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Balancing of Lossless and Passive Systems

Arjan van der Schaft

Abstract—Different balancing techniques are applied to lossless nonlinear systems, with open-loop balancing applied to their scattering representation. It is shown that they all lead to the same result: the pair of to-be-balanced functions is given by two copies of the physical energy function, yielding thus no information about the relative importance of the state components in a balanced realization. In particular, in the linear lossless case all balancing singular values and similarity invariants are equal to one. This result is extended to general passive systems, in which case the to-be-balanced functions are ordered into a single sequence of inequalities, and the similarity invariants are all less than or equal to one.

Index Terms—Balancing, lossless, nonlinear, passive, scattering.

I. INTRODUCTION

Modeling of technological or physical systems often leads to high-dimensional dynamical models. The same occurs if distributed-parameter models are spatially discretized. An important issue concerns model reduction of these high-dimensional systems, both for analysis and control.

Within the systems and control literature a popular and elegant tool for model reduction is *balancing*, dating back to [6]. A favorable property of model reduction based on balancing, as compared with other techniques such as modal analysis, is that the approximation of the system is based on its input-output properties.

In this paper we investigate various balancing methods for general (linear and nonlinear) systems which are *passive* or *lossless*. Passive and lossless systems, including their scattering representation, are treated in Section II. In Section III different balancing approaches

for lossless systems are investigated, leading to the result that all balancing functions coincide with the physical energy. In Section IV this is extended to passive systems, leading to a sequence of inequalities, and—in the linear case—to similarity invariants that are all less than or equal to one. Preliminary versions of the results obtained in this paper have been reported in [13].

II. PASSIVE AND LOSSLESS SYSTEMS

Consider the square nonlinear input-state-output system

$$\Sigma : \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) + d(x)u \end{cases} \quad (1)$$

where $u, y \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$ are local coordinates for an n -dimensional state space manifold \mathcal{X} . In such local coordinates $a(x)$ denotes an n -dimensional vector, $b(x)$ an $n \times m$ -dimensional matrix, $c(x)$ an m -dimensional vector, while $d(x)$ is an $m \times m$ -dimensional matrix. Throughout we assume the existence of a distinguished *equilibrium* x_0 , that is,

$$a(x_0) = 0, \quad c(x_0) = 0. \quad (2)$$

The system Σ is called *passive* [17] if there exists a function $H : \mathcal{X} \rightarrow \mathbb{R}$ with $H(x_0) = 0$ and $H(x) \geq 0$ for every x , such that

$$H(x(t_2)) - H(x(t_1)) \leq \int_{t_1}^{t_2} u^T(t)y(t)dt \quad (3)$$

for all solution trajectories $(u(\cdot), x(\cdot), y(\cdot))$ of the system Σ and all time instants $t_1 \leq t_2$. This means that we define the class \mathfrak{U} of *admissible input functions* $u : [t_0, \infty) \rightarrow \mathbb{R}^m$ for Σ in such a manner that $\dot{x} = a(x) + b(x)u$, $x(t_0) = x_0$, has a unique solution on $[t_0, \infty)$ for all t_0, x_0 , while the integral in the right-hand side of (3) is always well-defined.

The function H is called a *storage function* and the inequality (3) is called the *dissipation inequality*. The system is *lossless* if the inequality \leq in (3) is replaced by an equality

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} u^T(t)y(t)dt. \quad (4)$$

Remark II.1: The assumption $H(x_0) = 0$ is not needed within a general definition of passive and lossless systems (see [12], [17]), but will be made for simplicity throughout this paper.

Passive systems are abundant in physical modeling by equating H with the energy stored in the physical system and $u^T y$ with the power supplied to the system. Lossless systems result by assuming that there is no internal energy dissipation in the system. (In some cases, such as weakly damped mechanical structures, this may be a useful idealizing assumption, both for analysis and control.) Henceforth we call the inputs u together with the outputs y of a passive system the *power variables*.

