Seven-branes and instantons in type IIB supergravity

Hartong, Jelle

Chapter 1

Overview of type IIB supergravity

The conventions used throughout this thesis are presented in appendix A.

This chapter contains an overview of the properties of type IIB supergravity. In section 1.1 the $SU(1,1)$ covariant formulation of the theory is presented. In section 1.2 two scalars are defined T and χ' that will play an important role in this thesis. They parameterize the coset manifold $SL(2,\mathbb{R})/SO(2)$ and are related to the dilaton ϕ and RR axion χ by a field redefinition. Then in section 1.3 the local supersymmetry algebra is presented in terms of the scalars T and χ'.

1.1 The $SU(1,1)$ covariant formulation

The bosonic field content of type IIB supergravity consists of the metric, two scalars, two 2-form potentials and a chiral 4-form potential with a self-dual 5-form field strength making up a total of 128 bosonic degrees of freedom. The fermionic field content consists of two Majorana–Weyl dilatini as well as two Majorana–Weyl gravitini comprising a total of 128 fermionic degrees of freedom. These fields are organized into one 10-dimensional $N=2$ supergravity multiplet. The two scalars parameterize the coset space $SU(1,1)/U(1)$ that is isomorphic to $SL(2,\mathbb{R})/SO(2)$. The local supersymmetry algebra for this theory has been constructed in [19–21] and shown to only close on-shell.

The two scalars in the theory transform non-linearly under $SU(1,1)$ or $SL(2,\mathbb{R})$. This transformation can be made linear by introducing a third scalar as well as a local $U(1)$ gauge symmetry. Upon fixing the $U(1)$ gauge the third scalar is eliminated as well as the local $U(1)$ gauge transformation.

The R-symmetry group of the global type IIB supersymmetry algebra acts in the local theory as a local $U(1)$ transformation on the spinors. In the coset formulation this local $U(1)$ symmetry is identified with the local $U(1)$ symmetry of the coset description. After fixing the $U(1)$ gauge of the coset formulation the spinors still transform under the $U(1)$ R-symmetry group.

The $SU(1,1)/U(1)$ coset model is described in subsection 1.1.1. The bosonic fields and their equations of motion are given in subsections 1.1.2 and 1.1.3 and the local supersymmetry algebra (up to second order fermions) is given in subsection 1.1.4.

1.1.1 The $SU(1,1)/U(1)$ coset

Consider the group $SU(1,1)$ consisting of all complex two by two matrices g with $\det g = 1$ that satisfy $g^{-1} = \eta g^\dagger \eta$ where $\eta = \mathrm{diag}(1,-1)$. The generators of this group are H and $T_\pm$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_- \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.1.1}$$

and they satisfy the algebra

$$[H,H] = 0\,, \quad [H,T_\pm] = \pm 2T_\pm\,. \tag{1.1.2}$$

The matrix H generates the maximal compact subgroup $U(1)$.

The coset $SU(1,1)/U(1)$ consists of all $SU(1,1)$ matrices V that are identified under the transformations of the compact subgroup $U(1)$. If one takes V to depend on space-time points x then the equivalence under $U(1)$ becomes a gauge symmetry. A (left-)coset representative V transforms as

$$V(x) \to gV(x)h(x)\,, \tag{1.1.3}$$

where $g \in SU(1,1)$ and $h \in U(1)$. The coset representative V is parameterized as

$$V = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix}, \tag{1.1.4}$$

where $(V_\mp^1)^* = V_\pm^2$ and

$$V_-^\alpha V_+^\beta - V_-^\beta V_+^\alpha = \epsilon^{\alpha\beta} = \epsilon_{\alpha\beta} \tag{1.1.5}$$

with $\alpha,\beta = 1,2$ denoting $SU(1,1)$ indices and with $\epsilon^{12} = +1$.

Using the matrix V a left-invariant Lie algebra element of $SU(1,1)$ can be written as

$$V^{-1}dV = \begin{pmatrix} -iQ & P \\ \bar{P} & iQ \end{pmatrix}. \tag{1.1.6}$$

In terms of the components of V eq. (1.1.6) reads

$$P = -\epsilon_{\alpha\beta} V_+^\alpha dV_+^\beta\,, \quad (1.1.7)$$
$$\bar{P} = \epsilon_{\alpha\beta} V_-^\alpha dV_-^\beta\,, \quad (1.1.8)$$
$$Q = -i\epsilon_{\alpha\beta} V_-^\alpha dV_+^\beta\,. \quad (1.1.9)$$

Under a local $U(1)$ transformation $h = e^{i\alpha(x)H}$ with parameter $\alpha(x)$ the 1-forms Q and P transform as

$$\begin{aligned} Q &\to Q - d\alpha\,, \\ P &\to e^{-2i\alpha} P\,. \end{aligned} \quad (1.1.10)$$

Both P and Q are invariant under global $SU(1,1)$ transformations. The 1-form Q is real-valued and transforms as a composite $U(1)$ gauge connection under local $U(1)$ transformations. The 1-form P is complex-valued and is referred to as the coset Zweibein. The $U(1)$ weight w of P is $w = 2$ [1]. The gauge-covariant derivative, denoted by D, of P is defined in the standard way as $DP = dP - 2iQ \wedge P$. The Bianchi identity for P_μ is given by

$$DP = 0\,. \quad (1.1.12)$$

From eqs. (1.1.5), (1.1.7) and (1.1.8) it can be concluded that the covariant derivative D acting on $V_\pm^\alpha$ gives

$$DV_+^\alpha = V_-^\alpha P\,, \quad (1.1.13)$$
$$DV_-^\alpha = V_+^\alpha \bar{P}\,. \quad (1.1.14)$$

It is convenient to define the following $U(1)$ gauge-invariant (right-invariant) matrix of 1-forms p as follows

$$p = V \begin{pmatrix} 0 & P \\ \bar{P} & 0 \end{pmatrix} V^{-1}\,. \quad (1.1.15)$$

It can be shown that the components of p are the three Noether currents that derive from the global $SU(1,1)$ invariance of the scalar kinetic terms of the IIB Lagrangian. The Bianchi identity for p reads

$$dp - 2p \wedge p = 0\,. \quad (1.1.16)$$

[1] A field X is assigned a $U(1)$-weight $w(X)$ if it transforms under a local $U(1)$ transformation $h = e^{i\alpha H}$ as

$$X \to e^{-iw\alpha} X\,. \quad (1.1.11)$$

Using this definition it follows that $w(V_\pm^\alpha) = \pm 1$ and $w(P) = +2$.

1.1.2 The non-self-dual bosonic type IIB action

In the conventions of [22] the bosonic part of the IIB supergravity action is given by

$$S = \int_{\mathcal{M}_{10}} \left(\star 1 R - 2 \star P \wedge \bar{P} - \frac{1}{2} \star G_3 \wedge \bar{G}_3 - 4 \star F_5 \wedge F_5 + \frac{i}{2} F_5 \wedge \epsilon_{\alpha\beta} A_2^\alpha \wedge F_3^\beta \right) . \tag{1.1.17}$$

In order to obtain the type IIB bosonic field equations one must additionally impose the self-duality condition $F_5 = \star F_5$ on-shell[2] which is why (1.1.17) is referred to as the non-self-dual type IIB action. The forms P (defined in 1.1.1), G_3 and F_5 are defined via the Bianchi identities,

$$DP = dP - 2iQ \wedge P = 0 , \tag{1.1.18}$$

$$DG_3 = dG_3 - iQ \wedge G_3 = -P \wedge \bar{G}_3 , \tag{1.1.19}$$

$$dF_5 = -\frac{i}{8} G_3 \wedge \bar{G}_3 . \tag{1.1.20}$$

The solution to the Bianchi identity for G_3 is given by

$$G_3 = -\epsilon_{\alpha\beta} V_+^\alpha F_3^\beta \qquad \text{where} \qquad F_3^\beta = dA_2^\beta . \tag{1.1.21}$$

The 2-forms, A_2^α, transform as a doublet under $SU(1,1)$ and transform under gauge transformations as $\delta A_2^\alpha = d\Lambda_1^\alpha$. The solution to the Bianchi identity for F_5 is

$$F_5 = dA_4 + \frac{i}{16} \epsilon_{\alpha\beta} A_2^\alpha \wedge F_3^\beta . \tag{1.1.22}$$

