Seven-branes and instantons in type IIB supergravity

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Chapter 4

Instantons

4.1 Introduction

The objects that are electrically charged under χ' defined in (1.2.4) and that are dual to a Q7-brane are called Q-instantons [18]. These are BPS solutions of the Wick rotated Euclidean IIB supergravity. It is well-known that the object that is dual to the D7-brane is the D-instanton [61]. In this chapter a path integral analysis of the Q-instantons is presented providing one with their tunneling interpretation and their basic physical properties such as their charge and on-shell Euclidean action.

At first sight the existence of new half-supersymmetric instanton solutions to Euclidean IIB supergravity might be surprising since a simple analysis of the Killing spinor equations and field equations seems to lead to the unique D-instanton solution of [61], up to an $SL(2,\mathbb{Z})$ transformation. There remains however the possibility that the Q-instantons preserve the same supersymmetries as the D-instanton (just as Q7-branes and D7-branes preserve the same supersymmetries) but that they differ from the D-instanton in the source and boundary terms. This is in fact what happens and it leads to different on-shell actions for the D- and Q-instantons.

In [74] it was shown that the D-instanton contributes to higher order corrections to the string effective action in the form of $\mathcal{R}^4$ terms. Since Q-instantons preserve the same supersymmetries as the D-instanton, they are expected to contribute to the same $\mathcal{R}^4$ terms as well. This will be argued to be the case.

This chapter is organized as follows. In section 4.2 the construction and properties of the D-instanton are reviewed. These results are compared in section 4.3 with the Q-instanton source and boundary terms and its on-shell action. In section 4.4 a path integral analysis of the Q-instanton is performed and in section 4.5 the Q-instanton is argued to contribute to the $\mathcal{R}^4$ terms near the points $\tau_0 = i, \rho$ (and their $SL(2,\mathbb{Z})$ transforms) of the type IIB quantum moduli space.

When D-branes are added to the type IIB supergravity theory the duality group $SL(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the RR axion, i.e. the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$. This for example implies that all the D-brane actions are invariant under the shift of the RR axion (see e.g. the discussion on the 3-brane in section 2.3). Likewise, when a Q-brane is added to the type IIB supergravity theory the duality group $SL(2,\mathbb{R})$ is broken down to the subgroup that is generated by the shift symmetry of the χ', i.e. the $SO(2)$ subgroup of $SL(2,\mathbb{R})$. Hence, all the brane solutions of IIB supergravity are associated to fixed points of e^Q with either $\det Q = 0$ or $\det Q > 0$. The case $\det Q = 0$ corresponds to the (p', q') branes[1] with p' and q' relatively prime integers corresponding to the charges of the string that is ending on the brane. The case $\det Q > 0$ corresponds to Q-branes. The case $\det Q < 0$ does not arise because there are no fixed points of e^Q with $\det Q < 0$ that are part of the quantum moduli space (3.10.2) as explained in subsection 3.8.4.

As shown in subsection 3.7.4 the Q7-brane configurations can be described in terms of the variables T and χ' that are defined by the following relations

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} = e^{2i\sqrt{\det Q}\,\mathcal{T}}\,, \tag{4.1.1}$$

where $\mathcal{T}$ is given by

$$\mathcal{T} = \chi' + \frac{i}{4\sqrt{\det \mathrm{Q}}} \log \frac{T + 2\sqrt{\det \mathrm{Q}}}{T - 2\sqrt{\det \mathrm{Q}}}\,, \qquad \chi' \sim \chi' + \frac{\pi}{\sqrt{\det Q}}\,. \tag{4.1.2}$$

As follows from eq. (4.1.1) the values χ' and $\chi' + \pi/\sqrt{\det Q}$ are to be identified.

Table 3.9.1 on page 82, shows the values of p, q, r for each of the orbifold points τ_0 of figure 3.9.1, depicted on page 81 and the related $SL(2,\mathbb{Z})$ conjugacy classes. The periodicity of the axion χ' is determined by the value of $\pi/\sqrt{\det Q}$. The discrete isometries are the $PSL(2,\mathbb{Z})$ transformations generated by T and S. The periodicity of χ' follows from the fact that the transformations S and $T^{-1}S$ are, respectively, of order 2 and 3 in $PSL(2,\mathbb{Z})$.

In terms of the fields T and χ' the scalar kinetic terms of the IIB action (1.1.17) take the form of eq. (1.3.22). Following subsection 1.3.2 a 9-form field strength that is dual to $d\chi'$ is introduced

$$(T^2 - 4\mathrm{det}Q)\, d\chi' = \star(pF_9 + qH_9 + \frac{r}{2}G_9) \equiv \star\mathcal{F}_9\,, \tag{4.1.3}$$

where the 9-forms F_9, H_9, G_9 are organized in a triplet transforming in the adjoint of $SL(2,\mathbb{R})$. From the axion χ' equation of motion (when ignoring its coupling to the 2-forms and 6-forms) it follows that

$$d\mathcal{F}_9 = 0\,, \tag{4.1.4}$$

[1] The $SL(2,R)$ covariant actions which describe the (p', q') branes have been constructed in [26,27]. They can be regarded as the $SL(2,R)$ transformed Dp-brane actions.

so that locally

$$\mathcal{F}_9 = d\mathcal{A}_8\,, \tag{4.1.5}$$

with $\mathcal{A}_8$ given in eq. (1.3.27). The Q7-brane minimally couples[2] to $\mathcal{A}_8$ via the Wess–Zumino term

$$S^{Q7}_{min} = m\int \mathcal{A}_8\,, \tag{4.1.6}$$

where m is the Q7-brane electric charge with respect to $\mathcal{A}_8$, or magnetic charge associated with its axion dual χ'.

The dynamics of the 9-form $\mathcal{F}_9$ can be described by the following first order action

$$\begin{aligned} S\left[g_{\mu\nu},\chi',\mathcal{F}_9,T\right] \quad = \quad & \int_{\mathcal{M}_{9,1}}\Bigg(*1R - \frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT- \\ & \frac{1}{2}\frac{1}{T^2-4\det Q}\star\mathcal{F}_9\wedge\mathcal{F}_9 - \chi' d\mathcal{F}_9\Bigg)\,. \end{aligned} \tag{4.1.7}$$

In the action (4.1.7) the axion χ' appears (in a shift symmetry invariant way) as a Lagrange multiplier. Using the results of appendix B it can be seen that the term $d\mathcal{F}_9$ is parity odd as is χ'. The variation of (4.1.7) with respect to $\mathcal{F}_9$ gives the duality relation (4.1.3). If this relation is substituted back into the action the action for χ' is obtained. When (4.1.7) is varied with respect to χ' the Bianchi identity for $\mathcal{F}_9$ (4.1.4) is found. Substituting its solution (4.1.5) back into the action (4.1.7) a second order action for $\mathcal{A}_8$ results. The action (4.1.7) will be the starting point of the discussion on the Q-instantons.

4.2 D-instantons

Before discussing the new Q-instanton solution of type IIB supergravity, the derivation of the D-instanton solution [61] is briefly discussed. The D-instanton is a solution of the equations of motion of the axion and dilaton coupled to gravity in Euclidean space. The Wick rotation of the axion kinetic term is carried out by taking into account that the axion is an axial scalar and hence gets replaced with $i\chi$[3]. The Wick rotation thus changes the sign of the Einstein term and the dilaton kinetic term leaving intact the sign of the axion kinetic term. So the Euclidean action is

$$S = \int_{\mathcal{M}_{10}}\left(-\star 1R + \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{2\phi}\star d\chi\wedge d\chi\right)\,. \tag{4.2.1}$$

[2]For an early discussion on instantons and monopole-like configurations related to $(d-2)$-form gauge fields in d-dimensional space-time see [75].

[3]It is anticipated that in section 4.4 it will be shown that from the path integral point of view it is not allowed to send χ to $i\chi$ (or χ' to $i\chi'$ when it concerns the Q-instanton) under a Wick rotation. Since, in this and the next section only classical Euclidean field theory is discussed that only provides on-shell information about the saddle point approximation sending χ to $i\chi$ is harmless.

