Seven-branes and instantons in type IIB supergravity

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Appendix A

Conventions

A.1 Conventions for the bosonic sector of IIB supergravity

Lower case Greek indices refer to space-time indices. Space-time coordinates are denoted by x^μ with $\mu = 0, 1, \ldots, 9$ in which x^0 is a time-like coordinate (sometimes denoted by t) and in which the x^i for $i = 1, \ldots, 9$ are space-like coordinates. Early lower case Latin indices, such as $a, b, \ldots$ refer to flat tangent space-time indices [1]. The values taken by tangent space-time indices will be underlined, e.g. $a = \underline{0}, \underline{1}, \ldots, \underline{9}$. When space-time coordinates are given a specific symbol such as t to denote x^0 the flat index $\underline{0}$ will be denoted by $\underline{t}$. Early upper case Latin indices, such as $A, B, \ldots$ refer to world-volume indices of a p-brane. World-volume coordinates are denoted by σ^A with $A = 0, 1, \ldots, p$ in which σ^0 is a time-like coordinate and σ^I for $I = 1, \ldots, p$ are space-like coordinates. Underlined early upper case Latin indices, such as $\underline{A}, \underline{B}, \ldots$, refer to world-volume tangent space indices. Their values will also be underlined, so $\underline{A} = \underline{0}, \underline{1} \ldots, \underline{p}$. The conventions for the various indices are summarized on the next page in table A.1.1.

The tangent space to space-time has a metric that will be denoted by η_{ab}. The signature of this metric is chosen to be mostly plus, i.e.

$$\eta_{\underline{00}} = -1\,, \qquad \eta_{\underline{ii}} = +1\,. \tag{A.1.1}$$

The tangent space Levi–Cività tensor will be denoted by $e_{a_1 \ldots a_{10}}$. It is defined such that

$$e_{\underline{01}\ldots\underline{9}} = +1\,. \tag{A.1.2}$$

[1] In sections 2.6 and 3.11 early lower case Latin indices will be used to denote 8-dimensional space-time indices.

Index	Values
space-time indices $\mu, \nu, \ldots$	$(0, i) = (0, 1, \ldots, 9)$
tangent space indices $a, b, \ldots$	$(\underline{0}, \underline{i}) = (\underline{0}, \underline{1}, \ldots, \underline{9})$
world-volume (WV) indices $A, B, \ldots$	$(0, I) = (0, 1, \ldots, p)$
tangent space WV indices $\underline{A}, \underline{B}, \ldots$	$(\underline{0}, \underline{I}) = (\underline{0}, \underline{1}, \ldots, \underline{p})$

Table A.1.1: Summary of index notation.

The indices of $e_{a_1 \ldots a_{10}}$ are raised with the inverse tangent space metric η^{ab}, so that one has

$$e^{\underline{01} \ldots \underline{9}} = -1 \,. \tag{A.1.3}$$

The Zehnbein e_μ^a is introduced via

$$e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu} \,, \tag{A.1.4}$$

where $g_{\mu\nu}$ denotes the space-time metric. The determinant of the Zehnbein, $\det(e_\mu^a)$, will be denoted by e and the inverse Zehnbein will be denoted by e_a^μ.

The curved Levi–Cività tensor $\epsilon_{\mu_1 \ldots \mu_{10}}$ is defined through

$$e_{\mu_1}^{a_1} \cdots e_{\mu_{10}}^{a_{10}} e_{a_1 \ldots a_{10}} = e \, e_{\mu_1 \ldots \mu_{10}} \equiv \epsilon_{\mu_1 \ldots \mu_{10}} \,, \tag{A.1.5}$$

where $e_{\mu_1 \ldots \mu_{10}}$ is the Levi–Cività symbol defined as

$$e_{01 \ldots 9} = -e^{01 \ldots 9} = +1 \,. \tag{A.1.6}$$

The components of the curved Levi–Cività tensor $\epsilon_{\mu_1 \ldots \mu_{10}}$ are raised with the inverse metric $g^{\mu\nu}$. The result is

$$\epsilon^{\mu_1 \ldots \mu_{10}} = \frac{1}{e} e^{\mu_1 \ldots \mu_{10}} \,. \tag{A.1.7}$$

