On t-Motifs
Taelman, Lenny

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2007

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.
Chapter 4

Constant \( t \)-Motifs

4.1 Constant \( t \)-Motifs

4.1.1. Let us go back to the category of 1.1.2, whose objects are pairs \((V, \sigma)\) of a finite dimensional \( K \)-vector space \( V \) equipped with a non-degenerate semilinear map \( \sigma : V \rightarrow V \). We have seen in Theorem 1.1.2 that this category is equivalent to the category of \( k \)-linear continuous representations of \( G_K = \text{Gal}(K^s/K) \), a fact that we could rephrase as: the category of pairs \((V, \sigma)\) is \( k \)-linear neutral Tannakian with fundamental group \( G_K \). Note that we abusively write \( G_K \) for both the pro-finite group and the corresponding constant affine group scheme over \( k \) (obtained as the limit of the system of finite constant group schemes corresponding to the finite quotients of the pro-finite group.) Their categories of representations on finite dimensional \( k \)-vector spaces coincide.

4.1.2. A pair \((V, \sigma)\) induces an effective \( t \)-motif \( M(V) \overset{\text{def}}{=} V \otimes_K K[t] \) where the action of \( \sigma \) is induced from the action on \( V \).

We would like to interpret the collection of \( t \)-motifs \( M(V) \) as a Tannakian subcategory of \( t.M^e \), but there are of course many more morphisms \( M(V_1) \rightarrow M(V_2) \) than morphisms \( V_1 \rightarrow V_2 \) and the kernel and cokernel of a morphism from \( M(V_1) \) to \( M(V_2) \) are typically not of the form \( M(V) \).

**Proposition.** Let \( M \) be an effective \( t \)-motif over \( K \). The following are equiva-
lent:

- $M$ is isomorphic to a subquotient of $M(V)$ for some $V$,
- $M \otimes_K K^\sigma \approx n \mathbf{1}$ for some $n$.

An $M$ satisfying the equivalent conditions is called a constant $t$-motif.

**Proof of the Proposition.** If $M$ is a subquotient of $M(V)$ then $M_{\mathbb{K}^s}$ is a subquotient of $M(V_{\mathbb{K}^s}) \approx m \mathbf{1}$ and therefore $M_{\mathbb{K}^s} \approx n \mathbf{1}$.

Conversely, assume that $M_{\mathbb{K}^s}$ has a basis of $\sigma$-invariant vectors. There exists some finite extension $K'/K$ inside $K^\sigma$ such that this basis is already defined over $K'$. The natural map $K[t] \to K'[t]$ defines the structure of a $K[t]$-module on $M'$. Denote it by $R_{K'/K}M'$ in order to distinguish it from the $K'[t]$-module $M'$. It is clear that $R_{K'/K}M'$ is naturally an effective $t$-motif over $K$ of rank $\operatorname{rk}(M)[K' : K]$. (Call it the Weil restriction\(^{(1)}\) of $M'$ from $K'$ to $K$.) But, $M$ is a submodule of $R_{K'/K}M' \otimes_K K^\sigma$ and the latter is isomorphic to $M(R_{K'/K}W)$ with $W$ the sum of a number of copies of $K'$ with the diagonal action of $\sigma$, whence the Proposition. \qed

**4.1.3.** The full subcategory $t\mathcal{M}_{\text{cst}}^\circ(K)$ of $t\mathcal{M}^\circ(K)$ consisting of the constant (effective) $t$-motifs is rigid abelian $k(t)$-linear and has a fibre functor

$$M \leadsto (M \otimes_{K[t]} K^\sigma[t])^\sigma \otimes_{k[t]} k(t)$$

and with this fibre functor we have

**Proposition.** $t\mathcal{M}_{\text{cst}}^\circ(K)$ is neutral Tannakian with fundamental group $G_K$.

Note that it is not needed to use analytic methods to obtain a fibre functor on constant $t$-motifs and in particular it is not needed to demand that $k[t] \to K$ be injective.

**Proof of the Proposition.** The functor $M(V) \leadsto H(V) \otimes_k k(t)$ induces a fully faithful embedding of $t\mathcal{M}_{\text{cst}}^\circ(K)$ into the category of $k(t)$-linear representations of $G_K$. It will be essentially surjective as soon as every continuous $k(t)$-linear representation of $G_K$ is a subquotient of $H \otimes_k k(t)$

\(^{(1)}\)After §1.3 of [Weil 1982].
for some $k$-linear representation $H$. This is indeed so, since every (algebraic, or continuous) representation of $G_K$ factors though a finite group $G$ and every representation of $G$ is a subquotient of the direct sum of a number of copies of the regular representation $k(t)[G]$, which is nothing but the regular representation $k[G]$ over $k$, tensored with $k(t)$. □