Note that the action is invariant under the axion shift symmetry $\chi \to \chi + b$ with b a constant real parameter. The Einstein equations and the equations of motion of the axion and the dilaton, which follow from (4.2.1), have the form

$$R_{mn} - \frac{1}{2}\left(\partial_m\phi\partial_n\phi - e^{2\phi}\partial_m\chi\partial_n\chi\right) = 0\,, \tag{4.2.2}$$

$$\nabla_m(e^{2\phi}\nabla^m\chi) = 0\,, \tag{4.2.3}$$

$$\nabla_m\nabla^m\,\phi + e^{2\phi}\,(\partial\chi)^2 = 0\,. \tag{4.2.4}$$

The Ansatz imposed on the fields to get the D-instanton solution of (4.2.2) to (4.2.4) is

$$g_{mn} = \delta_{mn}, \quad d\chi = \pm e^{-\phi}\,d\phi = \mp de^{-\phi}. \tag{4.2.5}$$

Eq. (4.2.5) is nothing but the Bogomol'nyi bound saturation condition imposed on the axidilaton system in flat space. The upper and lower signs in (4.2.5) correspond to the D-instanton and anti-D-instanton, respectively. When (4.2.5) is imposed eqs. (4.2.2) to (4.2.4) reduce to

$$\partial_m(e^{2\phi}\partial^m\chi) = 0 \quad \to \quad \partial^2\,e^{\phi} = 0\,, \tag{4.2.6}$$

$$\partial_m\partial^m\phi + (\partial\phi)^2 = 0 \quad \to \quad e^{-\phi}\,\partial^2\,e^{\phi} = 0. \tag{4.2.7}$$

A spherically symmetric solution to the above equations that describes a single (anti-)instanton is

$$e^{\phi} = e^{\phi_\infty} + \frac{c}{r^8}, \quad \chi - \chi_\infty = \mp(e^{-\phi} - e^{\phi_\infty})\,, \tag{4.2.8}$$

where the upper sign stands for the instanton and the lower sign corresponds to the anti-instanton, ϕ_∞ and χ_∞ are the values of the dilaton and axion at $r = \sqrt{x^m x_m} = \infty$ and $c > 0$ is (roughly speaking) the instanton charge, namely,

$$c = \frac{2\pi|n|}{8\mathrm{Vol}(S^9)}\,, \tag{4.2.9}$$

with $\mathrm{Vol}(S^9)$ being the volume of a 9-sphere of a unit radius and n being an integer which manifests the instanton charge quantization [61]. Note that from (4.2.8) it follows that for the D-instanton $\chi + e^{-\phi}$ is constant and for the anti-D-instanton $\chi - e^{-\phi}$ is constant everywhere in the 10-dimensional Euclidean space.

The solution (4.2.8) is singular at $r = 0$ which implies that it is sourced by a point-like object (the instanton) sitting at $r = 0$. The (anti-)instanton contribution to the right hand side of the axion and dilaton field equations (4.2.6) to (4.2.7) is as follows

$$\partial_m(e^{2\phi}\partial^m\chi) = \mp 2\pi|n|\,\delta^{(10)}(\vec{x})\,, \qquad e^{-\phi}\,\partial^2\,e^{\phi} = -2\pi|n|\,e^{-\phi}\,\delta^{(10)}(\vec{x})\,. \tag{4.2.10}$$

Eqs. (4.2.10) can be obtained by varying the supergravity action (4.2.1) coupled to the instanton source

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) \\ &\quad + 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(e^{-\phi} \pm \chi \right) \star 1 \,, \end{aligned} \tag{4.2.11}$$

and imposing the Ansatz (4.2.5).

The presence of the instanton source term breaks the invariance of the action (4.2.11) under the shift symmetry $\chi \to \chi + b$. The invariance can be restored by adding to eq. (4.2.11) the boundary term

$$- \int_{\partial\mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi = - \int_{\mathcal{M}_{10}} d(\chi \, e^{2\phi} \star d\chi) = \int_{\mathcal{M}_{10}} d^{10}x \, \partial_m (\chi \, e^{2\phi} \, \partial^m \chi) \,, \tag{4.2.12}$$

such that

$$\int_{\partial\mathcal{M}_{10}} e^{2\phi} \star d\chi = \pm 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \star 1 = \pm 2\pi |n| \,. \tag{4.2.13}$$

Note that this boundary condition is compatible with eqs. (4.2.10).

The appearance of the boundary term (4.2.12) in the supergravity action can be understood best if one starts from the action that includes the field strength $F_9 = dA_8$ of the 8-form gauge field A_8 and then dualizes it into the axion action by adding the term $\int_{\mathcal{M}_{10}} \chi dF_9$ (compare with (4.1.7))

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi + \frac{1}{2} e^{-2\phi} \star F_9 \wedge F_9 \right) + \int_{\mathcal{M}_{10}} \chi \, dF_9 \,. \tag{4.2.14}$$

If in (4.2.14) the field F_9 is considered as the independent one (i.e. not a curl of A_8), the variation with respect to this field gives the duality relation $F_9 = e^{2\phi} \star d\chi$ that can be substituted back into the action (4.2.14) thus reducing it to

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial\mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \,. \tag{4.2.15}$$

The boundary term appeared as a result of the integration by parts of the last term in (4.2.14). To summarize, the shift symmetry invariant action for the IIB supergravity - D-instanton system is

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2} \star d\phi \wedge d\phi - \frac{1}{2} e^{2\phi} \star d\chi \wedge d\chi \right) - \int_{\partial\mathcal{M}_{10}} \chi \, e^{2\phi} \star d\chi \\ &\quad + 2\pi |n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(e^{-\phi} \pm \chi \right) \star 1 \,. \end{aligned} \tag{4.2.16}$$

The on-shell value of this action can be computed by substituting into (4.2.16) the instanton solution (4.2.5) and (4.2.8). Then the bulk part of the action vanishes because of the Bogomol'nyi bound saturation, the contribution from the boundary term gets cancelled by the χ part of the source term, and one is left with

$$S_D|_{\rm on-shell} = 2\pi|n|\, e^{-\phi_\infty} , \tag{4.2.17}$$

where e^{ϕ_∞} is the string coupling constant.

In section 4.4 it will be shown that the result (4.2.17) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between axion conjugate momentum eigenstates or, what is the same, between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi \to \chi + b$, that differ by n units. As shown in section 4.4 (see also [74]), in order to obtain a saddle point approximation between axion χ eigenstates, one must add to (4.2.17) the imaginary term

$$-2\pi n i \chi_\infty , \tag{4.2.18}$$

with $n > 0$ for the D-instanton and $n < 0$ for the anti-D-instanton. The axion that appears in (4.2.18) is the RR axion χ of the Lorentzian IIB theory (and not the Wick rotated one of this section). Thus the D-instanton action takes the form

$$S_D = -2\pi i |n| \tau_\infty . \tag{4.2.19}$$

4.3 Q-instantons

The analog of eq. (4.2.14), that should provide the relevant boundary term in the Q-instanton action, is the Euclidean version of the action (4.1.7), namely

$$S = \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 + \chi' d\mathcal{F}_9 \right) , \tag{4.3.1}$$

where χ' has been replaced by $i\chi'$. The $\mathcal{F}_9$ equation of motion gives the duality relation between $\mathcal{F}_9$ and the Wick rotated χ' (similar to (4.1.3)). Substituting the duality relation back into the action it is found that

$$\begin{aligned} S \;=\; & \int_{\mathcal{M}_{10}} \left(- \star 1 R + \frac{1}{2}\frac{1}{T^2 - 4\det Q} \star dT \wedge dT - \frac{1}{2}(T^2 - 4\det Q) \star d\chi' \wedge d\chi' \right) \\ & - \int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi' . \end{aligned} \tag{4.3.2}$$

To this action the one-half BPS Q-instanton source action

$$2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log \frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi' \right) \star 1 \tag{4.3.3}$$

(see table 2.2.1 on page 40) is coupled. The source term guarantees that the Q-instanton solution will be defined on the entire 10-dimensional Euclidean space. As in the D-instanton case, the Q-instanton charge is quantized ($|n| = 1, 2, 3, \cdots$) as will be demonstrated in the next subsection.