The reality properties of A_2^α and A_4 are

$$(A_2^1)^* = A_2^2 \qquad \text{and} \qquad A_4^* = A_4 . \tag{1.1.23}$$

The equations of motion that follow from the action (1.1.17) supplemented with the self-duality condition of the 5-form are:

$$R_{\mu\nu} = P_\mu \bar{P}_\nu + \bar{P}_\mu P_\nu + \frac{1}{6} F_{\sigma_1 \cdots \sigma_4 \mu} F^{\sigma_1 \cdots \sigma_4}{}_\nu + \frac{1}{8} \left(G_{\sigma_1\sigma_2\mu} \bar{G}^{\sigma_1\sigma_2}{}_\nu + \bar{G}_{\sigma_1\sigma_2\mu} G^{\sigma_1\sigma_2}{}_\nu - \frac{1}{6} g_{\mu\nu} \bar{G}_{\sigma_1\sigma_2\sigma_3} G^{\sigma_1\sigma_2\sigma_3} \right) \tag{1.1.24}$$

$$D \star P = \frac{1}{4} \star G_3 \wedge G_3 \tag{1.1.25}$$

$$D \star G_3 = P \wedge * \bar{G}_3 + 4i \star F_5 \wedge G_3 \tag{1.1.26}$$

$$\star F_5 = F_5 . \tag{1.1.27}$$

[2] It is possible to construct an action that incorporates the self-duality condition $F_5 = \star F_5$ [23]. Since in this thesis the emphasis will be on the gravity-scalar sector of the bosonic action the chiral 4-form action of [23] will not be needed.

1.1.3 The form fields

It is possible to dualize the 2-forms, A_2^α, to a doublet of 6-forms, A_6^α, with $(A_6^1)^* = A_6^2$, via the respective duality relation and Bianchi identity,

$$F_7^\alpha = i \star \left(V_-^\alpha G_3 - V_+^\alpha \bar{G}_3\right) , \tag{1.1.28}$$

$$dF_7^\alpha = 4F_3^\alpha \wedge F_5 . \tag{1.1.29}$$

Solving the Bianchi identity (1.1.29) gives

$$F_7^\alpha = dA_6^\alpha + \frac{4}{3} F_5 \wedge A_2^\alpha - \frac{8}{3} A_4 \wedge F_3^\alpha . \tag{1.1.30}$$

In verifying that (1.1.30) satisfies (1.1.29) one can use the following 2-form identity:

$$\epsilon_{\beta\gamma} F_3^\beta \wedge F_3^\gamma \wedge A_2^\alpha = -2\epsilon_{\beta\gamma} A_2^\beta \wedge F_3^\gamma \wedge F_3^\alpha . \tag{1.1.31}$$

The object G_7 that is defined by

$$G_7 = -\epsilon_{\alpha\beta} V_+^\alpha F_7^\beta \tag{1.1.32}$$

satisfies the Bianchi identity,

$$DG_7 + P \wedge \bar{G}_7 = 4G_3 \wedge F_5 . \tag{1.1.33}$$

From eq. (1.1.28) it follows that $G_7 = i \star G_3$.

It is further possible to introduce a triplet of 8-forms [24, 25], $A_8^{\alpha\beta} = A_8^{\beta\alpha}$, with $(A_8^{11})^* = A_8^{22}$ and $(A_8^{12})^* = A_8^{12}$, via the respective duality relation and Bianchi identity

$$F_9^{\alpha\beta} = i \star \left(V_+^\alpha V_+^\beta \bar{P} - V_-^\alpha V_-^\beta P\right) , \tag{1.1.34}$$

$$dF_9^{\alpha\beta} = \frac{1}{4} F_3^{(\alpha} \wedge F_7^{\beta)} . \tag{1.1.35}$$

Solving the Bianchi identity (1.1.35), $F_9^{\alpha\beta}$ can be written as

$$F_9^{\alpha\beta} = dA_8^{\alpha\beta} + \frac{1}{16} F_7^{(\alpha} \wedge A_2^{\beta)} - \frac{3}{16} F_3^{(\alpha} \wedge A_6^{\beta)} . \tag{1.1.36}$$

As follows from eq. (1.1.34) the field strengths $F_9^{\alpha\beta}$ satisfy the $SU(1,1)$ invariant constraint

$$\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} V_-^\alpha V_+^\beta F_9^{\gamma\delta} = 0 . \tag{1.1.37}$$

The p-form gauge fields ($p = 2, 4, 6, 8$) have the following gauge transformations:

$$\delta A_2^\alpha = d\Lambda_1^\alpha \,, \tag{1.1.38}$$

$$\delta A_4 = d\Lambda_3 - \frac{i}{16}\epsilon_{\alpha\beta}\, F_3^\alpha \wedge \Lambda_1^\beta \,, \tag{1.1.39}$$

$$\delta A_6^\alpha = d\Lambda_5^\alpha + \frac{4}{3}F_5 \wedge \Lambda_1^\alpha - \frac{8}{3}F_3^\alpha \wedge \Lambda_3 \,, \tag{1.1.40}$$

$$\delta A_8^{\alpha\beta} = d\Lambda_7^{\alpha\beta} + \frac{1}{16}F_7^{(\alpha} \wedge \Lambda_1^{\beta)} - \frac{3}{16}F_3^{(\alpha} \wedge \Lambda_5^{\beta)} \,. \tag{1.1.41}$$

This can also be written as:

$$\delta A_2^\alpha = d\Lambda_1^\alpha \,, \tag{1.1.42}$$

$$\delta A_4 = d\Sigma_3 + \frac{i}{16}\epsilon_{\alpha\beta}\, A_2^\alpha \wedge \delta A_2^\beta \,, \tag{1.1.43}$$

$$\delta A_6^\alpha = d\Sigma_5^\alpha + \frac{8}{3}A_2^\alpha \wedge \delta A_4 - \frac{4}{3}A_4 \wedge \delta A_2^\alpha + \frac{i}{12}A_2^\alpha \wedge \epsilon_{\gamma\delta}\, \delta A_2^\gamma \wedge A_2^\delta \,, \tag{1.1.44}$$

$$\begin{aligned}\delta A_8^{\alpha\beta} = {} & d\Sigma_7^{\alpha\beta} + \frac{3}{16}A_2^{(\alpha} \wedge \delta A_6^{\beta)} - \frac{1}{16}A_6^{(\alpha} \wedge \delta A_2^{\beta)} - \frac{1}{4}A_2^\alpha \wedge A_2^\beta \wedge \delta A_4 \\ & + \frac{1}{6}A_4 \wedge A_2^{(\alpha} \wedge \delta A_2^{\beta)} - \frac{1}{12}A_2^\alpha \wedge A_2^\beta \wedge \frac{i}{16}\epsilon_{\gamma\delta}\, \delta A_2^\gamma \wedge A_2^\delta \,,\end{aligned} \tag{1.1.45}$$

where the gauge transformation parameters Σ_p differ from Λ_p.

1.1.4 Supersymmetry transformation rules

Below are given the supersymmetry variations of the fields of type IIB supergravity as they are presented in [22] up to second order in fermions:

$$\delta g_{\mu\nu} = 2i\, \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + 2i\bar{\epsilon}_C\gamma_{(\mu}\psi_{C\,\nu)} \,, \tag{1.1.46}$$

$$\delta V_+^\alpha = V_-^\alpha \bar{\epsilon}_C \lambda \,, \tag{1.1.47}$$

$$\delta V_-^\alpha = V_+^\alpha \bar{\epsilon}\lambda_C \,, \tag{1.1.48}$$

$$\begin{aligned}\delta A_{\mu\nu}^\alpha = {} & V_-^\alpha \bar{\epsilon}\gamma_{\mu\nu}\lambda + V_+^\alpha \bar{\epsilon}_C\gamma_{\mu\nu}\lambda_C \\ & + 4iV_-^\alpha \bar{\epsilon}_C\gamma_{[\mu}\psi_{\nu]} + 4iV_+^\alpha \bar{\epsilon}\gamma_{[\mu}\psi_{C\,\nu]} \,,\end{aligned} \tag{1.1.49}$$

