Chapter 4

Constant $t$-Motifs

4.1 Constant $t$-Motifs

4.1.1. Let us go back to the category of 1.1.2, whose objects are pairs $(V,\sigma)$ of a finite dimensional $K$-vector space $V$ equipped with a non-degenerate semilinear map $\sigma: V \to V$. We have seen in Theorem 1.1.2 that this category is equivalent to the category of $k$-linear continuous representations of $G_K = \text{Gal}(K^s/K)$, a fact that we could rephrase as: the category of pairs $(V,\sigma)$ is $k$-linear neutral Tannakian with fundamental group $G_K$. Note that we abusively write $G_K$ for both the pro-finite group and the corresponding constant affine group scheme over $k$ (obtained as the limit of the system of finite constant group schemes corresponding to the finite quotients of the pro-finite group.) Their categories of representations on finite dimensional $k$-vector spaces coincide.

4.1.2. A pair $(V,\sigma)$ induces an effective $t$-motif $M(V) \overset{\text{def}}{=} V \otimes_K K[t]$ where the action of $\sigma$ is induced from the action on $V$.

We would like to interpret the collection of $t$-motifs $M(V)$ as a Tannakian subcategory of $t,M^\otimes$, but there are of course many more morphisms $M(V_1) \to M(V_2)$ than morphisms $V_1 \to V_2$ and the kernel and cokernel of a morphism from $M(V_1)$ to $M(V_2)$ are typically not of the form $M(V)$.

**Proposition.** Let $M$ be an effective $t$-motif over $K$. The following are equiva-
lent:
- $M$ is isomorphic to a subquotient of $M(V)$ for some $V$,
- $M \otimes_K K^\sigma \approx n1$ for some $n$.

An $M$ satisfying the equivalent conditions is called a constant $t$-motif.

**Proof of the Proposition.** If $M$ is a subquotient of $M(V)$ then $M_{K^s}$ is a subquotient of $M(V_{K^s}) \approx m1$ and therefore $M_{K^s} \approx n1$.

Conversely, assume that $M_{K^s}$ has a basis of $\sigma$-invariant vectors. There exists some finite extension $K'/K$ inside $K^\sigma$ such that this basis is already defined over $K'$. The natural map $K[t] \to K'[t]$ defines the structure of a $K[t]$-module on $M'$. Denote it by $R_{K'/K}M'$ in order to distinguish it from the $K'[t]$-module $M'$. It is clear that $R_{K'/K}M'$ is naturally an effective $t$-motif over $K$ of rank $\text{rk}(M)[K':K]$. (Call it the Weil restriction\(^{(1)}\) of $M'$ from $K'$ to $K$.) But, $M$ is a submodule of $R_{K'/K}M' \otimes_K K^\sigma$ and the latter is isomorphic to $M(R_{K'/K}W)$ with $W$ the sum of a number of copies of $K'$ with the diagonal action of $\sigma$, whence the Proposition. \(\square\)

4.1.3. The full subcategory $tM_{\text{cst}}^\circ(K)$ of $tM^\circ(K)$ consisting of the constant (effective) $t$-motifs is rigid abelian $k(t)$-linear and has a fibre functor

$$M \leadsto (M \otimes_{K[t]} K^\sigma[t])^{\sigma} \otimes_{k[t]} k(t)$$

(4.1)

and with this fibre functor we have

**Proposition.** $tM_{\text{cst}}^\circ(K)$ is neutral Tannakian with fundamental group $G_K$.

Note that it is not needed to use analytic methods to obtain a fibre functor on constant $t$-motifs and in particular it is not needed to demand that $k[t] \to K$ be injective.

**Proof of the Proposition.** The functor $M(V) \leadsto H(V) \otimes_k k(t)$ induces a fully faithful embedding of $tM_{\text{cst}}^\circ(K)$ into the category of $k(t)$-linear representations of $G_K$. It will be essentially surjective as soon as every continuous $k(t)$-linear representation of $G_K$ is a subquotient of $H \otimes_k k(t)$.

\(^{(1)}\)After §1.3 of [Weil 1982].
for some $k$-linear representation $H$. This is indeed so, since every (algebraic, or continuous) representation of $G_K$ factors though a finite group $G$ and every representation of $G$ is a subquotient of the direct sum of a number of copies of the regular representation $k(t)[G]$, which is nothing but the regular representation $k[G]$ over $k$, tensored with $k(t)$. □

4.1.4. Constant $t$-motifs are the $t$-counterparts of the algebro-geometric Artin motifs (named after Emil Artin.) Let $\mathbb{Z} \to K$ be any field. Consider the category of smooth and projective varieties $X$ over $K$ that are of dimension zero. These are the spectra of the finite étale $K$-algebras and by Grothendieck’s formulation of Galois theory the category of such $X$ is equivalent to the category of finite $G_K$-sets. The motifs that are subquotients of the $h(X, \mathbb{Q})$ for zero-dimensional $X$ are called Artin motifs. They form a category which is equivalent to the category of $\mathbb{Q}$-linear representations of $G_K$. (2)