As in the D-instanton case it will be assumed that for the solution under consideration the 10-dimensional Euclidean space is flat and, as follows from the Einstein equation, T and χ' are related by the following Bogomol'nyi bound saturation condition

$$d\chi' = \pm(T^2 - 4\det Q)^{-1}\, dT\,. \tag{4.3.4}$$

Upon adding (4.3.3) to eq. (4.3.2) the following action in flat Euclidean space is obtained

$$\begin{aligned} S \;=\; & \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2 - 4\det Q} dT \wedge \star dT - \frac{1}{2}(T^2 - 4\det Q) d\chi' \wedge \star d\chi'\right) \\ & - \int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q)\, \chi' \star d\chi' \\ & + 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \left(\frac{1}{4\sqrt{\det Q}} \log\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \pm \chi'\right) \star 1\,. \end{aligned} \tag{4.3.5}$$

As in the D-instanton case, due to the presence of the source term there is only one boundary, $\partial\mathcal{M}_{10}$, that is located at $r = \sqrt{x^m x_m} = \infty$. The action (4.3.2) is invariant under arbitrary shifts of the axion, $\chi' \to \chi' + b$ (where b is any real number) provided that

$$\int_{\partial\mathcal{M}_{10}} (T^2 - 4\det Q) \star d\chi' = \pm\, 2\pi|n| \int_{\mathcal{M}_{10}} \delta^{(10)}(\vec{x}) \star 1 \;=\; \pm\, 2\pi|n|. \tag{4.3.6}$$

The equations of motion of the fields χ' and T that follow from eq. (4.3.5) acquire the contribution of the instanton source term and take the following form, after imposing the Bogomol'nyi bound (4.3.4),

$$\partial_m((T^2 - 4\det Q)\, \partial^m \chi') = \mp\, 2\pi|n|\delta^{(10)}(\vec{x})\,, \tag{4.3.7}$$

$$\partial_m\partial^m T = -2\pi|n|\delta^{(10)}(\vec{x})\,. \tag{4.3.8}$$

The Q-instanton solution to eqs. (4.3.4) and (4.3.8) is

$$\begin{aligned} \chi' - \chi'_\infty \;&=\; \mp\left[\frac{1}{4\sqrt{\det Q}} \log\left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}}\right) - \frac{1}{4\sqrt{\det Q}} \log\left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}}\right)\right]\,, \\ T \;&=\; T_\infty + \frac{c}{r^8}\,, \end{aligned} \tag{4.3.9}$$

where c has been given in eq. (4.2.9). Substituting this solution into the action (4.3.5) the on-shell value of the Q-instanton action can be seen to be

$$S_Q \,|_{\text{on-shell}} = \frac{\pi|n|}{2\sqrt{\det \mathrm{Q}}} \log \frac{T_\infty + 2\sqrt{\det \mathrm{Q}}}{T_\infty - 2\sqrt{\det \mathrm{Q}}} \,. \tag{4.3.10}$$

In section 4.4 it will be shown that the result (4.3.10) corresponds to a saddle point approximation of a path integral that computes the transition amplitude between Noether charge eigenstates of the Noether current, associated to the shift symmetry $\chi' \to \chi' + b$, that differ by n units. As will also be shown in section 4.4, in order to obtain a saddle point approximation between axion χ' eigenstates one must add to (4.2.17) the imaginary term

$$-2\pi n i \chi'_\infty \,, \tag{4.3.11}$$

with $n > 0$ for a Q-instanton and $n < 0$ for an anti-Q-instanton. The axion that appears in (4.3.11) is the axion χ' of the Lorentzian IIB theory (and not the Wick rotated one of this section). The Q-instanton action thus acquires the form

$$S_Q = -2\pi i |n| \mathcal{T}_\infty \,. \tag{4.3.12}$$

The Q-instanton solution, eq. (4.3.9), has a singularity at the point $r = 0$. The singularity is similar to the one found in the case of the D-instanton, see eq. (4.2.8). In the D-instanton case, when going to the string frame the solution becomes a wormhole [61] in which the point $r = 0$ represents the asymptotic region of one end of the wormhole. In section 3.12 the notion of a Q-string whose tension is proportional to $\left(T^2 - 4\det Q\right)^{1/4}$ has been introduced. In the frame $ds_Q^2 = \left(T^2 - 4\det Q\right)^{1/4} ds_E^2$ the solution again becomes a wormhole with the point $r = 0$ representing the asymptotic region of one end of the wormhole.

4.3.1 Q-instanton charge quantization

The quantization of the Q-instanton charge, eq. (4.3.6), follows from the standard Dirac–Nepomechie–Teitelboim quantization condition [76–78] applied to the Q(-1)-brane (Q-instanton) and a Euclidean Q7-brane in a way similar to the D-instanton case [61]. Assume that the spatial volume of the 7-brane is compact with the topology of S^7. If one keeps one point on the S^7 surface fixed and transports the 7-brane along closed paths its world-volume will have the topology of S^8. The wave function of this compact 7-brane will acquire, due to its minimal coupling to the axion dual 8–form (4.1.6) (with $m = 1$), the following phase factor

$$e^{i\int_{\Sigma_8} \mathcal{A}_8} \,, \tag{4.3.13}$$

where Σ_8 is the world-volume of the compact 7-brane. Using Stokes' theorem it is possible to write

$$\int_{\Sigma_8} \mathcal{A}_8 = \int_S \mathcal{F}_9 = -\int_{S'} \mathcal{F}_9 \,, \tag{4.3.14}$$

where S and S' are the two capping surfaces of the world-volume $\Sigma_8 = S^8$. The single-valuedness of the wave function (4.3.13) requires that

$$\int_{S^9} \mathcal{F}_9 = 2\pi n \,, \tag{4.3.15}$$

where $S^9 = S \bigcup S'$. Taking now into account the duality relation between $\mathcal{F}_9$ and the Wick rotated axion χ', as follows by varying eq. (4.3.1) with respect to $\mathcal{F}_9$, eq. (4.3.6) is found that relates the value of the Q–instanton boundary term to its quantized charge.

4.3.2 The half-BPS condition

Here it will be shown that the Bogomol'nyi bound (4.3.4) also follows by analyzing the Killing spinor equations. Using the results from subsection 1.3.1, in the Lorentzian IIB theory with vanishing 3- and 5-form field strengths the Killing spinor equations can be written as

$$\delta\Psi'_m = (\nabla_m - \frac{i}{2}Q_m)\epsilon' \,, \tag{4.3.16}$$

$$\delta\lambda' = iP_m\gamma^m\epsilon'_C \,, \tag{4.3.17}$$

where the tilde on δ appearing in (1.3.21) and (1.3.20) has been dropped and in which

$$P_m = \frac{1}{2}\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} + \frac{i}{2}(T^2 - 4\det Q)^{1/2}\partial_m\chi' \,, \tag{4.3.18}$$

$$Q_m = \frac{T}{2}\partial_m\chi' \,. \tag{4.3.19}$$

Wick rotating eqs. (4.3.16) and (4.3.17) by sending χ' to $i\chi'$. Treating eqs. (4.3.16) and (4.3.17) and their complex conjugates separately one obtains

$$0 = \left(\nabla_m + \frac{T}{4}\partial_m\chi'\right)\epsilon' \,, \tag{4.3.20}$$

$$0 = \left(\nabla_m - \frac{T}{4}\partial_m\chi'\right)\epsilon'_C \,, \tag{4.3.21}$$

$$0 = \left(\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} - (T^2 - 4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon'_C \,, \tag{4.3.22}$$

$$0 = \left(\frac{\partial_m T}{(T^2 - 4\det Q)^{1/2}} + (T^2 - 4\det Q)^{1/2}\partial_m\chi'\right)\gamma^m\epsilon' \,, \tag{4.3.23}$$

where ϵ' and ϵ'_C are Wick rotated spinors.

The 1/2 BPS condition for the Q-instanton is

$$\partial_m \chi' = (T^2 - 4\det Q)^{-1}\partial_m T\,, \qquad \epsilon' = 0\,, \tag{4.3.24}$$

and for the anti-Q-instanton

$$\partial_m \chi' = -(T^2 - 4\det Q)^{-1}\partial_m T\,, \qquad \epsilon'_C = 0\,. \tag{4.3.25}$$

When either (4.3.24) or (4.3.25) holds the Q- or anti-Q-instanton source term (4.3.3) is half BPS. Using $g_{mn} = \delta_{mn}$ it follows that for the anti-Q-instanton the Killing spinor ϵ is given by

$$\epsilon' = (T^2 - 4\det Q)^{1/8}\epsilon'_0\,, \tag{4.3.26}$$

where ϵ'_0 is a constant spinor.

