Seven-branes and instantons in type IIB supergravity

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Chapter 3

Seven-branes

3.1 Introduction

This chapter will start by discussing the local properties of 7-branes. By local is meant the properties of a single 7-brane for which, as will be shown, there do not exist any globally well-defined solutions[1]. Starting in section 3.8 globally well-defined solutions that necessarily contain multiple 7-branes will be discussed. The discussion presented in this chapter is based on [16, 17].

3.2 Seven-brane sources

In subsection 1.1.3 it is shown that it is possible to introduce an $SU(1,1)$ triplet of 8-form potentials, $A_8^{\alpha\beta}$. A generic 8-form can be written as $q_{\alpha\beta}A_8^{\alpha\beta}$, where $q_{\alpha\beta}$ is the $SU(1,1)$ charge tensor defined in section 1.2. Knowing that there exists such a triplet of 8-forms a natural question to ask is: does $q_{\alpha\beta}A_8^{\alpha\beta}$ couple electrically to some 7-brane? The answer will prove to be yes as long as $\det Q \geq 0$, where the matrix Q is related to $q_{\alpha\beta}$ via eqs. (1.2.7) and (1.3.38). To establish the existence of these 7-branes, supergravity solutions for them will be constructed. In principle one can then from the solutions derive statements about dynamics through a study of the zero modes. In the following it will be assumed, unless stated otherwise, that the 2- and 4-form field strengths are zero.

The 7-brane world-volume is denoted by $\mathcal{M}_8$ in (3.2.3). The world-volume metric g_{AB} (whose determinant is denoted by $g_{(8)}$) and 8-form $q_{\alpha\beta}A^{\alpha\beta}_{A_1\ldots A_8}$ are the pull-backs

[1]The term 'single 7-brane' here means a 7-brane solution that has an electric description in terms of a single 8-form.

of the target space-time metric $g_{\mu\nu}$ and 8-form $q_{\alpha\beta}A^{\alpha\beta}_{\mu_1\ldots\mu_8}$,

$$g_{AB} = \frac{\partial X^\mu}{\partial\sigma^A}\frac{\partial X^\nu}{\partial\sigma^B}g_{\mu\nu}\,, \tag{3.2.1}$$

$$q_{\alpha\beta}A^{\alpha\beta}_{A_1\ldots A_8} = \frac{\partial X^{\mu_1}}{\partial\sigma^{A_1}}\cdots\frac{\partial X^{\mu_8}}{\partial\sigma^{A_8}}q_{\alpha\beta}A^{\alpha\beta}_{\mu_1\ldots\mu_8}\,, \tag{3.2.2}$$

where the $X^\mu(\sigma)$ are the embedding coordinates of the 7-brane. In section 2.2 it is shown that the following static 7-brane action

$$S = -\int_{\mathcal{M}_8} d^8\sigma\, T\sqrt{-g_{(8)}} + \int_{\mathcal{M}_8} q_{\alpha\beta}A_8^{\alpha\beta}\,, \tag{3.2.3}$$

where T is defined in eq. (1.2.3), preserves half of the IIB supersymmetries with a projector given by

$$P\epsilon = \frac{1}{2}\left(1 + i\gamma_{\underline{0}\cdots\underline{7}}\right)\epsilon = \frac{1}{2}\left(1 - i\gamma_{\underline{8}}\gamma_{\underline{9}}\right)\epsilon = 0\,, \tag{3.2.4}$$

for a 7-brane extended in the directions $x^1,\ldots,x^7$. Here the static gauge,

$$\frac{\partial X^\mu}{\partial\sigma^A} = \delta^\mu_A\,, \tag{3.2.5}$$

has been employed.

3.3 Electric coupling of 7-branes

The field strengths $F_9^{\alpha\beta}$ of the 8-forms $A_8^{\alpha\beta}$ are defined in eqs. (1.1.34) and (1.1.35). In section 1.2 it is shown that the 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ is dual to the scalar χ' defined in eq. (1.2.4). The duality relation between $q_{\alpha\beta}A_8^{\alpha\beta}$ and χ' is given in eq. (1.2.26). The equation of motion for $q_{\alpha\beta}F_9^{\alpha\beta}$ immediately follows from eq. (1.2.26) and reads

$$d\star\left[(T^2 - 4\det Q)^{-1}q_{\alpha\beta}F_9^{\alpha\beta}\right] = 0\,. \tag{3.3.1}$$

The equations of motion for T and χ' follow from the IIB action (1.1.17) in which G_3 and F_5 are put to zero and in which P is given by eq. (1.2.8).

The equations of motion for T and $q_{\alpha\beta}A_8^{\alpha\beta}$ can be obtained from the following action

$$S = \int_{\mathcal{M}_{10}}\left(\star 1\, R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\frac{\star q_{\alpha\beta}F_9^{\alpha\beta}\wedge q_{\alpha\beta}F_9^{\alpha\beta}}{T^2 - 4\det Q}\right)\,. \tag{3.3.2}$$

In order to couple the action to the 7-brane source action (3.2.3) one simply adds (3.2.3) to (3.3.2). This coupled system provides an electric coupling description. It will prove useful to write the source action (3.2.3) as a 10-dimensional bulk integral. To this end a 7-brane current 8-form with delta function support on the 7-brane world-volume, is introduced. This current is denoted by J_8 and it can be written as

$$J^{\mu_1\cdots\mu_8} = \frac{1}{\sqrt{-g}}\int_{\mathcal{M}_8} dX^{\mu_1}\wedge\cdots\wedge dX^{\mu_8}\,\delta(x-X(\sigma))\,. \tag{3.3.3}$$

The 2-form dual of the current is a closed form

$$d\star J_8 = 0\,. \tag{3.3.4}$$

The electrically coupled system is then described by the action

$$\begin{aligned} S \quad = \quad & \int_{\mathcal{M}_{10}}\left(\star 1\,R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2-4\det Q} - \frac{1}{2}\frac{\star q_{\alpha\beta}F_9^{\alpha\beta}\wedge q_{\alpha\beta}F_9^{\alpha\beta}}{T^2-4\det Q}\right) \\ & -\int_{\mathcal{M}_{10}} d^{10}x\int_{\mathcal{M}_8} d^8\sigma\,\delta(x-X(\sigma))\,T\sqrt{-g_{(8)}} \\ & +\int_{\mathcal{M}_{10}} q_{\alpha\beta}A_8^{\alpha\beta}\wedge\star J_8\,. \end{aligned} \tag{3.3.5}$$

The sourced version of eq. (3.3.1) is

$$d\star\left[(T^2-4\det Q)^{-1}q_{\alpha\beta}F_9^{\alpha\beta}\right] = \star J_8\,. \tag{3.3.6}$$

3.4 Magnetic coupling of 7-branes

In order to describe the 7-brane coupling magnetically a first order action is constructed for a 1-form F_1 that satisfies the Bianchi identity

$$dF_1 = 0\,, \tag{3.4.1}$$

so that locally $F_1 = d\chi'$. This Bianchi identity can be obtained from an action principle by introducing an 8-form $q_{\alpha\beta}A_8^{\alpha\beta}$ as a Lagrange multiplier field whose equation of motion is (3.4.1). The first order action for F_1 is

$$\begin{aligned} S \quad = \quad & \int_{\mathcal{M}_{10}}\left(\star 1\,R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2-4\det Q} - \frac{1}{2}\left(T^2-4\det Q\right)\star F_1\wedge F_1\right. \\ & \left. -F_1\wedge q_{\alpha\beta}dA_8^{\alpha\beta}\right)\,. \end{aligned} \tag{3.4.2}$$

The equation of motion for $q_{\alpha\beta}A_8^{\alpha\beta}$ gives (3.4.1). Substituting the solution, $F_1 = d\chi'$, to (3.4.1) back into the action (3.4.2) one finds the action for T and χ'. The equation of motion for F_1 gives the duality relation between F_1 and $q_{\alpha\beta}dA_8^{\alpha\beta}$. Substituting this back into the action (3.4.2) the action for $q_{\alpha\beta}A_8^{\alpha\beta}$, eq. (3.3.2), is found.

Once again, in order to couple this action to the 7-brane source term (3.2.3) one simply adds (3.2.3) to (3.4.2). The magnetically coupled system is described by the action

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \Big(\star 1\, R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\left(T^2 - 4\det Q\right)\star F_1\wedge F_1 \\ &\quad - F_1\wedge q_{\alpha\beta}dA_8^{\alpha\beta}\Big) - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, T\sqrt{-g_{(8)}} \\ &\quad + \int_{\mathcal{M}_{10}} q_{\alpha\beta}A_8^{\alpha\beta}\wedge\star J_8\,. \end{aligned} \tag{3.4.3}$$

The equations of motion for $q_{\alpha\beta}A_8^{\alpha\beta}$ and F_1 are

$$dF_1 = \star J_8 = d\star G_9 \quad\Rightarrow\quad F_1 = d\chi' + \star G_9\,, \tag{3.4.4}$$

and

$$\star q_{\alpha\beta}dA_8^{\alpha\beta} = \left(T^2 - 4\det Q\right)F_1\,, \tag{3.4.5}$$

respectively. These two equations implicitly define the axion χ'. In eq. (3.4.4) the object G_9 has been introduced. Since $\star J_8$ satisfies $d\star J_8 = 0$ it can, at least locally, be represented as the differential of a 1-form that will be called $\star G_9$. One has

$$\star J_8 = d\star G_9 \tag{3.4.6}$$

in which

$$G^{\mu_1\ldots\mu_9} = \frac{1}{\sqrt{-g}}\int_{\mathcal{M}_9} dX^{\mu_1}\wedge\cdots\wedge dX^{\mu_9}\,\delta(x - X(\xi))\,. \tag{3.4.7}$$

In the last equation the delta function has support on a 9-dimensional surface $\mathcal{M}_9$, parameterized by coordinates ξ, whose boundary is the world-volume $\mathcal{M}_8$ of the 7-brane. The 9-dimensional surface is associated with the world-volume of a Dirac 8-brane, which is a brane generalization of the Dirac string stemming from a monopole. In the present case the Dirac 8-brane is stemming from the 7-brane. It is by means of the Dirac 8-brane that the 7-brane will couple magnetically to the axion field strength $F_1 = d\chi'$. Substituting the solution (3.4.4) for F_1 back into the action (3.4.2) coupled to (3.2.3) one obtains

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \left(\star 1\, R - \frac{1}{2}\frac{\star dT\wedge dT}{T^2 - 4\det Q} - \frac{1}{2}\left(T^2 - 4\det Q\right)\star F_1\wedge F_1\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, T\,\sqrt{-g_{(8)}}\,, \end{aligned} \tag{3.4.8}$$

where $F_1 = d\chi' + \star G_9$ and the Wess–Zumino term has disappeared.

Using eqs. (1.2.8) and (1.2.9) the result for the magnetic coupling can be written as

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} \Big(\star 1\, R - 2 \star \hat{P} \wedge \hat{P}^*\Big) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, q_{\alpha\beta} V_-^\alpha V_+^\beta \sqrt{-g_{(8)}}\,, \end{aligned} \tag{3.4.9}$$

where $\hat{P}$ is

$$\hat{P} = P + \frac{i}{2} q_{\alpha\beta} V_+^\alpha V_+^\beta \star G_9\,. \tag{3.4.10}$$

In terms of $\hat{P}$ the duality relation between the 8-form $q_{\alpha\beta} A_8^{\alpha\beta}$ and the axion χ' in the presence of sources takes the form

$$q_{\alpha\beta} F_9^{\alpha\beta} = -i \star \Big(q_{\alpha\beta} V_-^\alpha V_-^\beta \hat{P} - q_{\alpha\beta} V_+^\alpha V_+^\beta \hat{P}^*\Big)\,. \tag{3.4.11}$$

This relates the equations of motion and the Bianchi identity of the 8-form $q_{\alpha\beta} A_8^{\alpha\beta}$ to the Bianchi identity and the equations of motion of the axion χ'. The Bianchi identity for $\hat{P}$ can be written as

$$\hat{D}\hat{P} \equiv d\hat{P} - 2i\hat{Q} \wedge \hat{P} = \frac{i}{2} q_{\alpha\beta} V_+^\alpha V_+^\beta \star J_8\,, \tag{3.4.12}$$

in which

$$\hat{Q} = Q - \frac{1}{2} q_{\alpha\beta} V_+^\alpha V_-^\beta \star G_9\,, \tag{3.4.13}$$

with Q the composite $U(1)$ gauge field defined in (1.1.9). Eq. (3.4.12) is the 'sourced' version of the Bianchi identity (1.1.12).

The magnetic coupling can also be described with respect to the axidilaton field $(\tau, \bar{\tau})$. To achieve this one chooses a $U(1)$ gauge for $V_\pm^\alpha$ appearing in (3.4.9). One such $U(1)$ gauge is given in eqs. (1.3.81) to (1.3.84). Then writing $q_{\alpha\beta}$ in terms of the real numbers p, q, r via eqs. (1.2.7) and (1.3.38) one finds

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} d^{10}x \sqrt{-g} \left(R - \frac{1}{2(\mathrm{Im}\,\tau)^2} \mid \partial_\mu \tau + (p + q\tau^2 + r\tau)\,(\star G_9)_\mu \mid^2\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \int_{\mathcal{M}_8} d^8\sigma\, \delta(x - X(\sigma))\, \frac{1}{\mathrm{Im}\tau} (p + q|\tau|^2 + r\mathrm{Re}\,\tau) \sqrt{-g_{(8)}}\,. \end{aligned} \tag{3.4.14}$$

If one takes $p = 1$ and $q = r = 0$ the action (3.4.14) describes the coupling of a D7-brane. The units used in (3.4.14) are

$$16\pi G_N^{(10)} = 1\,, \tag{3.4.15}$$

where $G_N^{(10)}$ is the 10-dimensional Newton's constant. To restore the factors of $G_N^{(10)}$ in the action (3.4.14) it is noted that the bulk part gets multiplied by a factor of $(16\pi G_N^{(10)})^{-1}$ and the tension of a D7-brane is normalized such that it is given by

$$\frac{e^{\phi}}{(2\pi)^7\, l_s^8\, g_s}\,. \qquad (3.4.16)$$

Here l_s is the string length, the square root of α' and g_s is the string coupling constant, the vacuum expectation value of e^{ϕ}. Customarily, the 10-dimensional Newton's constant is related to l_s via

$$16\pi G_N^{(10)} = (2\pi)^7\, l_s^8\, g_s^2\,. \qquad (3.4.17)$$

Hence, for the coupling of a D7-brane one would write

$$\begin{aligned} S \;&=\; \frac{1}{16\pi G_N^{(10)}}\int_{\mathcal{M}_{10}} d^{10}x\sqrt{-g}\left(R - \frac{1}{2(\mathrm{Im}\,\tau)^2}\mid \partial_\mu\tau + (\star G_9)_\mu\mid^2\right)\\ &\quad - \frac{g_s}{16\pi G_N^{(10)}}\int_{\mathcal{M}_{10}} d^{10}x\int_{\mathcal{M}_8} d^8\sigma\,\delta(x - X(\sigma))\,\frac{1}{\mathrm{Im}\tau}\sqrt{-g_{(8)}}\,. \end{aligned} \qquad (3.4.18)$$

It follows that the relative coefficient between bulk and source terms does not depend on l_s or α'. The factor of g_s in front of the source term can be rescaled away by shifting the value of the dilaton ϕ while at the same time rescaling the value of the RR axion.

3.4.1 Unobservability of the Dirac 8-brane

To describe the magnetic coupling of the 7-brane to the axion the Dirac 8-brane, eqs. (3.4.6) and (3.4.7), has been introduced into the action (3.4.8). As in the classical Dirac monopole problem, the Dirac brane is not a physical object, i.e. the dynamics of the system should not depend on the orientation of the Dirac 8-brane in the bulk. This is reflected in the fact that the 8-brane equations of motion are not independent. They are satisfied provided the axion field equations hold. To see this, let us derive the axion field equation and the equation of motion of the embedding coordinates of the Dirac 8-brane.

The field equation of χ' is

$$\partial_\mu\left[\sqrt{-g}\,\left(T^2 - 4\det Q\right)\,F^\mu\right] = 0\,, \qquad (3.4.19)$$

where F_1 satisfies the Bianchi identity (3.4.4). The variation of (3.4.8) with respect to the Dirac 8-brane world-volume coordinates $X^\mu(\xi)$, appearing in F_1 produces the equation

$$\partial_\mu\left[\sqrt{-g}\,\left(T^2 - 4\det Q\right)\,F^\mu\right]\mid_{\mathcal{M}_9} = 0\,, \qquad (3.4.20)$$

which is nothing but the χ' field equation, eq. (3.4.19), pulled-back on the Dirac 8-brane world-volume. Therefore, the Dirac brane is not physical. Its position can be anywhere in space-time and it is invisible provided the Dirac veto holds, which does not allow the Dirac brane to intersect the world-volumes of the objects coupled to χ' in an electric way. If such objects (which would be instantons) are present, their 'currents' contribute to the right hand side of eq. (3.4.19), while eq. (3.4.20) remains sourceless. The two equations are then consistent provided the world-volumes of the Dirac brane and the electrically charged objects never intersect[2]. In quantum theory, as is well known, the unobservability of the Dirac branes is guaranteed by the Dirac quantization condition which results in the quantization of corresponding fluxes.

3.5 Dirac strings and monodromy

The hatted 1-form fields $\hat{P}$ and $\hat{\mathcal{Q}}$ defined in (3.4.10) and (3.4.13) can be collected into the matrix-valued 1-form

$$V\begin{pmatrix} -i\hat{Q} & \hat{P} \\ \hat{P}^* & i\hat{Q} \end{pmatrix}V^{-1} = dVV^{-1} - SQS^{-1}\star G_9\,, \tag{3.5.1}$$

with S defined in eq. (1.2.7) and where V is the matrix defined in equation (1.1.4). Let us define

$$\hat{p} = p - SQS^{-1}\star G_9\,, \tag{3.5.2}$$

where p is defined in eq. (1.1.15). Then $\hat{p}$ satisfies the Bianchi identity

$$d\hat{p} - 2p\wedge p = -SQS^{-1}\star J_8\,. \tag{3.5.3}$$

Outside the source the Bianchi identity (3.5.3) is solved by

$$p = \frac{1}{2}(dC)C^{-1}\,, \tag{3.5.4}$$

where $C = VV^\dagger$. Alternatively, this solution can be written as

$$DC = 0 \qquad \text{with} \qquad D = d - 2p\,. \tag{3.5.5}$$

This equation can be interpreted as saying that C is parallel transported with respect to the flat connection p. Let $\gamma(\lambda)$ be some path parameterized by λ which runs from

[2] Additional complications and subtleties regarding the Dirac branes and corresponding singular terms in the action and equations of motion arise when the action contains Wess–Zumino terms with 'bare' electric and/or magnetic potentials. In such cases it becomes much less trivial to reconcile the Dirac veto with the physical field equations. This happens for example in the case of the M5-brane [55]. In [56–59] a consistent method was developed to resolve these problems and related problems of anomalies.

