Chapter 7

A Few Notes on Extensions

7.1 Extensions of Effective $t$-Motifs

7.1.1. Let $0 \to M_2 \to M \to M_1 \to 0$ be an exact sequence of $K[t, \sigma]$-modules. If $M_1$ and $M_2$ are effective $t$-motifs, then so is $M$. We say that $M$ is an extension of the effective $t$-motif $M_1$ by $M_2$. The set of isomorphism classes of extensions forms an abelian group $\text{Ext}_{tM_{\text{eff}}}(M_1, M_2)$ under the Baer sum\(^{(1)}\) and this group is naturally a $k[t]$-module.

7.1.2. This module can be given a more explicit description. Put $H \overset{\text{def}}{=} \text{Hom}_{K[t]}(M_1, M_2)$, the space of linear maps of $M_1$ to $M_2$, and write $H'$ for the space of semi-linear maps of $M_1$ to $M_2$. The $K[t]$-modules $H$ and $H'$ are isomorphic, since both are free of rank $\text{rk}(M_1) \text{rk}(M_2)$, but there is no natural isomorphism (unless $K = k$). Consider the $k[t]$-linear map

$$\delta : H \to H' : f \mapsto \sigma_2 \circ f - f \circ \sigma_1.$$ 

Note that $\ker(\delta) = \text{Hom}(M_1, M_2)$. We contend that

$$\text{coker}(\delta) = \text{Ext}_{tM_{\text{eff}}}(M_1, M_2). \quad (7.1)$$

In fact, if $M$ is an extension of $M_1$ by $M_2$ then as $K[t]$-modules $M \cong M_1 \oplus M_2$ and $\sigma(m_1, m_2) = (\sigma_1(m_1), \sigma_2(m_2) + \gamma(m_1))$ with $\gamma \in H'$. This

\(^{(1)}\)See \cite{Baer 1934} or Chapter XIV of \cite{Cartan and Eilenberg 1956}. 

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extension splits if and only if there exists a linear $f : M_1 \to M_2$ such that $\gamma = \delta(f)$.

**7.1.3.** The $k[t]$-module $\text{Ext}$ is almost a divisible module:

**Proposition.** If $K = \mathbb{K}^2$ and $\lambda \in k[t]$ has an invertible image under $k[t] \to K$ then multiplication with $\lambda$ is surjective on $\text{Ext}_{\mathcal{M}_{\text{eff}}}(M_1, M_2)$.

**Proof of the proposition.** Consider the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & H \\
\downarrow{\delta} & & \downarrow{\delta} \\
0 & \longrightarrow & H'
\end{array}
\quad
\begin{array}{ccc}
\longrightarrow & \longrightarrow & H/\lambda H \\
\longrightarrow & \longrightarrow & H'/\lambda H' \\
\longrightarrow & \longrightarrow & 0
\end{array}
$$

with exact rows. This gives an exact sequence of cokernels

$$
\cdots \to \text{Ext}(M_1, M_2) \xrightarrow{\lambda} \text{Ext}(M_1, M_2) \to \text{coker}(\bar{\delta}) \to 0.
$$

It remains to show that $\text{coker}(\bar{\delta}) = 0$, that is, that $\bar{\delta}$ is surjective. The vector space $H/\lambda H$ is of finite dimension over $K$. Since $t - \theta$ is invertible in $K[t]/\lambda K[t]$ (here the condition on $\lambda$ is needed), the map $\bar{\delta}$ is the sum of a non-degenerate semi-linear map $(\sigma_2 \circ f)$ and a linear isomorphism $(f \circ \sigma_1)$. It follows from Corollary b.2.2 that $\bar{\delta}$ is surjective.  

**7.2  \text{Ext}(1,1)**

**7.2.1.** As was already indicated in §5.2, knowledge about the group of extensions $\text{Ext}(1,1)$ in some Tannakian category leads to information on the underlying fundamental group. We have calculated $\text{Ext}(1,1)$ in the category of constant $t$-motifs (essentially 5.2.2) and deduced from it the Artin-Schreier Theorem on degree $p$ extensions in characteristic $p$. Similarly we calculated $\text{Ext}(1,1)$ in the category of interior $t$-motifs (5.2.3), to find that the abelianisation of the affine group scheme involved is of multiplicative type. In this section we will repeat the exercise and calculate $\text{Ext}(1,1)$ in the categories of $t$-motifs.
7.2.2. One should realise that a \( t \)-motif that is an extension of two effective \( t \)-motifs need not be effective. In fact, it follows from the definition of the category \( tM \) that for every pair \( M_1, M_2 \) of \( t \)-motifs

\[
\text{Ext}_{tM}(M_1, M_2) = \lim_{n \to \infty} \text{Ext}_{tM_{\text{eff}}}(M_1 \otimes C^n, M_2 \otimes C^n)
\]

where the limit is for increasing \( n \), starting at a sufficiently large value for the right-hand-side to make sense.