$$\delta A_{\mu\nu\rho\sigma} = \bar{\epsilon}\gamma_{[\mu\nu\rho}\psi_{\sigma]} - \bar{\epsilon}_C\gamma_{[\mu\nu\rho}\psi_{C\,\sigma]} - \frac{3i}{8}\epsilon_{\alpha\beta}A_{[\mu\nu}^\alpha \delta A_{\rho\sigma]}^\beta \,, \tag{1.1.50}$$

$$\begin{aligned}\delta A_{\mu_1\ldots\mu_6}^\alpha = {} & iV_-^\alpha \bar{\epsilon}\gamma_{\mu_1\ldots\mu_6}\lambda - iV_+^\alpha \bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_6}\lambda_C + 12V_-^\alpha \bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} \\ & -12V_+^\alpha \bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]} + 40A_{[\mu_1\ldots\mu_4}\delta A_{\mu_5\mu_6]}^\alpha \\ & -20\delta A_{[\mu_1\ldots\mu_4}A_{\mu_5\mu_6]}^\alpha - \frac{15i}{2}A_{[\mu_1\mu_2}^\alpha \epsilon_{\beta\gamma}A_{\mu_3\mu_4}^\beta \delta A_{\mu_5\mu_6]}^\gamma \,,\end{aligned} \tag{1.1.51}$$

$$
\begin{aligned}
\delta A^{\alpha\beta}_{\mu_1\ldots\mu_8} &= -iV^{\alpha}_{+}V^{\beta}_{+}\bar{\epsilon}\gamma_{\mu_1\ldots\mu_8}\lambda_C + iV^{\alpha}_{-}V^{\beta}_{-}\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_8}\lambda \\
&\quad +8V^{(\alpha}_{+}V^{\beta)}_{-}\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_7}\psi_{\mu_8]} - 8V^{(\alpha}_{+}V^{\beta)}_{-}\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_7}\psi_{C\,\mu_8]} \\
&\quad +\frac{21}{4}A^{(\alpha}_{[\mu_1\ldots\mu_6}\delta A^{\beta)}_{\mu_7\mu_8]} - \frac{7}{4}A^{(\alpha}_{[\mu_1\mu_2}\delta A^{\beta)}_{\mu_3\ldots\mu_8]} \\
&\quad -35A^{(\alpha}_{[\mu_1\mu_2}A^{\beta)}_{\mu_3\mu_4}\delta A_{\mu_5\ldots\mu_8]} + 70A_{[\mu_1\ldots\mu_4}A^{(\alpha}_{\mu_5\mu_6}\delta A^{\beta)}_{\mu_7\mu_8]} \\
&\quad -\frac{105i}{8}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta)}_{\mu_3\mu_4}\epsilon_{\gamma\delta}A^{\gamma}_{\mu_5\mu_6}\delta A^{\delta}_{\mu_7\mu_8]}\,, \qquad (1.1.52)
\end{aligned}
$$

$$
\delta\lambda = iP_\mu\gamma^\mu\epsilon_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon\,, \qquad (1.1.53)
$$

$$
\begin{aligned}
\delta\psi_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon \\
&\quad +\frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon_C\,. \qquad (1.1.54)
\end{aligned}
$$

The $U(1)$ weight of the complex Weyl dilatino, λ, is $\frac{3}{2}$ and the $U(1)$ weight of the complex Weyl gravitino, ψ_μ, is $\frac{1}{2}$. Under global $SU(1,1)$ transformations λ, ψ_μ and ϵ transform as scalars.

The commutator of two supersymmetry transformations closes on the bosonic fields up to local symmetries of the theory except for the 4-form in which case closure also requires the self-duality condition (1.1.27) to be imposed. The local symmetries are general coordinate transformations (or local Lorentz transformations), gauge transformations and the local $U(1)$ transformation of the coset model. The commutator of two supersymmetries acting on the spinors closes up to local Lorentz transformations, local $U(1)$ transformations and upon imposing the self-duality condition (1.1.27) as well as the fermionic equations of motion. The fermionic equations of motion are given by (up to first order in fermions) [19]

$$
\gamma^\mu D_\mu\lambda = \frac{i}{240}\gamma^{\mu_1\ldots\mu_5}F_{\mu_1\ldots\mu_5}\lambda\,, \qquad (1.1.55)
$$

$$
\gamma^{\mu\nu\rho}D_\nu\psi_\rho = -\frac{i}{2}\gamma^\rho\gamma^\mu P_\rho\lambda_C - \frac{i}{48}\gamma^{\nu\rho\lambda}\gamma^\mu\bar{G}_{\nu\rho\lambda}\lambda\,, \qquad (1.1.56)
$$

where D_μ contains the $U(1)$ connection Q_μ. By applying a supersymmetry transformation on the fermionic field equations the bosonic equations of motion (1.1.24) to (1.1.26) follow. Hence, the type IIB local supersymmetry algebra is an on-shell algebra.

The type IIB local supersymmetry algebra described above is unique up to the introduction of 10-form potentials [22, 26]. These 10-forms will be briefly considered in the next subsection.

1.1.5 Inclusion of ten-form potentials

As shown in [22] the type IIB supersymmetry algebra is compatible with the introduction of two types of 10-form potentials: a doublet and quadruplet. Ten-form potentials are interesting for the inclusion of space-time filling 9-branes in the IIB theory, see section 2.5. The quadruplet consists of 10-forms, $A^{\alpha\beta\gamma}$, symmetric in α, β and γ with reality conditions

$$(A^{111}_{10})^* = A^{222}_{10}\,, \qquad (A^{112}_{10})^* = A^{122}_{10}\,. \tag{1.1.57}$$

The supersymmetry algebra closes on $A^{\alpha\beta\gamma}_{10}$ provided they have the following gauge transformation:

$$\delta A^{\alpha\beta\gamma}_{10} = d\Lambda^{\alpha\beta\gamma}_9 - \frac{1}{15}F_9^{(\alpha\beta}\wedge\Lambda_1^{\gamma)} + \frac{4}{15}F_3^{(\alpha}\wedge\Lambda_7^{\beta\gamma)}\,. \tag{1.1.58}$$

Under supersymmetry the ten-form quadruplet transforms as

$$\begin{aligned}
\delta A^{\alpha\beta\gamma}_{\mu_1\ldots\mu_{10}} \;=\;& iV_+^{(\alpha}V_+^{\beta}V_-^{\gamma)}\bar\epsilon_C\gamma_{\mu_1\ldots\mu_{10}}\lambda_C - iV_-^{(\alpha}V_-^{\beta}V_+^{\gamma)}\bar\epsilon\gamma_{\mu_1\ldots\mu_{10}}\lambda\\
&+\frac{20}{3}V_+^{(\alpha}V_+^{\beta}V_-^{\gamma)}\bar\epsilon\gamma_{[\mu_1\ldots\mu_9}\psi_{C\,\mu_{10}]} - \frac{20}{3}V_-^{(\alpha}V_-^{\beta}V_+^{\gamma)}\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_9}\psi_{\mu_{10}]}\\
&-12A^{(\alpha\beta}_{[\mu_1\ldots\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + 3A^{(\alpha}_{[\mu_1\mu_2}\delta A^{\beta\gamma)}_{\mu_3\ldots\mu_{10}]}\\
&-\frac{63}{4}A^{(\alpha}_{[\mu_1\ldots\mu_6}A^{\beta}_{\mu_7\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + \frac{21}{4}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}\delta A^{\gamma)}_{\mu_5\ldots\mu_{10}]}\\
&-210A_{[\mu_1\ldots\mu_4}A^{(\alpha}_{\mu_5\mu_6}A^{\beta}_{\mu_7\mu_8}\delta A^{\gamma)}_{\mu_9\mu_{10}]} + 105A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}A^{\gamma)}_{\mu_5\mu_6}\delta A_{\mu_7\ldots\mu_{10}]}\\
&+\frac{315i}{8}A^{(\alpha}_{[\mu_1\mu_2}A^{\beta}_{\mu_3\mu_4}A^{\gamma)}_{\mu_5\mu_6}\epsilon_{\delta\tau}A^{\delta}_{\mu_7\mu_8}\delta A^{\tau}_{\mu_9\mu_{10}]}\,.
\end{aligned} \tag{1.1.59}$$