0 to 1. Then one has

$$C(\lambda = 1) = \mathcal{P}\exp[2\int_\gamma p]\;\; C(\lambda = 0)\;, \tag{3.5.6}$$

where $\mathcal{P}$ denotes the path ordering symbol. Since the connection is flat the quantity $\mathcal{P}\exp[2\int_\gamma p]$ for closed γ will only depend on the base point of the closed path. The location of the base point can be changed by a similarity transformation,

$$\mathcal{P}\mathrm{exp}[2\oint_\gamma p] \;\to\; H\,\mathcal{P}\mathrm{exp}[2\oint_\gamma p]\,H^{-1} \quad \text{where} \quad H = \mathcal{P}\mathrm{exp}[2\int_{\tilde{\gamma}} p]\;, \tag{3.5.7}$$

with the path $\tilde{\gamma}$ connecting the initial to the final base point. This means that the eigenvalues of the monodromy matrix, $\mathcal{P}\mathrm{exp}[2\oint_\gamma p]$, are preserved under shifting the position of the base point. Therefore a physical quantity that can be associated with the Bianchi identity $dp - 2p\wedge p = 0$ is the Wilson line[3]

$$\mathrm{Tr}\,\mathcal{P}\exp[2\oint_\gamma p]\,. \tag{3.5.8}$$

For 7-brane solutions the matrix p only depends on the two coordinates transverse to the brane. In that case the quantity $\mathcal{P}\mathrm{exp}[2\oint_\gamma p]$ will determine the monodromy of C and thus of the scalars that parameterize it.

The monodromy of the matrix $C = VV^\dagger$ is given by

$$C(\lambda = 1) = \mathcal{P}\exp[2\oint_\gamma p]\;\; C(\lambda = 0)\,. \tag{3.5.9}$$

Let us consider a path γ which encircles the 7-brane (point) source in the transverse space. Further it is assumed that γ encloses an area of infinitesimal size, denoted by D. Expanding the path-ordered expression up to second order it is found that

$$C(\lambda = 1) = \left(1 + \int_D (d\hat{p} - 2p\wedge p) + \cdots\right) C(\lambda = 0)\left(1 + \int_D (d\hat{p} - 2p\wedge p)^\dagger\right)\,, \tag{3.5.10}$$

where use has been made of the fact that $pC = Cp^\dagger$ and $\oint_\gamma p = \int_D d\hat{p}$. According to eq. (3.5.3) this can be written as

$$C(\lambda = 1) = \left(1 - SQS^{-1} + \cdots\right) C(\lambda = 0)\left(1 - (SQS^{-1})^\dagger + \cdots\right)\,. \tag{3.5.11}$$

Since the monodromy of $C = VV^\dagger$ when going at an infinitesimal distance around a 7-brane is known and since the parametrization of V in terms of τ, see eqs. (1.3.81) to

[3] The terminology is borrowed from Yang–Mills theory. Here p is not a gauge field. It is because of a mathematical similarity that this quantity is referred to as a Wilson line.

(1.3.84), is known the monodromy of τ can be obtained. It follows that τ transforms as

$$\tau \to e^{Q}\tau\,. \tag{3.5.12}$$

The Wilson line (3.5.8) when evaluated around the contour γ encircling a 7-brane at an infinitesimal distance can be evaluated and is equal to $\mathrm{Tr}\, e^{Q}$. Hence, the Wilson line computes what is called the $SL(2,\mathbb{R})$ conjugacy class.

In order to evaluate the Wilson line (3.5.8) up to the first nontrivial term the following general result [60] is used. Let $A(\epsilon)$ be an arbitrary $SU(1,1)$ matrix which tends to 1 as $\epsilon \to 0$ then one can write A as

$$A(\epsilon) = \exp\left[\epsilon\alpha_1^i T^i + \epsilon^2\alpha_2^i T^i + \cdots\right]\,, \tag{3.5.13}$$

where the T^i are the generators of the $SU(1,1)$ group and where the α_1^i etc. are arbitrary functions. Expanding $A(\epsilon)$ up to second order the trace of A (up to this order in ϵ) is given by

$$\mathrm{Tr}A(\epsilon) = 2 + \frac{1}{2}\mathrm{Tr}\left(\epsilon\alpha_1^i T^i\right)^2 + \cdots\,. \tag{3.5.14}$$

Applying this formula to eq. (3.5.8) with

$$\epsilon\alpha_1^i T^i = 2\int_D \left(d\hat{p} - 2p \wedge p\right) \tag{3.5.15}$$

the following result is found

$$\mathrm{Tr}\mathcal{P}\exp[2\oint_\gamma p] = 2 + \frac{1}{2}\mathrm{Tr}\,4(SQS^{-1})^2 + \cdots = 2 - 4\det Q + \cdots\,. \tag{3.5.16}$$

When the globally well-defined 7-brane solutions will be discussed in section 3.10 and onwards the base point dependence discussed in this section will play a role in characterizing the various 7-branes in the solution. It will turn out that one can only meaningfully speak of 7-branes as representative elements of $SL(2,\mathbb{Z})$ conjugacy classes, a result that is related to the base point independence of the Wilson line (3.5.16).

3.6 Supersymmetry and holonomy of the Killing spinor

In order to describe 7-brane solutions sourced by eq. (3.2.3) the Killing spinor equations must be solved imposing the supersymmetry projector (3.2.4). The Killing spinor equations are

$$\delta\lambda = -\frac{i}{\tau-\bar\tau}\,(\gamma^\mu\partial_\mu\tau)\,\epsilon_C = 0\,, \tag{3.6.1}$$

$$\delta\psi_\mu = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{4(\tau-\bar\tau)}\partial_\mu(\tau+\bar\tau)\right)\epsilon = 0\,. \tag{3.6.2}$$

These Killing spinor equations are obtained from (1.3.72) and (1.3.73) in which P and Q are taken to be (1.3.74) and (1.3.75).

Eqs. (3.6.1) and (3.6.2) transform covariantly under the $SL(2,\mathbb{R})$ transformations (1.3.39) and (1.3.103). Observe that, unlike τ, the Killing spinor ϵ does transform under $S^2 = -\mathbb{1}$ as $\epsilon \to i\,\epsilon$. Under $S^4 = \mathbb{1}$ one has $\epsilon \to -\epsilon$. Only S^8 acts as the identity on ϵ. This means that ϵ transforms under the double cover of $SL(2,\mathbb{Z})$.

From eq. (3.2.4) it follows that $(\gamma_{\underline{8}} - i\gamma_{\underline{9}})\,\epsilon = 0 = (\gamma_{\underline{8}} + i\gamma_{\underline{9}})\,\epsilon_C$. If it is assumed that τ and the metric do not depend on the world-volume coordinates $x^0,\ldots,x^7$ and a conformally flat transverse metric is chosen, then eq. (3.6.1) implies $(\partial_8 - i\partial_9)\,\bar\tau = 0$. The complex transverse coordinate $z = x^8 + ix^9$ is introduced so that now $\partial_z\bar\tau = 0$, i.e. τ is a holomorphic function. In complex coordinates the condition on ϵ can be written as $\gamma_{\underline{z}}\epsilon = 0$. Under these conditions, the most general 7-brane solution to eqs. (3.6.1) and (3.6.2) is given by [24, 61–63]

$$ds^2 = -dt^2 + d\vec{x}_7^{\,2} + (\mathrm{Im}\,\tau)|f|^2 dz d\bar z\,, \tag{3.6.3}$$

$$\tau = \tau(z)\,, \tag{3.6.4}$$

$$f = f(z)\,, \tag{3.6.5}$$

$$\epsilon = \left(\frac{\bar f}{f}\right)^{1/4}\epsilon_0\,, \tag{3.6.6}$$

where ϵ_0 is a constant spinor satisfying $\gamma_{\underline{z}}\epsilon_0 = 0$.

The functions τ and f are assumed to be defined on the Riemann sphere. The form of the solution is therefore fixed up to $SL(2,\mathbb{C})$ transformations

$$z \to \frac{az+b}{cz+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})\,. \tag{3.6.7}$$

These are the most general global coordinate transformations that do not change the structure of the branch cuts and singularities of τ and f in the complex z-plane. Note that locally (but not globally) one can always choose a basis in which $f(z) = 1$. This, however, has the disadvantage of introducing multi-valued coordinates w defined via $dw = f(z)dz$.

Although the configurations (3.6.3)–(3.6.6) are locally supersymmetric, they must satisfy further conditions to be globally well-defined and supersymmetric. The main issue here will be the possible multi-valuedness of $\tau(z)$ and $f(z)$, due to the presence of branch cuts.

Whenever a branch cut of the function τ is crossed τ and ϵ must transform as in eqs. (1.3.39) and (1.3.103). The transformation rule for the function f when crossing a branch cut can be obtained by requiring that the holonomy of ϵ be well-defined. The holonomy of the Killing spinor is computed with respect to the generalized connection in (3.6.2), which is the sum of the Lorentz connection and the $U(1)$ connection of the $SU(1,1)/U(1)$ coset model. The integrability condition of (3.6.2) requires that the total curvature vanishes but the Riemann curvature of the transverse space and the $U(1)$ curvature are, separately, non-trivial.

If the Killing spinor ϵ is parallel-transported using the connection in (3.6.2), evaluated on the solution, eqs. (3.6.3) and (3.6.4), from a base point b around a closed loop γ_b it can be shown that the holonomy (with respect to the Lorentz group) of ϵ is given by

$$\epsilon(b) \to \exp\left(-\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right)\epsilon(b)\,, \tag{3.6.8}$$

where the prime denotes differentiation with respect to z.

The holonomy phase factor will depend on the base point b but only through the homotopy class of γ_b due to the vanishing total curvature. The holonomy with respect to the generalized connection will be trivial if eq. (3.6.8) equals[4] an $SL(2,\mathbb{R})$ transformation as given in eq. (1.3.103)

$$\exp\left(-\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right) = e^{i\beta}\,. \tag{3.6.9}$$

Let γ_b be parameterized by $\lambda \in [0,1]$. Then

$$\exp\left(\frac{i}{2}\,\mathrm{Im}\oint_{\gamma_b}(\log f)'dz\right) = \left(\frac{f(\lambda=1)}{|f(\lambda=1)|)}\right)^{1/2}\left(\frac{|f(\lambda=0)|}{f(\lambda=0))}\right)^{1/2}\,. \tag{3.6.10}$$

The requirement (3.6.9) then leads to the following condition for the function f

$$f(\lambda=1) = (c\tau+d)f(\lambda=0)\,. \tag{3.6.11}$$

Thus, when crossing a branch cut it must be that

$$f \to (c\tau+d)f\,. \tag{3.6.12}$$

The $SL(2,\mathbb{R})$ transformation properties of τ, f and ϵ are summarized in table 3.6.1.

The metric $g_{\mu\nu}$ does not transforms under $SL(2,\mathbb{R})$ and must be single-valued modulo coordinate transformations. On the other hand, $\mathrm{Im}\,\tau$ appears explicitly in

[4]In general one can also allow for nontrivial spin structures.

Fields	Group	Order of S
τ	$PSL(2,\mathbb{R})$	2
f	$SL(2,\mathbb{R})$	4
ϵ	double cover of $SL(2,\mathbb{R})$	8

Table 3.6.1: Some $SL(2,\mathbb{R})$ transformation properties of τ, f and ϵ.

the expression (3.6.3) for $g_{\mu\nu}$ and it may transform into $|c\tau+d|^{-2}\text{Im}\,\tau$ when crossing a branch cut. In general, the extra factor $|c\tau+d|^{-2}$ cannot be eliminated by an $SL(2,\mathbb{C})$ transformation and, thus, it must be compensated by $f(z)$. From eq. (3.6.12) it is clear that the metric remains invariant when crossing a branch cut of the function τ.

3.7 Field equations and local 7-brane solutions

3.7.1 The equation of motion of X^μ

The 7-brane equation of motion obtained by varying (3.4.8) or (3.2.3) with respect to the world-volume field $X^\mu(\sigma)$ is

$$\begin{aligned} g_{\mu\nu}\nabla_A\left(T\sqrt{-g_8}\,g^{AB}\tfrac{\partial X^\mu}{\partial\sigma^B}\right) - \sqrt{-g_8}\,\tfrac{\partial T}{\partial X^\mu} = \\ \tfrac{1}{8!}e^{A_1\ldots A_8}\,\tfrac{\partial X^{\mu_1}}{\partial\sigma^{A_1}}\cdots\tfrac{\partial X^{\mu_8}}{\partial\sigma^{A_8}}\,\epsilon_{\mu_1\ldots\mu_8\mu\rho}\,\left(T^2-4\det Q\right)\,F^\rho\,, \end{aligned} \tag{3.7.1}$$

where F_1 is defined in eq. (3.4.4) and where

$$\nabla_A X^\mu_B = \partial_A X^\mu_B - \Gamma^C_{AB}\,X^\mu_C + \frac{\partial X^\rho}{\partial\sigma^A}\Gamma^\mu_{\rho\nu}\,X^\nu_B\,, \tag{3.7.2}$$

for some tensor X^μ_B in which Γ^C_{AB} is the Levi–Civitá connection of the pulled-back metric g_{AB} and in which $\Gamma^\mu_{\rho\nu}$ is the Levi–Civitá connection of the target space-time metric $g_{\mu\nu}$. The objects $e^{A_1\ldots A_8}$ and $\epsilon_{\mu_1\ldots\mu_8\mu\rho}$ are defined in appendix A. When varying eq. (3.2.3) with respect to $X^\mu(\sigma)$ one employs eq. (3.4.5) to bring the result in the form (3.7.1).

Employing the static gauge (3.2.5) the supersymmetric configuration (3.6.3) and (3.6.4) corresponds to a 7-brane that is static and does not fluctuate in the transverse directions, i.e. $X^{8,9} = \text{cst}$. From the point of view of the 2-dimensional transverse space the Dirac 8-brane has reduced to a Dirac string ending on a magnetic point source. The equation of motion for X^μ in static gauge (3.2.5) with $X^{8,9} = \text{cst}$ reduces to

$$\partial_i T = \left(T^2 - 4\det Q\right)\sqrt{-g}\,\epsilon_{01\ldots 7ij}\left(\partial^j\chi' - (\star G_9)^j\right)\,, \tag{3.7.3}$$

with $i, j = 8, 9$. Away from the source and the Dirac string one finds

$$\partial_8 T = (T^2 - 4\det \mathrm{Q})\partial_9\chi'\,, \quad \partial_9 \mathrm{T} = -(\mathrm{T}^2 - 4\det \mathrm{Q})\partial_8\chi'\,. \tag{3.7.4}$$

Specializing eqs. (3.7.4) to the cases $\det Q = 0$ and $\det Q > 0$ it is found that

$$\det Q = 0\ :\quad \frac{\partial}{\partial\bar z}\left(\chi' + \frac{i}{T}\right) = 0\,, \tag{3.7.5}$$

$$\det Q > 0\ :\quad \frac{\partial}{\partial\bar z}\left(\chi' + \frac{i}{4\sqrt{\det Q}}\log\frac{T + 2\sqrt{\det Q}}{T - 2\sqrt{\det Q}}\right) = 0\,, \tag{3.7.6}$$

with $z = x^8 + ix^9$. Expressing the conditions (3.7.5) and (3.7.6) in terms of the axidilaton field τ via eqs. (1.3.79) and (1.3.80) for the $\det Q = 0$ case and via eq. (1.3.42) for the $\det Q > 0$ case it is found that the τ must be holomorphic in agreement with eq. (3.6.4).

The conditions (3.7.5) and (3.7.6) under which the 7-brane can be considered static coincides with the condition that τ is a holomorphic function. As it follows from equation (3.7.3), along the Dirac string the holomorphicity fails, so the Dirac string plays the role of a branch cut of τ. Crossing of these branch cuts is related to the nontrivial monodromy of the function τ in agreement with the results of section 3.5.

3.7.2 The axidilaton equation of motion

Consider eq. (3.4.4). Let D denote a disk in the transverse space whose center is the location of a 7-brane. Integrating (3.4.4) over D gives

$$\int_D dF_1 = \int_D \star J_8 = \oint_{\partial D}(d\chi' + \star G_9)\ . \tag{3.7.7}$$

In eq. (3.7.7) $d\chi'$ is an exact 1-form and does not contribute to the integral. The non-zero value of the integral in (3.7.7) is due to the Dirac string. Alternatively, the Dirac string can be taken out of (3.7.7) granted the differential $d\chi'$ fails to be exact everywhere on the boundary ∂D. Taking the boundary to be a circle of radius r, the integral (3.7.7) can be written as

$$\int_D dF_1 = \oint_{\partial D} d\chi' = \int_0^{2\pi}\frac{d\chi'}{d\theta}d\theta = \int_D \star J_8 = 1\,. \tag{3.7.8}$$

Here θ is an angular variable parameterizing the circle ∂D. Its range is $0 \leq \theta < 2\pi$. The solution for χ' that satisfies (3.7.8) is

$$\chi' = \frac{\theta}{2\pi}\,. \tag{3.7.9}$$

The function χ' is single-valued on the cut plane where the cut is formed by the line $\theta = 0$. The solution to the equation of motion for the scalar T, that follows from the action (3.4.8), is constrained to satisfy the condition (3.7.5) or (3.7.6). It is concluded that in order to solve for the equations of motion for T and χ', or equivalently, for the equations of motion for τ, the Dirac strings $\star G_9$ in the actions (3.4.8) and (3.4.14) can be ignored at the price of working with functions τ that have branch cuts. The behavior of τ when crossing a branch cut is given in eq. (3.5.12).