It is well-known that if the function H is *differentiable* then the property of being passive is equivalent to the Willems-Hill-Moylan conditions [4], [17]

$$\begin{bmatrix} 2 \frac{\partial^T H}{\partial x}(x)a(x) & \frac{\partial^T H}{\partial x}(x)b(x) - c^T(x) \\ b^T(x) \frac{\partial H}{\partial x}(x) - c(x) & -d(x) - d^T(x) \end{bmatrix} \leq 0 \quad (5)$$

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for all x, u , while in the lossless case the inequality \leq is replaced by an equality. (Throughout $(\partial H/\partial x)(x)$ will denote the n -dimensional column vector of partial derivatives of H .) In the lossless case these conditions specialize to

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x)a(x) &= 0 \\ c(x) &= b^T(x) \frac{\partial H}{\partial x}(x) \\ d(x) &= -d^T(x) \end{aligned} \quad (6)$$

while in the passive case more explicit expressions are obtained by assuming either $d(x) + d^T(x) > 0$ (the *strictly passive* case) and taking Schur complements, or by assuming $d(x) = 0$, in which case (5) reduces to

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x)a(x) &\leq 0 \\ c(x) &= b^T(x) \frac{\partial H}{\partial x}(x). \end{aligned} \quad (7)$$

For a *linear* system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (8)$$

with distinguished equilibrium $x_0 = 0$ we may as well restrict to quadratic storage functions $H(x) = (1/2)x^T Qx$ with $Q = Q^T \geq 0$. In this case the Willems–Hill–Moylan conditions specialize to the Kalman–Yakubovich–Popov Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^T Q + QA & QB - C^T \\ B^T Q - C & -D - D^T \end{bmatrix} \leq 0 \quad (9)$$

which again specializes in the lossless case to

$$A^T Q + QA = 0, \quad C = B^T Q, \quad D = -D^T \quad (10)$$

and in the passive case with $D = 0$ to the LMI

$$A^T Q + QA \leq 0, \quad C = B^T Q. \quad (11)$$

For a passive system, there exists in general a *set* of storage functions, having the following interesting structure [17]. Define for any state x of Σ the *available storage* $S_a(x)$ as

$$S_a(x) = \sup_{u, T \geq 0} - \int_0^T u^T(t)y(t)dt \quad (12)$$

where the supremum is taken over all admissible input functions $u \in \mathcal{U}$. It can be shown [12], [17] that Σ is passive if and only if $S_a(x)$ is finite for all x , and that in this case S_a defines a storage function. Next, define the *required supply* $S_r(x)$ to reach x at $t = 0$ starting from x_0 as

$$S_r(x) = \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 u^T(t)y(t)dt \quad (13)$$

where again the infimum is over \mathcal{U} . If the system is reachable from x_0 then it can be shown [12], [17] that Σ is passive if and only if there exists a constant $K > -\infty$ such that $S_r(x) \geq K$ for all $x \in \mathcal{X}$, in which case S_r defines a storage function for Σ . Furthermore, it follows [12], [17] that for all x

$$S_a(x) \leq S(x) \leq S_r(x) \quad (14)$$

for *all* storage functions S , and thus S_a is the *minimal* and S_r the *maximal* storage function [12], [17]. In the case of a lossless system that is both reachable from and controllable to x_0 the inequality (14) reduces to the equality $S_a(x) = H(x) = S_r(x)$ [17], and H is the unique storage function.

In the case of a linear passive system Σ given by (8) with $D = 0$ the available storage S_a is given as $(1/2)x^T Q_a x$ where Q_a is the *minimal* solution to the LMI (11) while the required supply is $(1/2)x^T Q_r x$ where Q_r is the *maximal* solution to this same LMI.

Remark II.2: Note that for a passive system with non-zero internal energy dissipation we need not always get strict inequalities in (14). Indeed, consider the ubiquitous mass-spring-damper system

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad u = \text{force} \\ y &= \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \text{velocity} \end{aligned}$$

with physical energy $H(q, p) = (1/2m)p^2 + (1/2)kq^2$, and internal energy dissipation corresponding to the damper with coefficient $c > 0$. The LMI (11) takes the form

$$\begin{aligned} \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -\frac{c}{m} \end{bmatrix} \leq 0 \\ [0 \quad 1] \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = [0 \quad \frac{1}{m}]. \end{aligned}$$

The last equation yields $q_{12} = 0$ as well as $q_{22} = (1/m)$. Substituted in the inequality this yields as unique solution $q_{11} = k$, corresponding to a unique storage function $H(q, p) = (1/2m)p^2 + (1/2)kq^2$, which is equal to S_a and S_r . The reason for this at first sight perhaps surprising equality is the fact that the definitions of S_a and S_r involve *sup* and *inf* (instead of *max* and *min*).