7.2.3. For \( \text{Ext}(1,1) \) in \( tM^\circ \) we have:

**Proposition.** If \( K = K^\circ \) then

\[
\text{Ext}_{tM^\circ}(1,1) = \begin{cases} 
\bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t] & \text{if } k[t] \to K \text{ is injective} \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** By 7.1.2, the group of effective extensions of \( C^n \) by \( C^n \) is the cokernel of the map

\[
\delta : K[t] \to K[t] : f \mapsto (t - \theta)^n (\sigma(f) - f).
\]

By the Artin-Schreier Theorem (\( K \) is separably closed), the image of \( \delta \) is \( (t - \theta)^n K[t] \), hence in the category \( tM \) we have

\[
\text{Ext}_{tM}(1,1) = \bigcup_{n \geq 0} (t - \theta)^{-n} K[t]/K[t].
\]

From this the \( \text{Ext} \) in the category \( tM^\circ \) of \( t \)-motifs up to isogeny can be calculated: use that

\[
\text{Ext}_{tM^\circ}(-,-) = \text{Ext}_{tM}(-,-) \otimes_{k[t]} k(t)
\]

to obtain the modules as stated in the Proposition. \( \Box \)

7.2.4. If \( k[t] \to K \) is injective then the above calculation also yields the \( \text{Ext}(1,1) \) in the Tannakian category \( tM^\circ_{\text{a.t.}} \). In fact,

**Lemma.** Every extension of an analytically trivial \( t \)-motif by an analytically trivial \( t \)-motif is itself analytically trivial.
and therefore
\[ \text{Hom}(\Gamma, \mathbb{G}_{a,k(t)}) = \text{Ext}_{\mathcal{M}^0}^{\mathbb{A}^1}(1,1) = \text{Ext}_{\mathcal{M}^0}(1,1). \]

This gives a description of the ‘additive part’ of the abelianisation of the affine group scheme \( \Gamma \).

**Proof of the Lemma.** Let \( E \) be an extension of \( M_2 \) by \( M_1 \), where both \( M_i \) are analytically trivial. This yields an exact sequence
\[ 0 \to M_1\{t\} \to E\{t\} \to M_2\{t\} \to 0. \]
Both \( M_1\{t\} \) and \( M_2\{t\} \) have a \( \sigma \)-invariant basis, and these define a basis for \( E\{t\} \) on which \( \sigma \) acts by an upper triangular block matrix. To show that \( E\{t\} \) is analytically trivial (has an invariant basis) it hence suffices to show that the map
\[ K^+\{t\} \to K^+\{t\} : \sum_i a_i t^i \mapsto \sum_i (a^d_i - a_i) t^i \]
is surjective. This is immediate from the observation that if \( b \in K^+ \) with \( \|b\| \leq 1 \) then the equation
\[ x^d - x = b \]
has a (in fact, unique) solution with \( \|x\| = \|b\|. \)

### 7.3 \( t \)-Motifs over Finite Fields

**7.3.1.** The situation is quite a bit simpler when \( K \) is a finite field.

**Proposition.** Let \( K \) be a finite field and \( M_1 \) and \( M_2 \) be two effective \( t \)-motifs over \( K \). Then \( \text{Ext}_{\mathcal{M}^0}(M_1, M_2) \) is a finitely generated \( k[t] \)-module and
\[ \text{rk}_{k[t]} \text{Hom}(M_1, M_2) = \text{rk}_{k[t]} \text{Ext}_{\mathcal{M}^0}(M_1, M_2). \]

Note that while in \( t\mathcal{M}^0 \) the \( k(t) \)-space \( \text{Ext}(1,1) \) is one-dimensional when \( K \) is a finite field, it vanishes when \( K \) is the algebraic closure of a finite field (7.2.3).
Proof of the Proposition. Take $H$, $H'$ and $\delta$ as in 7.1.2. Then
\[
0 \to \text{Hom}(M_1, M_2) \to H \xrightarrow{\delta} H' \to \text{Ext}(M_1, M_2) \to 0
\]
is an exact sequence of $k[t]$-modules. Since $K$ is finite over $k$, the modules $H$ and $H'$ are free and of the same finite rank over $k[t]$, whence the claims of the Proposition.