In this subsection the equation of motion for τ that follows from varying the action (3.4.14) with respect to $\bar\tau$ will be solved with the understanding that the Dirac string will be replaced by branch cuts. Performing a variation of the action (3.4.14) with respect to $\bar\tau$ and substituting in the resulting equation of motion the metric (3.6.3) leads to the following equation of motion for τ

$$\partial\bar\partial\tau - 2\frac{\partial\tau\bar\partial\tau}{\tau-\bar\tau} = -\frac{i}{4}\delta(z-z_0,\bar z-\bar z_0)\left(p+q\tau^2+r\tau\right)\,, \tag{3.7.10}$$

where $\partial = \frac{\partial}{\partial z}$. Due to the presence of the delta function[5] it is not possible, at this stage, to assume that τ is a holomorphic function. Eq. (3.7.10) can be integrated as follows. Let D be an infinitesimal disk $|z-z_0| \leq \delta$ whose boundary is denoted by γ_δ. Integrating eq. (3.7.10) over D gives

$$\lim_{\delta\to 0}\int_D\left(\partial\bar\partial\tau - 2\frac{\partial\tau\bar\partial\tau}{\tau-\bar\tau}\right)\frac{i}{2}dz\wedge d\bar z = -\frac{i}{4}\lim_{\delta\to 0}\oint_{\gamma_\delta}\tau' dz = -\frac{i}{4}\left(p+q\tau^2+r\tau\right)_{z=z_0}\,, \tag{3.7.12}$$

where the prime denotes differentiation with respect to z. Green's theorem[6] has been used to relate the integral over D to the integral over the boundary γ_δ together with the fact that $\bar\partial\tau = 0$ along γ_δ (except on a set of measure zero).

Assuming that when $q, r \neq 0$ the limit $\lim_{z\to z_0}\tau$ exists one may write

$$2\pi i\tau(z_0) = \lim_{\delta\to 0}\oint_{\gamma_\delta}\frac{\tau}{z-z_0}dz\,. \tag{3.7.14}$$

[5] The delta function in (3.7.10) is defined via

$$\int \frac{i}{2}dz\wedge d\bar z\,\delta(z,\bar z) = 1\,. \tag{3.7.11}$$

[6] In complex notation Green's theorem for any real-analytic function F defined on $D/\{z_0\}$ reads

$$\int_D \partial\bar\partial F\frac{i}{2}dz\wedge d\bar z = \frac{i}{4}\left(\oint_{\partial D}\bar\partial F d\bar z - \oint_{\partial D}\partial F dz\right)\,. \tag{3.7.13}$$

Therefore one has

$$\lim_{\delta\to 0}\oint_{\gamma_\delta}\left(2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0}\right)dz = 0\,. \tag{3.7.15}$$

This form of the axidilaton equation of motion is convenient to derive an approximation of the solution close to the source terms at z_0. This derivation goes as follows. Assume that the integrand of (3.7.15) is an analytic function without any poles in the interior of γ_δ. Then it admits in D a Taylor expansion

$$2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0} = \sum_{n=0}^{\infty} a_n(z-z_0)^n\,. \tag{3.7.16}$$

In the limit $|z-z_0|\to 0$ the poles on the left hand side will dominate all the terms on the right hand side. In this approximation the right hand side of (3.7.16) can be put to zero, and one is left with the homogeneous version of eq. (3.7.16), i.e.

$$2\pi i\tau' - p\frac{1}{z-z_0} - q\frac{\tau^2}{z-z_0} - r\frac{\tau}{z-z_0} = 0\,. \tag{3.7.17}$$

The solutions to (3.7.17) are

$$e^{2\pi i\tau/p} = z-z_0 \qquad \text{for } \det Q = 0 \text{ with } q=r=0\,, \tag{3.7.18}$$

$$c\left(\frac{\tau-\tau_0}{\tau-\bar\tau_0}\right)^{\frac{\pi}{\sqrt{\det Q}}} = z-z_0 \qquad \text{for } \det Q > 0\,, \tag{3.7.19}$$

where $c\neq 0$ is a constant. All these solutions have a non-trivial monodromy that is in agreement with eq. (3.5.12). Solutions with $\det Q = 0$ and nonzero values for q and r can be obtained by performing $SL(2,\mathbb{R})$ transformations of (3.7.18).

With respect to the field χ' both solutions (3.7.18) and (3.7.19) have axion charge equal to one. The axion charge is defined as

$$\int_0^{2\pi}\frac{d\chi'}{d\theta}d\theta\,, \tag{3.7.20}$$

where $z-z_0 = re^{i\theta}$. This can be verified by rewriting the solutions (3.7.18) and (3.7.19) in terms of T and χ' for the cases $\det Q = 0$ and $\det Q > 0$ separately.

3.7.3 The Einstein equations

Varying the action (3.4.14) with respect to the metric (ignoring the Dirac string by working with a multi-valued τ) and substituting equations (3.6.3) and (3.6.4) one finds that the $z\bar z$ component of the Einstein equations is given by

$$\partial\bar\partial\log|f|^2 = -\frac{1}{2}\delta(z-z_0,\bar z-\bar z_0)\frac{i}{\tau-\bar\tau}\left(p+q|\tau|^2 + r\mathrm{Re}\,\tau\right)\,. \tag{3.7.21}$$

All other components of the Einstein equations are identically zero. The local expression for f must be such that when crossing a branch cut of the function τ, f transforms as in eq. (3.6.12).

Integrating eq. (3.7.21) over a disk D that is bounded by γ and using Green's theorem (see footnote 6) and that $\bar{\partial} f = 0$ along γ (except on a set of measure zero) it is found that

$$\mathrm{Im} \oint_\gamma (\log f)' dz = -\frac{i}{\tau - \bar{\tau}} \left(p + q|\tau|^2 + r\mathrm{Re}\,\tau \right)_{z=z_0} . \tag{3.7.22}$$

Substituting eqs. (3.7.18) and (3.7.19) on the right hand side of (3.7.22) it follows that

$$\mathrm{Im} \oint_\gamma (\log f)' dz = -\sqrt{\det Q}\,, \tag{3.7.23}$$

where e^Q is the monodromy matrix of τ measured when going around the contour γ.

The solutions for τ, eqs. (3.7.18) and (3.7.19) can be written as[7]

$$\tau = \frac{p}{2\pi i} \log(z - z_0) \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.24}$$

$$\tau = \mathrm{Re}\,\tau_0 + \mathrm{Im}\,\tau_0 \tan\left(\sqrt{\det Q}\, w + \frac{\pi}{2} \right) \quad \text{for } \det Q > 0\,, \tag{3.7.25}$$

where w is

$$w = \frac{1}{2\pi i} \log \frac{z - z_0}{c}\,. \tag{3.7.26}$$

When going around z_0 by sending $z - z_0 \to e^{2\pi i}(z - z_0)$, or what is the same $w \to w + 1$, the axidilaton τ transforms as (1.3.39) with a, b, c and d given by e^Q, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^Q = \cos\sqrt{\det Q}\,\mathbb{1} + \frac{\sin\sqrt{\det Q}}{\sqrt{\det Q}} Q\,. \tag{3.7.27}$$

Since the function f transforms as in (3.6.12) with c and d as given in (3.7.27) it is concluded that when crossing the branch cuts of τ, eqs. (3.7.24) and (3.7.25), f transforms as

$$f \to f \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.28}$$

$$f \to \left[\cos\sqrt{\det Q} - \sin\sqrt{\det Q} \tan\left(\sqrt{\det Q}\, w + \frac{\pi}{2} \right) \right] f \quad \text{for } \det Q > 0\,. \tag{3.7.29}$$

The unique functions f that transform as in (3.7.28) and (3.7.29) while satisfying eq. (3.7.23) are

$$f = 1 \quad \text{for } \det Q = 0 \text{ with } q = r = 0\,, \tag{3.7.30}$$

$$f = \cos\left(\sqrt{\det Q}\, w + \frac{\pi}{2} \right) \quad \text{for } \det Q > 0\,. \tag{3.7.31}$$

[7] This form of τ very closely represents the local 7-brane solutions discussed in [63].

The expression for f for the $\det Q > 0$ case can be rewritten in terms of τ as

$$f = \frac{\mathrm{Im}\,\tau_0}{(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}}\,. \tag{3.7.32}$$

3.7.4 Local geometry of a 7-brane

At this stage in the analysis the local characteristics of a (p, q, r) 7-brane are known. For the case $\det Q = 0$ with $q = r = 0$, i.e. a D7-brane of charge p the local geometry can be inferred from expressions (3.6.3), (3.6.6), (3.7.24) and (3.7.30). Putting all this together one finds

$$\begin{aligned} ds^2 &= -dt^2 + d\vec{x}_7^{\,2} + (\mathrm{Im}\,\tau)dzd\bar{z}\,, && (3.7.33)\\ \tau &= \frac{p}{2\pi i}\log(z-z_0)\,, && (3.7.34)\\ \epsilon &= \epsilon_0\,. && (3.7.35)\end{aligned}$$

For the case $\det Q > 0$ the 7-brane is referred to as a Q7-brane and the local geometry follows from (3.6.3), (3.6.6), (3.7.19) and (3.7.32). The resulting local Q7-brane solution is

$$\begin{aligned} ds^2 &= -dt^2 + d\vec{x}_7^{\,2} + (\mathrm{Im}\,\tau)|f|^2 dzd\bar{z}\,, && (3.7.36)\\ c\left(\frac{\tau-\tau_0}{\tau-\bar{\tau}_0}\right)^{\frac{\pi}{\sqrt{\det Q}}} &= z - z_0\,, && (3.7.37)\\ f &= \frac{\mathrm{Im}\,\tau_0}{(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}}\,, && (3.7.38)\\ \epsilon &= \left(\frac{(\tau-\tau_0)(\tau-\bar{\tau}_0)}{(\bar{\tau}-\bar{\tau}_0)(\bar{\tau}-\tau_0)}\right)^{1/8}\epsilon_0\,. && (3.7.39)\end{aligned}$$

The local Q7-brane solution (3.7.36) to (3.7.39) has been written down in terms of the fields as defined in subsection 1.3.4. This will prove a useful way of writing the Q7-brane solution when combining the local geometry of Q7-branes with that of D7-branes later in section 3.10. However, it is insightful to rewrite the local Q7-brane in terms of the fields T, χ' and ϵ' that are used in subsection 1.3.1. The relation between τ and T and χ' can be obtained using eq. (1.3.1) together with eqs. (1.2.22) to (1.2.25). The relation is

$$\frac{\tau-\tau_0}{\tau-\bar{\tau}_0} = \left(\frac{T-2\sqrt{\det Q}}{T+2\sqrt{\det Q}}\right)^{1/2} e^{2i\sqrt{\det Q}\,\chi'}\,. \tag{3.7.40}$$

From which it follows that (3.7.37) becomes

$$\chi' + \frac{i}{4\sqrt{\det Q}}\log\frac{T+2\sqrt{\det Q}}{T-2\sqrt{\det Q}} = \frac{1}{2\pi i}\log\frac{z-z_0}{c}\,. \tag{3.7.41}$$

Further, the metric (3.7.36) becomes

$$ds^2 = -dt^2 + d\vec{x}_7^2 + q\left(T^2 - 4\det Q\right)^{-1/2} dz d\bar{z}\,. \tag{3.7.42}$$

In order to find the expression for the Killing spinor ϵ' defined in eq. (1.3.12) one must perform the local $U(1)$ transformation (1.3.7) and (1.3.12). The Killing spinor ϵ, given in eq. (3.7.39), is written in the $U(1)$ gauge of subsection 1.3.1 in which $V_\pm^\alpha$ is given in eqs. (1.3.81) to (1.3.84). Using this $U(1)$ gauge choice together with the relation between $q_{\alpha\beta}$ and p, q, r, eqs. (1.2.7) and (1.3.38), one recognizes

$$\frac{q_{\alpha\beta}V_+^\alpha V_+^\beta}{q_{\gamma\delta}V_-^\gamma V_-^\delta} = \frac{(\tau-\tau_0)(\tau-\bar{\tau}_0)}{(\bar{\tau}-\bar{\tau}_0)(\bar{\tau}-\tau_0)}\,, \tag{3.7.43}$$

so that via eqs. (1.3.7) and (1.3.12) the Killing spinor ϵ' is given by

$$\epsilon' = \epsilon_0\,. \tag{3.7.44}$$

In terms of T, χ' and ϵ' the local geometry of a Q7-brane shows a lot of similarity with the local geometry of a D7-brane, eqs. (3.7.33) to (3.7.35). The only property that does not straightforwardly map into each other is the behavior of $\operatorname{Im}\tau$ for the D7-brane and that of T for the Q7-brane. Other then that the axions and Killing spinors $\operatorname{Re}\tau$ and ϵ for the D7-brane and χ' and ϵ' for the Q7-brane behave in the same way.

In order to further study the local geometry of a Q7-brane it is convenient to introduce polar coordinates r and θ via

$$\frac{z-z_0}{c} = re^{i\theta}\,, \tag{3.7.45}$$

where $0 < r < 1$. From eqs. (3.7.41) and (3.7.42) it can be concluded that the metric can be written as

$$ds^2 = -dt^2 + d\vec{x}_7^2 + \frac{|c|^2}{4\operatorname{Im}\tau_0}\left(r^{-\frac{\sqrt{\det Q}}{\pi}} - r^{\frac{\sqrt{\det Q}}{\pi}}\right)\left(dr^2 + r^2 d\theta^2\right)\,. \tag{3.7.46}$$

Consider the Q7-brane metric near $r = 0$. Then (3.7.46) behaves as

$$ds^2 \sim -dt^2 + d\vec{x}_7^2 + \frac{|c|^2}{4\operatorname{Im}\tau_0} r^{-\frac{\sqrt{\det Q}}{\pi}}\left(dr^2 + r^2 d\theta^2\right)\,. \tag{3.7.47}$$

This is the metric of a cone whose apex is at $r = 0$ with a deficit angle δ equal to $\sqrt{\det Q}$. This result is to be contrasted with the local geometry of a D7-brane, eq. (3.7.33), where the metric behaves logarithmically as one approaches the location of the D7-brane.

Instead of looking at the local form of the metric it is possible to calculate the deficit angle using the function f. This is because the orders of the poles of the function $f(z)$ at $z = z_0$ determine the deficit angle δ at the location of the source. Let γ be a closed circular contour that encircles the point z_0. Then one has

$$\delta = i \oint_\gamma (\log f)' dz \,. \tag{3.7.48}$$

Combining eqs. (3.7.23) and (3.7.48) the following expression for the deficit angle at the location of the source is found

$$\delta = \sqrt{\det Q} \,, \tag{3.7.49}$$

in agreement with eq. (3.7.47). There is no deficit angle at the position of a (p, q, r) 7-brane for which $\det Q = 0$.

Eqs. (3.7.41) and (3.7.42) can be used to construct the solution for the 8-form $q_{\alpha\beta} A_8^{\alpha\beta}$. The duality relation (1.2.26) expressed in terms of the polar coordinates (3.7.45) reads

$$q_{\alpha\beta} F^{\alpha\beta}_{01\cdots 7r} = \frac{1}{2\pi r} \left(T^2 - 4 \det Q \right) = \frac{dT}{dr} \,, \tag{3.7.50}$$

where the last equality follows from eq. (3.7.41). Assuming that $q_{\alpha\beta} A_8^{\alpha\beta}$ only depends on r it is found that

$$q_{\alpha\beta} A^{\alpha\beta}_{01\cdots 7} = T + k \,, \tag{3.7.51}$$

where k is an integration constant. Specializing to the case $\det Q = 0$ with $q = r = 0$ using the definitions (1.3.79) and (1.3.89) it follows that the RR 8-form C_8 for the case of a charge p D7-brane is given by

$$C_{01\cdots 7} = \frac{1}{\operatorname{Im} \tau} + k \,. \tag{3.7.52}$$

At the level of the solution the electric dual of χ', i.e. the 8-form $q_{\alpha\beta} A^{\alpha\beta}_{01\cdots 7}$, is equal to the scalar T.

In general the function f, as indicated in table 3.6.1 on page 68, transforms under $SL(2, \mathbb{R})$. Since, however, use has been made of the relation between $q_{\alpha\beta}$ and p, q, r, eqs. (1.2.7) and (1.3.38), implying that q and p are both always positive, the explicit form of f, eq. (3.7.38), transforms under $PSL(2, \mathbb{R})$.

3.8 Mass and the BPS equation

From the analysis of the previous section it is clear that a single 7-brane can never produce a metric as well as a function τ that is defined everywhere. The local geometry of a D7-brane and of a Q7-brane leads to a curvature singularity at a finite distance from the brane. This occurs at the value of the radial coordinate r for which $\mathrm{Im}\,\tau$ goes through zero.

The monodromy transformations of the functions τ and f have been obtained. They transform as $\tau \to e^{Q}\tau = \frac{a\tau+b}{c\tau+d}$ and $f \to (c\tau+d)f$ whenever τ crosses a Dirac string or equivalently a branch cut. The question is how to interpolate between the local solutions. This problem is essentially a problem of how to globally organize the positioning of the Dirac strings. When this has been determined the global transformation properties of τ and f are fixed and since they are analytic functions they are fixed uniquely by specifying their monodromy transformations.

Granted that globally well-defined solutions exist, the total mass for them can be computed. This will be the subject of this section. The explicit construction of the globally well-defined solutions will be discussed in section 3.10.

3.8.1 Mass of a 7-brane solution

There are two contributions the total mass of a 7-brane solution. The first contribution comes from the deficit angles at the locations of the Q7-branes (see subsection 3.7.4). The second contribution comes from the energy contained in the bulk field τ.

