On t-Motifs
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Chapter 4

Constant $t$-Motifs

4.1 Constant $t$-Motifs

4.1.1. Let us go back to the category of 1.1.2, whose objects are pairs $(V, \sigma)$ of a finite dimensional $K$-vector space $V$ equipped with a non-degenerate semilinear map $\sigma : V \to V$. We have seen in Theorem 1.1.2 that this category is equivalent to the category of $k$-linear continuous representations of $G_K = \text{Gal}(K^s/K)$, a fact that we could rephrase as: the category of pairs $(V, \sigma)$ is $k$-linear neutral Tannakian with fundamental group $G_K$. Note that we abusively write $G_K$ for both the pro-finite group and the corresponding constant affine group scheme over $k$ (obtained as the limit of the system of finite constant group schemes corresponding to the finite quotients of the pro-finite group.) Their categories of representations on finite dimensional $k$-vector spaces coincide.

4.1.2. A pair $(V, \sigma)$ induces an effective $t$-motif $M(V) \overset{\text{def}}{=} V \otimes_K K[t]$ where the action of $\sigma$ is induced from the action on $V$.

We would like to interpret the collection of $t$-motifs $M(V)$ as a Tannakian subcategory of $t\mathcal{M}$, but there are of course many more morphisms $M(V_1) \to M(V_2)$ than morphisms $V_1 \to V_2$ and the kernel and cokernel of a morphism from $M(V_1)$ to $M(V_2)$ are typically not of the form $M(V)$.

Proposition. Let $M$ be an effective $t$-motif over $K$. The following are equiva-
lent:

- $M$ is isomorphic to a subquotient of $M(V)$ for some $V$,
- $M \otimes_K K^\sigma \approx n1$ for some $n$.

An $M$ satisfying the equivalent conditions is called a constant $t$-motif.

**Proof of the Proposition.** If $M$ is a subquotient of $M(V)$ then $M_K$ is a subquotient of $M(V_K) \approx m1$ and therefore $M_K \approx n1$.

Conversely, assume that $M_K$ has a basis of $\sigma$-invariant vectors. There exists some finite extension $K'/K$ inside $K^\sigma$ such that this basis is already defined over $K'$. The natural map $K[t] \to K'[t]$ defines the structure of a $K[t]$-module on $M'$. Denote it by $R_{K'/K}'M'$ in order to distinguish it from the $K'[t]$-module $M'$. It is clear that $R_{K'/K}'M'$ is naturally an effective $t$-motif over $K$ of rank $rk(M)[K':K]$. (Call it the Weil restriction$^{(1)}$ of $M'$ from $K'$ to $K$.) But, $M$ is a submodule of $R_{K'/K}'M' \otimes_K K^\sigma$ and the latter is isomorphic to $M(R_{K'/K}'W)$ with $W$ the sum of a number of copies of $K'$ with the diagonal action of $\sigma$, whence the Proposition.

4.1.3. The full subcategory $t\mathcal{M}^\circ_{\text{cst}}(K)$ of $t\mathcal{M}^\circ(K)$ consisting of the constant (effective) $t$-motifs is rigid abelian $k(t)$-linear and has a fibre functor

$$M \leadsto (M \otimes_{K[t]} K^\sigma[t])^\sigma \otimes_{k[t]} k(t)$$

and with this fibre functor we have

**Proposition.** $t\mathcal{M}^\circ_{\text{cst}}(K)$ is neutral Tannakian with fundamental group $G_K$.

Note that it is not needed to use analytic methods to obtain a fibre functor on constant $t$-motifs and in particular it is not needed to demand that $k[t] \to K$ be injective.

**Proof of the Proposition.** The functor $M(V) \leadsto H(V) \otimes_k k(t)$ induces a fully faithful embedding of $t\mathcal{M}^\circ_{\text{cst}}(K)$ into the category of $k(t)$-linear representations of $G_K$. It will be essentially surjective as soon as every continuous $k(t)$-linear representation of $G_K$ is a subquotient of $H \otimes_k k(t)$

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$^{(1)}$After §1.3 of [Wile, 1982].
for some \( k \)-linear representation \( H \). This is indeed so, since every (algebraic, or continuous) representation of \( G_K \) factors though a finite group \( G \) and every representation of \( G \) is a subquotient of the direct sum of a number of copies of the regular representation \( k(t)[G] \), which is nothing but the regular representation \( k[G] \) over \( k \), tensored with \( k(t) \). \( \Box \)

4.1.4. Constant \( t \)-motifs are the \( t \)-counterparts of the algebro-geometric Artin motifs (named after Emil Artin.) Let \( Z \to K \) be any field. Consider the category of smooth and projective varieties \( X \) over \( K \) that are of dimension zero. These are the spectra of the finite étale \( K \)-algebras and by Grothendieck’s formulation of Galois theory the category of such \( X \) is equivalent to the category of finite \( G_K \)-sets. The motifs that are subquotients of the \( h(X, \mathbb{Q}) \) for zero-dimensional \( X \) are called Artin motifs. They form a category which is equivalent to the category of \( \mathbb{Q} \)-linear representations of \( G_K \). \(^{(2)}\)

