Chapter 1

Effective $t$-Motifs:
a Motivated Definition

1.1 Galois Representations

1.1.1. Let $K$ be a field and $K^s$ a separable closure of $K$. With a smooth and projective algebraic variety $X$ over $K$ one associates $\ell$-adic cohomology groups,

$$X \rightsquigarrow H^\bullet_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell) = H^\bullet_{\text{ét}}(X_{K^s}, \mathbb{Z}_\ell) \otimes \mathbb{Q},$$

for every prime number $\ell$ coprime to the characteristic of $K$. These are finite dimensional $\mathbb{Q}_\ell$-vector spaces equipped with a continuous action of the absolute Galois group $G_K \overset{\text{def}}{=} \text{Gal}(K^s/K)$. One knows that a number of invariants of these cohomology groups are independent of $\ell$—dimension, characteristic polynomials of Frobenius—yet there is no direct way of relating these various cohomologies.

Grothendieck has had the idea that the different $\ell$-adic cohomology theories could be mere manifestations of a more profound cohomology with $\mathbb{Q}$-coefficients. He conjectured a factorisation of the above functor as

$$X \rightsquigarrow h(X, \mathbb{Q}) \rightsquigarrow H^\bullet_{\text{ét}}(X_{K^s}, \mathbb{Q}_\ell)$$

through an object $h(X, \mathbb{Q})$, independent of $\ell$, which he baptised the motif of $X$. This object is conjecturally a finite dimensional $\mathbb{Q}$-linear represen-
tation of some universal affine group scheme over $\mathbf{Q}$, at least when $K$ is of characteristic zero.$^{(1)}$

1.1.2. Now let $k$ be a finite field of $q$ elements and $K$ any field containing $k$. Representations of $G_K$ with coefficients in $k$ are considerably more accessible than representations with characteristic zero coefficients, largely because of this:

**Theorem.** The following two categories are equivalent.

- Pairs $(H, \rho)$ of a finite dimensional $k$-vector space $H$ and a continuous homomorphism $\rho : G_K \to \text{GL}(H),$
- Pairs $(V, \sigma)$ of a finite dimensional $K$-vector space $V$ and an additive map $\sigma : V \to V$ satisfying $\sigma(xv) = x^q v$ and such that $K\sigma(V) = V.$

An equivalence is given by the mutually inverse functors

- $H \mapsto V(H) \overset{\text{def}}{=} (H \otimes_k K^s)^{G_K}$ (invariants under $G_K$), and,
- $V \mapsto H(V) \overset{\text{def}}{=} (V \otimes_K K^s)^{\sigma}$ (invariants under $\sigma$).

Note that if $K$ is not perfect $K\sigma(V)$ need not coincide with $\sigma(V)$, as can be seen already when $(V, \sigma) = (K, x \mapsto x^q)$.

The Theorem can be read as a characteristic $p$ Riemann-Hilbert correspondence: $\sigma$ has the traits of a connexion on the ‘bundle’ $V$, while $\rho$ makes $H$ into a local system of $k$-vector spaces for the étale topology on $\text{Spec } K$.

**Proof of the Theorem.**$^{(2)}$ Consider a pair $(H, \rho)$. The profinite group $G_K$ acts continuously on the $K^s$-vector space $H \otimes_k K^s$ through the simultaneous action on both factors of the tensor product. This space has an invariant $K^s$-basis by ‘Hilbert 90’ for $\text{GL}(n)$ (see the appendices, b.1.1). Thus the map

$$(H \otimes_k K^s)^{G_K} \otimes_K K^s \to H \otimes_k K^s$$

defined by

$$(h \otimes x) \otimes y \mapsto h \otimes xy$$

$^{(1)}$For background on Grothendieck’s theory of motifs, the reader is advised to consult the lecture [Serre 1991] or the monograph [André 2004].

$^{(2)}$See also Proposition 4.1 of [Pink and Traulsen 2006].
is an isomorphism. By construction it is equivariant under the action of \( \sigma \) and taking invariants one obtains an isomorphism:

\[
H(V(H)) = \left((H \otimes_k K^s)^{G_K} \otimes_k K^s\right)^{\sigma} \approx (H \otimes_k K^s)^{\sigma} = H.
\]