The action describing the coupling to the IIB supergravity axidilaton sector of n 7-branes is given by

$$\begin{aligned} S &= \int_{\mathcal{M}_{10}} d^{10}x \left(\star 1\, R - 2\star \hat{P}\wedge \hat{P}^*\right) \\ &\quad - \int_{\mathcal{M}_{10}} d^{10}x \sum_{j=1}^{n} \int_{\mathcal{M}_8^j} d^8\sigma_j\, \delta(x - X_j(\sigma_j)) q^j_{\alpha\beta} V_-^{\alpha} V_+^{\beta} \sqrt{-g^j_{(8)}}\,, \end{aligned} \quad (3.8.1)$$

with

$$\hat{P} = P + \frac{i}{2}\sum_{j=1}^{n} q^j_{\alpha\beta} V_+^{\alpha} V_+^{\beta} \star G_9^j\,. \quad (3.8.2)$$

The world-volume $\mathcal{M}_8^j$ of each 7-brane, carrying a charge $q^j_{\alpha\beta}$, is parameterized by σ_j^A and is located in target space at the point $X_j(\sigma_j)$. The world-volume metric is g^j_{AB} and the Dirac 8-brane stemming from the 7-brane is described by G_9^j.

Consider the 7-branes to be wrapped on a T^7 with radii $R^1, \cdots, R^7$ so they can be viewed as point-particles moving in a $1+2$-dimensional space-time. The total energy of a massive particle in $1+2$ dimensions is measured by the deficit angle at infinity via the formula [64]

$$m = \frac{1}{16\pi G_N^{(3)}} \int d^2x \sqrt{|\gamma|}\, R(\gamma)\,, \quad (3.8.3)$$

where $G_N^{(3)}$ is the (2+1)-dimensional Newton's constant, related to the 10-dimensional one by

$$G_N^{(3)} = \frac{G_N^{(10)}}{(2\pi)^7 R^1 \cdots R^7}\,, \tag{3.8.4}$$

and γ is the metric of the transverse space.

For static solutions in 2+1 dimensions one has $R(\gamma) = -T^0{}_0$, where $R(\gamma)$ is the Ricci scalar of the metric γ and where $T^0{}_0$ is the time-time component of the energy-momentum tensor. Hence the energy is given by

$$m = \frac{1}{16\pi G_N^{(3)}} \int d^2x \sqrt{|\gamma|}\, R(\gamma) = -\frac{1}{16\pi G_N^{(3)}} \int \frac{i}{2} dz \wedge d\bar{z} \sqrt{|\gamma|} T^0{}_0\,. \tag{3.8.5}$$

Using action (3.8.1) it can be seen that

$$T^0{}_0 = -\frac{1}{\sqrt{|\gamma|}} \frac{1}{(\mathrm{Im}\tau)^2} \partial\tau\bar{\partial}\bar{\tau} - \sum_j \frac{1}{\sqrt{|\gamma|}} \delta(z - z_j, \bar{z} - \bar{z}_j) \frac{1}{\mathrm{Im}\tau} \left(p + q|\tau|^2 + r\mathrm{Re}\,\tau\right)\,, \tag{3.8.6}$$

where j labels the points z_j where the particles are located, the mass m can be written as

$$m = \frac{1}{16\pi G_N^{(3)}} \left(\int \frac{i}{2} dz \wedge d\bar{z} \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\mathrm{Im}\tau)^2} + 2 \sum_j \delta_j \right)\,, \tag{3.8.7}$$

where δ_j is the deficit angle at the location of the jth particle at the point z_j. In obtaining eq. (3.8.7) use has been made of eqs. (3.7.21), (3.7.23) and (3.7.49).

3.8.2 BPS formula

In this subsection a BPS equation for 7-brane solutions relating the energy m to a suitably defined $U(1)$ charge will be constructed.

The asymptotic region of the transverse space, the region $|z| \to \infty$, corresponds to a single point on the Riemann sphere, e.g. the point $z = \infty$. The location of infinity is arbitrary. One could have chosen to place it at any other point, z_0 say, of the Riemann sphere. Generally speaking a point z_0 on the Riemann sphere is mapped to an asymptotic region of the transverse space when the physical distance from z_0 to any other point z diverges as $|z - z_0|^{1-4G_N^{(3)}m}$ while $m < 1/4G_N^{(3)}$ (that is for non-compact transverse spaces) [64].

Eqs. (3.7.23) and (3.7.49) show that the function f can be used to compute deficit angles. The total mass of the solution is measured by the deficit angle at infinity. The energy (3.8.7) can thus be alternatively computed as follows

$$m = \frac{1}{8\pi G_N^{(3)}} \mathrm{Im} \oint_{z=\infty} (\log f)' dz = \frac{\delta_\infty}{8\pi G_N^{(3)}}\,, \tag{3.8.8}$$

where the contour integral encircles the point $z = \infty$ (in a clockwise direction) and δ_∞ is the deficit angle at infinity. Equating expressions (3.10.11) and (3.8.8) gives

$$\int_{TS} \frac{i}{2} dz \wedge d\bar{z} \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\mathrm{Im}\tau)^2} = -2 \sum_j \delta_j + 2\delta_\infty \,. \tag{3.8.9}$$

The subscript 'TS' below the integral sign means transverse space. The left hand side of eq. (3.8.9) can be written as an integral over the 2-form dQ with Q the 1-form given in (1.3.100). This implies that the deficit angles at the locations of the 7-branes in the transverse space, including the deficit angle at infinity, can also be computed via line integrals of the 1-form Q, the $U(1)$ connection of the coset $SL(2,\mathbb{R})/SO(2)$, around the locations of the 7-branes or infinity in transverse space

$$\delta = \frac{1}{2} \oint Q \,. \tag{3.8.10}$$

This expression can be taken as the definition of $U(1)$ charge (see [65]). Applying this formula to compute the $U(1)$ charge at infinity, i.e. $\oint_{z=\infty} Q$ going around $z = \infty$ in a clockwise direction, becomes a relation between the mass and the $U(1)$ charge with the characteristic form of a (saturated) BPS bound,

$$16\pi G_N^{(3)}\, m = \oint_{z=\infty} Q \,. \tag{3.8.11}$$

As discussed in [65], all the solutions of the system under consideration that have the asymptotic behavior allowing to define mass and $U(1)$ charge are automatically supersymmetric and such that the mass and charge satisfy (3.8.10). In other words, there are no 7-brane solutions for which the mass exceeds the bound (3.8.10). This excludes, for example, 7-brane solutions with horizons.

3.8.3 Dividing out by $SL(2,\mathbb{Z})$

The total mass of a 7-brane solution will be finite provided the integral in expression (3.8.7) is finite. The integral in (3.8.7) is the pull-back of an integration over the moduli space, i.e. the set of inequivalent values for τ. If one considers the classical moduli space,

$$\frac{PSL(2,\mathbb{R})}{SO(2)} \,, \tag{3.8.12}$$

then the energy (3.8.7) will be infinite. It is necessary to divide out (3.8.12) by discrete subgroups of $PSL(2,\mathbb{R})$ in order to make the integral finite. The largest possible discrete subgroup of $PSL(2,\mathbb{R})$ is $PSL(2,\mathbb{Z})$. Subgroups of $PSL(2,\mathbb{Z})$ can also be used as was shown explicitly in [16]. In this thesis the possibility to work with subgroups of $PSL(2,\mathbb{Z})$ will not be discussed. Thus, τ and $PSL(2,\mathbb{Z})$ transformations of τ must be considered physically equivalent.

The metric (3.6.3) is by construction a singlet under $SL(2,\mathbb{Z})$. The Killing spinor ϵ, eq. (3.6.6), transforms under local $U(1)$ transformations and can therefore naturally exist everywhere on a 7-brane background. The pair (τ, f) transforms under $SL(2,\mathbb{Z})$ and values of (τ, f) and $SL(2,\mathbb{Z})$ transforms of (τ, f) are equivalent. This means that the IIB supergravity theory on a background of 7-branes is divided out by $SL(2,\mathbb{Z})$. The 2-forms of the IIB theory do not transform under a local $U(1)$, they transform as a doublet under $SL(2,\mathbb{Z})$ and thus have no way of surviving the truncation of dividing the IIB theory out by $SL(2,\mathbb{Z})$. The 6-forms that are dual to the 2-forms suffer the same fate as the 2-forms.

Perhaps, more unexpectedly, also the 8-forms are truncated from the spectrum of fields of IIB supergravity. This is not in contradiction with the fact that one can still have nontrivial axidilaton fields. It means that after dividing out by $SL(2,\mathbb{Z})$ the axidilaton field cannot be dualized into an 8-form anymore. Another way to see that 8-forms must drop out, is by noting that the three 8-forms transform in the adjoint of $SL(2,\mathbb{Z})$ (see eq. (1.3.53)) and hence by declaring the three 8-forms and all their $SL(2,\mathbb{Z})$ transformed versions equivalent necessarily forces them to be zero. It follows from this that there is no electric coupling description possible for systems with multiple 7-branes. They form inherently magnetic systems.

3.8.4 What about $\det Q < 0$?

The $SL(2,\mathbb{R})$ duality group has three subgroups: $\mathbb{R}$, $SO(1,1)$ and $SO(2)$. The transformations e^Q with $\det Q = 0$, $\det Q < 0$ and $\det Q > 0$ belong to these respective subgroups. In this subsection it will be argued that there are no 7-branes that correspond to the $SO(1,1)$ subgroup. One could, in principle include the case $\det Q < 0$ in the discussion of 7-brane solutions. The solution for the local form for τ whose monodromy around a point $z = z_0$ is of the form $\tau \to e^Q\tau$ with $\det Q < 0$ is given by

$$\left(\frac{\tau - \tau_0^+}{\tau - \tau_0^-}\right)^{\frac{i\pi}{\sqrt{-\det Q}}} = z - z_0\,, \tag{3.8.13}$$

where

$$\tau_0^{\pm} = -\frac{r}{2q} \pm \frac{1}{q}\sqrt{-\det Q}\,, \tag{3.8.14}$$

which is such that $e^Q\tau_0^{\pm} = \tau_0^{\pm}$.

There are two problems with this possibility. The first problem is that the solution (3.8.13) is ill-defined at the point $z = z_0$. It is not possible to add a source term to the IIB action for the case $\det Q < 0$. One cannot, for example, substitute the solution (3.8.13) into the right hand side of the sourced Einstein equation, eq. (3.7.21). The second and perhaps more severe problem is that when one considers the quantum moduli space

$$\frac{PSL(2,\mathbb{R})}{SO(2)\times PSL(2,\mathbb{Z})} \tag{3.8.15}$$

none of the orbifold points corresponds to $\tau_0^{\pm}$. It can be shown that after quantization from $SL(2,\mathbb{R})$ to $SL(2,\mathbb{Z})$ the points $\tau_0^{\pm}$ are irrational points on the real line $\mathrm{Im}\,\tau = 0$ and such points lie outside the quantum moduli space. There is therefore no way that one can ever construct a globally well-defined solution in which a $\det Q < 0$ 7-brane occurs[8].

3.9 The quantum moduli space

If one assumes that two τ's differing by a $PSL(2,\mathbb{Z})$ transformation are equivalent, then τ takes values in the following set

$$\{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}/PSL(2,\mathbb{Z})\,. \tag{3.9.1}$$

The equivalence relation,

$$\tau \sim \tau' \quad \text{where} \quad \tau' = \frac{a\tau+b}{c\tau+d} \quad \text{with} \quad a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad-bc=1\,, \tag{3.9.2}$$

defines a space, an orbifold, that can be parameterized by the following region in the upper half-plane [68],

$$F = \{\tau \in \mathbb{C} \mid -\tfrac{1}{2} \leq \mathrm{Re}\,\tau < \tfrac{1}{2},\ \ |\tau| > 1,\ \ \text{circle segment } |\tau| = 1 \text{ from } \rho \text{ to } i\}\,,$$

where $\rho = e^{2\pi i/3}$. This set is called the fundamental domain of the modular group $PSL(2,\mathbb{Z})$ and will be denoted by F. It will also be referred to as the type IIB quantum moduli space. This region is depicted in figure 3.9.1. The orbifold points are $i\infty$, i and ρ. The circle segments from ρ to i and from i to $\rho+1$ are identified under the transformation $S(\tau) = -\frac{1}{\tau}$. The lines $\mathrm{Re}\,\tau = \pm\frac{1}{2}$ above the points ρ and $\rho+1$ are identified under the transformation $T(\tau) = \tau + 1$. In matrix notation

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,. \tag{3.9.3}$$

Any element of $PSL(2,\mathbb{Z})$ can be written as some product of S and T matrices [68].

[8] In the context of F-theory on K3 this statement can be refined by saying that it is not possible to force F-theory 7-branes to coincide at a point z_0, say, such that the monodromy around z_0 is of the form e^Q with $\det Q < 0$. There do exist F-theory 7-brane configurations that cannot be collapsed to one point around which the monodromy is of the type e^Q with $\det Q < 0$ [66, 67].

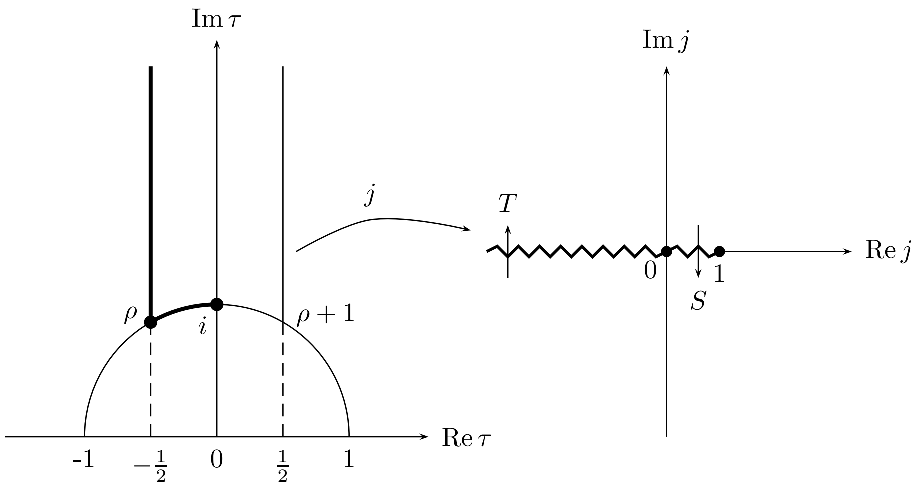

Figure 3.9.1: The fundamental domain and the mapping properties of j. The circle segments from ρ to i and from i to $\rho+1$ are identified as well as the lines $\mathrm{Re}\,\tau = \pm\frac{1}{2}$ above the points ρ and $\rho+1$.

Klein's modular function j maps F onto the Riemann sphere. It maps the line $\mathrm{Re}\,\tau = -\frac{1}{2}$ above the point $\tau = \rho$ onto the negative real axis, $(-\infty, 0)$, and the circle segment from $\tau = \rho$ to $\tau = i$ along $|\tau| = 1$ onto the interval $(0, 1)$. The points ρ and i are mapped to the points 0 and 1, respectively. The orbifold point at $\tau = i\infty$ gets mapped to the point ∞. Further, all points that are on the left/right of the line $\mathrm{Re}\,\tau = 0$ get mapped to points that lie above/below the real axis. The line $\mathrm{Re}\,\tau = 0$ itself maps to $(1, \infty)$ on the positive real axis. This mapping is conformal for all points strictly inside F. Further, the j-function is modular invariant, i.e. $j\left(\frac{a\tau+b}{c\tau+d}\right) = j(\tau)$. The inverse function j^{-1} maps the complex plane onto F.

The function $j(\tau)$ for $\mathrm{Im}\,\tau \to \infty$, i.e. near $\tau_0 = i\infty$, behaves as

$$j(\tau) \sim e^{-2\pi i\tau} \,. \tag{3.9.4}$$

Near $\tau_0 = i$ the behavior of $j(\tau)$ is

$$j(\tau) = 1 + c\left(\frac{\tau - i}{\tau + i}\right)^2 + \cdots \,, \tag{3.9.5}$$

with $c \neq 0$, that is $j(i) - 1$ is a second order zero point. Similarly, it is shown that

τ_0	(p,q,r)	$\pi/\sqrt{\det Q}$	$SL(2,\mathbb{Z})$
$i\infty$	$(1,0,0)$	∞	T
i	$(\frac{\pi}{2},\frac{\pi}{2},0)$	2	S
ρ	$(\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}},\frac{2\pi}{3\sqrt{3}})$	3	$T^{-1}S$

Table 3.9.1: Properties of the orbifold points $\tau_0 = i\infty, i, \rho$. The value ∞ for $\pi/\sqrt{\det Q}$ in the case $\tau_0 = i\infty$ means that this is a point of infinite order, i.e. there is no power of T such that it equals the identity. The last column indicates the transformations e^Q of which τ_0 is a fixed point. The entries of Q, the numbers p,q,r, given in the second column, are the smallest possible numbers such that τ_0 is a fixed point of e^Q.

near the point $\tau_0 = \rho$, which is a third order zero point, $j(\tau)$ behaves as

$$j(\tau) = c\left(\frac{\tau-\rho}{\tau-\bar{\rho}}\right)^3 + \cdots\,, \tag{3.9.6}$$

with $c \neq 0$. By the modular invariance of the j-function the expansions (3.9.4), (3.9.5) and (3.9.6) also provide expansions of $j(\Lambda\tau)$ around $\Lambda i\infty$, Λi and $\Lambda\rho$, respectively.

The orbifold points $\tau_0 = i\infty, i, \rho$ are fixed points of, respectively, T, S and $T^{-1}S$. By writing these transformations as e^Q with Q as given in (1.2.7) the numbers p,q,r that correspond to τ_0 can be obtained. Some data regarding the orbifold points τ_0 is summarized in table 3.9.1.

The branch cut of the function j^{-1} runs from $-\infty$ to 1 over the real axis. The branch cut is depicted by the zigzag lines of figure 3.9.1. The behavior of j^{-1} when crossing the branch cut is indicated by the arrows in figure 3.9.1. Crossing the branch cut between $-\infty$ and 0 from a point below the branch cut to a point above the branch cut corresponds to a T monodromy. Crossing the branch between 0 and 1 corresponds to an S monodromy. If one encircles the point $-\infty$ in the complex j-plane then the monodromy is T while when one encircles the point 1 the monodromy is S. Encircling the point 0 either gives $T^{-1}S$ when the base point lies below the real axis or ST^{-1} when the base point lies above the real axis. For the monodromy $ST^{-1} = S(T^{-1}S)S^{-1}$ the fixed point is not ρ but $\rho + 1 = S\rho$. One should consider the points $i\infty$, i and ρ merely as representative elements of the set of points that are fixed points under $\Lambda T\Lambda^{-1}$, $\Lambda S\Lambda^{-1}$ and $\Lambda T^{-1}S\Lambda^{-1}$, respectively, for arbitrary $\Lambda \in SL(2,\mathbb{Z})$, i.e. $i\infty$, i and ρ represent fixed point sets of the $SL(2,\mathbb{Z})$ conjugacy classes T, S and $T^{-1}S$. The $SL(2,\mathbb{Z})$ conjugacy classes are for $\det Q = 0$ given by

$\pm T^n$ (with $n = 0, 1, 2, \ldots$), while for $\det Q > 0$ they are given by $\pm S$, $\pm T^{-1}S$ and $\pm(T^{-1}S)^2$.