One should consider the Wick rotation as taking place in the context of a path integral. Schematically the path integral is

$$\int \mathcal{D}\psi \cdots e^{i\int_{\mathcal{M}_{9,1}} d^{10}x\mathcal{L}[\psi,\ldots]}\,, \tag{4.3.27}$$

where ψ is any irreducible IIB spinor. Since ψ is a Majorana–Weyl spinor it is possible to write the Lagrangian in terms of ψ only without using its complex conjugate. Further, one can explicitly use the Lorentzian signature gamma matrices writing everywhere explicitly the Zehnbeins. Then under Wick rotation the following happens. The path integral becomes

$$\int \mathcal{D}\psi \cdots e^{-\int_{\mathcal{M}_{10}} d^{10}x\mathcal{L}[\psi,\ldots]}\,, \tag{4.3.28}$$

in which the Lagrangian takes exactly the same form as in the Lorentzian case but with all the fields in principle complex-valued. The Lagrangian is holomorphic in all the IIB fields. The background $\mathcal{M}_{10}$ is taken to be flat Euclidean space which requires the component e_0^0 to be purely imaginary. In the complexified Euclidean action the Majorana condition on the spinors can no longer be imposed so that the spinors ϵ' and ϵ'_C are not related under charge conjugation. The one-half BPS condition $\epsilon' = 0$ or $\epsilon'_C = 0$ should thus be interpreted as a one-half BPS condition in the context of the complexified IIB action on a Euclidean space.

4.4 Path integral approach to Q-instantons

In this section the approach taken in section 4.3 will be justified by deriving the saddle point approximation of transition amplitudes between axion conjugate momentum eigenstates. Further, the imaginary part that, as was mentioned, should be added to the on-shell action, eq. (4.3.11), will be shown to follow from a Fourier transformation relating axion conjugate momentum eigenstates and axion field eigenstates. The discussions and arguments presented in this section are inspired by [79,80]. See [81–84] for related work in four dimensions.

4.4.1 Wick rotated path integrals and axions

In classical field theory when going from the Lorentzian IIB supergravity to Wick rotated Euclidean IIB supergravity χ' is replaced by $i\chi'$. Here, it will be shown that at the level of the path integral χ' does not get replaced by $i\chi'$ when Wick rotating the path integral.

Consider the path integral

$$\int \mathcal{D}T\,\mathcal{D}\mathcal{F}_9\,\mathcal{D}\chi'\,e^{iS\,[T,\mathcal{F}_9,\chi']}\,, \tag{4.4.1}$$

with $S\,[T,\mathcal{F}_9,\chi']$ as given in (4.1.7). The metric is not included in the discussion concerning the path integral since the metric for the instanton solutions is flat. The axion χ' in (4.4.1) can be integrated over using the identity

$$\int \mathcal{D}\chi'\,e^{-i\chi' d\mathcal{F}_9} = \delta[d\mathcal{F}_9]\,, \tag{4.4.2}$$

where $\delta[d\mathcal{F}_9]$ is a delta-functional, implying that $d\mathcal{F}_9 = 0$.

The Wick rotated version of (4.4.1) is

$$\int \mathcal{D}T\,\mathcal{D}\mathcal{F}_9\,\mathcal{D}\chi'\,e^{-S_E\,[T,\mathcal{F}_9,\chi']}\,, \tag{4.4.3}$$

where $S_E = -iS$(Wick rotated) is given by (leaving out the metric)

$$\begin{aligned} S_E\,[T,\mathcal{F}_9,\chi'] \;&=\; \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2 - 4\det Q}\star dT\wedge dT \right.\\ &\quad \left. + \frac{1}{2}\frac{1}{T^2-4\det Q}\star\mathcal{F}_9\wedge\mathcal{F}_9 + i\chi' d\mathcal{F}_9\right)\,. \end{aligned} \tag{4.4.4}$$

In the path integral (4.4.3) the integrations are over paths of field configurations with Dirichlet boundary conditions for the fields T and $\mathcal{F}_9$ while free or no boundary conditions are imposed for the field χ'. These boundary conditions are the same as those imposed on the variations of the action (4.1.7) with respect to T, $\mathcal{F}_9$ and χ'. The variation of χ' is entirely free without any boundary conditions because it appears in (4.4.4) without a derivative.

Notice that χ' in (4.4.4) has not been replaced by $i\chi'$. Now in the Euclidean path integral χ' can be again integrated out using the identity (4.4.2) which allows one to go to a second order formalism. If instead χ' had been replaced by $i\chi'$ this would have no longer been possible and the first order action in the Euclidean path integral would not have been equivalent to an 8-form gauge theory anymore since the Bianchi identity $d\mathcal{F}_9 = 0$ and its consequence $\mathcal{F}_9 = d\mathcal{A}_8$ would not arise.

4.4.2 The role of the moduli space

Consider the last term in (4.4.4)

$$i\int_{\mathcal{M}_{10}} \chi' d\mathcal{F}_9 = -i\int_{\mathcal{M}_{10}} d\chi' \wedge \mathcal{F}_9 + i\int_{\partial\mathcal{M}_{10}} \chi' \mathcal{F}_9 \,. \tag{4.4.5}$$

If it is required that the Euclidean path integral respects the standard IIB symmetry $\chi' \to \chi' + b$ where b is any real number then it is found that $\mathcal{F}_9$ should satisfy the following boundary condition

$$b\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z}\,. \tag{4.4.6}$$

Since b is arbitrary this means that $\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9$ has to vanish. This would mean that there is no instanton present. If instead one requires that the axion can undergo integer, in particular, unit shifts $\chi' \to \chi' + 1$ then it is found that

$$\int_{\partial\mathcal{M}_{10}} \mathcal{F}_9 = 2\pi n \qquad \text{with } n \in \mathbb{Z}\,. \tag{4.4.7}$$

It is concluded from this that instantons can only exist in axidilaton theories whose moduli space is given by (3.10.2). The situation with the 7-brane solutions is in this respect entirely analogous. There the arguments to use (3.10.2) are based on the requirement of having 7-brane solutions with finite energy [85] (see also subsection 3.8.3). The conclusion that one must factor the moduli space $SL(2,\mathbb{R})/SO(2)$ by $SL(2,\mathbb{Z})$ in order to even speak about instantons is clear from the path integral point of view and does not follow from the classical field theory approach of the previous two sections.

4.4.3 Integrating over $\mathcal{F}_9$

Instead of integrating out χ' it is also possible to integrate (4.4.3) over $\mathcal{F}_9$. This is achieved by defining a new 9-form $\mathcal{F}'_9$

$$\mathcal{F}'_9 = \mathcal{F}_9 + i(T^2 - 4\det Q) \star d\chi' \,. \tag{4.4.8}$$

A shift of $\mathcal{F}_9$ in the imaginary direction does not affect the integration in (4.4.3). The action (4.4.4) now becomes

$$\begin{aligned} S_E\,[T,\mathcal{F}_9',\chi'] &= \int_{\mathcal{M}_{10}} \left(\frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT + \frac{1}{2}\frac{1}{T^2-4\det Q}\star \mathcal{F}_9'\wedge\mathcal{F}_9' \right.\\ &\left. +\frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right) + i\int_{\partial\mathcal{M}_{10}}\chi'\mathcal{F}_9\,. \qquad (4.4.9)\end{aligned}$$

Even though $\mathcal{F}_9$ appears in the boundary term of (4.4.9) the $\mathcal{F}_9'$ integral is a Gaussian as the integration is over $\mathcal{F}_9'$ with Dirichlet boundary conditions. The $\mathcal{F}_9$ in the boundary term is not integrated over, but is fixed by the identification $\chi'\sim\chi'+1$, see eq. (4.4.7). Integrating over $\mathcal{F}_9'$ leads to the following path integral

$$\int_F (T^2-4\det Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\,e^{-\tilde{S}_E\,[T,\chi']}\,, \qquad (4.4.10)$$

where F below the integral sign means to indicate that only integrations over the paths of field configurations that are within the fundamental domain of the quantum moduli space (3.10.2) are to be performed. From now on this will always be assumed and the label F will be suppressed. The integration measure[4] now contains the factor $(T^2-4\det Q)^{1/2}$ and the Euclidean action $\tilde{S}_E\,[T,\chi']$ is given by

$$\begin{aligned} \tilde{S}_E\,[T,\chi'] &= \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT+\frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right)\\ &+i\int_{\partial\mathcal{M}_{10}}\chi'\mathcal{F}_9\,. \qquad (4.4.11)\end{aligned}$$

4.4.4 Splitting the χ' integration into bulk and boundary integrations

The integration over χ' can be split up into two pieces: the integration over bulk χ' field configurations and the integration over boundary χ_∂' field configurations. The bulk χ' field configurations will be denoted by the same symbol as was used in the previous subsections. Since now χ_∂' is written for the boundary values this should cause no confusion. This split is most easily done using Dirichlet boundary conditions for the paths appearing in the path integral over the bulk χ' field configurations. When this is done (4.4.10) can be written as

$$\int (T^2-4\det Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\mathcal{D}\chi_\partial'\,e^{-S_E\,[T,\chi',\chi_\partial']}\,, \qquad (4.4.12)$$

[4]In terms of τ and $\bar{\tau}$ the integration measure would be $(\mathrm{Im}\,\tau)^{-2}\mathcal{D}\tau\mathcal{D}\bar{\tau}$, which is $PSL(2,\mathbb{Z})$ invariant.

with Dirichlet boundary conditions on the integrations over T and χ'. The action appearing in (4.4.12) is given by

$$S_E\left[T,\chi',\chi'_\partial\right] = \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2-4\det Q}\star dT\wedge dT+\frac{1}{2}(T^2-4\det Q)\star d\chi'\wedge d\chi'\right) +i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9\,. \tag{4.4.13}$$

The variation of the bulk part of (4.4.13) with respect to T and χ' satisfying Dirichlet boundary conditions produces the standard (non-Wick rotated) IIB axidilaton equations of motion.