The covariant derivative with respect to general coordinate transformations and local Lorentz transformations is denoted by ∇_μ, acting on tensors ξ and spinors χ as

$$\nabla_\mu \xi = \partial_\mu \xi \,, \tag{A.1.8}$$

$$\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma_{\mu\rho}^\nu \xi^\rho \,, \tag{A.1.9}$$

$$\nabla_\mu \chi = \partial_\mu \chi + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \chi \,, \tag{A.1.10}$$

$$\nabla_\mu \chi^\nu = \partial_\mu \chi^\nu + \Gamma_{\mu\rho}^\nu \chi^\rho + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \, \chi^\nu \,. \tag{A.1.11}$$

Here $\Gamma^{\nu}_{\mu\rho}$ is the Levi–Cività connection and ${\omega_\mu}^{ab}$ is the spin connection defined by

$$ {\omega_\mu}^a{}_b = e^a_\nu\, e^\rho_b \Gamma^\nu_{\mu\rho} - e^\rho_b \partial_\mu e^a_\rho \,. \tag{A.1.12} $$

The gamma matrices $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$ will be discussed in section A.2. Symmetrization and anti-symmetrization are with weight one. The Riemann tensor is defined by

$$ {R^\rho}_{\mu\nu\sigma} = 2\partial_{[\nu}\Gamma^\rho_{\sigma]\mu} + 2\Gamma^\rho_{\lambda[\nu}\Gamma^\lambda_{\sigma]\mu} \,. \tag{A.1.13} $$

The following form notation is used:

$$ F_p = \frac{1}{p!} F_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, \tag{A.1.14} $$

$$ \star F_p = \frac{1}{(10-p)!\,p!} \epsilon_{\mu_1\ldots\mu_{10-p}\nu_1\ldots\nu_p} F^{\nu_1\ldots\nu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{10-p}} \,, \tag{A.1.15} $$

$$ \star 1 = \frac{1}{10!} \epsilon_{\mu_1\ldots\mu_{10}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{10}} \,, \tag{A.1.16} $$

$$ \star\star F_p = (-1)^{p+1} F_p \,, \tag{A.1.17} $$

$$ F_p \wedge G_q = \frac{1}{p!\,q!} F_{\mu_1\ldots\mu_p} G_{\mu_{p+1}\ldots\mu_{p+q}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+q}} \,, \tag{A.1.18} $$

$$ \star F_p \wedge F_p = \frac{1}{p!} F_{\mu_1\ldots\mu_p} F^{\mu_1\ldots\mu_p} \star 1 \,. \tag{A.1.19} $$

A.2 Conventions for the fermionic sector of IIB supergravity

The gamma matrices γ_a satisfy the Clifford algebra defined by

$$ \{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \,. \tag{A.2.1} $$

The representations of this Clifford algebra are taken to be unitary. This implies that Hermitean conjugation is given by

$$ \gamma_a^\dagger = \gamma_{\underline{0}}\gamma_a\gamma_{\underline{0}} \,. \tag{A.2.2} $$

If γ_a is a unitary representation of the Clifford algebra then so is γ_a^T. Therefore there exists a unitary matrix C, the charge conjugation matrix, such that

$$ \gamma_a^T = C\gamma_a C^{-1} \,. \tag{A.2.3} $$

It can be shown that the charge conjugation matrix C, defined through (A.2.3), in 10-dimensional Minkowski space-time can be taken to be symmetric. The matrix

γ_a^* is also a unitary representation of the Clifford algebra, but since $\gamma_a^* = (\gamma_a^\dagger)^T$ this representation follows from (A.2.2) and (A.2.3). Taking the transpose of (A.2.2) using (A.2.3) one has

$$\gamma_a^* = -B\gamma_a B^{-1}\,, \tag{A.2.4}$$

where B is a unitary matrix given by

$$B = -C\gamma_{\underline{0}}\,. \tag{A.2.5}$$

Actually, one finds that $-\gamma_a^*$ is equivalent to γ_a via a similarity transformation with a unitary matrix B. The Majorana representation[2] in which all the gamma matrices are purely imaginary can be obtained by taking the charge conjugation matrix C to be equal to $\gamma_{\underline{0}}$. It is straightforward to show that the complex conjugate of B satisfies $B^* = \gamma_{\underline{0}}C^{-1}$ so that one has:

$$BB^* = 1\,. \tag{A.2.6}$$

The chirality matrix γ_{11} is defined by

$$\gamma_{11} = \gamma_{\underline{0}}\gamma_{\underline{1}}\cdots\gamma_{\underline{9}}\,. \tag{A.2.7}$$

