Chapter 6

Weights and Purity

6.1 Dieudonné $t$-Modules

6.1.1. Let $k$ and $K$ be as usual. Denote by $\tau$ the continuous endomorphism of the field of Laurent series $K((t^{-1}))$ that fixes $t^{-1}$ and that restricts to the $q$-th power map on $K$.

**Definition.** A Dieudonné $t$-module$^{(1)}$ over $K$ is a pair $(V, \sigma)$ of

- a finite-dimensional $K((t^{-1}))$-vector space $V$ and
- an additive map $\sigma : V \to V$ satisfying $\sigma(fv) = \tau(f)\sigma(v)$ for all $f \in K((t^{-1}))$ and all $v \in V$,

such that $K\sigma(V) = V$.

A morphism of Dieudonné $t$-modules is of course a $K((t^{-1}))$-linear map commuting with $\sigma$.

6.1.2. Dieudonné $t$-modules are easily classified, at least over a separably closed field. The main ‘building blocks’ are the following modules:

**Definition.** Let $\lambda = s/r$ be a rational number with $(r, s) = 1$ and $r > 0$. The Dieudonné $t$-module $V_\lambda$ is defined to be the pair $(V_\lambda, \sigma)$ with

- $V_\lambda \overset{\text{def}}{=} K((t^{-1}))e_1 \oplus \ldots \oplus K((t^{-1}))e_r$

$^{(1)}$This the equal characteristic analogue of the $p$-adic object that is commonly called a Dieudonné module.
\( \sigma(e_i) \overset{\text{def}}{=} e_{i+1} \quad (i < r) \) and \( \sigma(e_r) \overset{\text{def}}{=} t^se_1 \)

The classification states:

**Proposition.** If \( V \) is a Dieudonné t-module over a separably closed field \( K \) then there exist rational numbers \( \lambda_1, \ldots, \lambda_n \) such that

- \( V \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \), and,
- the \( t^{-1} \)-adic valuations of the roots of the characteristic polynomial of \( \sigma \) expressed on any \( K((t^{-1})) \)-basis are \( \{-\lambda_i\}_{i} \), each counted with multiplicity \( \dim V_{\lambda_i} \).

If \( \lambda \neq \mu \) then \( \text{Hom}(V_{\lambda}, V_{\mu}) = 0 \). For all \( \lambda \), the ring \( \text{End}(V_{\lambda}) \) is a division algebra over \( k((t^{-1})) \). Its Brauer class is \( \lambda + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} = \text{Br}(k((t^{-1}))). \)

Note that this classification is formally identical to the classification of the classical (\( p \)-adic) Dieudonné modules.\(^{(2)}\)

**Proof.** This is shown in Appendix B of [Laumon 1996]. Although the statements therein are made only for a particular field \( K \), nowhere do the proofs make use of anything stronger then the separably closedness of \( K \). \( \square \)

6.1.3. The following characterisation of \( V_{\lambda} \) is useful.

**Proposition.** Let \( V \) be a Dieudonné t-module over a separably closed field \( K \) and \( \lambda \) a rational number. The following are equivalent:

- \( V \approx V_{\lambda} \oplus V_{\lambda} \oplus \cdots \oplus V_{\lambda} \);
- there exists a lattice \( \Lambda \subset V \) such that \( \sigma^r(\Lambda) = t^s\Lambda \) where \( r \) and \( s \) are coprime integers with \( \lambda = s/r \).

**Proof.** One \( \Rightarrow \) Two. If \( V = V_{\lambda} \) and \( (e_i) \) the basis that occurs in its definition (6.1.2) then the lattice generated by the same basis \( (e_i) \) has the required property. For \( V = V_{\lambda} \oplus \cdots \oplus V_{\lambda} \) it thus suffices to take the lattice \( \Lambda \oplus \cdots \oplus \Lambda \).

\(^{(2)}\) See [Dieudonné 1957].
Two ⇒ One. The operator $t^{-s} \sigma^r$ transforms a $K[[t^{-1}]]$-basis of $\Lambda$ into a new $K[[t^{-1}]]$-basis of $\Lambda$ and therefore has eigenvalues of valuation 0. □

6.2 Pure $t$-Motifs

6.2.1. Let $K$ be separably closed. Let $M$ be an effective $t$-motif over $K$. Then

$$M((t^{-1})) \overset{\text{def}}{=} M \otimes_{K[[t]]} K((t^{-1})) = M(t) \otimes_{K(t)} K((t^{-1}))$$

is a Dieudonné $t$-module. The displayed equality shows that it only depends on the isogeny class of $M$. By the classification of Dieudonné $t$-modules (6.1.2) there exist rational numbers $\lambda_1, \ldots, \lambda_n$ such that

$$M((t^{-1})) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}.$$  

We call these rational numbers the weights of $M$. If $K$ is not separably closed than we define the weights of an effective $t$-motif $M$ to be the weights of $M_{K'}$. This clearly does not depend on the choice of a separable closure.

We say that $M$ is pure of weight $\lambda$ if the only weight occurring is $\lambda$. By Proposition 6.1.3, this coincides with the definition as given in [Anderson 1986].