7.3.2. In particular, when $M_1$ and $M_2$ are pure of different weights, the module of homomorphisms is trivial (6.2.2) and therefore $\text{Ext}(M_1, M_2)$ is torsion. This is in line with Algebraic Geometry, where it is expected that every mixed motif over a finite field and with $\mathbb{Q}$-coefficients decomposes as a direct sum of pure motifs.\(^{(2)}\) Only, here we have to deal with the pathology that there exist $t$-motifs that do not have a filtration with pure quotients. We shall proceed immediately to exhibit an example.

7.3.3. Let $\theta = 0$, that is, $K$ has ‘characteristic $t$’. Consider the effective $t$-motif
\[
M = K[t]e_1 \oplus K[t]e_2 \quad \text{with} \quad \begin{cases} 
\sigma(e_1) = te_1 + e_2 \\
\sigma(e_2) = te_1
\end{cases}
\]

Proposition. $M$ has weights 0 and 1, yet $M_{K^s}$ has no proper pure sub-$t$-motifs.

Sketch of proof. On the given basis, the characteristic polynomial of $\sigma$ is $f(X) = X^2 - tX - t$. Using the Newton polygon, one verifies that the valuations of zeroes of $f$ are 0 and $-1$, whence the weights are 0 and 1.

If $M$ contains pure sub-$t$-motif then it is either isomorphic to 1 or to $C$. In other words, $M$ must contain a vector $v = ae_1 + be_2$ such that either $\sigma(v) = v$ or $\sigma(v) = tv$. The former can be excluded by an argument on the degrees of $a$ and $b$, the latter by a calculation modulo $t$. \(\square\)

7.4 Higher Ext

7.4.1. If an abelian category has sufficient injectives or projectives then functors $\text{Ext}^i(-,-)$ can be defined using resolutions. By \[\text{Yoneda 1954}\]

\(^{(2)}\) See Theorem 2.49 in \[\text{Milne 1994}\], where this is credited to Grothendieck.
these functors have a definition independent of the existence of resolutions, valid on any abelian category. This generalises the identification of $\text{Ext}^1$ with the group of extensions under Baer sum.

7.4.2. On the abelian category $tM^\circ$ they vanish:

**Theorem.** $\text{Ext}^i_{tM^\circ}(\cdot,\cdot) = 0$, for all $i > 1$.

**Corollary.** $\text{Ext}^i_{tM^\circ_{\ell}}(\cdot,\cdot) = 0$, for all $i > 1$. □

**Proof of the Theorem.** Clearly it suffices to show that the higher Ext are trivial on $tM^\circ_{\text{eff}}$. Denote by $\mathcal{C}$ the category of left modules over the ring

$$K[t,\sigma] \otimes_{k[t]} k(t).$$

The functor

$$M \mapsto M \otimes_{k[t]} k(t)$$

defines a fully faithful embedding of $tM^\circ_{\text{eff}}$ into $\mathcal{C}$. This induces for every pair $M_1, M_2$ of effective $t$-motifs natural maps

$$\phi^i : \text{Ext}^i_{tM^\circ_{\text{eff}}}(M_1, M_2) \rightarrow \text{Ext}^i_{\mathcal{C}}(M_1, M_2).$$

The category $\mathcal{C}$ has sufficient projectives and hence the target groups can be calculated using resolutions. In fact, using a $K[t]$-basis of an effective $t$-motif $M$ as a set of generators in $\mathcal{C}$ one sees that every effective $t$-motif $M$ has a free resolution of length at most 1 in $\mathcal{C}$: the free set of relations expresses the action of $\sigma$ on the basis. Thus for $i > 1$ the target groups of $\phi^i$ vanish.

Proposition 3.3 of [OORT 1964] asserts that if for some $i$ and all $M_1$ and $M_2$ the map $\phi^i$ is bijective, then (for all $M_1$ and $M_2$) the map $\phi^{i+1}$ is injective. Thus the Theorem will be shown as soon as $\phi^1$ is bijective.

This is indeed the case:

$$\text{Ext}^1_{tM^\circ_{\text{eff}}}(M_1, M_2) = \text{coker}(H \xrightarrow{\delta} H') \otimes_{k[t]} k(t)$$

$$= \text{coker}(H \otimes k(t) \xrightarrow{\delta \otimes k(t)} H' \otimes k(t))$$

$$= \text{Ext}^1_{\mathcal{C}}(M_1 \otimes k(t), M_2 \otimes k(t)),$$

using the flatness of the $k[t]$-module $k(t)$. □