Now start with a \((V, \sigma)\). The action of \( \sigma \) on extends to an action on the \( K^s \)-space \( V \otimes_k K^s \) by \( \sigma(v \otimes x) = \sigma(v) \otimes x^q \). By Lang’s ‘Hilbert 90’ Theorem (see b.2.1) \( V \otimes_k K^s \) has a \( \sigma \)-invariant basis and it follows that the map

\[
(V \otimes_k K^s)^{\sigma} \otimes_k K^s \to V \otimes_k K^s : (v \otimes x) \otimes y \mapsto yxv
\]

is an isomorphism. It is \( G_K \)-equivariant and taking invariants yields the isomorphism

\[
V(H(V)) = \left((V \otimes_k K^s)^{\sigma} \otimes_k K^s\right)^{G_K} \approx (V \otimes_k K^s)^{G_K} = V.
\]

Thus \( V(\_\_\_) \) and \( H(\_\_\_) \) are mutually inverse equivalences. \( \square \)

1.1.3. Assume given a free and finitely generated \( K[t] \)-module \( M \) together with a \( k[t] \)-linear map \( \sigma : M \to M \) satisfying \( \sigma(xm) = x^q m \) for all \( x \in K \). Assume also that the \( K \)-vector space \( K\sigma(M) \) is of finite codimension in \( M \).

Fix an irreducible monic polynomial \( \lambda \in k[t] \) and consider \( M/\lambda^n M \). This is a finite dimensional \( K \)-vector space and since \( \lambda \) and \( \sigma \) commute, the semi-linear action of \( \sigma \) on \( M \) carries over to an action on the quotient. For all but finitely many ‘bad’ \( \lambda \) the resulting action on the quotient \( M/\lambda^n M \) satisfies \( K\sigma(M/\lambda^n M) = M/\lambda^n M \). For a ‘good’ \( \lambda \) (and all \( n \)) one can thus apply the functor \( H \) of the Theorem. The resulting \( H(M/\lambda^n M) \) becomes a \( k[t]/\lambda^n k[t] \)-module by transport of structure and taking limits yields a functor

\[
M \rightsquigarrow H_\lambda(M) \overset{\text{def}}{=} \left( \lim_{\to} H(M/\lambda^n M) \right) \otimes_{k[t]} k(t) \quad \text{(1.2)}
\]

that associates with the pair \((M, \sigma)\) a continuous representation of \( G_K \) on a \( k(t)_{\lambda} \)-module of finite dimension equal to \( \text{rk}_{K[t]} M \). It is continuous for
the $\lambda$-adic topology since for every $n$ the representation on $H(M/\lambda^n M)$ is continuous.

It is tempting to interpret the collection of functors (1.2)—one for every good place $\lambda$—as an analogue to the second functor in (1.1) and we will see in the following chapters that there are in fact many similarities between the modules $(M, \sigma)$ and the conjectural motifs $h(X, \mathbb{Q})$.

1.1.4. Now let again $K$ be an arbitrary field and $\ell$ a prime number different from the characteristic of $K$. Denote by $\mu_{\ell^n}(K^s)$ the set of $\ell^n$-th roots of unity in $K^s$. The ring $\mathbb{Z}/\ell^n\mathbb{Z}$ acts naturally on $\mu_{\ell^n}(K^s)$ via $(m + \ell^n \mathbb{Z}, \zeta) \mapsto \zeta^m$. This action commutes with the obvious action of $G_K$. Taking the projective limit over $n$ one thus obtains a continuous representation of $G_{K}$ on the rank one $\mathbb{Z}_\ell$-module $\mu_{\ell^\infty}$ def $= \varprojlim \mu_{\ell^n}$. This $\ell$-adic representation is denoted by $\mathbb{Q}_\ell(1)$, and after extension of scalars by $\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, representations $\mathbb{Q}_\ell(i)$ are defined for all integers $i$ as $\mathbb{Q}_\ell(i) = \mathbb{Q}_\ell(1)^{\otimes i}$.

The representations $\mathbb{Q}_\ell(i)$ are intimately related to class field theory. For example, when $K = \mathbb{Q}$, the following property characterises $\mathbb{Q}_\ell(i)$:

**Proposition.** Let $p \neq \ell$ be a prime number, then $\mathbb{Q}_\ell(i)$ is unramified at $p$ and a Frobenius element $g_p \in G_{\mathbb{Q}}$ at $p$ acts on $\mathbb{Q}_\ell(i)$ as multiplication with $p^i$. $\square$

The $\mathbb{Q}_\ell(i)$ with $i \leq 0$ occur inside various $\ell$-adic cohomology groups. One has for example:

$$H^{2d}_{\text{ét}}(\mathbb{P}^d_{K^s}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-d).$$

The representation $\mathbb{Q}_\ell(i)$ with $i$ positive, however, cannot occur as a piece of the $\ell$-adic cohomology of some smooth and projective variety over, say, a number field, since on these groups the traces of geometric Frobenius—inverse to the arithmetic Frobenius of the above Proposition—must be algebraic integers.\(^{(3)}\)

It is most important to note that the prime $p$ plays two very distinct roles in the Proposition. It is staged as a prime of the base field $K = \mathbb{Q}$, \(^{(3)}\)

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\(^{(3)}\)By a Theorem of Deligne (the ‘Weil Conjectures’), see [Deligne 1974].
in the guise of the Frobenius $g_p$, and it features as an element of the coefficient field $\mathbb{Q}_p$, as multiplication with $p^i$.