Thus sets have come to play the role of \( k \)-vector spaces. But then, the field of constants of \( Z \) is the hypothetical field with one element and vector spaces over this folkloric field are nothing but sets. \(^{(3)}\)

4.2 The Connected Components of \( \Gamma \)

4.2.1. Suppose now that \( k[t] \to K \) is actually injective. Choose \( K^+ \supset K \) to be algebraically closed, complete and with \( \| \theta \| > 1 \). Let \( K^s \) be the separable closure of \( K \) inside \( K^+ \). For a constant \( t \)-motif \( M \) we have that

\[
(M \otimes_{K[t]} K^s[t])^\sigma = (M \otimes_{K[t]} K^+(\{t\}))^\sigma.
\]

That is to say, \( t\mathcal{M}(K)^{\text{cst}} \) is a full sub-category of \( t\mathcal{M}(K)^{\text{a.t.}} \) and the analytic fibre functor on the latter extends the algebraic fibre functor on the former.

**Proposition.** There is a short exact sequence

\[
0 \to \Gamma_{K^s} \to \Gamma_K \to G_K \to 0
\]

\(^{(2)}\) See §1.3 and §4.1 of [André 2004].

\(^{(3)}\) See §13 of [Tits 1957].
of affine group schemes over \( k(t) \).

4.2.2. Proof. The full subcategory \( t\mathcal{M}_{\text{est}}^\circ(K) \) of \( t\mathcal{M}_{\text{a.t.}}^\circ(K) \) is Tannakian with fundamental group \( G_K \) (4.1.3) and is closed under subquotients in \( t\mathcal{M}_{\text{a.t.}}^\circ \) by definition. This implies the existence of a faithfully flat, and hence surjective, morphism \( \Gamma_K \to G_K \) of affine group schemes.\(^{(4)}\)

If \( M \) is an effective \( t \)-motif over \( K \), then it has a model \( M' \) over a finite extension \( K' \) of \( K \). The \( t \)-motif \( M \) is a submotif of \( R_{K'/K}M' \otimes_K K^\delta \). Thus every \( t \)-motif over \( K^\delta \) is a submotif of a \( t \)-motif that is already defined over \( K \). It follows that the fully faithful functor \( M \sim M_{K^\delta} \) from \( t\mathcal{M}_{\text{a.t.}}^\circ(K) \) to \( t\mathcal{M}_{\text{a.t.}}^\circ(K^\delta) \) defines a closed immersion \( \Gamma_{K^\delta} \to \Gamma_K \).\(^{(5)}\)

The sequence is exact in the middle if and only if the representations of \( \Gamma_K \) on which \( \Gamma_{K^\delta} \) acts trivially are precisely those coming from a representation of \( G_K \). In other words, the exactness is equivalent with the statement that a \( t \)-motif \( M \) over \( K \) satisfies \( M_{K^\delta} \approx n1 \) for some \( n \) if and only if it is a constant \( t \)-motif. This was one of the equivalent definitions of the notion of a constant \( t \)-motif (see 4.1.2). \( \square \)

4.2.3. The following Theorem complements the Proposition.

**Theorem.** \( \Gamma_{K^\delta} \) has no finite quotients. In particular it is connected.

*First part of the proof.* Note that \( \Gamma \to \pi_0(\Gamma) \) is a pro-finite étale quotient, hence the second statement indeed follows from the first.

Let \( G \) be a finite quotient of \( \Gamma_{K^\delta} \). To this there corresponds a Tannakian subcategory \( \mathcal{C} \) of \( t\mathcal{M}_{\text{a.t.}}^\circ(K^\delta) \), equivalent to the category of representations of \( G \). Since \( G \) is finite, \( \mathcal{C} \) contains a \( t \)-motif \( M \) such that every \( t \)-motif in \( \mathcal{C} \) is a subquotient of \( nM \) for some \( n \). (It suffices to take the \( M \) corresponding to the regular representation of \( G \).) The algebraic group \( G \) is trivial if and only if \( M \) is constant.

Write \( M \) as \( (M', i) \) with \( i \) maximal. \( M \otimes M \) is a subquotient of \( nM \) for \( n \) sufficiently large. Equivalently, \( M' \otimes M' \otimes C^i \) is a subquotient of \( nM' \) and since subquotients of effective \( t \)-motifs are effective, it follows that \( M' \otimes M' \otimes C^i \) is effective. If \( i \) is negative, then this implies that the action

\(^{(4)}\) See for example Proposition 2.21 (a) of [Deligne and Milne 1982].

\(^{(5)}\) See Proposition 2.21 (b) of *loc. cit.*
of $\sigma$ on $M' \otimes M'$ is divisible by $t - \theta$ and hence also that the action of $\sigma$ on $M'$ is divisible by $t - \theta$, contradicting the maximality of $i$. Therefore $i \geq 0$ and $M = (M', i)$ is effective.

Using analytic methods, it will be shown in Chapter 8 that if $M$ is an analytically trivial effective $t$-motif so that $M \otimes M$ is a subquotient of $nM$ for some $n$, then $M$ is constant, hence $G$ trivial, and the Theorem follows... (to be continued in 8.4.2)