Chapter 4

Constant $t$-Motifs

4.1 Constant $t$-Motifs

4.1.1. Let us go back to the category of 1.1.2, whose objects are pairs $(V, \sigma)$ of a finite dimensional $K$-vector space $V$ equipped with a non-degenerate semilinear map $\sigma : V \to V$. We have seen in Theorem 1.1.2 that this category is equivalent to the category of $k$-linear continuous representations of $G_K = \text{Gal}(K^s/K)$, a fact that we could rephrase as: the category of pairs $(V, \sigma)$ is $k$-linear neutral Tannakian with fundamental group $G_K$. Note that we abusively write $G_K$ for both the pro-finite group and the corresponding constant affine group scheme over $k$ (obtained as the limit of the system of finite constant group schemes corresponding to the finite quotients of the pro-finite group.) Their categories of representations on finite dimensional $k$-vector spaces coincide.

4.1.2. A pair $(V, \sigma)$ induces an effective $t$-motif $M(V) \overset{\text{def}}{=} V \otimes_K K[t]$ where the action of $\sigma$ is induced from the action on $V$.

We would like to interpret the collection of $t$-motifs $M(V)$ as a Tannakian subcategory of $tM^0$, but there are of course many more morphisms $M(V_1) \to M(V_2)$ than morphisms $V_1 \to V_2$ and the kernel and cokernel of a morphism from $M(V_1)$ to $M(V_2)$ are typically not of the form $M(V)$.

Proposition. Let $M$ be an effective $t$-motif over $K$. The following are equiva-
lent:

- $M$ is isomorphic to a subquotient of $M(V)$ for some $V$,
- $M \otimes_K K^\sigma \approx n1$ for some $n$.

An $M$ satisfying the equivalent conditions is called a constant $t$-motif.

Proof of the Proposition. If $M$ is a subquotient of $M(V)$ then $M_{K^s}$ is a subquotient of $M(V_{K^s}) \approx m1$ and therefore $M_{K^s} \approx n1$.

Conversely, assume that $M_{K^s}$ has a basis of $\sigma$-invariant vectors. There exists some finite extension $K'/K$ inside $K^s$ such that this basis is already defined over $K'$. The natural map $K[t] \to K'[t]$ defines the structure of a $K[t]$-module on $M'$. Denote it by $R_{K'/K}M'$ in order to distinguish it from the $K'[t]$-module $M'$. It is clear that $R_{K'/K}M'$ is naturally an effective $t$-motif over $K$ of rank $\text{rk}(M)|K':K$. (Call it the Weil restriction\(^{(1)}\) of $M'$ from $K'$ to $K$.) But, $M$ is a submodule of $R_{K'/K}M' \otimes_K K^\sigma$ and the latter is isomorphic to $M(R_{K'/K}W)$ with $W$ the sum of a number of copies of $K'$ with the diagonal action of $\sigma$, whence the Proposition. \(\square\)

4.1.3. The full subcategory $tM^\circ_{\text{cst}}(K)$ of $tM^\circ(K)$ consisting of the constant (effective) $t$-motifs is rigid abelian $k(t)$-linear and has a fibre functor

$$M \leadsto (M \otimes_{K[t]} K^\sigma[t])^\sigma \otimes_{k[t]} k(t) \quad (4.1)$$

and with this fibre functor we have

Proposition. $tM^\circ_{\text{cst}}(K)$ is neutral Tannakian with fundamental group $G_K$.

Note that it is not needed to use analytic methods to obtain a fibre functor on constant $t$-motifs and in particular it is not needed to demand that $k[t] \to K$ be injective.

Proof of the Proposition. The functor $M(V) \leadsto H(V) \otimes_k k(t)$ induces a fully faithful embedding of $tM^\circ_{\text{cst}}(K)$ into the category of $k(t)$-linear representations of $G_K$. It will be essentially surjective as soon as every continuous $k(t)$-linear representation of $G_K$ is a subquotient of $H \otimes_k k(t)$

\(^{(1)}\)After §1.3 of [Weil 1982].
for some $k$-linear representation $H$. This is indeed so, since every (algebraic, or continuous) representation of $G_K$ factors though a finite group $G$ and every representation of $G$ is a subquotient of the direct sum of a number of copies of the regular representation $k(t)[G]$, which is nothing but the regular representation $k[G]$ over $k$, tensored with $k(t).

