Chapter 2

Duality and $t$-Motifs

In the present chapter the category of effective $t$-motifs will be extended to a slightly larger category which has internal homs. The objects in the resulting category will be called $t$-motifs.

2.1 Internal Hom

2.1.1. Let $M_1$ and $M_2$ be effective $t$-motifs over $K$. Inspired by the theory of linear representations of groups we could try to assign to $M_1$ and $M_2$ an effective $t$-motif of internal homomorphisms as

\[
\text{Hom}(M_1, M_2) \equiv \text{Hom}_{K[t]}(M_1, M_2) \text{ with } \sigma(f) \equiv \sigma \circ f \circ \sigma^{-1},
\]

where $\sigma \circ f \circ \sigma^{-1}$ is to be read as $\sigma_2 \circ f \circ \sigma_1^{-1}$. This does, however, not make sense, since $\sigma_1$ need not be invertible. First of all, $K$ need not be perfect, and secondly—more seriously—the determinant of $\sigma_1$ is $(t - \theta)^d$ up to a constant, and hence not invertible if $d > 0$.

2.1.2. This can be partially resolved. Write $K^d$ for some algebraic closure of $K$. Note that after extension of scalars from $K[t]$ to $K^d(t)$ the induced action of $\sigma$ on $M_1 \otimes_{K[t]} K^d(t)$ is invertible.

Proposition. For $n$ sufficiently large, the subgroup

\[
\text{Hom}_{K[t]}(M_1, M_2 \otimes C^n) \subset \text{Hom}_{K^d(t)}(M_1 \otimes K^d(t), M_2 \otimes C^n \otimes K^d(t))
\]
is stable under $f \mapsto \sigma \circ f \circ \sigma^{-1}$.

Proof. Choose bases and express $\sigma$ on $M_1$ and $M_2$ by matrices $S_1$ and $S_2$ respectively. Then $M_2 \otimes \mathbb{C}^n$ has a basis on which $\sigma$ is expressed by the matrix $(t - \theta)^n S_2$. The map $f \mapsto \sigma \circ f \circ \sigma^{-1}$ translates to a map

$$M(r_2 \times r_1, K^a(t)) \to M(r_2 \times r_1, K^a(t)) : F \mapsto F'$$

with

$$F' = (t - \theta)^n S_2 \tau(F \tau^{-1}(S_1^{-1}))$$

$$= (t - \theta)^n S_2 \tau(F) S_1^{-1} \quad (2.1)$$

The Proposition claims that $M(r_2 \times r_1, K[t])$ is mapped into itself. But since the determinant of $S_1$ is a power of $(t - \theta)$, the matrix $(t - \theta)^n S_1^{-1}$ has entries in $K[t]$ when $n$ is sufficiently large. This immediately implies that $M(r_2 \times r_1, K[t])$ is mapped into itself. 

2.1.3. It follows from the explicit formula (2.1) that $\sigma(f) \overset{\text{def}}{=} \sigma \circ f \circ \sigma^{-1}$ induces the structure of an effective $t$-motif on $\text{Hom}_{K[t]}(M_1, M_2 \otimes \mathbb{C}^n)$ for large $n$. We shall denote it by $\mathcal{H}om(M_1, M_2 \otimes \mathbb{C}^n)$. These internal homs are stable for growing $n$ in the sense that there are natural isomorphisms

$$\mathcal{H}om(M_1, M_2 \otimes \mathbb{C}^n) \otimes \mathbb{C} \to \mathcal{H}om(M_1, M_2 \otimes \mathbb{C}^{n+1}) \quad (2.2)$$

relating them.

2.2 $t$-Motifs

2.2.1. The previous section hints that the obstruction to having internal homs will be lifted as soon as the Carlitz $t$-motif is made invertible.\(^{(1)}\)

This can be done quite easily, because of the following:

\(^{(1)}\)Very reminiscent of the inversion of the Lefschetz motif in the construction of the category of pure motifs: If $X$ is a smooth and projective variety of dimension $d$ then $\ell$-adic Poincaré duality defines a perfect pairing

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \times H^{2d-i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-d)$$

which suggests that the motif $h^i(X, \mathbb{Q})$ is dual to $h^{2d-i}(X, \mathbb{Q})$ shifted by the $d$-th power of the Lefschetz motif. See §4.1 of [ANDRÉ 2004].
Lemma. If $M_1$ and $M_2$ are effective $t$-motifs, then the natural map

$$Hom_{\sigma}(M_1, M_2) \rightarrow Hom_{\sigma}(M_1 \otimes C^n, M_2 \otimes C^n)$$

that takes $f$ to $f \otimes id$ is an isomorphism.

