Chapter 3

Tannakian Aspects

3.1 Fibre Functors

3.1.1. Let $L$ be a field and $C$ a rigid abelian $L$-linear tensor category. Fix a field extension $L'$ of $L$. A faithful functor $\omega$ of $C$ to the category of finite dimensional $L'$-vector spaces that is exact and that respects the tensor product is called a fibre functor. If $L' = L$, one talks about a neutral fibre functor.

If such an $L'$ and an $\omega$ exist, the category $C$ is called Tannakian. If an $\omega$ exists with $L' = L$ it is called neutral Tannakian and if moreover a preferred $\omega$ is given one says that $(C, \omega)$ is a neutralised Tannakian category.\(^{(1)}\)

3.1.2. The rigid abelian $k(t)$-linear category $tM^\circ(K)$ is Tannakian for trivial reasons: the functor

\[(M, \sigma) \mapsto M \otimes_{K[t]} K(t)\]

is a fibre functor to the category of $K(t)$-vector spaces. It follows from a Theorem of DELIGNE\(^{(2)}\) that $tM^\circ(K)$ also has a fibre functor to $k(t)^a$-vector spaces.

\(^{(1)}\)See also Appendix c.

\(^{(2)}\)Corollaire 6.20 of [DELI GNE 1990].
3.1.3. If $k[t] \to K$ is not injective—equivalently: if $\theta \in K$ is algebraic over $k$—and if say $K = K^s$, then a neutral fibre functor on $tM^\circ(K)$ cannot exist. The reason is that there exist Drinfeld modules of rank $n$ whose endomorphism algebra is a division algebra of dimension $n^2$ over $k(t)$, and such an algebra cannot act on an $n$-dimensional $k(t)$-vector space when $n > 1$. (These Drinfeld modules are called super-singular.)

Take for example $\theta = 0$, that is, $K$ is of ‘characteristic $t$’. Consider the effective $t$-motif

$$M \overset{\text{def}}{=} K[t]e_1 \oplus \cdots \oplus K[t]e_n \quad \text{with} \quad \sigma(e_i) \overset{\text{def}}{=} \begin{cases} e_{i+1} & \text{if } i < n, \\ te_1 & \text{if } i = n. \end{cases}$$

One verifies that $\text{End}(M) \otimes_{K[t]} k(t)$ is a division algebra of dimension $n^2$, ramified at the places $t$ and $\infty$.

### 3.2 An Analytic Construction

In this section we construct a neutral fibre functor on a sub-category of $tM^\circ(K)$, where $k[t] \to K$ is assumed to be injective. This construction occurs already in [Anderson 1986] and is interpreted as a fibre functor in [Papanikolas 2005]. I do not know if there exists a neutral fibre functor on all of $tM^\circ$ (see Question 11.1.)

3.2.1. Let $K^\dagger$ be a field containing $K$ that is algebraically closed and complete with respect to a valuation $\| \cdot \|$. Denote by $K^\dagger\{t\} \subset K^\dagger[[t]]$ the subring of restricted power series, that is, those power series whose coefficients converge to 0. In particular, these series have a radius of convergence greater than or equal to 1. Note that $K^\dagger\{t\}$ is closed under $\tau$—raising all coefficients to the $q$-th power. A $\tau$-invariant power series has coefficients in the finite field $k$ and hence is restricted if and only if it is a polynomial in $t$. That is, $K^\dagger\{t\}^\tau = k[t]$. Denote by $K^\dagger(\{t\})$ the field of fractions of $K^\dagger\{t\}$. In the next paragraph we shall show that $K^\dagger(\{t\})^\tau = k(t)$.

\textsuperscript{(3)}This is the argument used by S\textsuperscript{e}rre to show that in Algebraic Geometry over a field of characteristic $p$ there can be no reasonable cohomology with coefficients in $\mathbb{Q}$.  