Besides the quadruplet of 10-forms there also exists a doublet of 10-form potentials A^α_{10} satisfying the reality condition: $(A^1_{10})^* = A^2_{10}$. In [22] the form of the supersymmetry transformation for A^α_{10} is given as

$$\begin{aligned}
\delta A^{\alpha}_{\mu_1\ldots\mu_{10}} \;=\;& V_-^\alpha\bar\epsilon\gamma_{\mu_1\ldots\mu_{10}}\lambda + V_+^\alpha\bar\epsilon_C\gamma_{\mu_1\ldots\mu_{10}}\lambda_C\\
&+20iV_+^\alpha\bar\epsilon\gamma_{[\mu_1\ldots\mu_9}\psi_{C\,\mu_{10}]} + 20iV_-^\alpha\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_9}\psi_{\mu_{10}]}\,.
\end{aligned} \tag{1.1.60}$$

The commutator of two supersymmetries acting on A^α_{10} produces the following gauge transformation

$$\delta A^\alpha_{10} = d\Lambda^\alpha_9\,. \tag{1.1.61}$$

However, in [26] it is shown, using superspace techniques, that the Bianchi identity for the 11-form field strength F^α_{11} is given by

$$dF^\alpha_{11} = a\left(i\epsilon_{\beta\gamma}F_3^\beta\wedge F_9^{\gamma\alpha} + 3F_5\wedge F_7^\alpha\right)\,, \tag{1.1.62}$$

where a is some nonzero real number. This Bianchi identity is solved by[3]

$$\begin{aligned} F_{11}^{\alpha} &= dA_{10}^{\alpha} + \frac{ai}{5}\epsilon_{\beta\gamma}A_2^{\beta} \wedge F_9^{\gamma\alpha} - \frac{4ai}{5}\epsilon_{\beta\gamma}F_3^{\beta} \wedge A_8^{\gamma\alpha} \\ &\quad + \frac{6a}{5}A_4 \wedge F_7^{\alpha} - \frac{9a}{5}F_5 \wedge A_6^{\alpha}\,, \end{aligned} \tag{1.1.63}$$

and implies the following gauge transformation for A_{10}^{α}

$$\begin{aligned} \delta A_{10}^{\alpha} &= d\tilde{\Lambda}_9^{\alpha} - \frac{ai}{5}\epsilon_{\beta\gamma}\Lambda_1^{\beta} \wedge F_9^{\gamma\alpha} - \frac{4ai}{5}\epsilon_{\beta\gamma}F_3^{\beta} \wedge \Lambda_7^{\gamma\alpha} \\ &\quad - \frac{6a}{5}\Lambda_3 \wedge F_7^{\alpha} - \frac{9a}{5}F_5 \wedge \Lambda_5^{\alpha}\,. \end{aligned} \tag{1.1.64}$$

There exists no redefinition of $\tilde{\Lambda}_9^{\alpha}$ appearing in (1.1.64) such that eqs. (1.1.61) and (1.1.64) are equivalent. There is thus a conflict situation if the 10-forms A_{10}^{α} in [22] and in [26] are the same. It is at present unclear what the resolution of the difference between eq. (1.1.61) and eq. (1.1.64) is (see also the discussion at the end of section (2.5)).

1.2 The scalars T and χ'

An $SU(1,1)$ charge tensor $q_{\alpha\beta} = q_{\beta\alpha}$ transforming in the adjoint of $SU(1,1)$ is introduced. Its components are raised and lowered with the $SU(1,1)$ invariant metric $\epsilon_{\alpha\beta}$,

$$q^{\alpha}{}_{\beta} = \epsilon^{\alpha\gamma}q_{\gamma\beta} \qquad \text{raising with the second index on } \epsilon\,, \tag{1.2.1}$$

$$q_{\alpha\beta} = \epsilon_{\gamma\alpha}q^{\gamma}{}_{\beta} \qquad \text{lowering with the first index on } \epsilon\,. \tag{1.2.2}$$

If one exponentiates the matrix $i\,q^{\alpha}{}_{\beta}$ (left upper index for the rows and the right lower index for the columns) the resulting matrix is an element of the group $SU(1,1)$. The charge tensor $q_{\alpha\beta}$ can be used to construct scalars that are invariant under local $U(1)$ transformations. Since the manifold $SU(1,1)/U(1)$ is two dimensional it can be parameterized using two scalars. Two scalars, denoted by T and χ', are defined as

$$T = q_{\alpha\beta}V_+^{\alpha}V_-^{\beta}\,, \tag{1.2.3}$$

$$d\chi' = -i\frac{P}{q_{\alpha\beta}V_+^{\alpha}V_+^{\beta}} + i\frac{\bar{P}}{q_{\alpha\beta}V_-^{\alpha}V_-^{\beta}}\,. \tag{1.2.4}$$

It is shown in appendix B that χ' is a parity odd scalar.

[3] In ten space-time dimensions it does not make sense to talk about a gauge invariant 11-form field strength since each term in (1.1.63) vanishes identically by itself. It is however useful as a tool to obtain the gauge transformation for A_{10}^{α}.

It will prove convenient to employ the following parametrization of $q_{\alpha\beta}$:

$$q_{\alpha\beta} = q_\alpha q_\beta + 4\det Q\, \tilde{q}_\alpha \tilde{q}_\beta\,, \tag{1.2.5}$$

where q_α and $\tilde{q}_\beta$ satisfy

$$q_{[\alpha}\tilde{q}_{\beta]} = \frac{i}{2}\epsilon_{\alpha\beta}\,. \tag{1.2.6}$$

The reality properties of q_α and $\tilde{q}_\alpha$ are $(q_1)^* = q_2$ and $(\tilde{q}_1)^* = \tilde{q}_2$. The parameter $\det Q$ is the determinant of a matrix Q defined by

$$Q = -\frac{i}{2}S^{-1}\mathbf{q}S\,, \qquad S = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, \tag{1.2.7}$$

where $\mathbf{q}$ is the matrix whose components are $q^\alpha{}_\beta$. The matrix S establishes the isomorphism between $SU(1,1)$ and $SL(2,\mathbb{R})$. The matrix e^Q is an element of $SL(2,\mathbb{R})$. When $\det Q = 0$ and $\det Q > 0$ the matrices e^Q form the $\mathbb{R}$ and $SO(2)$ subgroups of $SL(2,\mathbb{R})$, respectively[4]. In eq. (1.2.5) the value of $\det Q$ is independent of the parameters q_α and $\tilde{q}_\alpha$ so that when using (1.2.5) it can be freely send to zero. The limit $\det Q \to 0$ (through positive values of $\det Q$) corresponds to going from $SO(2)$ to $\mathbb{R}$.

It is possible to express dT using (1.1.13) and (1.1.14) in terms of P and $\bar{P}$. By combining the expressions for dT and $d\chi'$ the following expression for P, that is defined in eq. (1.1.7), in terms T and χ' can be obtained

$$P = \left(\frac{q_{\alpha\beta}V_+^\alpha V_+^\beta}{q_{\gamma\delta}V_-^\gamma V_-^\delta}\right)^{1/2}\left[\frac{1}{2}\frac{dT}{(T^2 - 4\det Q)^{1/2}} + \frac{i}{2}(T^2 - 4\det Q)^{1/2}d\chi'\right], \tag{1.2.8}$$

where the identity

$$q_{\alpha\beta}V_+^\alpha V_+^\beta\, q_{\gamma\delta}V_-^\gamma V_-^\delta = T^2 - 4\det Q \tag{1.2.9}$$

has been used. In deriving (1.2.9) one uses the completeness relation (1.1.5) and the parametrization (1.2.5). Since the left hand side of eq. (1.2.9) forms the modulus squared of a complex number the right hand side is non-negative. Hence it must be that $T^2 \geq 4\det Q$. Further, from the definition of T, eq. (1.2.3), and the parametrization of $q_{\alpha\beta}$, eq. (1.2.5), it follows that T can be written as the sum of two squares, so that $T \geq 0$. This and eq. (1.2.9) implies $T \geq 2\sqrt{\det Q}$. The factor in front of the square brackets in (1.2.8) is immaterial and forms a local $U(1)$ phase factor. It can thus be gauged away using (1.1.10).

The defining equation for χ', eq. (1.2.4), will be solved for the cases $\det Q = 0$ and $\det Q > 0$ separately.

[4]For the purposes of this thesis the case $\det Q < 0$ will not be interesting for reasons explained in subsection 3.8.4.