3.10 Constructing globally well-defined 7-brane solutions

This section discusses how to construct globally well-defined solutions. The focus will be on solutions with τ monodromy group $PSL(2, \mathbb{Z})$. The group $PSL(2, \mathbb{Z})$ is generated by T and S, defined in section 3.9. To show that one can also work with subgroups of $PSL(2, \mathbb{Z})$ the construction for the monodromy group $\Gamma_0(2)$ whose generators are T and ST^2S was included in [16]. Having chosen a monodromy group one can specify the functions τ and f by choosing them to transform as $\tau \to e^Q\tau = \frac{a\tau+b}{c\tau+d}$ and $f \to (c\tau + d)f$ whenever τ crosses a Dirac string or equivalently a branch cut.

3.10.1 The function τ

Since τ and $\Lambda\tau$ are identified there must exist a function, $j(\tau)$, that is monodromy neutral, i.e. $j(\tau)$ must be an automorphic function of the monodromy group,

$$j(\Lambda\tau) = j(\tau)\,, \tag{3.10.1}$$

where Λ is any element of the monodromy group. The local expansions of the function j around the fixed points τ_0 of Λ must reduce to the expressions (3.7.18) and (3.7.19).

A region of the complex upper half plane containing values of τ that are inequivalent under the monodromy group and that are related to all the points in the upper half plane is a fundamental domain of the monodromy group. For the monodromy group $PSL(2, \mathbb{Z})$ the moduli space (the space of inequivalent values of τ) is

$$\frac{PSL(2, \mathbb{R})}{SO(2) \cdot PSL(2, \mathbb{Z})}\,. \tag{3.10.2}$$

The space 3.10.2 is referred to as the quantum moduli space as this is the conjectured moduli space of the full quantum IIB theory [14]. Properties of this space and of the modular function $j(\tau)$, Klein's modular j-function, can be found in section 3.9. The space (3.10.2) is an orbifold whose orbifold points $i\infty$, i and ρ form representative elements of fixed point sets that are invariant under the $SL(2, \mathbb{Z})$ conjugacy classes formed by taking all the conjugated elements of the transformations T, S and $T^{-1}S$, respectively.

The function $j(\tau)$ maps the fundamental domain onto the Riemann sphere $\hat{\mathbb{C}}$ in a one-to-one fashion, so that the inverse function j^{-1} exists. The function $\tau(z)$ is then given by $\tau(z) = j^{-1}(z)$. To describe systems with many 7-branes an additional map from the Riemann sphere to N copies of itself is introduced. This map is given by the N to 1 automorphism $z \to P(z)/Q(z)$ for polynomials $P(z)$ and $Q(z)$. For $N = 1$ these polynomials are fixed by the requirement that the three orbifold points of the fundamental domain are mapped to three given points in the z-plane. For $N > 1$ the polynomials $P(z)$ and $Q(z)$ are fixed by the further requirement of how many branes are placed at the three points $z_{i\infty}, z_\rho$ and z_i where the subscript indicates the value of τ at that point (up to an $SL(2,\mathbb{Z})$ transformation). In the next section the explicit realizations of $P(z)$ and $Q(z)$ will be given. Summarizing, one has the sequence of maps

$$z \quad \overset{N\to 1}{\longrightarrow} \quad \frac{P(z)}{Q(z)} \quad \overset{j^{-1}}{\longrightarrow} \quad \tau(z) = j^{-1}\left(\frac{P(z)}{Q(z)}\right) . \qquad (3.10.3)$$

The inverse mapping j^{-1} which maps from the Riemann sphere $\hat{\mathbb{C}}$ onto the fundamental domain has branch cuts connecting the points $z_{i\infty}$ to z_ρ and z_ρ to z_i.

monodromy group	generators	orbifold points	area	modular function	$F(\tau)$
$PSL(2,\mathbb{Z})$	T, S	$i\infty,\ \rho,\ i$	$\pi/3$	$j(\tau)$	$\eta^2(\tau)$

Table 3.10.1: Properties of the groups $PSL(2,\mathbb{Z})$ and realizations of the functions τ and F. The role of the function $F(\tau)$ is explained in subsection 3.10.2. The column headed 'area' refers to the area of the fundamental domain of $PSL(2,\mathbb{Z})$.

3.10.2 The function f

The function f must be such that when crossing a branch cut of τ with τ transforming as $\tau \to e^Q\tau = \frac{a\tau+b}{c\tau+d}$ it transforms as $f \to (c\tau+d)f$. In table 3.10.2 the monodromies of τ and f are presented. The monodromies are measured when going around the points $z_{i\infty}$, z_ρ, z_i in a counter clockwise direction.

The function $f(z)$ can be written in the form

$$f(z) = F(\tau)h(z)\,, \qquad (3.10.4)$$

location	$SL(2,\mathbb{Z})$	(p,q,r)	monodromy f	$\sqrt{\det Q}$
$z_{i\infty}$	T	$(1,0,0)$	$f \to f$	0
z_i	S	$(\pi/2, \pi/2, 0)$	$f \to -\tau f$	$\pi/2$
z_ρ	$T^{-1}S$	$(\frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}}, \frac{2\pi}{3\sqrt{3}})$	$f \to -\tau f$	$\pi/3$

Table 3.10.2: The monodromies of τ and f, the p, q, r values and the value for $\sqrt{\det Q}$ for $\tau_0 = i\infty$, ρ, i.

where $F(\tau)$ will be a cusp form that transforms under $PSL(2,\mathbb{Z})$ as

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = e^{i\beta(a,b,c,d,\tau)}(c\tau+d)F(\tau)\,, \tag{3.10.5}$$

and $h(z)$ is a function of z that is chosen such that when going around a 7-brane it transforms as

$$h(z) \to e^{-i\beta(a,b,c,d,\tau(z))+\pi i\,k+2\pi i\,l}h(z)\,, \tag{3.10.6}$$

with $k, l \in \mathbb{Z}$. The term πk in (3.10.6) determines whether f transforms under $PSL(2,\mathbb{Z})$ (k = even) or under $-PSL(2,\mathbb{Z})$ (k = odd). The term $2\pi l$ in (3.10.6) is not seen by f, but is visible from the point of view of the Killing spinor ϵ, (3.6.6), because it transforms, a.o. with a factor $e^{-\pi i\,l} = (-1)^l$. Hence, the even or oddness of l relates to spinstructure.

The Dedekind eta function squared, $\eta^2(\tau)$, transforms as (3.10.5) for specific values of the phase β. The Dedekind eta function is a cusp form, i.e. it goes to zero as $\tau \to i\infty$ and is nonzero for values of τ that satisfy $0 < \mathrm{Im}\,\tau < \infty$. For $\mathrm{Im}\,\tau \to \infty$ one has

$$\eta(\tau) \to e^{2\pi i\tau/24}\,. \tag{3.10.7}$$

The transformations of the Dedekind η-function under the $PSL(2,\mathbb{Z})$-transformations T, S and $T^{-1}S$ are given by

$$T \quad : \quad \eta^2(\tau+1) = e^{\pi i/6}\eta^2(\tau)\,, \tag{3.10.8}$$

$$S \quad : \quad \eta^2\left(-\frac{1}{\tau}\right) = e^{-\pi i/2}\tau\,\eta^2(\tau)\,, \tag{3.10.9}$$

$$T^{-1}S \quad : \quad \eta^2\left(-\frac{\tau+1}{\tau}\right) = e^{-2\pi i/3}\tau\,\eta^2(\tau)\,. \tag{3.10.10}$$

Using the monodromies of this cusp form and the required monodromies of $f(z)$, given in table 3.10.2, the explicit form of the function $h(z)$ can be derived. In subsections 3.10.3 and 3.10.4 explicit expressions for $h(z)$ will be given.

For general N, defined in eq. (3.10.3), the mass formula (3.8.7) becomes

$$m = \frac{1}{16\pi G_N^{(3)}}\left(N \times \text{area fundamental domain} + \; 2\sum_j \delta_j\right) , \qquad (3.10.11)$$

where the area is measured with the area element

$$\frac{i}{2}\frac{d\tau \wedge d\bar{\tau}}{(\mathrm{Im}\,\tau)^2} , \qquad (3.10.12)$$

and is given in table 3.10.1.

This completes the general discussion of the construction of globally well-defined 7-brane solutions. Explicit examples will be discussed in the next subsection.

3.10.3 $N = 1$ solutions

Consider the following choice for $\tau(z)$

$$\text{I.a}: \quad j(\tau) = \frac{z_i - z_{i\infty}}{z_i - z_\rho}\frac{z - z_\rho}{z - z_{i\infty}} . \qquad (3.10.13)$$

At the points $z_{i\infty}$, z_i, z_ρ the j-functions takes the values ∞, 1 and 0, respectively. The monodromies of τ around the points $z_{i\infty}$, z_i and z_ρ can be found from the local expressions of the j-function as given in eqs. (3.9.4), (3.9.5) and (3.9.6). Since $j(\tau)$ is $PSL(2,\mathbb{Z})$ invariant the local monodromies of τ are determined up to $PSL(2,\mathbb{Z})$ transformations. Working with representative elements of $SL(2,\mathbb{Z})$ conjugacy classes one can take the monodromies around $z_{i\infty}$, z_i and z_ρ to be the respective transformations T, S and $T^{-1}S$ whose fixed points are $i\infty$, i and ρ, respectively. The monodromies of τ are elements of the group $PSL(2,\mathbb{Z})$. The function f transforms under $SL(2,\mathbb{Z})$ and therefore sees the difference between, e.g. an S or a $-S$ transformation. In this section the focus will be on those $SL(2,\mathbb{Z})$ transformations that when written as e^Q have $q > 0$. This guarantees that all contributions to the energy (3.8.7) are positive. In section 3.10.4 the analysis will be extended to the inclusion of objects with a negative tension. It will be shown that such objects play an important role in the construction of the so-called F-theory solutions.

Now that τ has been given in (3.10.13) and the $SL(2,\mathbb{Z})$ monodromies around $z_{i\infty}$, z_i and z_ρ have been chosen to be T, S and $T^{-1}S$ one is in a position to construct the function f. The required transformations for f around the points $z_{i\infty}$, z_i and z_ρ are given in table 3.10.2. The general form of f has been given in eq. (3.10.4). Using that $\eta^2(\tau)$ transforms as in eqs. (3.10.8), (3.10.9) and (3.10.10), it can be seen that f must be given by

$$\text{I.a}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_i)^{-1/4}(z - z_\rho)^{-1/6}\,. \tag{3.10.14}$$

For example, going around $z_{i\infty}$, i.e. sending $z - z_{i\infty} \to e^{2\pi i}(z - z_{i\infty})$ gives a factor $e^{\pi i/6}$ from the Dedekind eta function squared and a factor $e^{-\pi i/6}$ from the factor $(z - z_{i\infty})^{-1/12}$ appearing in (3.10.14), so that $f \to f$ as required. Going around z_i, i.e. sending $z - z_i \to e^{2\pi i}(z - z_i)$ gives $e^{-\pi i/2}\tau$ coming from η^2 and a factor $e^{-\pi i/2}$ coming from the factor $(z - z_i)^{-1/4}$ in (3.10.14), together this gives $-\tau$ as required. A similar calculation confirms the factor $(z - z_\rho)^{-1/6}$ in (3.10.14) for the point z_ρ. If one consider the function τ, eq. (3.10.13), near infinity, i.e. near $z = \infty$, it is found that the monodromy of τ around infinity is trivial. The functions f near $z = \infty$ behaves as $f \sim z^{-1/2}$. Going around $z = \infty$ is the same as going counter clockwise around $1/z$ near $z = 0$, i.e. $z \to e^{-2\pi i}z$ when going around $z = \infty$. Doing so leads to the monodromy $f \to -f$, so that the $SL(2,\mathbb{Z})$ monodromy around $z = \infty$ is equal to $-\mathbb{1}$.

The solution for τ and f, eqs. (3.10.13) and (3.10.14), can be pictorially represented as in figure 3.10.1. The branch cuts are those of the function j^{-1}, see figure 3.9.1 with the exception of the $-\mathbb{1}$ branch cut which is not a branch cut from the point of view of τ only.

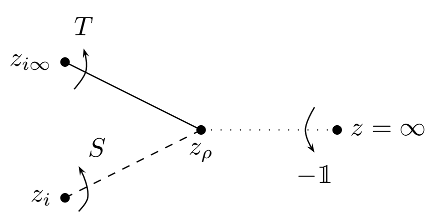

Figure 3.10.1: Pictorial representation of the solution I.a.

Consider next the following $N = 1$ choice for $\tau(z)$

$$\text{I.b}: \quad j(\tau) = \frac{z - z_\rho}{z - z_{i\infty}}\,. \tag{3.10.15}$$

This form for τ can be obtained from (3.10.13) by applying an $SL(2,\mathbb{C})$ transformation. The $SL(2,\mathbb{C})$ transformations are reparametrizations of the Riemann sphere.

The monodromies of τ and f are preserved by such transformations. Hence, in order to construct a second $N = 1$ solution, independent of the I.a solution, eqs. (3.10.13) and (3.10.14), the function f must be chosen such that its monodromies cannot be obtained from (3.10.14) via the above-mentioned $SL(2,\mathbb{C})$ transformation. The monodromies around $z_{i\infty}$ and z_ρ are taken to be the $SL(2,\mathbb{Z})$ transformations T and $T^{-1}S$, respectively, just as is the case in the I.a solution. However, taking the monodromy around $z_i = \infty$ to be $-S$ leads to a new solution. The function f is given by

$$\text{I.b}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_\rho)^{-1/6}\,. \tag{3.10.16}$$

Going around $z_i = \infty$ the Dedekind eta function squared transforms with a factor $e^{-\pi i/2}\tau$ while the factor $(z - z_{i\infty})^{-1/12}(z - z_\rho)^{-1/6}$ behaves as $z^{-1/4}$ leading to an additional factor of $e^{\pi i/2}$ when going around $z_i = \infty$, so that in total f transforms as $f \to \tau f$ implying that the $SL(2,\mathbb{Z})$ monodromy is $-S$. This can be verified by taking S as defined in (3.9.3) from which one reads off that $c = 1$ and $d = 0$ in the parametrization (3.7.27), so that indeed $f \to \tau f$. The I.b solution can be pictorially represented as in figure 3.10.2.

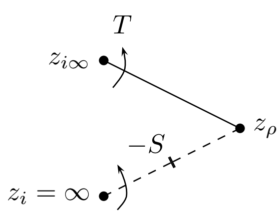

Figure 3.10.2: Pictorial representation of the solution I.b.

In a similar fashion two more independent solutions can be constructed for the $N = 1$ case these will be referred to as the I.c and I.d solutions and are given by

$$\text{I.c}: \quad j(\tau) = \frac{z_i - z_{i\infty}}{z - z_{i\infty}}\,, \tag{3.10.17}$$

$$\text{I.c}: \quad f(\tau) = \eta^2(\tau)(z - z_{i\infty})^{-1/12}(z - z_i)^{-1/4}\,, \tag{3.10.18}$$

$$\text{I.d}: \quad j(\tau) = \frac{z - z_\rho}{z_i - z_\rho}\,, \tag{3.10.19}$$

$$\text{I.d}: \quad f(\tau) = \eta^2(\tau)(z - z_i)^{-1/4}(z - z_\rho)^{-1/6}\,. \tag{3.10.20}$$

For the I.c solution the monodromy around $z_\rho = \infty$ is $-T^{-1}S$ and for the I.d solution the monodromy around $z_{i\infty} = \infty$ is $-T$. These solutions have the pictorial representations given in figures 3.10.3 and 3.10.4.

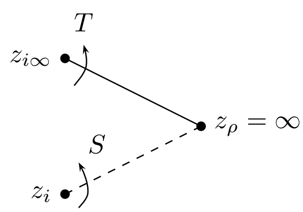

Figure 3.10.3: Pictorial representation of the solution I.c.

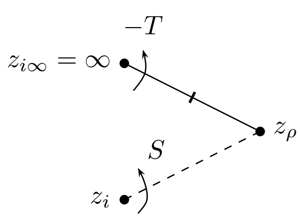

Figure 3.10.4: Pictorial representation of the solution I.d.

In drawing the figures representing the I.a to I.d solutions a certain notation has been introduced. The rules for drawing such figures are summarized in figure 3.10.5.

The I.a solution is one containing three 7-branes whereas the I.b, I.c and I.d solutions all contain two 7-branes. The I.b and I.c solutions both contain one D7-brane and one particular Q7-brane. The I.d solution contains no D7-branes, but two different Q7-branes.

The τ functions for the solutions I.a to I.d are related via $SL(2,\mathbb{C})$ transformations. This, however, is not true for the functions f. The monodromies of f differ from case

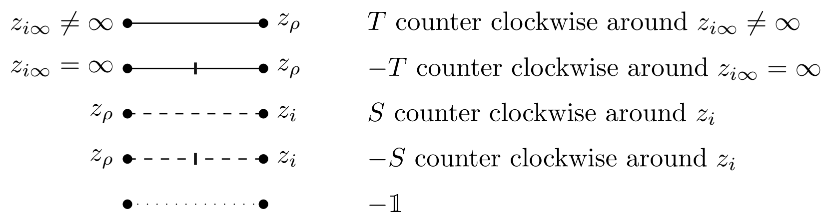

Figure 3.10.5: Notation for the pictorial representations of $SL(2,\mathbb{Z})$ monodromies of the pair (τ, f). Each point $z_{i\infty} \neq \infty$ is the endpoint of a T branch cut. A $-T$ monodromy is only allowed around the point $z_{i\infty} = \infty$. Points z_i (equal to ∞ or not) are the endpoints of either an S or a $-S$ monodromy. Points z_ρ (equal to ∞ or not) are the endpoints of two branch cuts one with T or $-T$ and one with S or $-S$ monodromy. A $-\mathbb{1}$ branch cut can go from any point $z_{i\infty}, z_i, z_\rho$ or $z = \infty$ (where τ takes any value not equal to $i\infty, i, \rho$) to any other such point.

to case by essential minus signs.