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3.2.2. Define the functors $H_{an}(-, k[t])$ and $H_{an}(-, k(t))$ on the category $t\mathcal{M}_{\text{eff}}$ of effective $t$-motifs as

$$H_{an}(M, k[t]) \overset{\text{def}}{=} (M \otimes K[t] K^\dagger \{t\})^\sigma$$

$$H_{an}(M, k(t)) \overset{\text{def}}{=} (M \otimes K[t] K^\dagger\{\{t\}\})^\sigma$$

The functors $H_{an}(-, k[t])$ and $H_{an}(-, k(t))$ are related:

**Proposition.** $H_{an}(M, k(t)) = H_{an}(M, k[t]) \otimes k(t)$.

Taking $M$ to be 1 yields

**Corollary.** $K^\dagger\{\{t\}\}^T = k(t)$.

**Proof of the Proposition.** The ring $K^\dagger\{t\}$ is a principal ideal domain and every non-zero ideal is of the form

$$(t - \alpha_1)(t - \alpha_2) \cdots (t - \alpha_n)K^\dagger\{t\} \subset K^\dagger\{t\}$$

with $\|\alpha_i\| \leq 1$ for all $i$.(4)

Take a $g \in H_{an}(M, k(t))$ and write it as $h^{-1}m$ with $m \in M \otimes K^\dagger\{t\}$ and $h$ a finite product $h = \prod(t - \alpha_i)$. Assume that the degree of $h$ is minimal. The invariance of $g = h^{-1}m$ gives

$$\tau(h)m = h\sigma(m) \in M \otimes K^\dagger\{t\},$$

hence the minimality of $h$ implies $\tau(h) = h$.(4)

3.2.3. The Carlitz $t$-motif is trivialised by $K^\dagger\{t\}$:

**Proposition.** If $\|\theta\| > 1$ then $H_{an}(C, k[t]) \approx k[t]$ and $H_{an}(C, k(t)) \approx k(t)$.

**Proof.** (5) Consider the product expansion

$$\Omega \overset{\text{def}}{=} \frac{1}{\sqrt[2^\theta]{\theta}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^i q_i}\right),$$

(4) See [Tate 1971] for the structure of the ring $K^\dagger\{t\}$.

(5) Compare with Lemma 2.5 of [Anderson and Thakur 1990].
where \( q - 1 \sqrt{\theta} \) is any root in \( K^\dagger \). (Any two such roots differ by a scalar in \( k^\times \).) The infinite product converges for all values of \( t \) and all zeroes have absolute value greater than or equal to \( \|\theta\| > 1 \), thus \( \Omega \in K\{t\}^\times \). By construction \( \Omega = (t - \theta) \tau(\Omega) \) and therefore \( H_{an}(C,k[t]) = k[t] \Omega e \). Similarly \( H_{an}(C,k(t)) = k(t) \Omega e \).

Henceforth, when considering the functors \( H_{an} \), we shall always assume that \( \|\theta\| > 1 \). This implies in particular that \( k[t] \to K \) is injective.

3.2.4. So far, we have only considered effective \( t \)-motifs. Shifting back and forth with powers of the Carlitz motif, we can extend the functors \( H_{an} \) to functors defined on all \( t \)-motifs as follows

\[
H_{an}( (M,i),k[t] ) \overset{\text{def}}{=} H_{an}(M,k[t]) \otimes_{k[t]} H_{an}(C,k[t]) \otimes^i
\]

\[
H_{an}( (M,i),k(t) ) \overset{\text{def}}{=} H_{an}(M,k(t)) \otimes_{k(t)} H_{an}(C,k(t)) \otimes^i.
\]

The resulting functor is well-defined by the canonical isomorphisms

\[
H_{an}(M \otimes C,k[t]) = H_{an}(M,k[t]) \otimes_{k[t]} H_{an}(C,k[t]).
\]

3.2.5. These functors are not faithful. In fact, we shall shortly see that there exist non-trivial \( M \) with \( H_{an}(M,k[t]) = 0 \).

**Definition.** A \( t \)-motif \( (M,i) \) over \( K \) is said to be **analytically trivial** if one of the following equivalent conditions holds:

- \( M \otimes_{K[t]} K^\dagger \{t\} \) has a \( \sigma \)-invariant \( K^\dagger \{t\} \)-basis,
- \( M \otimes_{K[t]} K^\dagger(\{t\}) \) has a \( \sigma \)-invariant \( K^\dagger(\{t\}) \)-basis,
- \( \operatorname{rk}_{k[t]} H_{an}(M,k[t]) = \operatorname{rk} M \),
- \( \operatorname{dim}_{k(t)} H_{an}(M,k(t)) = \operatorname{rk} M \).

Denote by \( tM_{\text{a.t.}} \subset tM \) and \( tM_{\text{a.t.}}^\circ \subset tM^\circ \) the full subcategories consisting of the analytically trivial objects.