6.2.2. We now collect a number of facts related to the notions of weights and purity. They are either immediate consequences of the definitions or well-known facts established in the literature.

**Proposition.** We have the following:

- If $M$ is pure of weight $\lambda$ then every subquotient of $M$ is pure of weight $\lambda$;
- If $M$ has a filtration in which all successive quotients are pure of weight $\lambda$, then $M$ is pure of weight $\lambda$;
- If the sets of weights of $M_1$ and $M_2$ are disjoint then $\text{Hom}(M_1, M_2) = 0$;
- Drinfeld modules of rank $r$ are pure of weight $1/r$ (in particular: $C$ is pure of weight $1$);
• The weights of $M_1 \otimes M_2$ are the sums of weights of $M_1$ with those of $M_2$.
• The weight of a pure effective $t$-motif $M$ is non-negative.

Proofs. One. If $M'$ is a subquotient of $M$ then $M'((t^{-1}))$ is a subquotient of $M((t^{-1}))$ and the claimed statement follows at once from the Classification 6.1.2.

Two. A normal series of $M$ induces a normal series of $M((t^{-1}))$ and again the contention follows from 6.1.2.

Three. $\text{Hom}(M_1, M_2)$ is a submodule of $\text{Hom}(M_1((t^{-1})), M_2((t^{-1})))$, which is zero by 6.1.2.

Four. See Proposition 4.1.1. of [Anderson 1986].

Five. Immediate since the zeroes of the characteristic polynomials are multiplied.

Six. Clear for rank one $M$, for a general $M$ take the top exterior power. $\square$

6.2.3. If $M$ is an effective $t$-motif and

$$M((t^{-1})) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$$

then by the Proposition

$$(M \otimes C)((t^{-1})) \approx V_{\lambda_{1}+1} \oplus \cdots \oplus V_{\lambda_{n}+1}.$$ 

It is thus natural to define the weights of a $t$-motif $(M, i)$ to be the set of $\lambda + i$ where $\lambda$ runs through the weights of $M$. To be consistent, a $t$-motif $(M, i)$ is then said to be pure of weight $\lambda$ if and only if $M$ is pure of weight $\lambda - i$.

### 6.3 A Digression on Brauer Groups

6.3.1. Let $M$ be a $t$-motif that is pure of weight $\lambda$. Thus $M((t^{-1})) \approx nV_\lambda$ and by Proposition 6.1.2 the endomorphism ring $\text{End}(M((t^{-1})))$ is a central simple algebra whose class in the Brauer group of $k((t^{-1}))$ is $\lambda + \mathbb{Z}$. Thus, the map

$$\{\text{weights}\} = \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} = \text{Br}(k((t^{-1})))$$

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has some kind of interpretation in terms of Dieudonné $t$-modules of pure $t$-motifs.

**6.3.2.** Classical motifs also have weights, and these weights form an infinite cyclic group. If we normalise things so that the Lefschetz motif has weight $1$, then the weight group is $\frac{1}{2}\mathbb{Z}$. A piece of the degree $i$ cohomology $h^i(X)$ then has weight $i/2$. All this seems to harmonise easily with the fact that the Brauer group of $\mathbb{R}$ is cyclic of order two—it suggests that the map

$$\{\text{weights}\} = \frac{1}{2}\mathbb{Z} \rightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z} = \text{Br}(\mathbb{R}),$$

can be interpreted in a fashion similar to the above.

**6.3.3.** We shall sketch one possible such interpretation, albeit a somewhat *ad hoc* one. Let $X$ be a smooth and projective variety over $\mathbb{C}$. Put $V = H^i(X(\mathbb{C}), \mathbb{C})$. The complex vector space $V$ comes equipped with a Hodge decomposition

$$V = \bigoplus_{p+q=i} H^{p,q}.$$

Let $\alpha$ be the anti-linear automorphism of the complexified co-tangent bundle of $X$ that is the composition of the linear automorphism ‘multiplication with $i$’ followed by complex conjugation. Then $\alpha$ induces an anti-linear automorphism $\alpha^*$ of $V$. On the Hodge decomposition it restricts to

$$\alpha^*: H^{p,q} \rightarrow H^{q,p}: c \mapsto i^{q-p} \overline{c}.$$  (6.1)

The endomorphisms of $V$ that commute with $\alpha^*$ form an $\mathbb{R}$-algebra denoted $\text{End}(V, \alpha^*)$. Starting from (6.1) an easy calculation yields

$$\text{End}(V, \alpha^*) \approx \begin{cases} M(n, \mathbb{R}) & \text{if } i \text{ even}, \\ M(n/2, \mathbb{H}) & \text{if } i \text{ odd}, \end{cases}$$

where $n$ stands for the dimension of $V$ and $\mathbb{H}$ for the algebra of Hamilton quaternions. We conclude that the two elements of the Brauer group of $\mathbb{R}$ correspond to the two weight classes modulo $\mathbb{Z}$.

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Footnote: This is *not* the customary normalisation—one usually assigns the weight 2 to the Lefschetz motif.