1.2.1 The $\det Q = 0$ case

When $\det Q = 0$ the solution to eq. (1.2.4) is

$$\chi' = \frac{2q_{(\alpha}\tilde{q}_{\beta)} V_+^\alpha V_-^\beta}{q_\gamma q_\delta V_+^\gamma V_-^\delta} \tag{1.2.10}$$

The $U(1)$ gauge can be fixed by taking

$$q_\alpha V_+^\alpha = q_\alpha V_-^\alpha \,. \tag{1.2.11}$$

It then follows from this gauge choice together with the completeness relation (1.1.5) and the identity (1.2.9) that

$$q_\alpha V_+^\alpha = q_\alpha V_-^\alpha = T^{1/2} \,, \tag{1.2.12}$$

$$\tilde{q}_\alpha V_+^\alpha = (\tilde{q}_\alpha V_-^\alpha)^* = \frac{T^{1/2}}{2}\left(\chi' + \frac{i}{T}\right) \,. \tag{1.2.13}$$

1.2.2 The $\det Q > 0$ case

The $U(1)$ gauge symmetry can be fixed by taking

$$\frac{q_{\alpha\beta} V_+^\alpha V_+^\beta}{q_{\gamma\delta} V_-^\gamma V_-^\delta} = e^{4i\sqrt{\det Q}}\chi' \,. \tag{1.2.14}$$

It is also possible to write an expression for χ' without fixing the $U(1)$ gauge. This goes by solving eq. (1.2.4) for χ'.

When $\det Q > 0$ both the parameters q_α and $\tilde{q}_\alpha$ can be parameterized in terms of $q_{\alpha\beta}$. This is not possible when $\det Q = 0$ because in that case $q_{\alpha\beta}$, as is clear from eq. (1.2.5), does not depend on $\tilde{q}_\alpha$ and eq. (1.2.6) is not sufficient to find $\tilde{q}_\alpha$ in terms of $q_{\alpha\beta}$. This fact will be of some relevance later in subsection 1.3.3 when q_α and $\tilde{q}_\alpha$ will be parameterized in terms of $q_{\alpha\beta}$. When q_α and $\tilde{q}_\alpha$ are parameterized in terms of $q_{\alpha\beta}$ the parameter $\det Q$ depends on q_α and $\tilde{q}_\alpha$. It is then no longer possible to send $\det Q$ to zero independently of q_α and $\tilde{q}_\alpha$.

Using the parametrization (1.2.5) of $q_{\alpha\beta}$ eq. (1.2.4) can be solved for χ' giving

$$e^{4i\sqrt{\det Q}}\chi' = e^{i\theta}\frac{\left(q_\alpha + 2i\sqrt{\det Q}\tilde{q}_\alpha\right) V_+^\alpha}{\left(q_\gamma - 2i\sqrt{\det Q}\tilde{q}_\gamma\right) V_+^\gamma}\,\frac{\left(q_\beta + 2i\sqrt{\det Q}\tilde{q}_\beta\right) V_-^\beta}{\left(q_\delta - 2i\sqrt{\det Q}\tilde{q}_\delta\right) V_-^\delta} \,, \tag{1.2.15}$$

where $\theta \in \mathbb{R}$ is an integration constant.

For notational convenience the following combinations of q_α and $\tilde{q}_\alpha$ are defined:

$$s_\alpha = e^{-i\theta/4}\left(q_\alpha - 2i\sqrt{\det Q}\tilde{q}_\alpha\right) \,, \tag{1.2.16}$$

$$t_\alpha = e^{i\theta/4}\left(q_\alpha + 2i\sqrt{\det Q}\tilde{q}_\alpha\right) \,. \tag{1.2.17}$$

These combinations satisfy the properties:

$$q_{\alpha\beta} = t_{(\alpha}s_{\beta)}\,, \tag{1.2.18}$$
$$t_{[\alpha}s_{\beta]} = 2\sqrt{\det Q}\,\epsilon_{\alpha\beta}\,, \tag{1.2.19}$$
$$(t_1)^* = s_2\,, \qquad (t_2)^* = s_1\,. \tag{1.2.20}$$

By comparing the gauge fixed expression (1.2.14) with the ungauge fixed expression (1.2.15) the gauge fixing condition (1.2.14) can be rewritten as

$$s_\alpha V_+^\alpha = t_\alpha V_-^\alpha\,. \tag{1.2.21}$$

In this gauge fixed setting $V_\pm^\alpha$ are parameterized as:

$$t_\alpha V_-^\alpha = \Big(T + 2\sqrt{\det Q}\Big)^{1/2}\,, \tag{1.2.22}$$
$$s_\alpha V_+^\alpha = \Big(T + 2\sqrt{\det Q}\Big)^{1/2}\,, \tag{1.2.23}$$
$$s_\alpha V_-^\alpha = \Big(T - 2\sqrt{\det Q}\Big)^{1/2} e^{-2i\sqrt{\det Q}\,\chi'}\,, \tag{1.2.24}$$
$$t_\alpha V_+^\alpha = \Big(T - 2\sqrt{\det Q}\Big)^{1/2} e^{2i\sqrt{\det Q}\,\chi'}\,. \tag{1.2.25}$$

When the charges $q_{\alpha\beta}$ are not quantized it is possible to imagine the parameter $\sqrt{\det Q}$ to be infinitesimally small. If one expands the formulae of this subsection in the parameter $\sqrt{\det Q}$ and keeps only terms that are of first order in $\sqrt{\det Q}$ all the expressions of subsection 1.2.1 are recovered (assuming that the integration constant θ goes to zero as $\det Q \to 0$).

The 1-form $d\chi'$ is the Hodge dual of the 9-form $q_{\alpha\beta}F_9^{\alpha\beta}$ as is clear from eqs. (1.1.34) and (1.2.9),

$$q_{\alpha\beta}F_9^{\alpha\beta} = \star(T^2 - 4\det Q)d\chi'\,. \tag{1.2.26}$$

1.3 Supersymmetry transformation rules in terms of T and χ'

It is desirable to express the supersymmetry transformation rules, eqs. (1.1.46) to (1.1.54), in terms of the scalars T and χ' for the cases $\det Q > 0$ and $\det Q = 0$ separately. As the $\det Q = 0$ case can be treated via a limit procedure from the $\det Q > 0$ case, the latter is treated first.

1.3.1 The $\det Q > 0$ case

It is not possible to apply the $U(1)$ gauge fixing condition (1.2.21) straightforwardly to the supersymmetry transformation rules (1.1.47) and (1.1.48) because the supersymmetry transformations (1.1.47) and (1.1.48) do not respect the gauge choice (1.2.21). This problem can be cured by redefining the local supersymmetry transformation δ with a local $U(1)$ transformation as

$$\tilde{\delta} = \delta + \delta_{U(1)} \,. \tag{1.3.1}$$

Then under $\tilde{\delta}$ the objects $V_\pm^\alpha$ transform as

$$\tilde{\delta} V_+^\alpha = V_-^\alpha \bar{\epsilon}_C \lambda - iS\, V_+^\alpha \,, \tag{1.3.2}$$

$$\tilde{\delta} V_-^\alpha = V_+^\alpha \bar{\epsilon} \lambda_C + iS\, V_-^\alpha \,, \tag{1.3.3}$$

where the second terms in eqs. (1.3.2) and (1.3.3) correspond to infinitesimal $U(1)$ transformations in which for S the following Ansatz is taken

$$S = a \bar{\epsilon}_C \lambda + a^* \bar{\epsilon} \lambda_C \,. \tag{1.3.4}$$

The condition that is imposed on the redefined supersymmetry transformation $\tilde{\delta}$ in order that it preserves the gauge choice (1.2.21) is

$$s_\alpha \tilde{\delta} V_+^\alpha = t_\alpha \tilde{\delta} V_-^\alpha \,. \tag{1.3.5}$$

The condition (1.3.5) requires a in (1.3.4) to be

$$a = -\frac{i}{2} \frac{s_\alpha V_-^\alpha}{s_\beta V_+^\beta} \,. \tag{1.3.6}$$

The redefined supersymmetry transformation $\tilde{\delta}$ only acts non-trivially on the scalars, i.e. on all other fields $\tilde{\delta} = \delta$ (up to second order in fermions). The reason is that the non-scalar bosonic fields do not transform under local $U(1)$ and the fermions do not see this local $U(1)$ as long as one works up to second order in fermions.