Proof. Note that $f \mapsto f \otimes id$ defines a natural isomorphism

$$Hom_{K[t]}(M_1, M_2) \rightarrow Hom_{K[t]}(M_1 \otimes C^n, M_2 \otimes C^n)$$

of $K[t]$-modules. An element $g = f \otimes 1$ of the latter is a morphism of effective $t$-motifs if and only if it satisfies

$$(t - \theta)^n \sigma_2 \circ f = f \circ (t - \theta)^n \sigma_1$$

which is equivalent with $\sigma_2 \circ f = f \circ \sigma_1$, that is, with $f$ being an element of $Hom_{\sigma}(M_1, M_2)$.

2.2.2. Now we are ready to make the following definition.

Definition. A $t$-motif is a pair $(M, i)$ consisting of an effective $t$-motif $M$ and an integer $i \in \mathbb{Z}$. Morphisms between $t$-motifs are defined by

$$Hom_{\sigma}((M_1, i_1), (M_2, i_2)) \overset{\text{def}}{=} Hom_{\sigma}(M_1 \otimes C^{n+i_1}, M_2 \otimes C^{n+i_2}),$$

for $n$ sufficiently large. The resulting category is denoted by $tM(K)$ or simply by $tM$.

It suffices to take $n \geq \max(-i_1, -i_2)$ in the definition. The module of morphisms is independent of $n$ by the preceding Lemma.

The functor $M \mapsto (M, 0)$ is fully faithful and we will identify $tM_{\text{eff}}$ with its image in $tM$.

2.2.3. The natural isomorphism between $M \otimes C^{n+1}$ and $M \otimes C^n \otimes C$ defines a distinguished isomorphism of $t$-motifs

$$(M, i + 1) = (M \otimes C, i).$$

In particular, we can identify $C^i$ with $(1, i)$. But note that $(1, i)$ is an object in $tM$ even when $i$ is negative.
2.2.4. The operations $\oplus$ and $\otimes$ and $\mathcal{H}om$ extend from the category of effective $t$-motifs—or parts thereof—to the full category of $t$-motifs:

$$(M_1, i_1) \oplus (M_2, i_2) \overset{\text{def}}{=} (M_1 \otimes C^{n+i_1} \oplus M_2 \otimes C^{n+i_2}, -n)$$

$$(M_1, i_1) \otimes (M_2, i_2) \overset{\text{def}}{=} (M_1 \otimes M_2, i_1 + i_2)$$

$${\mathcal{H}om}((M_1, i_1), (M_2, i_2)) \overset{\text{def}}{=} ({\mathcal{H}om}(M_1, M_2 \otimes C^{i_1-i_2+n}), -n)$$

The occurrences of $n$ in these definitions should be read ‘with $n$ sufficiently large’. Using the isomorphisms (2.2) and (2.3), one verifies that these are independent of $n$ and coincide with the operations on effective $t$-motifs, whenever defined.

2.2.5. From now on we will often drop the integer $i$ from the notation and write $M$ for a $t$-motif, effective or not.

As usual, we define the dual of a $t$-motif $M$ to be $M^\vee \overset{\text{def}}{=} \mathcal{H}om(M, 1)$. The operations of direct sum, tensor product, duality and internal hom satisfy the expected relations—those familiar from the theory of linear representations of groups. In particular, there is an adjunction formula

$$\mathcal{H}om(M_1 \otimes M_2, M_3) = \mathcal{H}om(M_1, \mathcal{H}om(M_2, M_3)).$$

(2.4)

Also, taking duals is reflexive: the natural morphism

$$M \rightarrow (M^\vee)^\vee$$

is an isomorphism. And finally, $\mathcal{H}om$ is distributive over $\otimes$ in the sense that the natural morphism

$$\mathcal{H}om(M_1, M_3) \otimes \mathcal{H}om(M_2, M_4) \rightarrow \mathcal{H}om(M_1 \otimes M_2, M_3 \otimes M_4),$$

(2.6)

is an isomorphism. Proofs of these three statements are given in the coming paragraphs. Assuming them for now, we can show:

**Theorem.** $t\mathcal{M}$ is a rigid $k[t]$-linear pre-abelian tensor category.\(^{(2)}\)

\(^{(2)}\)That is: a rigid $k[t]$-linear $\otimes$-category that is also pre-abelian. Appendix C reviews terminology on $\otimes$-categories.
Proof. The category $t\mathcal{M}$ is evidently a $k[t]$-linear tensor category.