4.1.4. Constant $t$-motifs are the $t$-counterparts of the algebro-geometric Artin motifs (named after Emil ARTIN.) Let $Z \to K$ be any field. Consider the category of smooth and projective varieties $X$ over $K$ that are of dimension zero. These are the spectra of the finite étale $K$-algebras and by GROTHENDIECK’s formulation of Galois theory the category of such $X$ is equivalent to the category of finite $G_K$-sets. The motifs that are subquotients of the $h(X, Q)$ for zero-dimensional $X$ are called Artin motifs. They form a category which is equivalent to the category of $Q$-linear representations of $G_K$.\(^{(2)}\)

Thus sets have come to play the role of $k$-vector spaces. But then, the field of constants of $Z$ is the hypothetical field with one element and vector spaces over this folkloric field are nothing but sets.\(^{(3)}\)

4.2 The Connected Components of $\Gamma$

4.2.1. Suppose now that $k[t] \to K$ is actually injective. Choose $K^\dagger \supset K$ to be algebraically closed, complete and with $\|\theta\| > 1$. Let $K^s$ be the separable closure of $K$ inside $K^\dagger$. For a constant $t$-motif $M$ we have that

$$\left(M \otimes_{K[t]} K^s[t]\right)^\sigma = \left(M \otimes_{K[t]} K^\dagger(t)\right)^\sigma.$$  

That is to say, $t\mathcal{M}(K)_{\text{cst}}$ is a full sub-category of $t\mathcal{M}(K)_{\text{a.t.}}$ and the analytic fibre functor on the latter extends the algebraic fibre functor on the former.

**Proposition.** There is a short exact sequence

$$0 \to \Gamma_{K^s} \to \Gamma_K \to G_K \to 0$$

\(^{(2)}\)See §1.3 and §4.1 of [ANDRÉ 2004].

\(^{(3)}\)See §13 of [TITS 1957].
of affine group schemes over \( k(t) \).

**4.2.2. Proof.** The full subcategory \( t\mathcal{M}_\text{est}^\circ(K) \) of \( t\mathcal{M}_{a.t.}^\circ(K) \) is Tannakian with fundamental group \( G_K \) (4.1.3) and is closed under subquotients in \( t\mathcal{M}_{a.t.}^\circ \) by definition. This implies the existence of a faithfully flat, and hence surjective, morphism \( \Gamma_K \to G_K \) of affine group schemes.(4)

If \( M \) is an effective \( t \)-motif over \( K \), then it has a model \( M' \) over a finite extension \( K' \) of \( K \). The \( t \)-motif \( M \) is a submotif of \( R_{K'/K}M' \otimes_K K' \). Thus every \( t \)-motif over \( K' \) is a submotif of a \( t \)-motif that is already defined over \( K \). It follows that the fully faithful functor \( M \rightsquigarrow M_{K'} \) from \( t\mathcal{M}_{a.t.}^\circ(K) \) to \( t\mathcal{M}_{a.t.}^\circ(K') \) defines a closed immersion \( \Gamma_{K'} \to \Gamma_K \). (5)

The sequence is exact in the middle if and only if the representations of \( \Gamma_K \) on which \( \Gamma_{K'} \) acts trivially are precisely those coming from a representation of \( G_K \). In other words, the exactness is equivalent with the statement that a \( t \)-motif \( M \) over \( K \) satisfies \( M_{K'} \approx n1 \) for some \( n \) if and only if it is a constant \( t \)-motif. This was one of the equivalent definitions of the notion of a constant \( t \)-motif (see 4.1.2).

**4.2.3.** The following Theorem complements the Proposition.

**Theorem.** \( \Gamma_{K'} \) has no finite quotients. In particular it is connected.

**First part of the proof.** Note that \( \Gamma \to \pi_0(\Gamma) \) is a pro-finite étale quotient, hence the second statement indeed follows from the first.

Let \( G \) be a finite quotient of \( \Gamma_{K'} \). To this there corresponds a Tannakian subcategory \( C \) of \( t\mathcal{M}_{a.t.}^\circ(K') \), equivalent to the category of representations of \( G \). Since \( G \) is finite, \( C \) contains a \( t \)-motif \( M \) such that every \( t \)-motif in \( C \) is a subquotient of \( nM \) for some \( n \). (It suffices to take the \( M \) corresponding to the regular representation of \( G \).) The algebraic group \( G \) is trivial if and only if \( M \) is constant.

Write \( M \) as \( (M', i) \) with \( i \) maximal. \( M \otimes M \) is a subquotient of \( nM \) for \( n \) sufficiently large. Equivalently, \( M' \otimes M' \otimes C^i \) is a subquotient of \( nM' \) and since subquotients of effective \( t \)-motifs are effective, it follows that \( M' \otimes M' \otimes C^i \) is effective. If \( i \) is negative, then this implies that the action

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(4) See for example Proposition 2.21 (a) of [Deligne and Milne 1982].

(5) See Proposition 2.21 (b) of loc. cit.
of $\sigma$ on $M' \otimes M'$ is divisible by $t - \theta$ and hence also that the action of $\sigma$ on $M'$ is divisible by $t - \theta$, contradicting the maximality of $i$. Therefore $i \geq 0$ and $M = (M', i)$ is effective.

Using analytic methods, it will be shown in Chapter 8 that if $M$ is an analytically trivial effective $t$-motif so that $M \otimes M$ is a subquotient of $nM$ for some $n$, then $M$ is constant, hence $G$ trivial, and the Theorem follows... (to be continued in 8.4.2)