The objects P, Q and G_3 as defined in (1.1.7), (1.1.8) and (1.1.21), respectively can be expressed in terms of T and χ'. For this purpose one can use eq. (1.2.19) giving the relation between $\epsilon_{\alpha\beta}$, that appears in the definitions of P, Q and G_3, and t_α and s_α. Then one can employ the parametrization of $V_\pm^\alpha$ as given in eqs. (1.2.22) to (1.2.25) and the freedom to perform the local $U(1)$ transformation $P \to e^{-2i\alpha} P$, $Q \to Q - d\alpha$ and $G_3 \to e^{-i\alpha} G_3$ with parameter α given by

$$e^{-2i\alpha} = \left(\frac{q_{\alpha\beta} V_-^\alpha V_-^\beta}{q_{\gamma\delta} V_+^\gamma V_+^\delta} \right)^{1/2} \,, \tag{1.3.7}$$

to find

$$\begin{aligned} P &= \frac{1}{2}\frac{dT}{(T^2-4\det Q)^{1/2}} + \frac{i}{2}(T^2-4\det Q)^{1/2}d\chi'\,, &(1.3.8)\\ Q &= \frac{T}{2}\,d\chi'\,, &(1.3.9)\\ G_3 &= \frac{1}{4\sqrt{\det Q}}\left[\left(T+2\sqrt{\det Q}\right)^{1/2} e^{-i\sqrt{\det Q}\,\chi'}t_\alpha F_3^\alpha - \right.\\ &\quad \left.\left(T-2\sqrt{\det Q}\right)^{1/2} e^{i\sqrt{\det Q}\,\chi'}s_\alpha F_3^\alpha\right]\,, &(1.3.10)\\ F_5 &= dA_4 + \frac{i}{64\sqrt{\det Q}}\left(t_\alpha A_2^\alpha\wedge s_\beta F_3^\beta - s_\alpha A_2^\alpha\wedge t_\beta F_3^\beta\right)\,, &(1.3.11) \end{aligned}$$

where also the 5-form F_5 is given in terms of the 2-forms $s_\alpha A_2^\alpha$ and $t_\alpha A_2^\alpha$. Further also the spinors λ, ψ_μ and ϵ are redefined as

$$\lambda' = e^{-3i\alpha/2}\lambda\,,\qquad \psi'_\mu = e^{-i\alpha/2}\psi_\mu\,,\qquad \epsilon' = e^{-i\alpha/2}\epsilon\,. \qquad (1.3.12)$$

In terms of these primed spinors and the redefined supersymmetry transformation $\tilde\delta$ the supersymmetry transformation rules in terms of T and χ' read

$$\begin{aligned} \tilde\delta g_{\mu\nu} &= 2i\,\bar\epsilon'\gamma_{(\mu}\psi'_{\nu)} + 2i\,\bar\epsilon'_C\gamma_{(\mu}\psi'_{C\,\nu)}\,, &(1.3.13)\\ \tilde\delta T &= \left(T^2-4\det Q\right)^{1/2}\left(\bar\epsilon'_C\lambda' + \bar\epsilon'\lambda'_C\right)\,, &(1.3.14)\\ \tilde\delta\chi' &= \frac{i}{\left(T^2-4\det Q\right)^{1/2}}\left(\bar\epsilon'_C\lambda' - \bar\epsilon'\lambda'_C\right)\,, &(1.3.15)\\ t_\alpha\tilde\delta A^\alpha_{\mu\nu} &= e^{i\sqrt{\det Q}\,\chi'}\left[\left(T+2\sqrt{\det Q}\right)^{1/2}\bar\epsilon'\gamma_{\mu\nu}\lambda'\right.\\ &\quad + \left(T-2\sqrt{\det Q}\right)^{1/2}\bar\epsilon'_C\gamma_{\mu\nu}\lambda'_C\\ &\quad +4i\left(T+2\sqrt{\det Q}\right)^{1/2}\bar\epsilon'_C\gamma_{[\mu}\psi'_{\nu]}\\ &\quad \left.+4i\left(T-2\sqrt{\det Q}\right)^{1/2}\bar\epsilon'\gamma_{[\mu}\psi'_{C\,\nu]}\right]\,, \\ & &(1.3.16)\\ \delta A_{\mu\nu\rho\sigma} &= \bar\epsilon'\gamma_{[\mu\nu\rho}\psi'_{\sigma]} - \bar\epsilon'_C\gamma_{[\mu\nu\rho}\psi'_{C\,\sigma]} +\\ &\quad \frac{3i}{32\sqrt{\det Q}}\left(s_\alpha A^\alpha_{[\mu\nu}\,t_\beta\tilde\delta A^\beta_{\rho\sigma]} - t_\alpha A^\alpha_{[\mu\nu}\,s_\beta\tilde\delta A^\beta_{\rho\sigma]}\right)\,, &(1.3.17) \end{aligned}$$

$$
\begin{aligned}
t_\alpha \tilde{\delta} A^\alpha_{\mu_1\ldots\mu_6} &= e^{i\sqrt{\det Q}\,\chi'}\Big[i\left(T + 2\sqrt{\det Q}\right)^{1/2} \bar{\epsilon}'\gamma_{\mu_1\ldots\mu_6}\lambda' \\
&\quad - i\left(T - 2\sqrt{\det Q}\right)^{1/2} \bar{\epsilon}'_C\gamma_{\mu_1\ldots\mu_6}\lambda'_C \\
&\quad + 12\left(T + 2\sqrt{\det Q}\right)^{1/2} \bar{\epsilon}'_C\gamma_{[\mu_1\ldots\mu_5}\psi'_{\mu_6]} \\
&\quad - 12\left(T - 2\sqrt{\det Q}\right)^{1/2} \bar{\epsilon}'\gamma_{[\mu_1\ldots\mu_5}\psi'_{C\,\mu_6]}\Big] \\
&\quad + 40 A_{[\mu_1\ldots\mu_4}\, t_\alpha \tilde{\delta} A^\alpha_{\mu_5\mu_6]} - 20\tilde{\delta} A_{[\mu_1\ldots\mu_4}\, t_\alpha A^\alpha_{\mu_5\mu_6]} \\
&\quad - \frac{15i}{8\sqrt{\det Q}}\, t_\alpha A^\alpha_{[\mu_1\mu_2}\left(t_\beta A^\beta_{\mu_3\mu_4}\, s_\gamma \tilde{\delta} A^\gamma_{\mu_5\mu_6]} - s_\beta A^\beta_{\mu_3\mu_4}\, t_\gamma \tilde{\delta} A^\gamma_{\mu_5\mu_6]}\right)\,,
\end{aligned} \tag{1.3.18}
$$

$$
\begin{aligned}
q_{\alpha\beta}\tilde{\delta} A^{\alpha\beta}_{\mu_1\ldots\mu_8} &= -i\left(T^2 - 4\det Q\right)^{1/2}\left(\bar{\epsilon}'\gamma_{\mu_1\ldots\mu_8}\lambda'_C - \bar{\epsilon}'_C\gamma_{\mu_1\ldots\mu_8}\lambda'\right) \\
&\quad + 8T\left(\bar{\epsilon}'\gamma_{[\mu_1\ldots\mu_7}\psi'_{\mu_8]} - \bar{\epsilon}'_C\gamma_{[\mu_1\ldots\mu_7}\psi'_{C\,\mu_8]}\right) \\
&\quad + \frac{21}{8}\, t_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\, s_\beta \tilde{\delta} A^\beta_{\mu_7\mu_8]} + \frac{21}{8}\, s_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\, t_\beta \tilde{\delta} A^\beta_{\mu_7\mu_8]} \\
&\quad - \frac{7}{8}\, t_\alpha A^\alpha_{[\mu_1\mu_2}\, s_\beta \tilde{\delta} A^\beta_{\mu_3\ldots\mu_8]} - \frac{7}{8}\, s_\alpha A^\alpha_{[\mu_1\mu_2}\, t_\beta \tilde{\delta} A^\beta_{\mu_3\ldots\mu_8]} \\
&\quad + \frac{70}{2} A_{[\mu_1\ldots\mu_4}\, t_\alpha A^\alpha_{\mu_5\mu_6}\, s_\beta \tilde{\delta} A^\beta_{\mu_7\mu_8]} + \frac{70}{2} A_{[\mu_1\ldots\mu_4}\, s_\alpha A^\alpha_{\mu_5\mu_6}\, t_\beta \tilde{\delta} A^\beta_{\mu_7\mu_8]} \\
&\quad - 35\, t_\alpha A^\alpha_{[\mu_1\mu_2}\, s_\beta A^\beta_{\mu_3\mu_4}\tilde{\delta} A_{\mu_5\ldots\mu_8]} \\
&\quad - \frac{105i}{64\sqrt{\det Q}}\left(t_\alpha A^\alpha_{[\mu_1\mu_2}\, s_\beta A^\beta_{\mu_3\mu_4} + s_\alpha A^\alpha_{[\mu_1\mu_2}\, t_\beta A^\beta_{\mu_3\mu_4}\right)\times \\
&\quad \times\left(t_\gamma A^\gamma_{\mu_5\mu_6}\, s_\delta \tilde{\delta} A^\delta_{\mu_7\mu_8]} - s_\gamma A^\gamma_{\mu_5\mu_6}\, t_\delta \tilde{\delta} A^\delta_{\mu_7\mu_8]}\right)\,,
\end{aligned} \tag{1.3.19}
$$