4.1.4. Constant $t$-motifs are the $t$-counterparts of the algebro-geometric Artin motifs (named after Emil Artin.) Let $Z \to K$ be any field. Consider the category of smooth and projective varieties $X$ over $K$ that are of dimension zero. These are the spectra of the finite étale $K$-algebras and by Grothendieck’s formulation of Galois theory the category of such $X$ is equivalent to the category of finite $G_K$-sets. The motifs that are subquotients of the $h(X, \mathbb{Q})$ for zero-dimensional $X$ are called Artin motifs. They form a category which is equivalent to the category of $\mathbb{Q}$-linear representations of $G_K$. (2)

Thus sets have come to play the role of $k$-vector spaces. But then, the field of constants of $Z$ is the hypothetical field with one element and vector spaces over this folkloric field are nothing but sets. (3)

4.2 The Connected Components of $\Gamma$

4.2.1. Suppose now that $k[t] \to K$ is actually injective. Choose $K^+ \supset K$ to be algebraically closed, complete and with $\|\theta\| > 1$. Let $K^s$ be the separable closure of $K$ inside $K^+$. For a constant $t$-motif $M$ we have that

$$(M \otimes_{K[t]} K^s[t])^\sigma = (M \otimes_{K[t]} K^+(\{t\}))^\sigma.$$ 

That is to say, $t\mathcal{M}(K)^{\circ}_{\text{cst}}$ is a full sub-category of $t\mathcal{M}(K)^{\circ}_{\text{a.t.}}$ and the analytic fibre functor on the latter extends the algebraic fibre functor on the former.

Proposition. There is a short exact sequence

$$0 \to \Gamma_{K^s} \to \Gamma_K \to G_K \to 0$$

(2) See §1.3 and §4.1 of [André 2004].

(3) See §13 of [Tits 1957].
of affine group schemes over $k(t)$.

4.2.2. Proof. The full subcategory $tM^\circ_{\text{est}}(K)$ of $tM^\circ_{a.t.}(K)$ is Tannakian with fundamental group $G_K$ (4.1.3) and is closed under subquotients in $tM^\circ_{a.t.}$ by definition. This implies the existence of a faithfully flat, and hence surjective, morphism $\Gamma_K \to G_K$ of affine group schemes.\(^{(4)}\)

If $M$ is an effective $t$-motif over $K$, then it has a model $M'$ over a finite extension $K'$ of $K$. The $t$-motif $M$ is a submotif of $R_{K'/K}M' \otimes_K K^e$. Thus every $t$-motif over $K^e$ is a submotif of a $t$-motif that is already defined over $K$. It follows that the fully faithful functor $M \rightsquigarrow M_{K^e}$ from $tM^\circ_{a.t.}(K)$ to $tM^\circ_{a.t.}(K^e)$ defines a closed immersion $\Gamma_{K^e} \to \Gamma_K$.\(^{(5)}\)

The sequence is exact in the middle if and only if the representations of $\Gamma_K$ on which $\Gamma_K$ acts trivially are precisely those coming from a representation of $G_K$. In other words, the exactness is equivalent with the statement that a $t$-motif $M$ over $K$ satisfies $M_{K^e} \approx n1$ for some $n$ if and only if it is a constant $t$-motif. This was one of the equivalent definitions of the notion of a constant $t$-motif (see 4.1.2).

4.2.3. The following Theorem complements the Proposition.

**Theorem.** $\Gamma_{K^e}$ has no finite quotients. In particular it is connected.

*First part of the proof.* Note that $\Gamma \to \pi_0(\Gamma)$ is a pro-finite étale quotient, hence the second statement indeed follows from the first.

Let $G$ be a finite quotient of $\Gamma_{K^e}$. To this there corresponds a Tannakian subcategory $C$ of $tM^\circ_{a.t.}(K^e)$, equivalent to the category of representations of $G$. Since $G$ is finite, $C$ contains a $t$-motif $M$ such that every $t$-motif in $C$ is a subquotient of $nM$ for some $n$. (It suffices to take the $M$ corresponding to the regular representation of $G$.) The algebraic group $G$ is trivial if and only if $M$ is constant.

Write $M$ as $(M', i)$ with $i$ maximal. $M \otimes M$ is a subquotient of $nM$ for $n$ sufficiently large. Equivalently, $M' \otimes M' \otimes C^i$ is a subquotient of $nM'$ and since subquotients of effective $t$-motifs are effective, it follows that $M' \otimes M' \otimes C^i$ is effective. If $i$ is negative, then this implies that the action

\(^{(4)}\)See for example Proposition 2.21 (a) of [Deligne and Milne 1982].

\(^{(5)}\)See Proposition 2.21 (b) of loc. cit.
of $\sigma$ on $M' \otimes M'$ is divisible by $t - \theta$ and hence also that the action of $\sigma$ on $M'$ is divisible by $t - \theta$, contradicting the maximality of $i$. Therefore $i \geq 0$ and $M = (M', i)$ is effective.

Using analytic methods, it will be shown in Chapter 8 that if $M$ is an analytically trivial effective $t$-motif so that $M \otimes M$ is a subquotient of $nM$ for some $n$, then $M$ is constant, hence $G$ trivial, and the Theorem follows... (to be continued in 8.4.2) \hfill \Box