4.4.5 Tunneling interpretation

In this subsection it will be discussed what is precisely computed by the Euclidean path integral (4.4.3), i.e. by (4.4.12). One would like to interpret (4.4.12) in terms of matrix elements describing a tunneling process from an initial ($t=-\infty$) time-like hypersurface Σ_i to a final ($t=+\infty$) time-like hypersurface Σ_f. The time-like hypersurfaces Σ_i and Σ_f constitute surfaces on which field operator states exist. In order to describe this within the space-time $\mathcal{M}_{9,1}$ spatial infinity is added to it as a point giving rise to the manifold $\mathcal{M}_{9,1}\cup\{r=\infty\}$, where r is a radial coordinate. The topology of this one-point compactified space-time is given by $\mathbb{R}\times S^9$ whose boundary $\partial\left(\mathcal{M}_{9,1}\cup\{r=\infty\}\right)$ is given by the disjoint union $\Sigma_i\cup\Sigma_f$ where the initial and final time-like hypersurfaces have the topology of S^9.

The instanton charge $\int_{\partial\mathcal{M}_{10}}\mathcal{F}_9$ that appears in the imaginary part of eq. (4.4.13) is equal to $\int_{\Sigma_f}\mathcal{F}_9^f-\int_{\Sigma_i}\mathcal{F}_9^i$. Multiplying this equality by χ'_∂ gives

$$i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9 = i\int_{\Sigma_f}\chi'_\partial\,\mathcal{F}_9^f - i\int_{\Sigma_i}\chi'_\partial\,\mathcal{F}_9^i\,, \tag{4.4.14}$$

where the values of the axion χ' on the initial and final timelike hypersurfaces Σ_i and Σ_f are the same: $\chi'_i=\chi'_f=\chi'_\partial$. In the following χ'_∂ and χ'_∞ will be identified. Further, it is possible to write

$$\int\mathcal{D}\chi'_\partial e^{-i\int_{\partial\mathcal{M}_{10}}\chi'_\partial\,\mathcal{F}_9} = \int\mathcal{D}\chi'_i\mathcal{D}\chi'_f\delta(\chi'_i-\chi'_f)e^{-i\int_{\Sigma_f}\chi'_f\,\mathcal{F}_9^f+i\int_{\Sigma_i}\chi'_i\,\mathcal{F}_9^i}\,. \tag{4.4.15}$$

The boundary states in (4.4.12) at $\Sigma_{i,f}$ satisfy (4.4.7) and (4.4.14) and so the $\mathcal{F}_9^{i,f}$ boundary data are on-shell.

Next, it will be shown that one can use the duality relation (4.1.3) restricted to the surfaces $\Sigma_{i,f}$ to interpret the boundary data $\mathcal{F}_9^{i,f}$ of eq. (4.4.15) in terms of the axion momentum, or equivalently, in terms of the Noether charge density associated with the axion shift symmetry. Note that the time component of the Noether current (charge density) is equal to the axion χ' canonical momentum π' obtained by varying the Lagrangian containing the χ' kinetic term with respect to $\partial_0\chi'$

$$\pi' = \frac{\delta\mathcal{L}}{\delta(\partial_0\chi')} = (T^2 - 4\det Q)\,\partial_0\chi' = J_N^0\,. \tag{4.4.16}$$

Consider tunneling between canonical momentum eigenstates of the axion χ' (or equivalently between its Noether charge eigenstates) from the initial surface Σ_i to the final surface Σ_f. These are described by the following matrix element

$$\lim_{\Delta T\to\infty}\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i\rangle\,, \tag{4.4.17}$$

where ΔT is the Wick rotated time interval between Σ_i and Σ_f, H is the axidilaton Hamiltonian and $\pi'_{i,f}$ are the initial and final momenta of the axion.

The matrix element (4.4.17) is related by a Fourier transformation to the matrix element describing the transition between two boundary eigenstates χ'_i and χ'_f of the axion. Namely, (for $\Delta T\to\infty$) one has

$$\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i\rangle = \int \mathcal{D}\chi'_i\mathcal{D}\chi'_f e^{-i\int_{\Sigma_f}\chi'_f\,\pi'_f + i\int_{\Sigma_i}\chi'_i\,\pi'_i}\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i\rangle\,, \tag{4.4.18}$$

where

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i\rangle = \langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f\rangle\delta(\chi'_i - \chi'_f)\,. \tag{4.4.19}$$

Hence, no tunneling takes place between vacua for which $\chi'_i \neq \chi'_f$. This means that the value of χ' at, say, $t = +\infty$ acts as a superselection parameter, like the theta parameter in Yang–Mills theory. Therefore, physical processes in vacua with different values of χ'_f are not correlated.

The matrix element appearing on the right hand side of eq. (4.4.19) is given by

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f\rangle = \int (T^2 - 4\det Q)^{1/2}\mathcal{D}T\,\mathcal{D}\chi'\,e^{-S_E\,[T,\chi']}\,, \tag{4.4.20}$$

with Dirichlet boundary conditions on the integrations over T and χ' and

$$S_E\,[T,\chi'] = \int_{\mathcal{M}_{10}}\left(\frac{1}{2}\frac{1}{T^2 - 4\det Q}\star dT\wedge dT + \frac{1}{2}(T^2 - 4\det Q)\star d\chi'\wedge d\chi'\right)\,. \tag{4.4.21}$$

Compare eqs. (4.4.18) to (4.4.21) with (4.4.12) to (4.4.15). Eqs. (4.4.18) to (4.4.21) taken together provide a closed expression for the matrix element on the left hand side of eq. (4.4.18). On the other hand eqs. (4.4.12) to (4.4.15) provide an expression for the path integral in (4.4.12). In order that (4.4.12) computes a physical quantity, namely the matrix element of (4.4.18), the boundary values of $\mathcal{F}_9$ are taken to be associated with the boundary values of the χ' canonical momentum (4.4.16) via the duality relation 4.1.3

$$\int_{\Sigma_{i,f}} \mathcal{F}_9^{i,f} = \int_{\Sigma_{i,f}} \star(T^2 - 4\det Q)d\chi' = \int_{\Sigma_{i,f}} J_N^0 \, d\Omega_9 = \int_{\Sigma_{i,f}} \pi'_{i,f} \, d\Omega_9 \,, \qquad (4.4.22)$$

where $d\Omega_9$ denotes the integration measure of the unit 9-sphere.

Using the inverse Fourier transform gives

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_i \rangle = \int \mathcal{D}\pi'_i \mathcal{D}\pi'_f e^{i\int_{\Sigma_f} \chi'_f \, \pi'_f - i\int_{\Sigma_i} \chi'_i \, \pi'_i} \langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle \,, \qquad (4.4.23)$$

where now

$$\langle \pi'_f \mid e^{-H\Delta T} \mid \pi'_i \rangle = \int \mathcal{D}T \, \mathcal{D}\mathcal{F}_9 \, \delta[d\mathcal{F}_9] \, e^{-S_E\,[T,\mathcal{F}_9]} \,, \qquad (4.4.24)$$

with Dirichlet boundary conditions imposed on the integrations over T and $\mathcal{F}_9$ and where

$$S_E\,[T, \mathcal{F}_9] = \int_{\mathcal{M}_{10}} \left(\frac{1}{2} \frac{1}{T^2 - 4\det Q} \star dT \wedge dT + \frac{1}{2} \frac{1}{T^2 - 4\det Q} \star \mathcal{F}_9 \wedge \mathcal{F}_9 \right) . \qquad (4.4.25)$$