$$
\tilde{\delta}\lambda' = iP_\mu\gamma^\mu\epsilon'_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon'\,, \tag{1.3.20}
$$

$$
\begin{aligned}
\tilde{\delta}\psi'_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon' + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon' \\
&\quad + \frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon'_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon'_C\,,
\end{aligned} \tag{1.3.21}
$$

with P, Q, G_3 and F_5 as given in eqs. (1.3.8) to (1.3.11). The supersymmetry transformation rules for $s_\alpha A^\alpha_{2,6}$ follow by taking the complex conjugate of eqs. (1.3.16) and (1.3.18).

1.3.2 The scalars η and φ and the p-forms $\mathcal{A}_p$

The scalar kinetic terms for T and χ' that follow from the term $2\star P\wedge\bar{P}$ in the action (1.1.17) are given by

$$2\star P\wedge\bar{P}=\frac{1}{2}\frac{\star dT\wedge dT}{T^2-4\det Q}+\frac{1}{2}\left(T^2-4\det Q\right)\star d\chi'\wedge d\chi'\,. \tag{1.3.22}$$

The kinetic term for T is not canonically normalized. Further, below eq. (1.2.9) it is concluded that $T\geq 2\sqrt{\det Q}$. The scalar $2\sqrt{\det Q}\,\chi'$ for the case $\det Q>0$ is an angular variable with period 2π as is clear from eq. (1.2.24) or (1.2.25): $2\sqrt{\det Q}\,\chi'$ and $2\sqrt{\det Q}\,\chi'+2\pi$ are to be identified. In order to work with canonically normalized scalars the scalars η and φ are introduced via

$$\frac{T}{2\sqrt{\det Q}} = \cosh\eta\,, \tag{1.3.23}$$

$$2\sqrt{\det Q}\,\chi' = \varphi\,, \tag{1.3.24}$$

where $0\leq\eta<\infty$ and $0\leq\varphi<2\pi$. Eq. (1.3.22) then becomes equal to

$$\frac{1}{2}\star d\eta\wedge d\eta+\frac{1}{2}\sinh^2\eta\star d\varphi\wedge d\varphi\,. \tag{1.3.25}$$

The geometrical interpretation of η and φ as coordinates on the moduli space will be given in section 1.4.

The supersymmetry transformation rules eqs. (1.3.13) to (1.3.21) can readily be rewritten in terms of η and φ. When doing so it is, for notational purposes, convenient to write the supersymmetry transformations rules for the form fields in terms of the following 2,6,8-forms, denoted by $\mathcal{A}_{2,6,8}$, and defined via

$$\mathcal{A}_{2,6} = \frac{t_\alpha A^\alpha_{2,6}}{2(\det Q)^{1/4}}\,, \tag{1.3.26}$$

$$\mathcal{A}_8 = q_{\alpha\beta}A_8^{\alpha\beta}\,. \tag{1.3.27}$$

The 4-form A_4 does require such a redefinition. The 2- and 6-forms $\mathcal{A}_{2,6}$ are complex-valued while the 8-form $\mathcal{A}_8$ is real-valued. When working with the fields η, φ, $\mathcal{A}_{2,6,8}$ it is assumed that $\det Q$ is some fixed non-zero number. In this formulation the following symmetry is manifest:

$$\eta \to \eta\,,\quad (T\to T) \tag{1.3.28}$$

$$\varphi \to \varphi+2\sqrt{\det Q}\,,\quad (\chi'\to\chi'+1) \tag{1.3.29}$$

$$\mathcal{A}_{2,6} \to e^{i\sqrt{\det Q}}\mathcal{A}_{2,6}\,, \tag{1.3.30}$$

$$\mathcal{A}_8 \to \mathcal{A}_8\,, \tag{1.3.31}$$

$$\lambda' \to \lambda'\,, \tag{1.3.32}$$

$$\psi'_\mu \to \psi'_\mu\,,\qquad \epsilon'\to\epsilon'\,. \tag{1.3.33}$$

This set of transformation forms the $SO(2)$ subgroup of the classical $SL(2,\mathbb{R})$ duality group.

It is straightforward to write down the IIB action or the equations of motion in terms of the scalars η and φ and the form fields $\mathcal{A}_2$ and A_4. This can be achieved by substituting the definitions (1.3.23), (1.3.24) and (1.3.26) into the expressions for P, G_3 and F_5 as given in eqs. (1.3.8), (1.3.10) and (1.3.11) in terms of which the action (1.1.17) is written. The result is:

$$P = \frac{1}{2}d\eta + \frac{i}{2}\sinh\eta\, d\varphi\,, \tag{1.3.34}$$

$$Q = \frac{1}{2}\cosh\eta\, d\varphi\,, \tag{1.3.35}$$

$$G_3 = \cosh\tfrac{\eta}{2}\, e^{-i\varphi/2} d\mathcal{A}_2 - \sinh\tfrac{\eta}{2}\, e^{i\varphi/2} d\bar{\mathcal{A}}_2\,, \tag{1.3.36}$$

$$F_5 = dA_4 + \frac{i}{16}\left(\mathcal{A}_2 \wedge d\bar{\mathcal{A}}_2 - \bar{\mathcal{A}}_2 \wedge d\mathcal{A}_2\right)\,. \tag{1.3.37}$$

1.3.3 The (p,q,r) parametrization

In eq. (1.2.7) the matrix Q was defined in terms of the numbers $q_{\alpha\beta}$. The traceless $SL(2,\mathbb{R})$ algebra valued charge matrix Q can be parameterized by three numbers, that will be denoted by p,q,r, as

$$Q = \begin{pmatrix} r/2 & p \\ -q & -r/2 \end{pmatrix}\,. \tag{1.3.38}$$

By writing p and q in terms of t_α and s_α using eqs. (1.3.38), (1.2.7) and (1.2.5) it can be shown that $p,q \geq 0$.

The purpose of introducing the numbers p,q,r is two-fold. On the one hand it will enable one to relate the variables η and φ to the complex axidilaton field τ in terms of which IIB supergravity is often formulated. On the other hand from the point of view of quantization the $SL(2,\mathbb{R})$ covariant formulation is more convenient than the $SU(1,1)$ covariant formulation.