The pictorial representation of the I.a to I.d solutions used the rules given in figure 3.10.5. These rules will also be used later when solutions with $N > 1$ are discussed.

In principle one could next consider solutions with $N > 1$, but for this purpose there is still one piece of information missing. The point is that the primary information comes from τ which, from the $SL(2,\mathbb{Z})$ point of view, is the most insensitive function. For example when the τ monodromy is trivial the function f can transform under $\pm\mathbb{1}$. When it transforms as $-\mathbb{1}$ this could be because it so happens that the monodromy is measured around a point where two S branch cuts meet. When the monodromy is $+\mathbb{1}$ this could be because the monodromy is measured around a point where a S and a $-S$ branch cut meet. There are no values of $p, q \geq 0$ such that e^Q equals $-S$. Therefore, this is not a Q7-brane. If the monodromy of f is $+\mathbb{1}$ branch cuts are still needed and this is due to the essential path dependence of monodromies. Even though around a point where an S and a $-S$ branch cut meet the monodromy is trivial, the S and $-S$ branch cuts may have a role to play in the monodromies around other points.

It will be shown in the next subsection that points with $-S$ monodromy and also those with $-T^{-1}S$ monodromy correspond to objects with a negative tension. In order not to have explicit negative energy sources in the solution they will always be taken coincident with positive tension Q7-branes so that the local deficit angles are always non-negative. This is the most conservative viewpoint that one can adopt. Less conservative would be to allow for the presence of explicit negative energy sources in the solution giving rise to negative deficit angles. Although this in principle can be done and is not ruled out by supersymmetry it will not be considered here.

3.10.4 Including negative tension objects

When writing S or $T^{-1}S$ as e^Q in both cases one finds that $q > 0$ (implying $p > 0$ since $\det Q = qp - r^2/4 > 0$). This means that it is possible to use the formulae derived in section 3.7. The reason is that the results are based on the relation between $q_{\alpha\beta}$ and p, q, r, a relation implying $p, q > 0$ as explained below eq. (1.3.38).

When writing $-S$ as e^Q one would find $p = q = -\frac{\pi}{2}$ and $r = 0$ while for $-T^{-1}S$ one would find $p = q = r = -\frac{4\pi}{3\sqrt{3}}$. This implies that for $-T^{-1}S$ one would have $\sqrt{\det Q} = \frac{2\pi}{3}$. This should be contrasted with the case $T^{-1}S$ for which $\sqrt{\det Q} = \frac{\pi}{3}$. The axidilaton field τ should not see the difference between $T^{-1}S$ or $-T^{-1}S$. It is clear that applying, for example, formula (3.7.19) to the case $-T^{-1}S$ with $\sqrt{\det Q} = \frac{2\pi}{3}$ would give the wrong answer. Hence, there is no source term interpretation for points with monodromies $-S$ or $-T^{-1}S$.

On the other hand it is not too difficult to construct solutions containing points z_i or z_ρ around which the monodromy is $-S$ or $-T^{-1}S$, respectively[9]. The local behavior for τ near z_i and z_ρ will be the same as for 7-branes with S and $T^{-1}S$ monodromy. The function f, does see the difference between, e.g. $T^{-1}S$ and $-T^{-1}S$. Under $T^{-1}S$ the function f transforms as $f \to -\tau f$, a transformation that is fully induced by the transformation of τ under $T^{-1}S$ as is shown in 3.7.3, while under $-T^{-1}S$ the function f transforms as $f \to \tau f$. The latter transformation of f cannot be interpreted as being fully generated by its $PSL(2,\mathbb{Z})$ part. In this case instead of inserting a factor $(z - z_\rho)^{-1/6}$ as is done for the $T^{-1}S$ case one inserts the factor $(z - z_\rho)^{-1/6}(z - z_\rho)^{1/2} = (z - z_\rho)^{1/3}$ into the function f whenever a $-T^{-1}S$ object is placed at z_ρ. Similarly, a factor $(z - z_i)^{-1/4}(z - z_i)^{1/2} = (z - z_i)^{1/4}$ is inserted for each $-S$ object placed at z_i. The fact that factors $(z - z_\rho)^{1/3}$ and $(z - z_i)^{1/4}$ have a positive power of 1/3 and 1/4, respectively, means that the metric at the location of the $-T^{-1}S$ and $-S$ objects has a negative deficit angle. Hence, these objects have a negative tension and contribute negatively to the total energy of the solution. As mentioned before, in order not to have explicit negative energy sources in the solution they will always be taken coincident with positive tension Q7-branes so that the local deficit angles are always positive.

To summarize, the behavior of τ is the same for a $-T^{-1}S$ or a $-S$ negative tension object as it is for a $T^{-1}S$ or an S (positive tension) Q7-brane. The behavior of f due

[9] Points $z_{i\infty}$ around which the monodromy is $-T$ will not be considered except when $z_{i\infty} = \infty$ in which case there is no object whose monodromy is $-T$, but only an overall monodromy that happens to be $-T$.

to the presence of a $-T^{-1}S$ or a $-S$ object is given by

$$-T^{-1}S: \quad f(\tau) = \eta^2(\tau)(z - z_\rho)^{1/3} \cdots, \tag{3.10.21}$$

$$-S: \quad f(\tau) = \eta^2(\tau)(z - z_i)^{1/4} \cdots. \tag{3.10.22}$$

The nature of the negative tension objects with $-S$ or $-T^{-1}S$ monodromy is not clear, but it will be shown that they have a role to play in the construction of, e.g. the F-theory solutions.

The results of this subsection can be used to derive the asymptotic form of the metric for the I.b, I.c and I.d solutions discussed in subsection 3.10.3. In these three cases the monodromy around $z = \infty$ is of the form $-e^Q$ where Q is the usual matrix with $q, p \geq 0$. The minus sign in the monodromy follows from a square root branch cut in the function f so that near $z = \infty$ one has for the function f

$$f_{-e^Q} = f_{e^Q}\, z^{-1/2}, \tag{3.10.23}$$

where f_{e^Q} is the form f takes when the monodromy is e^Q, which has been derived in subsection 3.7.3. For a T monodromy $f = 1$ and for an S or a $T^{-1}S$ monodromy f is given in terms of τ in eq. (3.7.32). Therefore one has

$$f_{-T} = z^{-1/2}, \tag{3.10.24}$$

$$f_{-S} = \frac{1}{(\tau - i)^{1/2}(\tau + i)^{1/2}} z^{-1/2}, \tag{3.10.25}$$

$$f_{-T^{-1}S} = \frac{\frac{1}{2}\sqrt{3}}{(\tau - \rho)^{1/2}(\tau - \bar{\rho})^{1/2}} z^{-1/2}, \tag{3.10.26}$$

in which the dependence of $\tau(z)$ near $z = \infty$ follows from expressions (3.10.15), (3.10.17) and (3.10.19) depending on which solution, I.b to I.d, one is considering. The asymptotic form for the metric then follows from combining this result for f with the asymptotic form for τ and eq. (3.6.3). It follows that for a $-T$ monodromy around $z = \infty$ the metric behaves for $|z| \to \infty$ as $\frac{\log|z|}{|z|}$ while for $-S$ and $-T^{-1}S$ the metric goes as $|z^{-1/4}|^2$ and $|z^{-1/3}|^2$, respectively.

The rules for representing the pair (τ, f) are given in figure 3.10.5. These rules follow from the $N = 1$ solutions of the previous subsection together with the negative tension objects of this subsection.

3.10.5 Seven-branes and $SL(2,\mathbb{Z})$ conjugacy classes

The $SL(2,\mathbb{Z})$ conjugacy classes are for $\det Q = 0$ given by $\pm T^n$ (with $n = 0, 1, 2, \ldots$), while for $\det Q > 0$ they are given by $\pm S$, $\pm T^{-1}S$ and $\pm(T^{-1}S)^2$ [66, 67]. These $SL(2,\mathbb{Z})$ conjugacy classes can be interpreted as follows. The conjugacy class T^n is formed by n coincident D7-branes. The conjugacy class $-T^n$ does not correspond to any object, but can arise as the monodromy measured around $z = \infty$ (see for example the I.d solution shown in figure 3.10.4). The conjugacy classes S and $T^{-1}S$ correspond to Q7-branes associated with the orbifold points $\tau_0 = i$ and $\tau_0 = \rho$, respectively. The $SL(2,\mathbb{Z})$ conjugacy classes $-S$ and $-T^{-1}S$ correspond to certain negative tension objects. The conjugacy class $(T^{-1}S)^2$ can be interpreted as formed out of two coincident $T^{-1}S$ Q7-branes. Finally, the conjugacy class $-(T^{-1}S)^2$ can be read as the result of putting a $T^{-1}S$ Q7-brane on top of a negative tension $-T^{-1}S$ object.

Instead of interpreting $-T^{-1}S$ as a negative tension object one could also say that it corresponds to a set of four coincident $T^{-1}S$ Q7-branes, based on the $SL(2,\mathbb{Z})$ identity $-T^{-1}S = (T^{-1}S)^4$. The two interpretations are not physically equivalent. For the case of four coincident $T^{-1}S$ Q7-branes one must insert a factor $(z - z_\rho)^{-4/6}$ into the function f. It follows that under $(T^{-1}S)^4$ the function f transforms as $f \to e^{-2\pi i}\tau f$ whereas under $-T^{-1}S$ was shown to transform as $f \to \tau f$. The difference between these two interpretations lies in the behavior of the Killing spinor ϵ, eq. (3.6.6). For $(T^{-1}S)^4$ and for $-T^{-1}S$ the transformation of ϵ differs a sign, i.e. the spinstructure for a $-T^{-1}S$ object differs from the spinstructure of four coincident $T^{-1}S$ Q7-branes. It is also clear that, unless there are negative tension objects, one cannot have more than eight $T^{-1}S$ Q7-branes in a solution (coincident or not) because they make up a total deficit angle of 4π as follows from eq. (3.8.7) (see also section 3.11). Similar statements apply to S Q7-branes. A $-S$ monodromy can also be read as coming from three coincident S Q7-branes, based on the $SL(2,\mathbb{Z})$ identity $-S = S^3$. Also in this case the difference between the two interpretations lies in the behavior of the Killing spinor. Further, unless there are negative tension objects, one cannot have more than six S Q7-branes in a solution (see section 3.11).

3.10.6 General solution with N and $\tau(z = \infty)$ arbitrary

For the $N = 1$ case four distinct solutions could be constructed. It will be clear that the number of distinct solutions grows fast as N becomes bigger. A considerable limitation would be to consider only those configurations for which the asymptotic value of τ can be arbitrary but not equal to $i\infty, i$ or ρ. In the $N = 1$ case there is only one solution that has this property, namely the $N = 1a$ solution. There is good physical motivation to make this restriction. The asymptotic value for τ is the coupling constant and when it can be arbitrary it is possible to consider the asymptotic regime as the perturbative starting point for some underlying theory. This could be ordinary type IIB superstring theory whose coupling constant is the asymptotic value of $(\mathrm{Im}\,\tau)^{-1}$ or it could be perturbative with respect to another coupling constant (see for example the hypothetical Q-string tension of section 3.12). Hence, from now on only those solutions for which τ at $z = \infty$ is a free parameter of the solution, will be considered.

The form of τ for arbitrary N and $\tau(z=\infty)$ is

$$j(\tau)=\frac{P_N}{P_N+Q_N}\,, \tag{3.10.27}$$

where P_N and Q_N are arbitrary polynomials of degree N. Writing P_N and Q_N as

$$P_N = c_P(z-z_\rho^1)\cdots(z-z_\rho^N)\,, \tag{3.10.28}$$
$$Q_N = c_Q(z-z_i^1)\cdots(z-z_i^N)\,, \tag{3.10.29}$$

with c_P and c_Q nonzero complex constants, the general form taken by the function f is

$$f(\tau)=\eta^2(\tau)\,(P_N+Q_N)^{-1/12}\,(z-z_\rho^1)^{-s_1}\cdots(z-z_\rho^N)^{-s_N}(z-z_i^1)^{-t_1}\cdots(z-z_i^N)^{-t_N}\,, \tag{3.10.30}$$

in which s_j and t_j, $j=1,\ldots,N$, are as given in table 3.10.3.

location	monodromy of ϵ	function h	(k,l)
z_i^j	S	$s_j=-1/4$	$(k=\text{even}, l=\text{even})$
z_i^j	$-S$	$s_j=1/4$	$(k=\text{odd}, l=\text{even})$
z_ρ^j	$T^{-1}S$	$t_j=-1/6$	$(k=\text{even}, l=\text{even})$
z_ρ^j	$-T^{-1}S$	$t_j=1/3$	$(k=\text{odd}, l=\text{even})$

Table 3.10.3: Form of the function h defined in eq. (3.10.4) for Q7-branes and negative tension objects. The integers k and l are defined in eq. (3.10.6).

Any other monodromy such as S^3 or $-(T^{-1}S)^4$ can be obtained by combining the monodromies given in table 3.10.3. As explained at the end of the previous subsection monodromies such as $-S$ or S^3 are inequivalent within the double cover of $SL(2,\mathbb{Z})$.

The positive values for s_j and t_j in (3.10.30) lead to negative deficit angles. In order not to have negative energy sources in the solution one must take sufficient Q7-branes coincident with the negative tension objects so that the deficit angle becomes positive. It is not necessary to include negative tension objects in order to write down fully well-defined solutions, however, they can be used to undo the presence of certain Q7-branes if desired.

3.11 F-theory and Q7-branes

The well-known 7-brane configurations of F-theory have the properties that the monodromy of τ close to the points z_i, z_ρ is the identity in $PSL(2,\mathbb{Z})$ and T around $z_{i\infty}$. Further, the function f has no poles or zeros, which means that there are no local sources of energy. The function f does have a zero at infinity and the order of the zero determines the energy through eq. (3.8.8). This implies that the monodromy of f around any point not being infinity is $f \to f$. The above conditions are satisfied if and only if τ and f are of the following form

$$j(\tau) = \frac{P^3}{P^3+Q^2}\,, \qquad (3.11.1)$$

$$f(\tau) = \eta^2(\tau)\left(P^3+Q^2\right)^{-1/12}\,, \qquad (3.11.2)$$

where P^3+Q^2 is a polynomial of order N whose zeros are the locations of the $\det Q = 0$ (p,q,r) 7-branes.

Solutions of this type exist whenever N can be divided by either 2 or 3. Thus they exist for $N = 2,3,4,6,8,9,10,12,24$. The deficit angle at infinity is given by $2\pi N/12$. For $N = 12$ this corresponds to a cylinder and for $N = 24$ to a 2-sphere. Values of $12 < N < 24$ do not correspond to a regular space. Requiring additionally that the value τ at infinity is arbitrary singles out $N = 6,12,24$ as the only possible solutions. Solutions with $N = 2,3,\ldots$ also exist but for these the asymptotic value of τ is equal to i or ρ and so for these solutions there does not exist a perturbative regime at infinity making these solutions inherently non-perturbative. The number of free complex parameters for the $N = 6,12,24$ cases is $3,8,18$, respectively. These numbers can be found by counting the number of free complex parameters in P and Q and subtracting four complex parameters that are associated to the $SL(2,\mathbb{C})$ coordinate freedom and to the scale symmetry $P \to \lambda^4 P$ and $Q \to \lambda^6 Q$ that leaves invariant τ and scales f with a factor λ^{-1}. The scaling of f implies that the transverse space metric scales with $|\lambda|^{-2}$ which is harmless in that the overall scale of the transverse space metric is a free real parameter. Further the scaling of f implies a shift of the overall phase of the Killing spinor that can be absorbed in a redefinition of the constant spinor ϵ_0 in (3.6.6).

The case $N = 6$ is the first instance in which solutions with only $\det Q = 0$ (p,q,r) 7-branes and an arbitrary value for τ at infinity are possible. The $N = 6$ case of eqs. (3.11.1) and (3.11.2) has the pictorial representation given in figure 3.11.1 using the rules given in figure 3.10.5.

As is clear from figure 3.11.1 the $SL(2,\mathbb{Z})$ monodromy around small loops encircling only z_i^1, z_i^2, z_i^3, z_ρ^1 or z_ρ^2 is $+\mathbb{1}$. At the points z_i^j with $j = 1,2,3$ there is an S Q7-brane coincident with a $-S$ negative tension object such that locally the mass and charge cancel. At the points z_ρ^j with $j = 1,2$ there are two $T^{-1}S$ Q7-branes

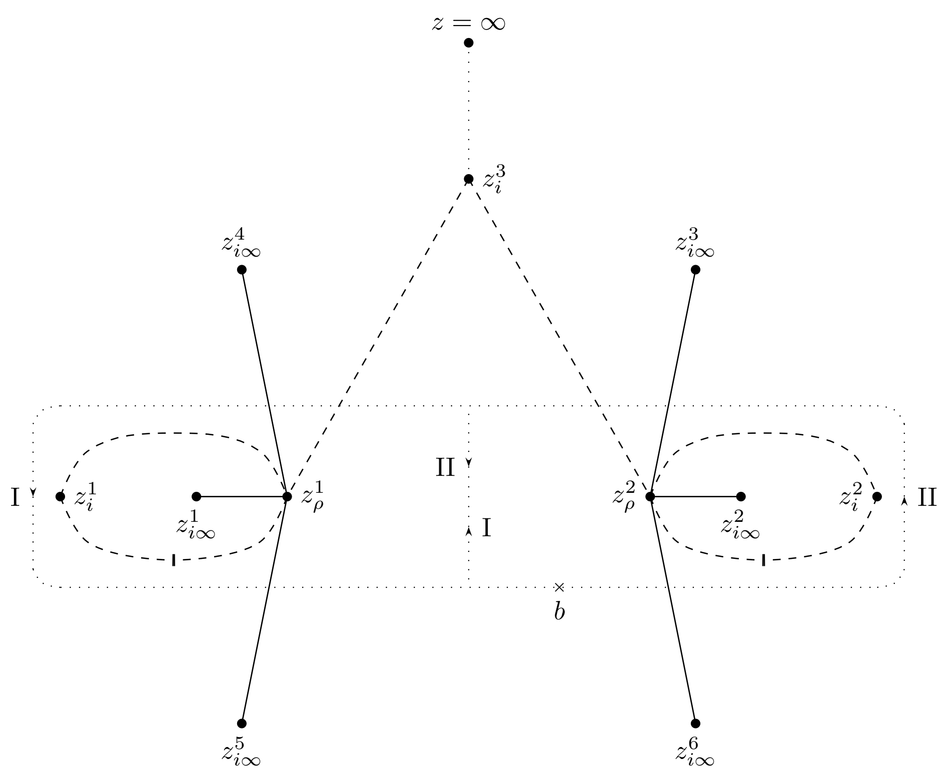

Figure 3.11.1: Pictorial representation of the F-theory $N = 6$ solution. The monodromy of the O7-plane and of the split O7-plane is measured with respect to the base point b. The O7-plane monodromy, $-T^{-4}$, is measured along I+II. The monodromy around II starting in b is ST^{-2}. To measure the monodromy around loop I starting in b one must first go around I+II and then around II in opposite direction. The monodromy measured going around loop I starting in b is T^2ST^{-4}.

coincident with one $-T^{-1}S$ negative tension object so that the masses and charges cancel.