For $t\mathcal{M}$ to be pre-abelian it needs to have kernels and cokernels. All morphisms in $t\mathcal{M}$ become morphisms of effective $t$-motifs after an appropriate shift with a tensor power of the Carlitz motif. It is thus sufficient to show that $t\mathcal{M}_{\text{eff}}$ has kernels and cokernels.

Let $M_1 \to M_2$ be a morphism of effective $t$-motifs. Its group-theoretic kernel is automatically a $t$-motif and a kernel in the category $t\mathcal{M}_{\text{eff}}$. The cokernel of $f$ in the pre-abelian category of free $K[t]$-modules—the ordinary cokernel modulo torsion—inherits an action of $\sigma$ and one verifies that this defines an effective $t$-motif and a cokernel of $f$ in $t\mathcal{M}_{\text{eff}}$. Hence $t\mathcal{M}$ is pre-abelian.

That it is rigid means by definition that there is a bifunctor $\mathcal{H}om$ for which the stated adjunction formula, the reflexivity and the distributivity hold.

2.2.6. Proof of the adjunction formula. After a shift by powers of the Carlitz motif, we may assume that the $M_1$, $M_2$, and $M_3$ occurring in (2.4) are effective $t$-motifs and that the $\mathcal{H}om$ that occurs in the adjunction formula is well defined in the sense of 2.1.2.

There is certainly a natural isomorphism of $K[t]$-modules

$$\text{Hom}_{K[t]}(M_1 \otimes M_2, M_3) = \text{Hom}_{K[t]}(M_1, \text{Hom}(M_2, M_3)), \quad (2.7)$$

mapping an element $f$ of the left hand side to

$$g : m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2)).$$

The map $f$ is a morphism of $t$-motifs when it satisfies

$$f \circ (\sigma_1 \otimes \sigma_2) = \sigma_3 \circ f \quad (2.8)$$

while $g$ is a morphism of $t$-motifs when it satisfies

$$g(\sigma_1(m_1)) \circ \sigma_2 = \sigma_3 \circ g(m_1). \quad (2.9)$$

Observe that (2.8) is verified if and only if (2.9) is, and hence that the bijection (2.7) restricts to the claimed adjunction formula (2.4).
2.2.7. Proof of the reflexivity and distributivity. First of all, the existence of the maps (2.5) and (2.6) is a formal consequence of the adjunction formula. To see that they are isomorphisms, it suffices to note that on the level of \( K[t] \)-modules, these are just the ordinary reflexivity and distributivity homomorphisms and hence they are isomorphisms of \( K[t] \)-modules. The claims then follow at once since a morphism of \( t \)-motifs is an isomorphism if and only if it is an isomorphism on the underlying \( K[t] \)-modules.

2.3 Isogenies

2.3.1. An isogeny between two effective \( t \)-motifs \( M_1 \) and \( M_2 \) is by definition a morphism \( f \in \text{Hom}_\sigma(M_1, M_2) \) such that there exists a \( g \in \text{Hom}_\sigma(M_2, M_1) \) and a nonzero \( h \) in \( k[t] \) with \( fg = h \text{id} = gf \).

The category whose objects are effective \( t \)-motifs over \( K \) and whose hom-sets are the modules \( \text{Hom}_\sigma(-, -) \otimes_{k[t]} k(t) \) is denoted by \( t\mathcal{M}^\circ_{\text{eff}}(K) \). Sometimes we will refer to its objects as effective \( t \)-motifs up to isogeny.

2.3.2. Denote by \( M(t) \) the \( K(t) \)-module \( M \otimes_{K[t]} K(t) \). The action of \( \sigma \) on \( M \) extends naturally and makes \( M(t) \) into a \( K(t)[\sigma] \)-module.

**Proposition.** The natural map

\[
\text{Hom}_\sigma(M_1, M_2) \otimes_{k[t]} k(t) \rightarrow \text{Hom}_{K(t)[\sigma]}(M_1(t), M_2(t))
\]

is an isomorphism.

Hence the functor \( M \rightsquigarrow M(t) \) is fully faithful on \( t\mathcal{M}^\circ_{\text{eff}} \). We shall identify \( t\mathcal{M}^\circ_{\text{eff}} \) with its image in the category of \( K(t)[\sigma] \)-modules. If we take \( M_1 \) and \( M_2 \) in the Proposition to be the unit \( t \)-motif \( 1 \), we obtain that the field of invariants \( K(t)^\sigma \) equals \( k(t) \).