1.1.5. Now put $K = k(\theta)$, the function field of the projective line over the finite field $k$. A prime of $K$ is either ‘infinity’ or a monic irreducible polynomial $f \in k[\theta]$, which we shall call a finite prime. In contrast with the double role played by $p$ in the previous paragraph, here primes of the base $K$ will be polynomials in $\theta$, while elements of the coefficient fields $k(t)_\lambda$ will be (germs of) functions in $t$.

Denote by $C = (C, \sigma)$ the pair

$$ C \overset{\text{def}}{=} K[t]e \quad \text{with} \quad \sigma(f)e \overset{\text{def}}{=} \tau(f)(t - \theta)e, $$

where $\tau(f)$ is the polynomial in $K[t]$ obtained by raising all coefficients of $f$ to the $q$-th power. The following property of $C$ can be shown to characterise $C$ amongst those $(M, \sigma)$ with $M$ of rank one. It is essentially due to Carlitz.\(^{(4)}\)

**Proposition.** Let $\lambda \in k[t]$ be monic and irreducible. If $f = f(\theta) \in k[\theta]$ is a finite prime of $K$ with $f(t) \neq \lambda$ then $H_\lambda(C)$ is unramified at $f$ and a Frobenius element $g_f \in G_K$ at $f$ acts by multiplication with $f(t)^{-1}$.

**Proof.** Denote the degree of $f$ by $d$. The representation $H_\lambda(C)$ is constructed from the $H(C/\lambda^nC)$. Writing out their definition gives

$$ H(C/\lambda^nC) = (K^s[t]e/\lambda^nK^s[t])e^\sigma $$

$$ = \{ x \in K^s[t]/\lambda^nK^s[t] \mid \tau(x)(t - \theta) = x \}. $$

Iterate the action of $\tau$ on an $x$ in this set and obtain

$$ \tau^d(x)(t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}}) = x. $$

The defining property of the Frobenius element $g_f$ is that it equals $\tau^d$ after reduction modulo $f(\theta)$. Moreover, it acts on the $k[t]$-module $H(C/\lambda^nC)$ by endomorphisms. Thus, in order to conclude it suffices to observe that

$$ f(t) = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}}) $$

as polynomials in $t$ over the finite field $k[\theta]/f(\theta)k[\theta]$. \(\square\)

\(^{(4)}\)See [Carlitz 1935].
Define an action of $\sigma$ on the tensor product $C \otimes C$ diagonally:

$$C \otimes C \overset{\text{def}}{=} C \otimes_{K[t]} C \quad \text{with} \quad \sigma(a \otimes b) \overset{\text{def}}{=} \sigma(a) \otimes \sigma(b),$$

and similarly for the higher tensor powers $C^i$ (the number of factors $i$ being positive).\(^5\) The previous Proposition generalises in a straightforward manner. Take $\lambda \neq f(t)$, then

**Proposition.** $g_f$ acts on $H_\lambda(C^i)$ as multiplication with $f(t)^{-i}$. \(\square\)

### 1.2 Effective $t$-Motifs

1.2.1. As before, $k$ is a field of $q$ elements and $K$ a field containing $k$. Fix a homomorphism $k[t] \to K$ of $k$-algebras and denote the image of $t$ by $\theta$. This homomorphism is to play the role of the canonical homomorphism from $\mathbb{Z}$ into an arbitrary field. Its kernel replaces the notion of the characteristic of a field—$K$ itself is of course always of positive characteristic $p$. We shall frequently refer to ‘the field $K$’, this is silently understood to contain the structure homomorphism $k[t] \rightarrow K$.

Denote by $\tau$ the endomorphism of $K[t]$ determined by $\tau(x) = x^q$ for all $x \in K$ and $\tau(t) = t$.

The following definition goes back to [Anderson 1986], although here a slightly less restrictive form is used.