4.4.6 Saddle point approximation

The saddle point approximation of $\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle$ can be obtained using eqs. (4.4.23), (4.4.24) and (4.4.25). For a single instanton of charge n the saddle point approximation gives

$$\langle \chi'_f \mid e^{-H\Delta T} \mid \chi'_f \rangle \simeq N e^{i\int_{\Sigma_f} \chi'_f \, \pi'_f - i\int_{\Sigma_i} \chi'_f \, \pi'_i} e^{-S_E\,[T,\mathcal{F}_9]} \,|_{\text{on-shell}} \,, \qquad (4.4.26)$$

where N is a prefactor that will not be evaluated. On the mass shell it holds true that

$$\int_{\Sigma_f} \chi'_f \, \pi'_f d\Omega_9 - \int_{\Sigma_i} \chi'_f \, \pi'_i d\Omega_9 = 2\pi n \chi'_\infty \,, \qquad (4.4.27)$$

which follows from eqs. (4.4.7) and (4.4.14). Further, on-shell and outside the Q-instanton source $S_E\,[T, \mathcal{F}_9] = S_E\,[T, d\mathcal{A}_8]$. The on-shell action can be written as the sum of a quadratic term and a rest term as

$$S_E\,[T, d\mathcal{A}_8] = \frac{1}{2} \int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} \star (dT \mp \star\mathcal{F}_9) \wedge (dT \mp \star\mathcal{F}_9) \pm G \,, \qquad (4.4.28)$$

with G given by

$$G = \int_{\mathcal{M}_{10}} \frac{1}{T^2 - 4\det Q} dT \wedge \mathcal{F}_9 = -\int_{\partial\mathcal{M}_{10}} \frac{1}{4\sqrt{\det Q}} \log \left(\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}} \right) \mathcal{F}_9 \,, \qquad (4.4.29)$$

where $\partial\mathcal{M}_{10} = \partial\mathcal{M}_\infty + \partial\mathcal{M}_0$. The boundaries $\partial\mathcal{M}_\infty$ and $\partial\mathcal{M}_0$ are, respectively, the 9-sphere at infinity and around the origin where the field strength F_9 fails to be exact (the location of its magnetic source). However because at $|\vec{x}| = 0$ the field T blows up, the value of G is zero at this point and only the boundary at infinity contributes. The first term in the action (4.4.28) is positive definite. One thus has the following Bogomol'nyi bound for field configurations respecting the symmetries of the Q(-1)-brane solution

$$S_I \geq \pm G\,. \tag{4.4.30}$$

Solutions that satisfy the Bogomol'nyi bound must have the property that

$$dT = \pm \star \mathcal{F}_9\,. \tag{4.4.31}$$

For such configurations the on-shell value of the action is given by

$$S_E\left[T, d\mathcal{A}_8\right]|_{\text{on-shell}} = -G = \frac{\pi|n|}{2\sqrt{\det Q}}\log\left(\frac{T_\infty + 2\sqrt{\det Q}}{T_\infty - 2\sqrt{\det Q}}\right)\,, \tag{4.4.32}$$

where $T_\infty > 2\sqrt{\det Q}$ is the asymptotic value of T. The result (4.4.32) agrees with (4.3.10) and provides a saddle point approximation of the matrix element of a transition between axion charge eigenstates (or conjugate momentum eigenstates).

Using eqs. (4.4.27) and (4.4.32) the saddle point approximation (4.4.26) becomes,

$$\langle\chi'_f \mid e^{-H\Delta T} \mid \chi'_f\rangle \simeq N e^{2\pi n i\chi'_\infty - 2\pi|n|\mathrm{Im}\mathcal{T}_\infty} = \begin{cases} N e^{2\pi n i\mathcal{T}_\infty} & \text{for } n > 0\,, \\ N e^{2\pi n i\bar{\mathcal{T}}_\infty} & \text{for } n < 0\,. \end{cases} \tag{4.4.33}$$

The case $n > 0$ corresponds to the Q-instanton whereas $n < 0$ corresponds to the anti-Q-instanton. Thus, adding the term (4.3.11) to the action (4.3.10) leads to a saddle point approximation of the matrix element of the transition between axion eigenstates $\chi'_i = \chi'_f = \chi'_\infty$. The result (4.4.33) will be used in the next section to argue that the $\mathcal{R}^4$ terms near the points i and ρ of figure 3.9.1 on page 81, receive contributions from Q-instantons.

4.5 Q-instanton contributions to the $\mathcal{R}^4$ terms

The $\mathcal{R}^4$ terms are those terms in the effective action that are of order $(\alpha')^3$ relative to the Einstein–Hilbert term. In [74] it is argued that the part of the $\mathcal{R}^4$ terms that only contains derivatives of the metric is multiplied by a $PSL(2,\mathbb{Z})$ invariant real-analytic modular form, a generalized Eisenstein series. Such functions are eigenfunctions of the Laplace operator on the hyperbolic plane. In [86] it is shown that this picture is confirmed by requiring supersymmetry at the order $(\alpha')^3$ relative to the Einstein–Hilbert term. The $\mathcal{R}^4$ terms contain besides derivatives of the metric also contributions involving terms with derivatives of the other bosonic fields of the type IIB theory. For the NSNS fields and the RR 0-form a conjectured $SL(2,\mathbb{Z})$ invariant $\mathcal{R}^4$ term is proposed in [87]. Here only the part of the $\mathcal{R}^4$ terms that involves derivatives of the metric and that can be obtained by considering on-shell amplitudes for four graviton scattering will be considered. Consider [87]

$$\begin{aligned}\mathcal{R}^4 &= f(\tau,\bar\tau)\left(t_8^{abcdefgh}t_8^{mnpqrstu}+\frac{1}{8}\epsilon_{10}^{abcdefghij}\epsilon_{10}^{mnpqrstu}{}_{ij}\right)\times \\ &\quad\times R_{abmn}R_{cdpq}R_{efrs}R_{ghtu}+\ldots\,,\end{aligned} \tag{4.5.1}$$

where t_8 is defined in [88], ϵ_{10} is the 10-dimensional Levi-Cività tensor and R_{abmn} is the Riemann tensor. The dots indicate that there are more contributions to $\mathcal{R}^4$.

The function $f(\tau,\bar\tau)$, a generalized Eisenstein series, has the form

$$f(\tau,\bar\tau)=\sum_{(p,n)\neq(0,0)}\frac{\tau_2^{3/2}}{|p+n\tau|^3}\,, \tag{4.5.2}$$

where $\tau=\tau_1+i\tau_2$ and the sum is over all integers $p,n\in\mathbb{Z}$ except when both p and n are zero. In order to see the contributions coming from single multiply charged D- and anti-D-instantons one writes f as a Fourier series in $\tau_1=\chi$. One has [28]

$$\begin{aligned}f(\tau,\bar\tau) &= 2\zeta(3)\tau_2^{3/2}+\frac{2\pi^2}{3}\tau_2^{-1/2}+8\pi\tau_2^{1/2}\sum_{m\neq 0}\sum_{n=1}^{\infty}|\frac{m}{n}|e^{2\pi imn\tau_1}\,K_1(2\pi|mn|\tau_2) \\ &= 2\zeta(3)\tau_2^{3/2}+\frac{2\pi^2}{3}\tau_2^{-1/2} \\ &\quad +8\pi\tau_2^{1/2}\sum_{k=1}^{\infty}k\,\sigma_{-2}(k)\left(e^{2\pi ik\tau_1}+e^{-2\pi ik\tau_1}\right)\,K_1(2\pi k\tau_2)\,,\end{aligned} \tag{4.5.3}$$

with $\sigma_{-2}(k)$ given by

$$\sigma_{-2}(k)=\sum_{d|k}d^{-2}\,, \tag{4.5.4}$$

where the sum is over all positive divisors d of k. The expression (4.5.3) is a cosine series with coefficients $16\pi\tau_2^{1/2}\,k\,\sigma_{-2}(k)\,K_1(2\pi k\tau_2)$, where K_1 is the modified Bessel function of the second kind. The τ_1 independent terms in (4.5.3) do not come from D-instantons, instead they come from an $(\alpha')^3$ tree level and a one-loop effect in the four graviton amplitude [74].

In order to see the contribution from single multiply charged D- and anti-D-instantons one considers (4.5.3) close to $\tau_0=i\infty$, i.e. in the limit $\tau_2\to\infty$. Using that for $x\to\infty$ the function $K_1(x)$ behaves as $K_1(x)=\sqrt{\frac{\pi}{2x}}e^{-x}(1+\ldots)$ it follows that at the leading order in the limit $\tau_2 \to \infty$

$$f(\tau,\bar\tau) \approx 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-1/2} + 4\pi\sum_{k=1}^{\infty} k^{1/2}\,\sigma_{-2}(k)\left(e^{2\pi i k\tau} + e^{-2\pi i k\bar\tau}\right) . \qquad (4.5.5)$$

In the exponents of (4.5.5) one recognizes the D-instanton action (4.2.19).