Of the six zeros of $P^3 + Q^2$ two, the points $z_{i\infty}^1$ and $z_{i\infty}^2$, are non-perturbatively related to the point $z = \infty$ due to the fact that $z_{i\infty}^1$ and $z_{i\infty}^2$ are enclosed by S branch cuts. Suppose that the arbitrary value of τ at infinity is taken to be close to the point $\tau = i\infty$, so that the $(1, 0)$ string coupling constant $(\mathrm{Im}\,\tau)^{-1}$ is close to zero, then there exist loops that start at $z = \infty$, encircle any of the points $z^j_{i\infty}$ for $j = 3, 4, 5, 6$ once and return to $z = \infty$. Since these loops only cross a T branch cut, the 4-four 7-branes at $z^j_{i\infty}$ for $j = 3, 4, 5, 6$ can be considered to be $(1, 0)$ 7-branes, and they will have this interpretation as long as $\tau(z = \infty)$ is close to $i\infty$ and a restricted set of loops is considered. This solution can thus describe at most four D7-branes.

Consider again figure 3.11.1. There is another loop around which the monodromy has a perturbative interpretation. This is the loop indicated by $I+II$ in the caption of figure 3.11.1. The $SL(2, \mathbb{Z})$ monodromy measured around this loop is $-T^{-4}$, which is the monodromy of a single O7-plane. The figure also shows that if one leaves the perturbative regime the notion of an O7-plane breaks down. The O7-plane monodromy can be decomposed into the monodromy measured around loop I and loop II [10]. Loop I and II encircle the points $z^5_{i\infty}$ and $z^6_{i\infty}$, but always in a way that involves crossing an S branch cut. The monodromy measured around loop I and II is T^2ST^{-4} and ST^{-2}, respectively. These monodromies can be written as $T^2ST^{-4} = M_1TM_1^{-1}$ and $ST^{-2} = M_2TM_2^{-1}$ with

$$M_1 = \pm \begin{pmatrix} 3 & 3\lambda_1 - 1 \\ 1 & \lambda_1 \end{pmatrix} \quad \text{and} \quad M_2 = \pm \begin{pmatrix} 1 & \lambda_2 \\ 1 & 1 + \lambda_2 \end{pmatrix} . \tag{3.11.3}$$

Due to the above relations these branes can be viewed as $SL(2, \mathbb{Z})$-transformed versions of D7-branes. Instead of $\tau \to i\infty$ the complex scalar goes to the real line in their vicinity and henceforth IIB perturbative string theory is not valid there. The D7-brane has a $(1, 0)$ string ending on it. Using the transformation (2.3.10) with Λ equal to either M_1 or M_2 it follows that the 7-brane with monodromy ST^{-2} has a $(1, 1)$ string ending on it while the 7-brane with monodromy T^2ST^{-4} has a $(3, 1)$ string ending on it.

The $N = 24$ solution can be constructed by taking four copies of the $N = 6$ solution. For the $N = 24$ case the form of $j(\tau)$, eq. (3.11.1), is identical to (2.6.3) (the numbers 4 and 27 appearing in (2.6.3) have in (3.11.1) been absorbed into the definitions of P and Q). In fact F-theory on K3, discussed in subsection 2.6.3, is precisely described by eqs. (3.11.1) and (3.11.2), where $fdzd\tau$ is the holomorphic $(2, 0)$-form of the K3 manifold. The base manifold over which the 2-torus with modular parameter τ is elliptically fibered is given by eq. (3.6.3). The metric behaves logarithmically near an F-theory 7-brane (a zero of $P^3 + Q^2$ of order one that is not also a zero of P and/or Q) where the fiber becomes singular. Since the total deficit angle at infinity is 4π the base manifold is a 2-sphere with 24 F-theory 7-branes. Out of the 24 7-branes 16 can be taken to be of type $(1, 0)$ with monodromy T, 4 to be of type $(1, 1)$ with monodromy ST^{-2} and another 4 to be of type $(3, 1)$ with monodromy T^2ST^{-4}.

[10] This splitting of the O7-plane was predicted in [46] by using arguments from field theory whose validity can be understood by using 3-branes as probes and by studying what happens on the world-volume of the 3-brane [69].

In the orientifold limit, $P^3 = cQ^2$ with $c \neq 0$, so that τ is an arbitrary constant and $P = R^2$ and $Q = R^3$ in which R is a 4th order polynomial, this $N = 24$ solution has the metric

$$ds^2 = -dt^2 + d\vec{x}_7^{\,2} + |R^{-1/2}|^2 dz d\bar{z}\,. \tag{3.11.4}$$

The polynomial R has four zeros at each of which the metric has deficit angle π. This is the metric of $T^2/\mathbb{Z}_2$. The function f transforms as $f \to -f$ when going around an orbifold point of $T^2/\mathbb{Z}_2$, which is the transformation of f under the $-\mathbb{1}$ element of $SL(2,\mathbb{Z})$. The solution is interpreted as modding out $\text{Mink}_{1,7} \times T^2$ by $\mathbb{Z}_2 = \{1, I_{89}(-1)^{F_L}\Omega\}$ [46] or equivalently as F-theory on $T^4/\mathbb{Z}_2$ (see also subsections 2.6.2 and 2.6.1).

It was realized in [54] that apart from the $T^4/\mathbb{Z}_2$ orbifold limit of the K3 with 24 7-branes one can further consider the orbifold limits $T^4/\mathbb{Z}_n$ for $n = 3, 4, 6$ in which τ is not an arbitrary constant but equal to either i or ρ[11]. Consider the case $\tau = i$ in (3.11.1) which means that $Q = 0$ and $f \propto P^{-1/4}$. The polynomial P has eight different zeros. Since $P^{-1/4} = P^{-3/12}$ with each factor $(z - z_i)^{-1/12}$ coming from an F-theory 7-brane there are at each zero of P three coincident F-theory 7-branes. The number of complex parameters in P is 9 and 4 of them can be fixed using the $SL(2,\mathbb{C})$ coordinate freedom plus the scale symmetry mentioned above, so that there are 9-4=5 free complex parameters. For the case $\tau = \rho$ it must be that $P = 0$ and so $f \propto Q^{-1/6}$ in which case Q has twelve zeros and at each zero of Q two F-theory 7-branes are coinciding. In this case the number of free complex parameters is 13-4=9.

The five free complex parameters in the case $\tau = i$ can be fixed by grouping the 8 different zeros of P into three groups of zeros of order 3, 3 and 2, i.e. $f \propto (z - z_i^1)^{-3/4}(z - z_i^2)^{-3/4}(z - z_i^3)^{-1/2}$. The three deficit angles are $3\pi/2$, $3\pi/2$ and π which are the three deficit angles of the metric for $T^2/\mathbb{Z}_4$. The orbifold $T^2/\mathbb{Z}_4$ is defined by a complex parameter z which is such that $z \sim z + 1$, $z \sim z + i$ and $z \sim iz$. The complex structure modulus of the base manifold is fixed to be equal to i. The size of the base is still one free real modulus. The fixed points are $z = 0, \frac{1}{2}, \frac{1+i}{2}$. The group $\mathbb{Z}_4$ consists of four generators that are of the form $(R(\frac{\pi}{2}))^n$ with $n = 0, 1, 2, 3$ and in which $R(\frac{\pi}{2})$ denotes a rotation over $\frac{\pi}{2}$ (acting on z as multiplication by i). The fixed points 0 and $\frac{1+i}{2}$ are fixed points of order four because they are invariant under the action of $R(\frac{\pi}{2})$ (up to the torus identifications $z \sim z + 1$ and $z \sim z + i$) which is an order four element of $\mathbb{Z}_4$. The point $\frac{1}{2}$ is order 2 since it is invariant under $R(\frac{\pi}{2})^2 = -1$ (up to the torus identifications $z \sim z + 1$ and $z \sim z + i$) which is of order two in $\mathbb{Z}_4$. Knowing the orders of the fixed points the deficit angles (that must add up to 4π) $3\pi/2$, $3\pi/2$ and π follow. When going around the order four fixed points the function f transforms under $S^3 = -S$ while when going around the orbifold point of order two f transforms under $S^2 = -\mathbb{1}$. The complex structure modulus τ of the fibre is fixed to be equal to i. Given the $SL(2,\mathbb{Z})$ monodromies when going around

[11] See for example [70] for details on 2-dimensional orbifolds.

the three fixed points of the base it is concluded that the fibre is the orbifold $T^2/\mathbb{Z}_4$ in which $\mathbb{Z}_4$ has as its elements $(-S)^n$ for $n = 0, 1, 2, 3$. The F-theory description is thus $T^4/\mathbb{Z}_4$ and this should be equal to type IIB on the orientifold $T^2/\mathbb{Z}_4$ in which the orientifold actions form the group $\mathbb{Z}_4$ given by [54]

$$\mathbb{Z}_4 = \{1, -SR(\tfrac{\pi}{2}), (-SR(\tfrac{\pi}{2}))^2, (-SR(\tfrac{\pi}{2}))^3\}\,, \tag{3.11.5}$$

in which $(-SR(\frac{\pi}{2}))^2 = (-1)^{F_L}\Omega I_{89}$.

Reducing IIB over a point of order 4 gives the following 8-dimensional fields: a metric g_{ab}, one real scalar (the size of $T^2/\mathbb{Z}_4$), one 2-form (coming from the 4-form) and one complex vector $A_a = (B + iC)_{az}$. There are no Kaluza–Klein vectors and no free complex structure modulus of $T^2/\mathbb{Z}_4$ since this is fixed to be i. These fields form an $N = 1$, $d = 8$ supergravity multiplet. It is shown in [54] that near a point of order 4 the 7-brane gauge group is E_7 [12]. The orbifold $T^2/\mathbb{Z}_4$ has three fixed points of order 4, 4 and 2, so that the complete gauge group is $E_7 \times E_7 \times SO(8)$ which is of rank 18. The number 18 is also the number of free complex parameters in F-theory on K3, and therefore forms the highest possible rank of a 7-brane gauge group. Hence, the gauge symmetry enhancement has involved all the free 7-brane moduli of F-theory on K3 with 24 non-coincident 7-branes and preserved the rank of the gauge group. This should be contrasted with the case of IIB on the orientifold $T^2/\mathbb{Z}_2$ discussed earlier. In the latter case the 7-brane gauge group was $(SO(8))^4 \times (U(1))^2$ (times $(U(1))^2$ coming from the Kaluza–Klein vectors). It was mentioned that there can be at most 4 D7-branes in the $N = 6$ solution and so there can be at most 16 D7-branes in the $N = 24$ solution. In the realization of the gauge group $(SO(8))^4$ all the D7-branes (plus the four orientifold planes) play a role. The T^2ST^{-4} and ST^{-2} 7-branes that are shown in figure 3.11.1 on page 96, thus did not play a role in the symmetry enhancement for the case of IIB on $T^2/\mathbb{Z}_2$. For the case of IIB on $T^2/\mathbb{Z}_4$ this is different since the 16 D7-branes alone could never produce a gauge group of rank 18. Therefore the T^2ST^{-4} and ST^{-2} 7-branes must have played a role in the enhancement. In fact in [71] (see [72] for an account of the $SO(8)$ gauge group via open strings) it is shown how exceptional gauge groups can arise by considering not only open strings with two endpoints but also multi-pronged strings that have more than two endpoints in particular 3-string junctions play an important role in the analysis of [71]. The orbifold limits $T^4/\mathbb{Z}_3$ and $T^4/\mathbb{Z}_6$ of F-theory on K3 can be discussed analogously to the $T^4/\mathbb{Z}_4$ case [54].

So far two extreme cases have been considered. The first extreme case is the situation in which none of the 24 F-theory 7-branes are coincident and the second

[12] This also follows by consulting table 2.6.1, page 53, using that at an orbifold point of order 4 (a third order zero of P) nine F-theory 7-branes are coincident so that the third order zero of P becomes a nineth order zero of $P^3 + Q^2$ with $Q = 0$ by construction. Since the polynomial Q does not play any role one can assume that it has a fifth order zero at the same point where P has a third order zero. With this data table 2.6.1 yields the gauge group E_7.

extreme case is the situation in which large numbers of F-theory 7-branes coincide to the extent that τ must be a constant. An intermediate case will be to consider coincident F-theory 7-branes with a non-constant τ. It will be shown that these solutions have exactly the properties of non-constant τ solutions involving Q7-branes. In fact the Q7-branes such as the S or $T^{-1}S$ Q7-branes will have a direct interpretation in terms of coincident F-theory 7-branes.

Before continuing I would like to stress that the remaining part of this section is based on preliminary results and ideas that are not contained in [16, 17] and which can be regarded as work in progress.

Consider the following 7-brane solution containing eight D7-branes (these are all $(1,0)$ 7-branes as can be seen be drawing the pictorial representation of the solution given in (3.11.6) and (3.11.7)) and eight $T^{-1}S$ Q7-branes and which furthermore has $\tau(z=\infty)$ arbitrary

$$j(\tau) = \frac{P_8}{P_8+Q_4^2}\,, \tag{3.11.6}$$

$$f(\tau) = \eta^2(\tau)\left(P_8+Q_4^2\right)^{-1/12} P_8^{-1/6}\,. \tag{3.11.7}$$

The total deficit angle at $z=\infty$ is equal to 4π and hence the transverse space of the 7-branes is compact. The polynomials P and Q are of order eight and four, respectively, as indicated by the subscript 8 and 4. The eight zeros of P_8 produce eight places where the metric has a deficit angle. Around each of the zeros of P_8 the pair (τ, f) transforms under $T^{-1}S$. These are thus the locations of the eight $T^{-1}S$ Q7-branes. Setting $P_8 = cQ_4^2$ for some nonzero c forces τ to be constant, but further arbitrary. The resulting solution is the orbifold $T^4/\mathbb{Z}_2$ discussed earlier. Setting τ equal to i in eqs. (3.11.6) and (3.11.7) implies that $Q_4=0$ while $f \propto P_8^{-1/4}$ just as in the case discussed above. The limit $\tau=\rho$ however does not exist since it is not possible to set P_8 equal to zero in f. The solution (3.11.6) and (3.11.7) contains the two orbifold limits $T^4/\mathbb{Z}_n$ with $n=2,4$ suggesting that (3.11.6) and (3.11.7) is a special case of the generic configuration: F-theory on K3 with 24 non-coincident 7-branes. For convenience the situation with 24 non-coincident 7-branes is repeated here as

$$j(\tau) = \frac{P_8^3}{P_8^3+Q_{12}^2}\,, \tag{3.11.8}$$

$$f(\tau) = \eta^2(\tau)\left(P_8^3+Q_{12}^2\right)^{-1/12}\,, \tag{3.11.9}$$

with the orders of the polynomials P and Q written as subscripts. Next, consider a situation in which eight of the twelve zeros of Q_{12} coincide with the eight zeros of P_8, i.e. take

$$Q_{12} = P_8 Q_4\,, \tag{3.11.10}$$

where Q_4 is an arbitrary polynomial of order four. Substituting eq. (3.11.10) into eqs. (3.11.8) and (3.11.9) gives eqs. (3.11.6) and (3.11.7). The choice (3.11.10) implies that $P_8^3 + Q_{12}^2 = P_8^2(P_8 + Q_4^2)$ so that out of the originally 24 non-coincident F-theory 7-branes 8 are coincident with one other 7-brane and 8 remain non-coincident. Since the solution (3.11.6) and (3.11.7) is interpreted as consisting of 8 D7-branes and 8 $T^{-1}S$ Q7-branes it follows that each of the $T^{-1}S$ Q7-branes is formed out of two coinciding F-theory 7-branes. Further, it must be that the coinciding F-theory 7-branes forming a $T^{-1}S$ Q7-brane have different strings ending on them or what is the same their monodromies must have been different[13]. In the F-theory literature one speaks of mutually non-local F-theory 7-branes as being F-theory 7-branes that are separated by branch cuts in the sense that they cannot, with respect to the same base point, have the same monodromy. It is precisely this type of F-theory 7-branes that are forced to coincide by taking Q_{12} as in eq. (3.11.10). When writing (3.11.8) it is implicitly assumed that the polynomials in the numerator and denumerator have no common factors. One can still allow for common factors with the understanding that then the fibre becomes singular. The fact that (3.11.10) (or any of the orbifold constructions for that matter) forms a singular limit makes it ambiguous to state precisely which of the 24 F-theory 7-branes are coinciding and which are not - the ambiguity resides in the fact that one can only properly view a 7-brane as a representative element of an $SL(2,\mathbb{Z})$ conjugacy class.