**Definition.** An effective $t$-motif of rank $r$ over $K$ is a pair $M = (M, \sigma)$ consisting of

- a free and finitely generated $K[t]$-module $M$ of rank $r$, and,
- a map $\sigma : M \to M$ satisfying $\sigma(fm) = \tau(f)\sigma(m)$ for all $f \in K[t]$ and $m \in M$,

such that the determinant of $\sigma$ with respect to some (and hence any) $K[t]$-basis of $M$ is a power of $t - \theta$ up to a unit in $K$.

\(^5\)We use the exponential notation for tensor products ($V^2 \overset{\text{def}}{=} V \otimes V$) and the multiplicative notation for direct sums ($2V \overset{\text{def}}{=} V \oplus V$).
The condition on the determinant guarantees that the Galois representation $H_{\lambda}(M)$ is well-defined for all but at most one place $\lambda$, the possible exception being the kernel of $k[t] \to K$.

A morphism of effective $t$-motifs is a morphism of $K[t]$-modules making the obvious square commute. The group of morphisms is denoted $\text{Hom}_\sigma(M_1, M_2)$. The resulting category of effective $t$-motifs over $K$ is denoted by $t\mathcal{M}_{\text{eff}}(K)$ or simply by $t\mathcal{M}_{\text{eff}}$.

1.2.2. For any field $k[t] \to K$, the pair

$$C \overset{\text{def}}{=} K[t]e \quad \text{with} \quad \sigma(f e) \overset{\text{def}}{=} \tau(f)(t - \theta)e$$

that we already encountered is an effective $t$-motif and we will call it the Carlitz $t$-motif.

1.2.3. Define the tensor product of two effective $t$-motifs as

$$M_1 \otimes M_2 \overset{\text{def}}{=} M_1 \otimes_{K[t]} M_2 \quad \text{with} \quad \sigma(m_1 \otimes m_2) \overset{\text{def}}{=} \sigma(m_1) \otimes \sigma(m_2).$$

This is again an effective $t$-motif.

The pair $(K[t], \tau)$ is an effective $t$-motif which we shall denote $\mathbf{1}$. We call it the unit $t$-motif, since for every $M$, one has natural isomorphisms $M \otimes \mathbf{1} = M$ and $\mathbf{1} \otimes M = M$.

1.2.4. If $M_1$ and $M_2$ are effective $t$-motifs then $\text{Hom}(M_1, M_2)$ is naturally a $k[t]$-module.

**Proposition.** $\text{Hom}(M_1, M_2)$ is free and finitely generated over $k[t]$.

**Proof.** (6) It is sufficient to show that the natural map

$$\text{Hom}_\sigma(M_1, M_2) \otimes_k K \to \text{Hom}_{K[t]}(M_1, M_2)$$

is injective. Assume that this is not the case, that is, that there exist $k$-linearly independent $f_i \in \text{Hom}_\sigma(M_1, M_2)$ and scalars $x_i \in K$ with

$$f_0 + x_1 f_1 + \cdots + x_n f_n = 0 \quad \text{in} \quad \text{Hom}_{K[t]}(M_1, M_2). \quad (1.3)$$

(6) See Theorem 2 of [Anderson 1986].
and assume \( n \) to be minimal. The \( f_i \) are \( \sigma \)-equivariant, thus
\[
f_0(\sigma m) + x_1^q f_1(\sigma m) + \cdots + x_n^q f_n(\sigma m) = 0 \quad \forall m \in M_1.
\]
and since \( \sigma M_1 \subset M_1 \) is free and of finite codimension
\[
f_0 + x_1^q f_1 + \cdots + x_n^q f_n = 0 \quad \text{in } \text{Hom}_{K[t]}(M_1, M_2). \tag{1.4}
\]
Subtract (1.4) from (1.3) to obtain
\[
(x_1 - x_1^q)f_1 + \cdots + (x_n - x_n^q)f_n = 0
\]
and deduce from the minimality of \( n \) that all coefficients vanish, thus that the \( x_i \) are in \( k \), which contradicts the independence of the \( f_i \).

**1.2.5.** To write down an effective \( t \)-motif of rank \( r \), choose a \( K[t] \)-basis \( e = (e_i) \) of \( M \) and express the action of \( \sigma \) as \( \sigma(e) = Se \) for some \( S \in M(r \times r, K[t]) \) whose determinant equals \( x(t - \theta)^d \) with \( x \in K^\times \) and \( d \) a non-negative integer. Every such matrix \( S \) determines an effective \( t \)-motif.