For a passive system with H continuously differentiable it immediately follows from (5) that $(\partial^T H/\partial x)(x)a(x) \leq 0$. Thus if H is *positive definite*, that is, $H(x) > 0$ for every $x \neq x_0$, then H in this case is a valid Lyapunov function; implying that the equilibrium x_0 is at least *stable*. Furthermore, in the lossless case H is conserved along trajectories of the system for $u = 0$, showing that the system is *not* asymptotically stable around x_0 . Hence standard open-loop balancing cannot be directly applied to lossless systems, while its applicability to passive systems will depend on the 'pervasiveness' of internal energy dissipation (and its outcome may critically depend on—sometimes unknown—dissipation parameters).

A. Scattering Representation

In order to overcome this asymptotic stability obstacle and to relate open-loop balancing to physical energy considerations it is useful to switch to the well-known [1], [3] *scattering representation* Σ_s of Σ , which is obtained by the following transformation of the power variables u and y :

$$\begin{aligned} v &= \frac{1}{\sqrt{2}}(u + y) \\ z &= \frac{1}{\sqrt{2}}(-u + y) \end{aligned} \quad (15)$$

with inverse

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(v - z) \\ y &= \frac{1}{\sqrt{2}}(v + z). \end{aligned} \quad (16)$$

Substitution of these last expressions into Σ with $d(x) = 0$ yields the scattering representation Σ_s [1], [12]

$$\begin{aligned} \dot{x} &= a(x) - b(x)c(x) + \sqrt{2}b(x)v \\ \Sigma_s : \quad z &= \sqrt{2}c(x) - v \end{aligned} \quad (17)$$

which can be regarded as an input-state-output system with input v (the 'incoming wave') and output z (the 'outgoing wave').

Remark II.3: Similar, but more involved, formulas can be derived for the case $d \neq 0$ under the assumption that the matrix $I - d(x)$ is invertible.

For subsequent use we collect the following equalities relating the power variables u, y with the wave variables v, z :

$$\begin{aligned} \frac{1}{2} \|v\|^2 - \frac{1}{2} \|z\|^2 &= u^T y \\ \frac{1}{2} \|u\|^2 + \frac{1}{2} \|y\|^2 &= \|z\|^2 + u^T y = \|v\|^2 - u^T y \\ \|v\|^2 + \|z\|^2 &= \|u\|^2 + \|y\|^2 \end{aligned} \quad (\text{parallelogram identity}). \quad (18)$$

The first equality represents the basic relation between the power variables and the wave variables. Indeed, using the first equality we obtain the following equivalent characterization of passivity in terms of the scattering representation Σ_s :

$$H(x(t_2)) - H(x(t_1)) \leq \int_{t_1}^{t_2} \frac{1}{2} \|v(t)\|^2 - \frac{1}{2} \|z(t)\|^2 dt \quad (19)$$

for all solution trajectories $(v(\cdot), x(\cdot), z(\cdot))$ of the system Σ_s and all time instants $t_1 \leq t_2$. In the lossless case the inequality in (19) is replaced by an equality. The term $(1/2) \|v(t)\|^2$ equals the incoming power (due to the incoming wave v), while $(1/2) \|z(t)\|^2$ is the outgoing power (due to the outgoing wave z).

Remark II.4: This amounts to the well-known fact (see e.g., [3]) that the scattering representation of a passive system has L_2 -gain ≤ 1 , and in the linear case [1] to the correspondance between *positive-real* and *bounded-real* transfer matrices under the scattering transformation.

The transformation to the scattering representation has the following implications for asymptotic stability. We concentrate on the, critical, lossless case. If H is continuously differentiable and positive definite then we obtain for $v = 0$

$$\frac{d}{dt} H(x(t)) = -\frac{1}{2} \|z(t)\|^2 = -\|y(t)\|^2 \quad (20)$$

ensuring asymptotic stability if Σ_s is *zero-state detectable* (with x_0 representing the zero-state) [12]. Similarly, the *time-reversed* system Σ_s for $z = 0$ satisfies

$$\frac{d}{d(-t)} H(x(t)) = -\frac{1}{2} \|v(t)\|^2 = -\|y(t)\|^2. \quad (21)$$

This motivates the following assumption:

Assumption II.1: Consider the passive system Σ . The equilibrium x_0 is globally asymptotically stable for Σ_s with $v = 0$ and for the time-reversed system Σ_s with $z = 0$.

Remark II.5: From (17) we conclude that Assumption II.1 amounts to assuming that x_0 is globally asymptotically for $\dot{x} = a(x) - b(x)c(x)$ as well as for $\dot{x} = -[a(x) + b(x)c(x)]$. In the linear lossless case this is guaranteed by requiring that the pair (C, A) is *detectable*.