One defines $\gamma_{a_1\ldots a_n}$ to denote the anti-symmetrized products of gamma matrices:

$$\gamma_{a_1\ldots a_n} \equiv \gamma_{[a_1}\cdots\gamma_{a_n]}\,. \tag{A.2.8}$$

The objects $\gamma_{a_1\ldots a_n}$ and $\gamma_{b_1\ldots b_{10-n}}$ are related by duality,

$$\gamma_{a_1\ldots a_n} = -(-1)^{\frac{1}{2}n(n-1)}\frac{1}{(10-n)!}\epsilon_{a_1\ldots a_n b_1\ldots b_{10-n}}\gamma^{b_1\ldots b_{10-n}}\gamma_{11}\,. \tag{A.2.9}$$

The matrices $\gamma_{a_1\ldots a_n}C^{-1}$ and $\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1}$ have the following symmetry properties:

$$(\gamma_{a_1\ldots a_n}C^{-1})^T = (-1)^{n(n-1)/2}\gamma_{a_1\ldots a_n}C^{-1}\,, \tag{A.2.10}$$

$$(\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1})^T = (-1)^{n+1}(-1)^{n(n-1)/2}\gamma_{11}\gamma_{a_1\ldots a_n}C^{-1}\,. \tag{A.2.11}$$

Irreducible spinor representations of the Lorentz group $SO(9,1)$ are Majorana–Weyl spinors. On a spinor χ the Majorana condition reads

$$\chi^* = \alpha_\chi B\chi\,, \tag{A.2.12}$$

where α_χ is a yet to be determined phase factor. In order that χ satisfying the Majorana condition (A.2.12) is a consistent spinor representation of the Lorentz group in ten dimensions it must be that

$$\delta\chi^* = (\delta\chi)^* \qquad \text{with} \qquad \delta\chi = \frac{1}{2}\omega^{ab}\gamma_{ab}\chi\,, \tag{A.2.13}$$

[2] Had the charge conjugation matrix C be defined via $\gamma_a^T = -C\gamma_a C^{-1}$, which would have equally been possible since both γ_a^T and $-\gamma_a^T$ form representations of the Clifford algebra, then C would have been anti-symmetric and (A.2.4) would have been replaced by $\gamma_a^* = B\gamma_a B^{-1}$ with $B = -C\gamma_{\underline{0}}$. In this case the Majorana representation, $C = \gamma_{\underline{0}}$, implies that the gamma matrices are real.

with $\frac{1}{2}\gamma_{ab}$ the generators of $SO(9,1)$ and with ω^{ab} a set of infinitesimal parameters. It can be verified that the Majorana condition (A.2.12) is consistent with eq. (A.2.13). Further, requiring that $(\chi^*)^* = \chi$ forces α_χ to be a phase factor. Besides a Majorana condition one can, in even space-time dimensions, impose a chirality condition. The chirality operator will be denoted by $P_\pm = \frac{1}{2}(1 \pm \gamma_{11})$. The Hermitean operator $P_\pm$ is a projector, satisfying $P_\pm P_\pm = P_\pm$, $P_\pm P_\mp = 0$ and $P_+ + P_- = 1$. The latter property allows one to decompose an arbitrary spinor χ into two pieces each of which forms an eigenspinor of $P_\pm$, i.e. one has

$$\chi = P_+\chi + P_-\chi \qquad \text{with} \qquad \gamma_{11} P_\pm \chi = \pm P_\pm \chi \,. \tag{A.2.14}$$

In order for $P_\pm\chi$ to form representations of the Lorentz group it must be that

$$\delta(P_\pm\chi) = P_\pm \delta\chi \qquad \text{with} \qquad \delta\chi = \frac{1}{2}\omega^{ab}\gamma_{ab}\chi \,. \tag{A.2.15}$$

Since $P_\pm$ commutes with γ_{ab} the condition (A.2.15) is satisfied. In ten-dimensional Minkowski space-time one can have spinors that are both Majorana and Weyl. A Majorana–Weyl spinor is a spinor $P_\pm\chi$ satisfying the Majorana condition (A.2.12), i.e.