**Proof of the Proposition.**\(^{(4)}\) Note that the map is \( k(t) \)-linear. Injectivity is clear.

\(^{(3)}\)See §1 of [Deligne and Milne 1982].

\(^{(4)}\)See also the pre-print [Papanikolas 2005].
To show surjectivity, choose $K[t]$-bases for $M_1$ and $M_2$ and express the action of $\sigma$ on them through matrices $S_1$ and $S_2$ (as in 1.2.5.) Expressed on the induced bases for $M_1(t)$ and $M_2(t)$, a $K(t)[\sigma]$-homomorphism from $M_1(t)$ to $M_2(t)$ is a matrix $F$ over $K(t)$ that satisfies

$$S_2^{-1}FS_1 = \tau(F).$$

(2.10)

Let $h$ be the minimal common denominator of the entries of $F$, that is, the minimal monic polynomial in $K[t]$ with the property that $hF$ has entries in $K[t]$. The minimal common denominator of the entries of the right-hand-side $\tau(F)$ is $\tau(h)$ and the minimal common denominator of the left hand side is $(t - \theta)^r h$ for some $r$. Equating them yields $r = 0$ and $\tau(h) = h$, hence the Proposition.

2.3.3. This Proposition has an important consequence:

**Corollary.** $\mathcal{M}_{\text{eff}}^\circ$ is an abelian $k(t)$-linear tensor category. $\mathcal{M}^\circ$ is a rigid abelian $k(t)$-linear tensor category.

Note that $\mathcal{M}_{\text{eff}}$ is certainly not abelian. Indeed, take for example the multiplication-by-$t$ map from $1$ to $1$. Even though both its cokernel and kernel are trivial in $\mathcal{M}_{\text{eff}}$, it is not an isomorphism.

**Proof of the Corollary.** The kernels and cokernels in $\mathcal{M}_{\text{eff}}^\circ$ are just the ordinary group-theoretic kernels and cokernels in the category of left $K(t)[\sigma]$-modules, and it is clear that a morphism whose kernel and cokernel vanish is an isomorphism. Hence $\mathcal{M}_{\text{eff}}^\circ$ is abelian.

That $\mathcal{M}^\circ$ is abelian is implied by the abelianness of $\mathcal{M}_{\text{eff}}^\circ$ and that it is rigid is implied by the rigidity of $\mathcal{M}$, the required properties of $\text{Hom}$ are preserved under extension of scalars from $k[t]$ to $k(t)$.

2.3.4. Let $K$ be algebraically closed. Clearly the category $\mathcal{C}$ of finite dimensional $K(t)$-modules equipped with a surjective semi-linear endomorphism $\sigma$ is a rigid $k(t)$-linear $\otimes$-category. By Proposition 2.3.2 the functor

$$M \mapsto M(t)$$

is fully faithful on $\mathcal{M}_{\text{eff}}^\circ$. Thus we could have defined $\mathcal{M}^\circ$ to be the rigid $\otimes$-subcategory of $\mathcal{C}$ generated by $\mathcal{M}_{\text{eff}}^\circ$. 

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In [Papanikolas 2005] a rigid $\otimes$-category of $t$-motifs is defined as the rigid subcategory of $C$ generated by the effective $t$-motifs that are finitely generated over $K[\sigma]$. The resulting category embeds naturally in $tM^\circ$. I do not know if this embedding is an equivalence.\(^{(5)}\)

2.4 Summary

2.4.1. We have a ‘commutative square’ of categories

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{-\otimes k(t)} & \mathcal{M}^\circ \\
\downarrow & & \uparrow \\
\mathcal{M}_{\text{eff}} & \xrightarrow{-\otimes k(t)} & \mathcal{M}_{\text{eff}}^\circ
\end{array}
\]

The vertical arrows are fully faithful embeddings and the horizontal arrows denote extension of scalars on the Hom modules. These categories have the following properties:

- $\mathcal{M}_{\text{eff}}$ is a pre-abelian $k[t]$-linear tensor category,
- $\mathcal{M}$ is a pre-abelian rigid $k[t]$-linear tensor category,
- $\mathcal{M}_{\text{eff}}^\circ$ is an abelian $k(t)$-linear tensor category,
- $\mathcal{M}^\circ$ is a rigid abelian $k(t)$-linear tensor category.

\(^{(5)}\) Added in proof: It seems that this is indeed an equivalence.