III. BALANCING OF LOSSLESS SYSTEMS

In this section we will apply various balancing procedures to lossless systems. First we apply nonlinear open-loop balancing [10] to the scattering representation Σ_s . This involves the computation of the *observability function* (the subscript "s" standing for "scattering")

$$O_s(x) := \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt \quad (22)$$

with $v = 0$ and initial condition $x(0) = x$. Because $(1/2) \|z(t)\|^2$ is the outgoing power, the observability function $O_s(x)$ equals the *outgoing physical energy*. Since Σ_s is lossless it follows from (19) with equality, together with $H(x_0) = 0$ and Assumption II.1, that O_s is well-defined (that is, $z \in L^2[0, \infty)$), and $O_s(x) = H(x)$.

Secondly, open-loop balancing for Σ_s involves the computation of the *controllability function*

$$C_s(x) := \inf_v \int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 dt \quad (23)$$

where the infimum is taken over all L^2 input functions $v : (-\infty, 0) \rightarrow \mathbb{R}^m$ taking the state from x_0 at $t = -\infty$ to x at $t = 0$ (more accurately, the time-reversed controlled system starting from x at time $t = 0$ converges for $t \rightarrow -\infty$ to x_0). Thus $C_s(x)$ is the minimal physical energy that is needed to transfer the state from x_0 to x . Applying the first line of (18) and (4) together with $H(x_0) = 0$ we see that for any such v

$$\begin{aligned} &\int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 dt \\ &= \int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t) dt \\ &= \int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + H(x) \end{aligned} \quad (24)$$

implying that $v \in L^2$ if and only if $z \in L^2$. Furthermore, it implies

$$C_s(x) = \inf_v \left[\int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt \right] + H(x) \quad (25)$$

leading (using Assumption II.1) to the optimal input $v \in L^2$ being such that $z = 0$, while $C_s(x) = H(x)$. In fact, we conclude that the minimal energy $\int_{-\infty}^0 (1/2) \|v(t)\|^2 dt$ to reach x at $t = 0$ is achieved by letting v to be such that the *outgoing* wave vector on $(-\infty, 0)$ is zero. Therefore the minimal input energy is equal to $H(x)$. This is 'dual' to the computation of the observability function for $x(0) = x$, where we already *start* from the assumption that the *ingoing* wave v equals zero, resulting in an output energy equal to $H(x)$. (Note that alternatively we could have started from the *minimization* of $\int_0^\infty (1/2) \|z(t)\|^2 dt$ under the constraint $x(\infty) = x_0$ and *deriving* as the optimal input $v = 0$!) We conclude that

$$O_s = H = C_s. \quad (26)$$

Hence, since open-loop balancing is based on comparing O_s and C_s , no information is obtained about the relative importance of the state components in a balanced realization.

Remark III.1: For a linear lossless system in scattering representation Σ_s the equality (26) amounts to the fact that the observability Gramian M_s , which is the unique solution to

$$(A - BC)^T M_s + M_s (A - BC) = -2C^T C$$

and the inverse of the controllability Gramian W_s , which is the unique solution to

$$(A - BC)W_s + W_s(A - BC)^T = -2BB^T$$

are both equal to Q . Hence $M_s W_s$ equals the identity matrix, and the Hankel singular values are all equal to one.

Next we consider LQG-balancing or *closed-loop balancing* as introduced in [5] for linear systems, and its extension to the nonlinear case, see [11]. Thus we define for Σ the *future energy function* E_f [5], [11], [16] as

$$E_f(x) := \inf_u \int_0^\infty \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|y(t)\|^2 dt \quad (27)$$

where the infimum is taken over all L^2 input functions $u : [0, \infty) \rightarrow \mathbb{R}^m$ taking the system from state x at $t = 0$ to x_0 at time $t = \infty$ (more accurately, the controlled system converges for $t \rightarrow \infty$ to x_0). Because of the second equality in the second line of (18) it follows that:

$$\begin{aligned} E_f(x) &= \inf_u \left[\int_0^\infty (\|v(t)\|^2 - u^T(t)y(t)) dt \right] \\ &= \inf_u \left[\int_0^\infty \|v(t)\|^2 dt - \int_0^\infty u^T(t)y(t) dt \right] \\ &= \inf_u \left[\int_0^\infty \|v(t)\|^2 dt \right] + H(x) \end{aligned} \quad (28)$$

where the last equality follows from (4) for $t_1 = 0$ and $t_2 = \infty$ together with $x(\infty) = x_0$ and $H(x_0) = 0$. This infimum is obviously attained at u being such that $v = 0$, leading to the equality $E_f(x) = H(x)$. (Note that by Assumption II.1 x_0 is globally asymptotically stable for Σ_s with $v = 0$, while $v = 0$ corresponds to $u = -y$ and hence to $\int_0^\infty \|u(t)\|^2 dt = \int_0^\infty (1/2) \|u(t)\|^2 + (1/2) \|y(t)\|^2 dt = H(x)$, showing that the minimizing u is in L^2 .)