$$(P_\pm\chi)^* = \alpha_\chi B P_\pm \chi \,. \tag{A.2.16}$$

Given a spinor χ the charge conjugated spinor, χ_C, is defined by

$$\chi_C = \alpha_\chi^{-1} B^{-1} \chi^* \,, \tag{A.2.17}$$

so that a Majorana spinor is equal to its charge conjugated spinor. The Dirac conjugated spinor, $\bar{\chi}$, is defined by

$$\bar{\chi} = \chi^\dagger \gamma_{\underline{0}} \alpha_\chi^{-1} \,. \tag{A.2.18}$$

The type IIB supergravity theory contains two Majorana–Weyl dilatini, λ_1 and λ_2, of the same chirality that is taken to be

$$\gamma_{11}\lambda_1 = \lambda_1 \,, \qquad \gamma_{11}\lambda_2 = \lambda_2 \,, \tag{A.2.19}$$

and it contains two Majorana–Weyl gravitini, $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$ of the same chirality, but opposite to that of the dilatini, i.e.

$$\gamma_{11}\psi_{1\,\mu} = -\psi_{1\,\mu} \,, \qquad \gamma_{11}\psi_{2\,\mu} = -\psi_{2\,\mu} \,. \tag{A.2.20}$$

Because λ_1 and λ_2 as well as $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$ have the same chirality it is possible to consider the (reducible) complex Weyl spinors, λ and ψ_μ, defined by

$$\lambda = \lambda_1 + i\lambda_2 \,, \qquad \psi_\mu = \psi_{1\,\mu} + i\psi_{2\,\mu} \,, \tag{A.2.21}$$

together with the charge conjugated spinors, λ_C and $\psi_{C\,\mu}$,

$$\lambda_C = \lambda_1 - i\lambda_2\,, \qquad \psi_{C\,\mu} = \psi_{1\,\mu} - i\psi_{2\,\mu}\,. \tag{A.2.22}$$

Because the theory possesses $N = 2$ supersymmetry, there are also two Majorana–Weyl supersymmetry transformation parameters ϵ_1 and ϵ_2. These two parameters have the same chirality and reality properties as the gravitini $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$. The reality properties of the spinors λ_1 and λ_2 as well as $\psi_{1\,\mu}$ and $\psi_{2\,\mu}$, according to eq. (A.2.12), depend on an as yet undetermined phase factors α_{λ_1}, α_{λ_2}, α_{ψ_1} and α_{ψ_2}, that in principle could all be different. For the supersymmetry parameters ϵ_1 and ϵ_2, the phase factors α_{ϵ_1} and α_{ϵ_2} are taken to be equal to α_{ψ_1} and α_{ψ_2}, respectively. The reality properties of the spinors must be consistent with the reality properties of the bosons because the two are related via supersymmetry (see e.g. eqs. (1.1.46) to (1.1.54)). The following phases are chosen:

$$\alpha_{\lambda_1} = \alpha_{\lambda_2} = \alpha_{\psi_1} = \alpha_{\psi_2} = \alpha_{\epsilon_1} = \alpha_{\epsilon_2} \equiv \alpha = i\,. \tag{A.2.23}$$

This section is ended with some comments regarding spinor bilinears. Under complex conjugation the product of two spinors χ_1 and χ_2 is taken to behave as

$$(\chi_1\chi_2)^* = \chi_2^*\chi_1^*\,. \tag{A.2.24}$$

The spinors are anticommuting Grassmann variables. One then has the following two properties satisfied by arbitrary spinors χ_1 and χ_2:

$$(\bar\chi_1\gamma_{a_1\ldots a_n}\chi_2)^* = (-1)^n\bar\chi_{1C}\gamma_{a_1\ldots a_n}\chi_{2C}\,, \tag{A.2.25}$$

$$\bar\chi_1\gamma_{a_1\ldots a_n}\chi_2 = (-1)^{n(n+1)/2}\bar\chi_{2C}\gamma_{a_1\ldots a_n}\chi_{1C}\,. \tag{A.2.26}$$