The Q-instantons preserve the same supersymmetries as the D-instanton. It is therefore expected that they will also contribute to the generalized Eisenstein series (4.5.2). To justify this argument, in the remainder of this section the generalized Eisenstein series will be Fourier expanded in terms of χ' and the Fourier coefficients that are functions of T will be computed. This will result in an exact expression for f that is analogous to eq. (4.5.3). Schematically one can write

$$f(T,\chi') = \sum_{n=-\infty}^{\infty} c_n(T) e^{2\pi i n\chi'} , \qquad (4.5.6)$$

where $c_n(T)$ are the Fourier coefficients. This series is manifestly invariant under $\chi' \to \chi' + 1$ and describes the behavior of f near $\tau = \tau_0 = i, \rho$. It will be shown that f consists of a χ' independent part and of a cosine series that corresponds to an infinite sum of single multiply charged Q- and anti-Q-instantons.

The function f will be expanded around the fixed points $\tau_0 = i, \rho$ of the axidilaton moduli space. To this end it will prove convenient to work with the (η, φ) variables that are related to τ via eq. (1.4.3) and to χ' and $\mathrm{Im}\,\mathcal{T}$ via

$$\varphi = 2\sqrt{\det Q}\,\chi' \quad \text{where} \quad 0 \le \varphi < 2\pi , \qquad (4.5.7)$$

$$\tanh\frac{\eta}{2} = e^{-2\sqrt{\det Q}\,\mathrm{Im}\,\mathcal{T}} \quad \text{where} \quad 0 < \eta < \infty . \qquad (4.5.8)$$

Substituting (1.4.3) into (4.5.2) gives

$$f(\eta,\varphi) = \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\tau_0)^{3/2}}{|p+n\tau_0|^3} \frac{1}{\left(\cosh\eta + \sinh\eta\cos(\varphi + \beta(p,n;\tau_0))\right)^{3/2}} , \qquad (4.5.9)$$

where $\beta(p,n;\tau_0)$ is defined by

$$\cos\beta(p,n;\tau_0) = \frac{n^2(\mathrm{Im}\,\tau_0)^2 - (p+n\mathrm{Re}\,\tau_0)^2}{n^2(\mathrm{Im}\,\tau_0)^2 + (p+n\mathrm{Re}\,\tau_0)^2} , \qquad (4.5.10)$$

$$\sin\beta(p,n;\tau_0) = \frac{2n\,\mathrm{Im}\,\tau_0\,(p+n\mathrm{Re}\,\tau_0)}{n^2(\mathrm{Im}\,\tau_0)^2 + (p+n\mathrm{Re}\,\tau_0)^2} . \qquad (4.5.11)$$

From the definition of φ in terms of χ' it follows that the invariance of $f(T,\chi')$ under $\chi' \to \chi' + 1$ implies the invariance of $f(\eta,\varphi)$ under $\varphi \to \varphi + 2\sqrt{\det Q}$. Hence, the following Fourier series decomposition of $f(\eta,\varphi)$ is made

$$f(\eta,\varphi) = \sum_{m=-\infty}^{\infty} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) e^{\frac{\pi}{\sqrt{\det Q}} m i\varphi} . \tag{4.5.12}$$

The Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ are given by

$$a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) = \frac{1}{2\sqrt{\det Q}} \int_0^{2\sqrt{\det Q}} d\varphi\, f(\eta,\varphi) e^{-\frac{\pi}{\sqrt{\det Q}} m i\varphi} . \tag{4.5.13}$$

By using (4.5.9) and by shifting the integration over φ in (4.5.13) to an integration over $\theta = \varphi + \beta(p,n;\tau_0)$ the Fourier coefficients $a_{\frac{\pi m}{\sqrt{\det Q}}}$ can be written as

$$\begin{aligned} a_{\frac{\pi m}{\sqrt{\det Q}}}(\eta) \quad &= \quad \frac{1}{2\sqrt{\det Q}} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\tau_0)^{3/2}}{|p+n\tau_0|^3} e^{\frac{\pi}{\sqrt{\det Q}} m i \beta(p,n;\tau_0)} \times \\ &\times \int_{\beta(p,n;\tau_0)}^{2\sqrt{\det Q}+\beta(p,n;\tau_0)} d\theta \frac{e^{-\frac{\pi}{\sqrt{\det Q}} m i\theta}}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}} . \end{aligned} \tag{4.5.14}$$

Eq. (4.5.14) will be further evaluated for the cases $\tau_0 = i$ and $\tau_0 = \rho$ separately.

The case $\tau_0 = i$ is treated first. In table 3.9.1 on page 82 some data regarding the orbifold points $\tau_0 = i, \rho$ is presented. For $\tau_0 = i$ one has $2\sqrt{\det Q} = \pi$. From eqs. (4.5.10) and (4.5.11) specified to the case $\tau_0 = i$ the following two identities can be derived

$$\beta(-n,p;i) \quad = \quad \pi + \beta(p,n;i) , \tag{4.5.15}$$

$$\beta(n,p;i) \quad = \quad \pi - \beta(p,n;i) . \tag{4.5.16}$$

Using the identity (4.5.15) one can write

$$\begin{aligned} &\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_{\beta(p,n;i)}^{\pi+\beta(p,n;i)} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = \\ &\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_{\pi+\beta(p,n;i)}^{2\pi+\beta(p,n;i)} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} = \\ &\frac{1}{2} \sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_0^{2\pi} d\theta \frac{e^{-2mi\theta}}{(\cosh\eta+\sinh\eta\cos\theta)^{3/2}} , \end{aligned} \tag{4.5.17}$$

where in the last equality the average of the first two lines has been taken and the property $\int_\beta^{2\pi+\beta} = \int_\beta^0 + \int_0^{2\pi} + \int_{2\pi}^{2\pi+\beta} = \int_0^{2\pi}$, that follows from the fact that the integrand is periodic with the period 2π, has been used. Thus, the Fourier coefficients

(4.5.14) take the form

$$a_{2m}(\eta) = \frac{1}{2\pi} \sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \int_0^{2\pi} d\theta \frac{\cos(2m\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}} \,, \qquad (4.5.18)$$

where the integral from 0 to 2π that involves $\sin(2m\theta)$ vanished.

The identity (4.5.16) can be used to show that the sum preceding the integral in (4.5.18) satisfies

$$\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} = \sum_{(p,n)\neq(0,0)} \frac{e^{-2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}} \,, \qquad (4.5.19)$$

so that $a_{2m}(\eta) = a_{-2m}(\eta)$. The latter property implies that the Fourier expansion (4.5.12) becomes the following cosine series

$$f(\eta,\varphi) = a_0(\eta) + \sum_{m=1}^{\infty} a_{2m}(\eta)\left(e^{2mi\varphi} + e^{-2mi\varphi}\right) . \qquad (4.5.20)$$

The integral in (4.5.18) is the integral representation (up to a factor) of a toroidal function, denoted by $P^{2m}_{1/2}(\cosh\eta)$. Toroidal or ring functions are special cases of the associated Legendre functions. One has [89]

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}} = 2\pi(-1)^n \frac{\Gamma(\frac{3}{2}-n)}{\Gamma(\frac{3}{2})} P^n_{1/2}(\cosh\eta) \,. \qquad (4.5.21)$$

The functions $P^n_{1/2}(\cosh\eta)$ for $n = 0,1,2,\dots$ can be written in terms of a hypergeometric function[5] as follows [89]

$$P^n_{1/2}(\cosh\eta) = \frac{1}{2^n} \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)} \sinh^n\eta\, F(\tfrac{n}{2}+\tfrac{3}{4}, \tfrac{n}{2}-\tfrac{1}{4}; n+1; -\sinh^2\eta) \,. \quad (4.5.23)$$

Substituting eqs. (4.5.21) and (4.5.23) into (4.5.18) the function $f(\eta,\varphi)$, eq.

[5] The hypergeometric function $F(a,b;c;z)$ is defined for $c \neq 0,-1,-2,\cdots$ as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{z^n}{n!} \,. \qquad (4.5.22)$$

If $a,b \neq 0,-1,2,\cdots$ then (4.5.22) converges absolutely for $|z|<1$, see e.g. [89] for more details.