In terms of the 24 F-theory 7-branes, that can be thought of as having monodromies T (16 times), ST^{-2} (4 times) and T^2ST^{-4} (4 times) the $T^{-1}S$ Q7-brane (as a representative element of the $T^{-1}S$ conjugacy class) can be thought of as consisting of one T and one $T^{-2}S$ 7-brane (giving $TST^{-2} = T^2(T^{-1}S)T^{-2}$ or, depending on the order, $T(T^{-1}S)T^{-1}$ monodromy) or as one T and one T^2ST^{-4} 7-brane (giving $T^4(T^{-1}S)T^{-4}$ or $T^3(T^{-1}S)T^{-3}$). Further, the mass of a $T^{-1}S$ Q7-brane, contributing $\frac{\pi}{6}$ to the total deficit angle, is equal to the sum of the masses of the F-theory 7-branes each of which contributes $\frac{\pi}{12}$ to the total deficit angle.

Now, that it is known how to embed (3.11.6) and (3.11.7) into the context of F-theory on K3, it becomes possible to use table 2.6.1 on page 53 to find out about the Q7-brane gauge group. The location of any of the eight $T^{-1}S$ Q7-branes forms a first order zero of Q_{12}, a first order zero of P_8 and a second order zero of $P_8^3 + Q_{12}^2$. From table 2.6.1 it is concluded that this gives rise to the Argyres–Douglas singularity H_0 and that in this case there is no singularity type, i.e. the gauge group must be

[13]One may wonder if the statement that a $T^{-1}S$ Q7-brane consist of two F-theory 7-branes that when separated have a different monodromy has an interpretation in terms of the 8-forms to which the different 7-branes couple. However, due to the identifications under $SL(2,\mathbb{Z})$ there are no 8-forms in the solution that can be different from zero. The electric description only works locally and was worked out in section 3.7. Since the F-theory 7-branes forming a $T^{-1}S$ Q7-brane when separated from each other must be separated by a branch cut there is no electric picture of a $T^{-1}S$ Q7-brane as consisting of two other 7-branes simply because the electric description breaks down at the global level.

Abelian. The Tate algorithm only provides information about the non-Abelian part of the gauge group or the non-existence thereof. Therefore, at the moment it is not clear whether the $T^{-1}S$ Q7-brane gauge group is $U(1)$ or e.g. $(U(1))^2$ (after all it is formed out of two coinciding F-theory 7-branes each with a $U(1)$ gauge group). In order to find the number of $U(1)$ factors it may be instructive to study the zero modes of the solution (3.11.6) and (3.11.7). In particular the 2-form zero modes that are vectors may give information about the number of $U(1)$ factors. Even though the 2-forms themselves must be zero since the solution identifies $SL(2,\mathbb{Z})$ transformed IIB fields, the 2-form zero modes that will be functions of τ and f can still exist.

In the case of F-theory with 24 non-coincident 7-branes the 7-brane gauge group was found to be $(U(1))^{18}$ a result that followed from the fact that there are in total 18 free complex parameters describing the relative positions of the 7-branes. In the case of the solution (3.11.6) and (3.11.7) the number of free complex parameters is 10. So in going from (3.11.8) and (3.11.9) to (3.11.6) and (3.11.7) 8 parameters have been identified with 8 others. The embedding of the solution (3.11.6) and (3.11.7) into (3.11.8) and (3.11.9) through (3.11.10) implies that the eight fixed parameters describe the relative positions of the two F-theory 7-branes that make up a $T^{-1}S$ Q7-brane. These parameters should therefore show up as world-volume degrees of freedom on the world-volume of the $T^{-1}S$ Q7-brane. However, since they are not free parameters of the solution (3.11.6) and (3.11.7) it is difficult to formulate precisely how these scalar moduli show up in the world-volume theory of the Q7-brane[14]. For example the source term (3.2.3) presupposes that the scalars describing the relative position of the two F-theory 7-branes are fixed to be equal to some constant. There are furthermore two world-volume scalar degrees of freedom that describe the position of the $T^{-1}S$ Q7-brane in the transverse space and these are free parameters of the solution. There are thus in total 4 scalar degrees of freedom on the world-volume of the $T^{-1}S$ Q7-brane. World-volume supersymmetry then requires there to be two BI vectors. This latter fact is natural if one considers the problem of how to assign the 18 $U(1)$'s of the F-theory solution to the 7-branes of (3.11.6) and (3.11.7). At most 8 of the total of 18 $U(1)$'s can be assigned to the 8 D7-branes of (3.11.6) and (3.11.7). This leaves one with 10 $U(1)$'s that must be assigned to 8 $T^{-1}S$ Q7-branes. Since each $T^{-1}S$ Q7-brane is identical to any other $T^{-1}S$ Q7-brane they must all have the same number of $U(1)$'s. Obviously, assigning to each of them a single $U(1)$ is insufficient, and so the next possibility is to attribute a $(U(1))^2$ to each of them. The condition is that the rank of the 7-brane gauge group is not allowed to exceed 18 as this is the highest rank in the case of the 24 F-theory 7-branes in (3.11.8) and (3.11.9). The highest rank in the solution (3.11.6) and (3.11.7) is obtained by assigning 8 of the 10 free parameters to the 8 $T^{-1}S$ Q7-branes and the remaining two to D7-branes. This

[14]This is not a problem that is special to the case at hand but is generic in supergravity. There does not, so far, exist a technique to obtain the world-volume action of coinciding branes directly from the properties of the solution.

clearly gives a gauge group of rank 18.

In the following arguments will be given that support, but not prove the above mentioned assertion that the $T^{-1}S$ Q7-brane gauge group is $(U(1))^2$. In section 2.4 it was mentioned that the gauge invariant coupling to an 8-form that couples to a Q7-brane requires the introduction of two BI vectors. It is expected that when the 2-form zero modes are considered in the local neighborhood of a $T^{-1}S$ Q7-brane that two 2-form zero modes (BI vectors) will be present simultaneously. At the linearized level the two BI vectors are independent. Assume that their Lagrangian is required to be invariant under the Q7-brane monodromy. Let $\mathcal{V}$ denote a column vector of two BI vectors that transforms under $SL(2,\mathbb{Z})$ just as the 2-forms do in eq. (1.3.51), i.e. as $\mathcal{V} \to \Lambda\mathcal{V}$. An $SL(2,\mathbb{Z})$ invariant and quadratic action for $\mathcal{V}$ is given by $\star(d\mathcal{V})^T \wedge SQd\mathcal{V}$, which is easily seen to be unique. The $T^{-1}S$ Q7-brane monodromy can be written as the product of single F-theory 7-brane monodromies. Writing the F-theory 7-brane monodromies as e^{Q_1} and e^{Q_2} with Q_1 and Q_2 two zero determinant traceless matrices it follows that the charge matrix Q of the Q7-brane can be written in terms of Q_1 and Q_2 as follows

$$e^{Q_1}e^{Q_2} = e^{Q} \quad \text{with} \quad Q = c\left(Q_1 + Q_2 + \frac{1}{2}[Q_1, Q_2]\right), \tag{3.11.11}$$

where c is some real constant. For the case of a $T^{-1}S$ Q7-brane Q_2 is related to Q_1 by an $SL(2,\mathbb{Z})$ transformation. The expression (3.11.11) can be derived using the Baker–Campbell–Hausdorff formula. The matrices Q_1 and Q_2 can be written in terms of the string charges p' and q' that were introduced in eq. (2.3.8) and below as

$$Q_1 = S\begin{pmatrix} -q_1' \\ p_1' \end{pmatrix}\begin{pmatrix} -q_1' \\ p_1' \end{pmatrix}^T, \tag{3.11.12}$$

and similarly for Q_2. Using Q as in eq. (3.11.11) the proposed quadratic zero mode kinetic term becomes

$$\star(d\mathcal{V})^T \wedge SQd\mathcal{V} = c\star(d\mathcal{V})^T \wedge SQ_1 d\mathcal{V} + c\star(d\mathcal{V})^T \wedge SQ_2 d\mathcal{V}\,. \tag{3.11.13}$$

The commutator $[Q_1, Q_2]$ cancels at this order in the BI vectors. The leading term in the WZ term can be written as

$$\int_{\Sigma_8} \mathrm{Tr}\, Q \begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8 \end{pmatrix}, \tag{3.11.14}$$

and here the commutator $[Q_1, Q_2]$ remains. Therefore gauge invariance of (3.11.14) under the 8-form gauge transformation (1.1.45) will introduce interactions between the two BI vectors and the form of these interactions will be dictated by the commutator $[Q_1, Q_2]$.

As mentioned earlier the compact solution containing eight $T^{-1}S$ Q7-branes, eqs. (3.11.6) and (3.11.7), was obtained by fixing eight parameters in the compact F-theory solution containing 24 7-branes. These parameters can be interpreted as being the moduli describing the relative positions of the two F-theory 7-branes that make up a $T^{-1}S$ Q7-brane[15]. The moduli describing the position of the two coinciding F-theory 7-branes are still free. The bosonic world-volume degrees of freedom of a $T^{-1}S$ Q7-brane are thus expected to consist of 4 scalars (two for the relative motion of the two F-theory 7-branes and two for their center of mass motion) and two (interacting) BI vectors, so that the world-volume action should be formed out of two interacting $N = 1$, $d = 8$ vector multiplets and the gauge group is expected to be $(U(1))^2$ [16]. That is to say, if the assumptions about the independence of the two BI vectors and the invariance of the zero mode action under the Q7-brane monodromy are correct[17].

So far for the $T^{-1}S$ Q7-branes. Consider next a solution containing only S Q7-branes that is compact and has $\tau(z = \infty)$ arbitrary. Such a solution is given by

$$j(\tau) = \frac{P_2^3}{P_2^3 + Q_6}\,, \qquad (3.11.15)$$

$$f(\tau) = \eta^2(\tau)\left(P_2^3 + Q_6\right)^{-1/12} Q_6^{-1/4}\,. \qquad (3.11.16)$$

The solution, eqs. (3.11.15) and (3.11.16), follows from eqs. (3.11.8) and (3.11.9) by taking

$$P_8 = P_2 Q_6\,, \qquad Q_{12} = Q_6^2\,. \qquad (3.11.17)$$

The six S Q7-branes are located at the zeros of Q_6. Since $P_8^3 + Q_{12}^2 = Q_6^3(P_2^3 + Q_6)$ the S Q7-brane consists of three coincident F-theory 7-branes. At any of the zeros of Q_6 the orders of the zeros of P_8, Q_{12} and $P_8^3 + Q_{12}^2$ are 1, 2 and 3, so that according to table 2.6.1 one is dealing with the Argyres–Douglas singularity H_1 in which case the gauge group is $SU(2)$. This means that an S Q7-brane must have $SU(2)$ gauge group. In terms of the 24 F-theory 7-branes, that can be thought of as having monodromies T (16 times), ST^{-2} (4 times) and T^2ST^{-4} (4 times) the S Q7-brane (as a representative elements of the S conjugacy class) can be thought of as consisting of two T and one $T^{-2}S$ 7-brane (giving $TTT^{-2}S = S$ or, depending on the order, $T^{-2}(S)T^2$, $T^{-1}(S)T$ monodromy) or as two T and one T^2ST^{-4} 7-brane (giving $T^4(S)T^{-4}$, $T^3(S)T^{-3}$ or T^2ST^{-2}). The non-Abelian gauge group $SU(2)$ can be understood to result from

[15] The Wess–Zumino term (3.11.14) as well as the action (3.2.3) should be interpreted in the limit in which the two scalars describing the relative positions of the two branes are fixed to be equal to some constant.

[16] From the point of view of a probe Q3-brane the two vector zero modes may be related by an electro-magnetic duality transformation, so that on a single probe Q3-brane world-volume they are not independent degrees of freedom.

[17] It is stressed here once again that the statements regarding the number of $U(1)$'s in the $T^{-1}S$ Q7-brane gauge group rest on the embedding of the solution (3.11.6) and (3.11.7) into (3.11.8) and (3.11.9) through (3.11.10) as providing the correct interpretation of the $T^{-1}S$ Q7-brane.

the fact that there are two coincident D7-branes involved. Again, it is expected that there will be additional $U(1)$ factors coming from the local properties of the 2-form zero modes near an S Q7-brane. The argument will be the same as for the case of a $T^{-1}S$ Q7-brane leading to a local gauge group $SU(2) \times (U(1))^2$. The rank of the S Q7-brane gauge group is three. Since the solution (3.11.15) and (3.11.16) has 6 D7-branes and 6 S Q7-branes and has 6 free complex parameters the maximal rank is 18 which occurs when all the 6 free parameters are assigned to the positions of the 6 S Q7-branes. The maximal rank agrees with the rank of the 7-brane gauge group in the case of the solution (3.11.8) and (3.11.9).

It will be clear that one can by suitably choosing P_8 and Q_{12} construct many more solutions containing for example both S and $T^{-1}S$ Q7-branes and/or coincident $T^{-1}S$ Q7-branes, etc.

Apart from the compact solutions with total deficit angle equal to 4π there exist also non-compact solutions containing Q7-branes such as the I.a solution of subsection 3.10.3. It can be shown that the I.a solution is a special case of the F-theory $N = 6$ solution. In fact any solution with no explicit negative energy sources and for which $\tau(z) \neq \text{cst}$ and with $\tau(z = \infty)$ is arbitrary (but not equal to $i\infty, i$ or ρ) containing Q7-branes can be considered a special case of the F-theory solutions with either $N = 6, 12$ or 24 in which the numerator and denumerator in (3.11.1) are allowed to have common factors.

3.12 Open strings stretched between 7-branes

The solution near a D7-brane is singular, the metric behaves logarithmically. Near a Q7-brane there appears a deficit angle. Often when singularities appear in the solution in the Einstein frame they disappear when considering the solution from the point of view of the frame of the closed IIB superstring. However, if one considers the metric near a D7-brane in the string frame it is still singular. This may well be related to the fact that a single D7-brane in string theory is an inconsistent background with uncancelled tadpoles. There exists another frame in which the solution appears regular. For a general (p, q, r) 7-brane define the frame $d\tilde{s}^2$ by

$$d\tilde{s}^2 = dwd\bar{w} \quad \text{with} \quad dw = q^{1/2}(\tau - \tau_0)^{1/2}(\tau - \bar{\tau}_0)^{1/2} f dz \,. \tag{3.12.1}$$

Setting $\det Q = 0$ (3.12.1) one obtains

$$d\tilde{s}^2 = \frac{|\sqrt{q}\tau + \sqrt{p}|^2}{\operatorname{Im}\tau}\, ds_E^2 \,, \tag{3.12.2}$$

where ds_E^2 is the Einstein frame metric and the prefactor is the square of the tension of a (p', q') string with $p'^2 = p$ and $q'^2 = q$. It is clear that

$$\int_\gamma |dw| \tag{3.12.3}$$

measures the mass of a string stretched between two identical (p', q') 7-branes or stretched between a probe 3-brane, parallel with the 7-branes and a (p', q') 7-brane along a contour γ that lies in the complex z-plane not crossing any branch cuts [73]. The mass of the string then translates into certain BPS mass formulae (realized for straight contours in the w-plane) for the world-volume theory of the 3-brane.

The natural question is what about the Q7-branes viewed from the point of view of the frame (3.12.1)? The metric near a Q7-brane is fully regular in the frame (3.12.1). The relation with the Einstein frame for $\det Q > 0$ is

$$d\tilde{s}^2 = \frac{|q^{1/2}(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}|^2}{\operatorname{Im}\tau} ds_E^2 \,. \tag{3.12.4}$$

The prefactor in (3.12.4) can be rewritten as

$$\frac{|q^{1/2}(\tau-\tau_0)^{1/2}(\tau-\bar{\tau}_0)^{1/2}|^2}{\operatorname{Im}\tau} = \left(T^2 - 4\det Q\right)^{1/2} = 2\sqrt{\det Q}\sinh\eta \,. \tag{3.12.5}$$

The geometrical interpretation of η is given in section 1.4. It measures the Poincaré distance along a geodesic (semi-circles whose center is on the real axis) connecting τ and τ_0. By comparison the dilaton ϕ measures the Poincaré distance along a geodesic (straight semi-infinite vertical lines) connecting τ and $\tau = i\infty$ or τ_0 with $\operatorname{Im}\tau_0 = 0$.

Purely hypothetically, in analogy with the $\det Q = 0$ case, one could consider

$$\int_\gamma |dw| \tag{3.12.6}$$

for the case $\det Q > 0$ to compute the mass a new type of string whose tension is proportional to $\left(T^2 - 4\det Q\right)^{1/4}$. Since the metric $d\tilde{s}^2 = dw d\bar{w}$ is flat such a string can be BPS by choosing a straight contour in the w-plane. Since the background cannot have any nonzero 2-forms, due to the identifications under $SL(2,\mathbb{Z})$, these strings as well as the massive (p', q') strings are chargeless. One could refer to such a string as a Q-string since it ends on a Q7-brane.

3.13 Discussion

The most general 7-brane configuration that is globally well-defined with finite energy is necessarily supersymmetric, see subsection 3.8.2. The value of τ at infinity, $z = \infty$, can be fixed to be equal to either $i\infty, i$ or ρ or it is taken to be a free parameter not equal to either $i\infty, i$ or ρ. Besides that one can choose to allow for negative energy sources (only then the number of 7-branes can exceed 24), or negative energy sources that are (locally) cancelled by a Q7-brane. Those solutions for which $\tau(z = \infty)$ is a free parameter not equal to either $i\infty, i$ or ρ and in which any negative energy source is locally cancelled, containing some number of Q7-branes can be obtained from the F-theory $N = 6, 12$ or 24 solutions assuming that the numerator and denumerator in (3.11.1) are allowed to have common factors.

The embedding of compact Q7-brane solutions in the context of F-theory on K3 makes it possible to interpret the Q7-branes as follows. The $T^{-1}S$ Q7-brane is formed by forcing together two mutually non-local F-theory 7-branes. The S Q7-brane is formed by taking three F-theory 7-branes coincident of which two must be mutually non-local. The $T^{-1}S$ Q7-brane gauge group is suggested to be $(U(1))^2$ and similarly the gauge group of an S Q7-brane is suggested to be $SU(2) \times (U(1))^2$.

The Q7-branes, from the point of view of the local analysis, i.e. in the limit in which the scalars describing the relative positions of the various F-theory 7-branes that make up a Q7-brane are constant, seem to be living in their own world, so to speak, with their own 8-form to which they couple and a metric (3.7.42) that is hinting towards the existence of an open stretched BPS Q-string that stretches from one Q7-brane to another (see section 3.12).