \)

7.3.2. In particular, when \( M_1 \) and \( M_2 \) are pure of different weights, the module of homomorphisms is trivial (6.2.2) and therefore \( \text{Ext}(M_1, M_2) \) is torsion. This is in line with Algebraic Geometry, where it is expected that every mixed motif over a finite field and with \( \mathbb{Q} \)-coefficients decomposes as a direct sum of pure motifs.\(^{(2)}\) Only, here we have to deal with the pathology that there exist \( t \)-motifs that do not have a filtration with pure quotients. We shall proceed immediately to exhibit an example.

7.3.3. Let \( \theta = 0 \), that is, \( K \) has ‘characteristic \( t' \). Consider the effective \( t \)-motif

\[
M = K[t]e_1 \oplus K[t]e_2 \quad \text{with} \quad \begin{cases} 
\sigma(e_1) = te_1 + e_2 \\
\sigma(e_2) = te_1
\end{cases}
\]

Proposition. \( M \) has weights 0 and 1, yet \( M_K \) has no proper pure sub-\( t \)-motifs.

Sketch of proof. On the given basis, the characteristic polynomial of \( \sigma \) is \( f(X) = X^2 - tX - t \). Using the Newton polygon, one verifies that the valuations of zeroes of \( f \) are 0 and \(-1\), whence the weights are 0 and 1.

If \( M \) contains pure sub-\( t \)-motif then it is either isomorphic to \( 1 \) or to \( C \). In other words, \( M \) must contain a vector \( v = ae_1 + be_2 \) such that either \( \sigma(v) = v \) or \( \sigma(v) = tv \). The former can be excluded by an argument on the degrees of \( a \) and \( b \), the latter by a calculation modulo \( t \). \( \square \)

7.4 Higher \( \text{Ext} \)

7.4.1. If an abelian category has sufficient injectives or projectives then functors \( \text{Ext}^j(-, -) \) can be defined using resolutions. By [YONEDA 1954]

\( ^{(2)} \)See Theorem 2.49 in [MILNE 1994], where this is credited to GROTHENDIECK.

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these functors have a definition independent of the existence of resolutions, valid on any abelian category. This generalises the identification of Ext\(^1\) with the group of extensions under Baer sum.

7.42. On the abelian category \(tM^\circ\) they vanish:

**Theorem.** \(\text{Ext}^i_{tM^\circ}(\cdot,\cdot) = 0\), for all \(i > 1\).

**Corollary.** \(\text{Ext}^i_{tM^\circ_{\text{eff}}}(\cdot,\cdot) = 0\), for all \(i > 1\).

**Proof of the Theorem.** Clearly it suffices to show that the higher Ext are trivial on \(tM^\circ_{\text{eff}}\). Denote by \(\mathcal{C}\) the category of left modules over the ring

\[
K[t, \sigma] \otimes_{k[t]} k(t).
\]

The functor

\[
M \mapsto M \otimes_{k[t]} k(t)
\]

defines a fully faithful embedding of \(tM^\circ_{\text{eff}}\) into \(\mathcal{C}\). This induces for every pair \(M_1, M_2\) of effective \(t\)-motifs natural maps

\[
\phi^i : \text{Ext}^i_{tM^\circ_{\text{eff}}}(M_1, M_2) \to \text{Ext}^i_{\mathcal{C}}(M_1, M_2).
\]

The category \(\mathcal{C}\) has sufficient projectives and hence the target groups can be calculated using resolutions. In fact, using a \(K[t]\)-basis of an effective \(t\)-motif \(M\) as a set of generators in \(\mathcal{C}\) one sees that every effective \(t\)-motif \(M\) has a free resolution of length at most 1 in \(\mathcal{C}\): the free set of relations expresses the action of \(\sigma\) on the basis. Thus for \(i > 1\) the target groups of \(\phi^i\) vanish.

Proposition 3.3 of [OORT 1964] asserts that if for some \(i\) and all \(M_1\) and \(M_2\) the map \(\phi^i\) is bijective, then (for all \(M_1\) and \(M_2\)) the map \(\phi^{i+1}\) is injective. Thus the Theorem will be shown as soon as \(\phi^1\) is bijective.

This is indeed the case:

\[
\begin{align*}
\text{Ext}^1_{tM^\circ_{\text{eff}}}(M_1, M_2) &= \text{coker}(H \to H') \otimes_{k[t]} k(t) \\
&= \text{coker}(H \otimes k(t) \xrightarrow{\delta \otimes k(t)} H' \otimes k(t)) \\
&= \text{Ext}^1_{\mathcal{C}}(M_1 \otimes k(t), M_2 \otimes k(t)),
\end{align*}
\]

using the flatness of the \(k[t]\)-module \(k(t)\). \(\square\)