The analytic triviality of a \( t \)-motif \( M \) depends on the embedding of \( K \) into \( K^\dagger \)—see 3.2.8 for an example. When we are dealing with the category \( tM_{\text{a.t.}} \), we will assume that such an embedding has been fixed.
Proof of equivalence. By Proposition 3.2.2 the first condition is equivalent with the second, and the third with the fourth. Clearly the first implies the third. To conclude the converse (that the third implies the first), it suffices to show that for all effective $t$-motifs $M$ the natural map

$$(M \otimes K^t\{t\})^\sigma \otimes_{k[t]} K^t\{t\} \to M \otimes K^t\{t\}$$

is injective. This can be done exactly as in 1.2.4, using $K^t\{t\}^\sigma = k[t]$. \qed

3.2.6. Some immediate consequences of the definition are:

**Proposition.** The class of analytically trivial $t$-motifs is closed under tensor product and duality. Moreover

- $H_{an}(\cdot, k[t])$ is a faithful $k[t]$-linear $\otimes$-functor on $t\mathcal{M}_{a.t}$,
- $H_{an}(\cdot, k(t))$ is an exact, faithful, $k(t)$-linear $\otimes$-functor on $t\mathcal{M}_{a.t}$.

In particular $t\mathcal{M}_{a.t}^\circ(K)$ is Tannakian with fibre functor $H_{an}(\cdot, k(t))$. \qed

Denote the associated affine group scheme over $k(t)$ by $\Gamma_K$. Thus $t\mathcal{M}_{a.t}^\circ(K)$ is equivalent with the category of $k(t)$-linear representations of $\Gamma_K$.

But note that $\Gamma_K$ depends on the chosen valuation on $K$. We shall often tacitly assume that a valuation (with $\|\theta\| > 1$) has been fixed.

3.2.7. As an example, we now show that Drinfeld modules are analytically trivial. This will be generalised slightly in §9.2.

**Proposition.** Every effective $t$-motif $M$ that is free of rank one over $K[\sigma]$ is analytically trivial.

**Sketch of proof.** If $M$ is defined over a locally compact field, then the Proposition follows from Proposition 4.1.1 of [Anderson 1986].

In general, $M$ is defined over a field $K$ that is finitely generated over $k((\theta^{-1}))$. It has a model $\tilde{M}$ over a finitely generated $k((\theta^{-1}))$-algebra $R \subset K$ with quotient field $K$.

By the above remark, the specialisations $\tilde{M}_x$ of this family are analytically trivial. It then follows from the results of [Böckle and Hartl 2005] that $M$ is itself analytically trivial. \qed
3.2.8. Not all $t$-motifs are analytically trivial. Consider for example, the rank 2 effective $t$-motif

$$M_\xi = K[t]e_1 + K[t]e_2$$

with \begin{align*}
\sigma(e_1) &= \xi te_1 + e_2 \\
\sigma(e_2) &= e_1
\end{align*}

depending on a parameter $\xi \in K$.

**Proposition.** $M_\xi$ is analytically trivial if and only if $\|\xi\| < 1$.

In particular there exists no valuation on $K$ for which $M_\xi$ is analytically trivial when $\xi$ is algebraic over $k$.

**Proof.** Assume that $K^+ t e_1 + K^+ t e_2$ has an invariant vector $ae_1 + be_2$, with $a, b$ in $K^+ t$. Expressing the invariance under $\sigma$ gives

$$\begin{cases}
a = \tau(a) \xi t + \tau^2(a) \\
b = \tau(a)
\end{cases}$$

Expand $a = a_0 + a_1 t + \cdots$ with $a_i \in K^+$. Then it follows that $a_0^2 = a_0$, that is, $a_0$ lies in the quadratic extension $l/k$ inside $K^+$, and in particular $\|a_0\| = 1$ (assuming $a_0 \neq 0$). The higher $a_i$ satisfy the recurrence equation

$$a_n - a_n^2 = \xi a_{n-1}^q.$$  \hspace{1cm} (3.1)

If $\|\xi\| \geq 1$ then $\|a_n\| \geq 1$ for all $n$ and the series $a_0 + a_1 t + \cdots$ is therefore not a restricted series, confirming one direction of the Proposition. If on the other hand $\|\xi\| < 1$ then define an $a$ recursively by taking at every step the unique solution $a_n$ of (3.1) that has $\|a_n\| < 1$. This produces a restricted power series for every $a_0 \in l^\times$ and it suffices to take two $a_0$'s independent over $k$ to obtain two independent invariant vectors for $M_\xi \otimes_{K[t]} K^+ t$. \hfill $\square$