Analogously, we define the *past energy function* E_p as

$$E_p(x) := \inf_u \int_{-\infty}^0 \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|y(t)\|^2 dt \quad (29)$$

where the infimum is taken over all input functions $u : (-\infty, 0) \rightarrow \mathbb{R}^m$ taking the system from state x_0 at $t = -\infty$ to x at time $t = 0$. Because of the first equality in the second line of (18) it follows that

$$\begin{aligned} E_p(x) &= \inf_u \left[\int_{-\infty}^0 (\|z(t)\|^2 + u^T(t)y(t)) dt \right] \\ &= \inf_u \left[\int_{-\infty}^0 \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t) dt \right] \\ &= \inf_u \left[\int_{-\infty}^0 \|z(t)\|^2 dt \right] + H(x) \end{aligned} \quad (30)$$

where the last equality follows from (4) for $t_1 = -\infty$ and $t_2 = 0$ together with $x(-\infty) = x_0$ while $H(x_0) = 0$. This infimum is obviously attained at u being such that $z = 0$, leading to the equality $E_p(x) = H(x)$. (Note that by Assumption II.1 x_0 is globally asymptotically stable for the time-reversed Σ_s with $z = 0$.)

In conclusion, both the future and past energies E_f and E_p are equal to H :

$$E_f = H = E_p. \quad (31)$$

Thus, like in the case of open-loop balancing for Σ_s , nonlinear closed-loop balancing does not provide information about the relative importance of the state components.

Remark III.2: When specialized to a linear lossless system (8) with $D = 0$ the above result amounts to the fact that the stabilizing solution P to the Control Algebraic Riccati Equation (CARE)

$$A^T P + PA + C^T C - PBB^T P = 0$$

and the inverse of the stabilizing solution S to the Filter Algebraic Riccati Equation (FARE)

$$AS + SA^T + BB^T - SC^T C S = 0$$

are both equal to Q , because Q satisfies (10). In particular the closed-loop similarity invariants [5], [16] are all equal to 1. (It should be noted that in [5] (Theorem 3) the stronger result has been proved that the similarity invariants are all equal to 1 if and only if any balanced realization $(\tilde{A}, \tilde{B}, \tilde{C})$ satisfies $\tilde{A} + \tilde{A}^T = 0$ and $\tilde{B}\tilde{B}^T = \tilde{C}^T\tilde{C}$. Hence, not only are the closed-loop similarity invariants of a lossless linear system equal to 1, but, conversely, any system with closed-loop similarity invariants equal to 1 is lossless, up to an orthonormal transformation of the input and output space.)

Furthermore, the optimal LQG compensator [5] for Σ is seen to reduce to

$$\dot{\hat{x}} = A\hat{x} + B(y - \hat{y}) + Bu, \quad \hat{y} := C\hat{x}.$$

Hence, if the initial estimate $\hat{x}(0)$ is correct, that is, $\hat{x}(0) = x(0)$, then for all $t \geq 0$ we will have $\hat{y}(t) = y(t)$, implying that $u(t) = -y(t)$ for all $t \geq 0$, which equals the minimizing input $u = -y$ for the future energy function E_f .

Nonlinear closed-loop balancing applied to the *scattering representation* Σ_s yields the same result. Indeed, due to the third line of (18) (the parallelogram identity), the future and past energy functions E_f and E_p for the scattering representation are *equal* to the future and past energy functions for the power variable representation.

Finally, another form of balancing is based on comparing the available storage function S_a and the required supply function S_r of Σ . For the linear case (usually under the additional assumption of an invertible feedthrough matrix D , implying strict passivity) this so-called *positive-real balancing* has been studied in [7]–[9], [15], and more recently in [2], [14]. The approach can be applied to any nonlinear passive system. In the lossless case, however, whenever the system is reachable from and controllable to x_0 the functions S_a and S_r are equal [17]

$$S_a = H = S_r. \quad (32)$$

Remark III.3: In the linear lossless case the minimal and maximal solution Q_a and Q_r of the LMI (11) are equal, and thus $Q_a Q_r^{-1} = I$, implying that the similarity invariants of positive-real balancing are all equal to one.