Thus sets have come to play the role of $k$-vector spaces. But then, the field of constants of $\mathbb{Z}$ is the hypothetical field with one element and vector spaces over this folkloric field are nothing but sets. (3)

4.2 The Connected Components of $\Gamma$

4.2.1. Suppose now that $k[t] \to K$ is actually injective. Choose $K^+ \supset K$ to be algebraically closed, complete and with $\|\theta\| > 1$. Let $K^s$ be the separable closure of $K$ inside $K^+$. For a constant $t$-motif $M$ we have that

\[
(M \otimes_{K[t]} K^s[t])^\sigma = (M \otimes_{K[t]} K^+(\{t\}))^\sigma.
\]

That is to say, $tM(K)^cst$ is a full sub-category of $tM(K)_{a.t.}$ and the analytic fibre functor on the latter extends the algebraic fibre functor on the former.

**Proposition.** There is a short exact sequence

\[
0 \to \Gamma_{K^s} \to \Gamma_K \to G_K \to 0
\]

(2) See §1.3 and §4.1 of [André 2004].
(3) See §13 of [Tits 1957].
of affine group schemes over $k(t)$.

4.2.2. Proof. The full subcategory $t\mathcal{M}^\circ_{\text{est}}(K)$ of $t\mathcal{M}^\circ_{\text{a.t.}}(K)$ is Tannakian with fundamental group $G_K$ (4.1.3) and is closed under subquotients in $t\mathcal{M}^\circ_{\text{a.t.}}$ by definition. This implies the existence of a faithfully flat, and hence surjective, morphism $\Gamma_K \to G_K$ of affine group schemes.$^{(4)}$

If $M$ is an effective $t$-motif over $K_s$, then it has a model $M'$ over a finite extension $K'$ of $K$. The $t$-motif $M$ is a submotif of $R_{K'/K} M' \otimes_K K^\circ$. Thus every $t$-motif over $K^\circ$ is a submotif of a $t$-motif that is already defined over $K$. It follows that the fully faithful functor $M \rightsquigarrow M_{K_s}$ from $t\mathcal{M}^\circ_{\text{a.t.}}(K)$ to $t\mathcal{M}^\circ_{\text{a.t.}}(K^\circ)$ defines a closed immersion $\Gamma_{K_s} \to \Gamma_K$.$^{(5)}$

The sequence is exact in the middle if and only if the representations of $\Gamma_K$ on which $\Gamma_{K_s}$ acts trivially are precisely those coming from a representation of $G_K$. In other words, the exactness is equivalent with the statement that a $t$-motif $M$ over $K$ satisfies $M_{K_s} \approx n1$ for some $n$ if and only if it is a constant $t$-motif. This was one of the equivalent definitions of the notion of a constant $t$-motif (see 4.1.2).

4.2.3. The following Theorem complements the Proposition.

**Theorem.** $\Gamma_{K_s}$ has no finite quotients. In particular it is connected.

*First part of the proof.* Note that $\Gamma \to \pi_0(\Gamma)$ is a pro-finite étale quotient, hence the second statement indeed follows from the first.

Let $G$ be a finite quotient of $\Gamma_{K_s}$. To this there corresponds a Tannakian subcategory $C$ of $t\mathcal{M}^\circ_{\text{a.t.}}(K^\circ)$, equivalent to the category of representations of $G$. Since $G$ is finite, $C$ contains a $t$-motif $M$ such that every $t$-motif in $C$ is a subquotient of $nM$ for some $n$. (It suffices to take the $M$ corresponding to the regular representation of $G$.) The algebraic group $G$ is trivial if and only if $M$ is constant.

Write $M$ as $(M',i)$ with $i$ maximal. $M \otimes M$ is a subquotient of $nM$ for $n$ sufficiently large. Equivalently, $M' \otimes M' \otimes C^i$ is a subquotient of $nM'$ and since subquotients of effective $t$-motifs are effective, it follows that $M' \otimes M' \otimes C^i$ is effective. If $i$ is negative, then this implies that the action

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$^{(4)}$See for example Proposition 2.21 (a) of [Deligne and Milne 1982].

$^{(5)}$See Proposition 2.21 (b) of loc. cit.
of $\sigma$ on $M' \otimes M'$ is divisible by $t - \theta$ and hence also that the action of $\sigma$ on $M'$ is divisible by $t - \theta$, contradicting the maximality of $i$. Therefore $i \geq 0$ and $M = (M', i)$ is effective.

Using analytic methods, it will be shown in Chapter 8 that if $M$ is an analytically trivial effective $t$-motif so that $M \otimes M$ is a subquotient of $nM$ for some $n$, then $M$ is constant, hence $G$ trivial, and the Theorem follows... (to be continued in 8.4.2)