(4.5.20), around the point $\tau_0 = i$ can be written as the following Fourier series

$$\begin{aligned} f(\eta,\varphi) &= \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}}\, F(\tfrac{3}{4},-\tfrac{1}{4};1;-\sinh^2\eta) \\ &+ \sum_{m=1}^{\infty}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\frac{1}{2^{2m}}\frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})}\sinh^{2m}\eta\,\times \\ &\times F(m+\tfrac{3}{4},m-\tfrac{1}{4};2m+1;-\sinh^2\eta)\left(e^{2mi\varphi}+e^{-2mi\varphi}\right). \end{aligned} \qquad (4.5.24)$$

From eqs. (4.5.7) and (4.5.8) it is known that

$$\varphi = \pi\chi' \qquad \text{and} \qquad \sinh^2\eta = \frac{T^2 - 4\det Q}{4\det Q} \qquad \text{with} \qquad \sqrt{\det Q} = \frac{\pi}{2}\,. \qquad (4.5.25)$$

The Fourier series (4.5.24) in terms of χ' and $T^2 - 4\det Q$ associated with the fixed point $\tau_0 = i$ is analogous to the Fourier series expansion (4.5.3) around the point $\tau_0 = i\infty$ in terms of $\tau_1 = \chi$ and $\tau_2 = \operatorname{Im}\tau = e^{-\phi}$.

In order to make manifest the Q- and anti-Q-instanton contributions to the function f consider the expansion (4.5.24) at the leading order around the point $\eta = 0$ (that corresponds to a singular point of the associated Legendre function $P^{2m}_{1/2}(\cosh\eta)$). Note that, by virtue of the relation (1.4.3), the point $\eta = 0$ corresponds to $\tau = i$. Using that at leading order

$$\sinh^2\eta \approx 4\, e^{-2\pi \operatorname{Im}\mathcal{T}}\,, \qquad (4.5.26)$$

it follows that at this order[6]

$$\begin{aligned} f(\mathcal{T},\bar{\mathcal{T}}) &\approx \sum_{(p,n)\neq(0,0)} \frac{1}{(p^2+n^2)^{3/2}} + \sum_{m=1}^{\infty}\sum_{(p,n)\neq(0,0)} \frac{e^{2mi\beta(p,n;i)}}{(p^2+n^2)^{3/2}}\,\times \\ &\times \frac{\Gamma(\frac{3}{2}+2m)}{\Gamma(2m+1)\Gamma(\frac{3}{2})}\left(e^{2\pi mi\mathcal{T}} + e^{-2\pi mi\bar{\mathcal{T}}}\right). \end{aligned} \qquad (4.5.28)$$

The form of the sum over $m = 1, 2, \ldots$ in eq. (4.5.28), by its analogy to the D-instanton case, supports the assertion that it reproduces the contribution of single multiply charged Q- and anti-Q-instantons as one can see by comparing (4.5.28) with eq. (4.4.33). The first term in (4.5.28) does not correspond to an instanton contribution. Its origin is yet to be understood.

[6] This can alternatively be derived by using that $P^n_{1/2}(\cosh\eta)$ can also be written as

$$P^n_{1/2}(\cosh\eta) = \frac{\Gamma(\frac{3}{2}+n)}{\Gamma(n+1)\Gamma(\frac{3}{2}-n)}\tanh^n\tfrac{\eta}{2}\, F(-\tfrac{1}{2},\tfrac{3}{2};1+n;-\sinh^2\tfrac{\eta}{2})\,. \qquad (4.5.27)$$

Then using that for $\tau_0 = i$ one has $\tanh^2\frac{\eta}{2} = e^{-2\pi\operatorname{Im}\mathcal{T}}$ and $n = 2m$ the result eq. (4.5.28) follows.

This section is ended by briefly discussing the Fourier series expansion of f around $\tau_0 = \rho$. The starting point is eq. (4.5.14) with $\tau_0 = \rho$ and $\sqrt{\det Q} = \frac{\pi}{3}$ (see table 3.9.1 on page 82). Using eqs. (4.5.10) and (4.5.11) the following three identities are obtained

$$\begin{aligned} \beta(n, n-p;\rho) &= \frac{2\pi}{3} + \beta(p,n;\rho)\,, && (4.5.29)\\ \beta(p-n, p;\rho) &= \frac{4\pi}{3} + \beta(p,n;\rho) && (4.5.30)\\ -\beta(n,p;\rho) &= \frac{4\pi}{3} + \beta(p,n;\rho)\,. && (4.5.31) \end{aligned}$$

Using (4.5.29) and (4.5.30) one can show, in a way which is very similar to the derivation of eq. (4.5.18) for $\tau_0 = i$, that the Fourier coefficients $a_{3m}(\eta)$ are given by

$$a_{3m}(\eta) = \frac{1}{2\pi} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \int_0^{2\pi} d\theta \frac{\cos(3m\theta)}{(\cosh\eta + \sinh\eta\cos\theta)^{3/2}}\,. \tag{4.5.32}$$

It follows by employing eq. (4.5.31) that $a_{3m}(\eta) = a_{-3m}(\eta)$. Hence, using the Fourier decomposition (4.5.12) and eqs. (4.5.32), (4.5.21) and (4.5.23) it is found that for $\tau_0 = \rho$ the function f is expanded as

$$\begin{aligned} f(\eta,\varphi) \;=\; & \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} F(\tfrac{3}{4}, -\tfrac{1}{4}; 1; -\sinh^2\eta) \\ & + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} (-1)^m \frac{1}{2^{3m}} \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \times \\ & \times \sinh^{3m}\eta\, F(\tfrac{3m}{2}+\tfrac{3}{4}, \tfrac{3m}{2}-\tfrac{1}{4}; 3m+1; -\sinh^2\eta) \left(e^{3mi\varphi} + e^{-3mi\varphi}\right)\,. \end{aligned} \tag{4.5.33}$$

At leading order it holds true that

$$\sinh^3\eta \approx 8\, e^{-2\pi \mathrm{Im}\,\mathcal{T}}\,, \tag{4.5.34}$$

so that at this order near $\tau_0 = \rho$ eq. (4.5.33) becomes

$$\begin{aligned} f(\mathcal{T},\bar{\mathcal{T}}) \;\approx\; & \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} + \sum_{m=1}^{\infty} \sum_{(p,n)\neq(0,0)} \frac{(\mathrm{Im}\,\rho)^{3/2}}{|p+n\rho|^3} e^{3mi\beta(p,n;\rho)} \times \\ & \times (-1)^m \frac{\Gamma(\frac{3}{2}+3m)}{\Gamma(3m+1)\Gamma(\frac{3}{2})} \left(e^{2\pi mi\mathcal{T}} + e^{-2\pi mi\bar{\mathcal{T}}}\right)\,, \end{aligned} \tag{4.5.35}$$

where $\varphi = \frac{2\pi}{3}\chi'$ has been used.

The expressions (4.5.28) for $\tau_0 = i$ and (4.5.35) for $\tau_0 = \rho$ can be contrasted with the leading order result for $\tau_0 = i\infty$, eq. (4.5.5). The results (4.5.28) and (4.5.35) differ from (4.5.5) most notably in the axion-independent parts. At this moment it is not clear what kind of processes would account for for the χ' independent pieces of (4.5.28) and (4.5.35).

4.6 Discussion

In this chapter new 1/2 BPS instanton solutions to the Wick rotated Euclidean IIB supergravity theory have been constructed. These new instantons are referred to as Q-instantons. It was shown that Q-instantons form the electric partners of the Q7-branes of the previous chapter.

The path integral approach to the Q-instantons shows the existence of new vacua and a new superselection parameter χ'_∞. Further, it was argued that the Q-instantons contribute to the $\mathcal{R}^4$ terms near the points $\tau_0 = i, \rho$ of the quantum moduli space $SO(2)\backslash PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$. The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ contains terms that do not depend on χ' and for which an interpretation is yet to be found.

The results of this and the previous chapter lend support to the idea that type IIB supergravity provides, at least in the case of the 7-branes and instantons, a valid field theory approximation of some underlying quantum theory near each of the orbifold points of the quantum axidilaton moduli space depicted in figure 3.9.1 on page 81.

The expansion of the generalized Eisenstein series around the points $\tau_0 = i, \rho$ can be given in terms of expansions in the function $\sinh\eta$ (or $(T^2 - 4\det Q)^{1/2}$). In section 3.12 this function was suggested to form the square of the tension of an open Q-string. A better understanding of the origin of the Q7-brane world-volume dynamics in terms of strings may also shed some light on the role of $\sinh\eta$ in the case of the Q-instantons and potentially on the axion χ' independent parts of the $\mathcal{R}^4$ term expanded around $\tau_0 = i, \rho$.