Summarizing we may collect the above results in the following

Theorem III.4: Consider a lossless nonlinear system Σ that is both reachable from and controllable to x_0 , and satisfying Assumption II.1. Then

$$E_f = O_s = S_a = H = S_r = C_s = E_p. \quad (33)$$

IV. PASSIVE CASE

What happens with the series of equalities (33) in the passive case? First of all, recall that in the passive case *with* internal energy dissipation there may be a *gap* between the available storage S_a and the required supply S_r :

$$S_a \leq H \leq S_r. \quad (34)$$

What happens with the other balancing functions ? First, we note that by the first line of (18) with $v = 0$

$$\begin{aligned} O_s(x) &= \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt = - \int_0^\infty u^T(t)y(t)dt \\ &\leq \sup_{u, T \geq 0} - \int_0^T u^T(t)y(t)dt = S_a(x) \end{aligned} \quad (35)$$

showing that $O_s \leq S_a$. Furthermore, using the expression for E_f in the scattering representation we trivially obtain (take $v = 0$)

$$\begin{aligned} E_f(x) &= \inf_v \int_0^\infty \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|z(t)\|^2 dt \\ &\leq \int_0^\infty \frac{1}{2} \|z(t)\|^2 dt = O_s(x). \end{aligned} \quad (36)$$

On the other hand, by making use of the representation of the controllability function C_s obtained in (25), we obtain (recall that the infimum is taken over all functions v such that the time-reversed system starting from x at time $t = 0$ converges to x_0 for $t \rightarrow -\infty$)

$$\begin{aligned} C_s(x) &= \inf_v \left[\int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt + \int_{-\infty}^0 u^T(t)y(t)dt \right] \\ &\geq \inf_v \int_{-\infty}^0 u^T(t)y(t)dt \\ &\geq \inf_{u, T \geq 0, x(-T)=x_0} \int_{-T}^0 u^T(t)y(t)dt = S_r(x) \end{aligned} \quad (37)$$

and hence $C_s \geq S_r$. Furthermore

$$\begin{aligned} E_p(x) &= \inf_v \int_{-\infty}^0 \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|z(t)\|^2 dt \\ &\geq \inf_v \int_{-\infty}^0 \frac{1}{2} \|z(t)\|^2 dt = C_s(x) \end{aligned} \quad (38)$$

showing that $E_p \geq C_s$. Collecting all these inequalities we obtain

Theorem IV.1: For any passive system Σ that is reachable from x_0 and satisfying Assumption II.1

$$E_f \leq O_s \leq S_a \leq H \leq S_r \leq C_s \leq E_p. \quad (39)$$

Remark IV.2: In the linear case this sequence of inequalities reduces to (using notation previously introduced)

$$P \leq M_s \leq Q_a \leq Q \leq Q_r \leq W_s^{-1} \leq S^{-1}. \quad (40)$$

Balancing of the pair Q_a, Q_r amounts to choosing a basis such that $Q_a = Q_r^{-1} = \Pi$, with Π a diagonal matrix. Since $Q_a \leq Q_r$ it immediately follows that the diagonal elements of Π (the square roots of the similarity invariants of $Q_a Q_r^{-1}$) are all ≤ 1 . The same reasoning for the pairs (M_s, W_s^{-1}) and (P, S^{-1}) implies that the eigenvalues of $M_s W_s$, respectively of PS , are all ≤ 1 .

V. CONCLUSION

We have shown that open-loop balancing, closed-loop balancing, and positive-real balancing of general lossless systems all lead to the same result: the two balancing functions obtained are equal to the physical energy. For proving these results the scattering representation turned out to be instrumental, and, in fact, open-loop balancing was performed in this representation. In the passive case we obtained instead a sequence of inequalities, which gives room for obtaining useful information regarding the relative importance of state components. It would be of interest to

investigate when the inequalities in (39) are actually strict, and how the similarity invariants diverge from each other.

As a result, all balancing methods for passive systems primarily seem to compare the amount of dissipation that is present in the system equations, as opposed to, e.g., modal analysis for weakly damped mechanical systems. A preliminary attempt to reconcile these two approaches is given in [13], where first the passive system is written into port-Hamiltonian form, and then the balancing transformations are constrained to the class of transformations that leave the interconnection structure of the port-Hamiltonian system invariant.

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