Appendix B

Parity properties

In this appendix the parity properties of the bosonic and fermionic fields are discussed. It is assumed that the background metric is Minkowski space-time. The behavior of the fields of IIB supergravity under parity can be inferred from the supersymmetry transformation rules provided the behavior of the supersymmetry parameter ϵ under parity is specified. Under parity ϵ is taken to transform as:

$$P\,\epsilon(x^0, x^i) = i\gamma_{\underline{0}}\,\epsilon_C(x^0, -x^i)\,, \tag{B.1}$$

where the coordinates x^i form a Cartesian coordinate system, so that the space inversion means $x^i \to -x^i$ for $i = 1, \ldots 9$. The supersymmetry parameter ϵ in (B.1) is the one of subsection 1.1.4. Next, it is required that the local $U(1)$ transformation, $\epsilon \to e^{-i\alpha/2}\epsilon$, is consistent with the definition (B.1) in that the $U(1)$ transformed parameter $\epsilon' = e^{-i\alpha/2}\epsilon$ transforms under parity as

$$P\,\epsilon'(x^0, x^i) = i\gamma_{\underline{0}}\,\epsilon'_C(x^0, -x^i)\,. \tag{B.2}$$

Eq. (B.2) is consistent with eq. (B.1) provided the parameter α is odd under parity. This in turn implies the following parity properties of P and Q:

$$Q_0 \to -Q_0\,, \quad Q_i \to Q_i\,, \quad P_0 \to \bar{P}_0\,, \quad P_i \to -\bar{P}_i\,. \tag{B.3}$$

This follows from the behavior of P and Q under local $U(1)$ transformations (1.1.10). In the $U(1)$ fixed frames (1.2.14) and (1.2.21) this implies that the scalar χ' in both cases must be parity odd while T is parity even[1]. This fact in turn implies that λ'

[1]The scalars T and χ' for $\det Q > 0$ are related to the dilaton ϕ and RR axion χ via eq. (1.3.42) in which $\tau = \chi + ie^{-\phi}$ and eqs. (1.2.25) and (1.2.23). It can be checked that this relation between (T, χ') and (ϕ, χ) is such that parities of one pair of scalars implies the parities of the other pair in accordance with the findings of this subsection.

Field	Parity	Field	Parity
T	even	A_{ijkl}	odd
χ'	odd	$q_\alpha A^\alpha_{0i_1\ldots i_5}$	odd
$q_\alpha A^\alpha_{0i}$	even	$q_\alpha A^\alpha_{i_1\ldots i_6}$	even
$q_\alpha A^\alpha_{ij}$	odd	$\tilde{q}_\alpha A^\alpha_{0i_1\ldots i_5}$	even
$\tilde{q}_\alpha A^\alpha_{0i}$	odd	$\tilde{q}_\alpha A^\alpha_{i_1\ldots i_6}$	odd
$\tilde{q}_\alpha A^\alpha_{ij}$	even	$q_{\alpha\beta} A^{\alpha\beta}_{0i_1\ldots i_7}$	even
A_{0ijk}	even	$q_{\alpha\beta} A^{\alpha\beta}_{i_1\ldots i_8}$	odd

Table B.1: Parity properties of the bosonic fields of type IIB supergravity separated into electric (one time index) and magnetic components (no time index) on Minkowski space-time.

transforms under parity as follows:

$$P\,\lambda'(x^0, x^i) = i\gamma_{\underline{0}}\,\lambda'_C(x^0, -x^i)\,. \tag{B.4}$$

This can be obtained from the supersymmetry transformation rules for T and χ'. Requiring that the spinor bilinears in the supersymmetry transformations of the 2-form fields, eq. (1.3.16), that are multiplied by the same scalar function, e.g. $\bar{\epsilon}'\gamma_{\mu\nu}\lambda'$ and $\bar{\epsilon}'_C\gamma_{[\mu}\psi'_{\nu]}$ in (1.3.16), transform in the same way forces the gravitino to transform under parity as

$$P\,\psi'_\mu(x^0, x^i) = i\gamma_{\underline{0}}\,\psi'_{C\,\mu}(x^0, -x^i)\,, \tag{B.5}$$

It is now straightforward to deduce the parity properties of the form fields using eqs. (1.3.16) to (1.3.19). The results are summarized in table B.1 distinguishing electric (one time index) and magnetic components (no time index).