Given an \( M_1 \) of rank \( r_1 \) described on a basis by a matrix \( S_1 \) as above, as well as \( M_2, r_2 \) and \( S_2 \), we have
\[
\text{Hom}_r(M_1, M_2) = \{ F \in M(r_2 \times r_1, K[t]) \mid FS_1 = S_2 \tau(F) \}
\]
and in particular, two matrices \( S_1 \) and \( S_2 \) determine the same effective \( t \)-motif if and only if \( r_1 = r_2 = r \) and there exists an \( F \in \text{GL}(r, K[t]) \) such that \( S_1 = F^{-1}S_2 \tau(F) \).

**1.2.6.** Denote by \( K[\sigma] \) the ring whose elements are polynomial expressions of the form
\[
x_0 + x_1\sigma + \cdots + x_n\sigma^n
\]
and where multiplication is determined by the rule
\[
\sigma x = x^q \sigma.
\]
Effective \( t \)-motifs are naturally left \( K[\sigma] \)-modules.\(^{(7)}\)

\(^{(7)}\)More on the structure of the ring \( K[\sigma] \) is in Appendix a.
An effective $t$-motif may or may not be finitely generated over $K[\sigma]$. For instance: $C$ is but 1 is not. When an effective $t$-motif $M$ is finitely generated over $K[\sigma]$ then it is automatically free.\(^{(8)}\)

In that case one can equip $M$ with a free $K[\sigma]$-basis $f = (f_j)$ and express the action of $t$ by $tf = Tf$ for some $T \in M(d \times d, K[\sigma])$. A matrix $T \in M(d \times d, K[\sigma])$ determines an effective $t$-motif if and only if $T \equiv \theta I_d + N$ modulo $\sigma$, where $N \in M(d \times d, K)$ is a nilpotent matrix. To recover the rank of $M$ from $T$ one cannot just take the determinant of $T$ since $K[\sigma]$ is a non-commutative ring, but the degree of the Dieudonné determinant is well-defined and one has $\text{rk}(M) = \text{deg det } T$ (see Appendix a).

Given two such effective $t$-motifs $M_1, M_2$, equipped with $K[\sigma]$-bases on which the action of $t$ is expressed by $T_1$ and $T_2$ respectively, we have

\[
\text{Hom}_\sigma(M_1, M_2) = \{ G \in M(d_2 \times d_1, K[\sigma]) \mid GT_1 = T_2G \}
\]

and in particular, $M_1$ and $M_2$ are isomorphic if and only if $d_1 = d_2 = d$ and there exists a $G \in \text{GL}(d, K[\sigma])$ such that $T_1 = G^{-1}T_2G$.

### 1.3 Example: Drinfeld Modules

1.3.1. Let $M$ be an effective $t$-motif that is free of rank 1 as $K[\sigma]$-module. Denote by $Z$ the centre of the endomorphism ring of $M$. Thus $Z$ is a $k[t]$-algebra of finite rank. The following is shown in [DRINFEL'D 1974]:

**Theorem.** There exists a projective curve $X$ over $k$ and a closed point $\infty \in X$ such that $Z \approx H^0(X - \infty, O_X)$. If moreover $k[t] \to K$ is injective, then $\text{End}(M) = Z$. \(\square\)

1.3.2. Next, fix a $k[t]$-algebra $A$ such that the spectrum of $A$ is $X - \infty$ for some smooth projective curve $X$ and a closed point $\infty \in X$. We call a

\(^{(8)}\)Lemma 1.4.5 of [ANDERSON 1986]. Note that in Anderson’s paper only modules that are finitely generated over $K[\sigma]$ are considered. In fact, what Anderson calls a ‘$t$-motive’ is in our language an ‘effective $t$-motif that is finitely generated over $K[\sigma]$.’
Drinfeld A-module\(^{(9)}\) an effective \(t\)-motif \(M\) that is free of rank one over \(K[\sigma]\) together with an injective homomorphism \(A \to \text{End}(M)\). Such an \(M\) is a projective and finitely generated \(A \otimes K\)-module. The rank of \(M\) over \(A \otimes K\) is called (abusively) the \(A\)-rank of \(M\). The Carlitz motif \(C\) is a Drinfeld \(k[t]\)-module of \(k[t]\)-rank 1, but its higher tensor powers are not Drinfeld modules. Also shown in loc. cit. is:

**Theorem.** Over a separably closed field \(K\), Drinfeld A-modules of arbitrary \(A\)-rank exist.

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\(^{(9)}\)Introduced in [Drinfel’d 1974], where the term ‘elliptic module’ is used. Warning: one usually studies Drinfeld modules in the category opposite to the category of effective \(t\)-motifs.