Consider the complex axidilaton field $\tau = \chi + ie^{-\phi}$ with χ the RR scalar and ϕ the dilaton that transforms as

$$\Lambda\tau = \frac{a\tau+b}{c\tau+d} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Lambda = e^Q\,. \tag{1.3.39}$$

A point τ_0 is a fixed point under the e^Q transformation if it satisfies the equation

$$e^Q\tau_0 = \tau_0\,. \tag{1.3.40}$$

The fixed points τ_0 of e^Q with $0 \leq \operatorname{Im}\tau_0 < \infty$ for $\det Q \geq 0$ are given by

$$\tau_0 = -\frac{r}{2q} + \frac{i}{q}\sqrt{\det Q}\,. \tag{1.3.41}$$

The relation between (η,φ) (or (T,χ')) and $(\tau,\bar\tau)$ is established by taking, within the $U(1)$ gauge choice (1.2.21),

$$\frac{t_\alpha V^\alpha_+}{s_\beta V^\beta_+}=\frac{\tau-\tau_0}{\tau-\bar\tau_0}\,. \tag{1.3.42}$$

The left hand side of eq. (1.3.42) can be related to T and χ' via eqs. (1.2.23) and (1.2.25). Transforming T and χ' as in eqs. (1.3.28) and (1.3.29) can be seen to be equivalent to transforming $\tau\to e^Q\tau$ with $\det Q>0$ under which the right hand side of eq. (1.3.42) transforms as

$$\frac{\tau-\tau_0}{\tau-\bar\tau_0}\to e^{2i\sqrt{\det Q}}\,\frac{\tau-\tau_0}{\tau-\bar\tau_0}\,, \tag{1.3.43}$$

Next, for the 2- 6- and 8-forms, the combinations, denoted by B_2, C_2, C_6, B_6, C_8, B_8 and D_8 that are defined via

$$B_2 = \frac{1}{2}(A_2^1+A_2^2)\,, \tag{1.3.44}$$
$$C_2 = \frac{i}{2}(A_2^1-A_2^2)\,, \tag{1.3.45}$$
$$C_6 = \frac{1}{2}(A_6^1+A_6^2)\,, \tag{1.3.46}$$
$$B_6 = \frac{i}{2}(A_6^1-A_6^2)\,, \tag{1.3.47}$$
$$C_8 = A_8^{11}+A_8^{22}+2A_8^{12}\,, \tag{1.3.48}$$
$$B_8 = -A_8^{11}-A_8^{22}+2A_8^{12}\,, \tag{1.3.49}$$
$$D_8 = 2iA_8^{11}-2iA_8^{22}\,. \tag{1.3.50}$$

are taken to transform under $SL(2,\mathbb{R})$ as:

$$\begin{pmatrix} C_2 \\ B_2\end{pmatrix} \to \Lambda\begin{pmatrix} C_2 \\ B_2\end{pmatrix}\,, \tag{1.3.51}$$
$$\begin{pmatrix} B_6 \\ C_6\end{pmatrix} \to \Lambda\begin{pmatrix} B_6 \\ C_6\end{pmatrix}\,, \tag{1.3.52}$$
$$\begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8\end{pmatrix} \to \Lambda\begin{pmatrix} \frac{1}{2}D_8 & -B_8 \\ C_8 & -\frac{1}{2}D_8\end{pmatrix}\Lambda^{-1}\,, \tag{1.3.53}$$

where

$$\Lambda=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\,, \qquad \text{with} \qquad \det\Lambda=1\,. \tag{1.3.54}$$

The 2- and 6-forms B_2 and B_6 denote the NSNS 2- and 6-forms, respectively. While the 2-, 6- and 8-forms C_2, C_6 and C_8, represent the RR 2-, 6- and 8-forms, respectively[5]. Finally, the 8-form D_8 is needed in order to have explicit $SL(2,\mathbb{R})$ covariance. The 2- and 6-forms transform non-trivially under the full $SL(2,\mathbb{R})$ group while the 8-forms only transform non-trivially under the $PSL(2,\mathbb{R})$ group.

In terms of the NSNS and RR fields the form fields $\mathcal{A}_{2,6,8}$ can be written as:

$$\begin{aligned}
\mathcal{A}_2 &= \frac{-i}{(\mathrm{Im}\,\tau_0)^{1/2}}\left(-C_2+\tau_0 B_2\right), && (1.3.55)\\
\mathcal{A}_6 &= \frac{-i}{(\mathrm{Im}\,\tau_0)^{1/2}}\left(-B_6+\tau_0 C_6\right), && (1.3.56)\\
\mathcal{A}_8 &= pC_8+qB_8+\tfrac{r}{2}D_8\,, && (1.3.57)
\end{aligned}$$

where the relation between the parameter t_α appearing in (1.3.26) and in (1.3.27) (through eqs. (1.2.18) and (1.2.20)) and the parameter τ_0 is given by

$$\begin{aligned}
\frac{t_1}{2(\det Q)^{1/4}} &= \frac{-i}{2(\mathrm{Im}\,\tau_0)^{1/2}}(\tau_0-i)\,, && (1.3.58)\\
\frac{t_2}{2(\det Q)^{1/4}} &= \frac{-i}{2(\mathrm{Im}\,\tau_0)^{1/2}}(\tau_0+i)\,. && (1.3.59)
\end{aligned}$$

If one then transforms $\mathcal{A}_{2,6,8}$ as given in eqs. (1.3.55), (1.3.56) and (1.3.57) under the $SL(2,\mathbb{R})$ transformations (1.3.51), (1.3.52) and (1.3.53) with $\Lambda = e^Q$ the transformation rules (1.3.30) and (1.3.31) are recovered for the case $\det Q > 0$.

Associated with the matrix of 8-form potentials transforming in the adjoint of $SL(2,\mathbb{R})$, eq. (1.3.53), there is a matrix of 9-form field strengths that can be seen to be dual to $-2S^{-1}pS$, i.e.

$$-2S^{-1}pS = \star\begin{pmatrix} \frac{1}{2}dD_8 & -dB_8 \\ dC_8 & -\frac{1}{2}dD_8 \end{pmatrix} + \cdots \qquad (1.3.60)$$

with S as defined in (1.2.7) and with p as defined in (1.1.15). The dots in (1.3.60) denote contributions from wedge products of lower rank potentials that can be obtained using eq. (1.1.36) and eqs. (1.3.48) to (1.3.50). The matrix $-2S^{-1}pS$ is the matrix of Noether currents that derive from the $SL(2,\mathbb{R})$ invariance of the IIB action.

1.3.4 The $\det Q = 0$ case

The supersymmetry transformation rules in terms of T and χ' for the $\det Q = 0$ case can be obtained by taking the formulae (1.3.13) to (1.3.21) and treating $\sqrt{\det Q}$ as an

[5] From the point of view of the gauge transformations the fields $C_{6,8}$ (as well as the 4-form A_4) are not yet quite the RR gauge potentials. One can redefine the potentials with wedge products of lower rank potentials such that the gauge transformations are those of the RR potentials. The same comment applies to the NSNS 6-form B_6. These redefinitions with lower rank potentials are given in [27].

infinitesimally small parameter keeping only terms that are of first order in $\sqrt{\det Q}$. In order to take the $\det Q \to 0$ limit for the 2-forms it is convenient to take the limit of the following real combinations:

$$\frac{1}{2}\left(t_\alpha \tilde{\delta} A^\alpha_{\mu\nu} + s_\alpha \tilde{\delta} A^\alpha_{\mu\nu}\right) , \tag{1.3.61}$$

$$\frac{1}{2i}\left(t_\alpha \tilde{\delta} A^\alpha_{\mu\nu} - s_\alpha \tilde{\delta} A^\alpha_{\mu\nu}\right) , \tag{1.3.62}$$

and likewise for the 6-forms. Before stating the full result of the limit procedure it is mentioned that the primed spinors as defined in (1.3.12), at zeroth order in $\sqrt{\det Q}$, are equal to the unprimed spinors and that the first order relations between the primed and unprimed spinors in taking the $\det Q \to 0$ of (1.3.13) to (1.3.21) are never needed, so that the prime on λ, ψ_μ and ϵ can be dropped. The resulting supersymmetry transformation rules are:

$$\begin{aligned}
\tilde{\delta} g_{\mu\nu} &= 2i\,\bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + 2i\,\bar{\epsilon}_C\gamma_{(\mu}\psi_{C\,\nu)} \,, && (1.3.63)\\
\tilde{\delta} T &= T\bar{\epsilon}_C\lambda + T\bar{\epsilon}\lambda_C \,, && (1.3.64)\\
\tilde{\delta}\chi' &= \frac{i}{T}\,\bar{\epsilon}_C\lambda - \frac{i}{T}\,\bar{\epsilon}\lambda_C \,, && (1.3.65)\\
q_\alpha\tilde{\delta} A^\alpha_{\mu\nu} &= T^{1/2}\bar{\epsilon}\gamma_{\mu\nu}\lambda + T^{1/2}\bar{\epsilon}_C\gamma_{\mu\nu}\lambda_C + 4iT^{1/2}\bar{\epsilon}_C\gamma_{[\mu}\psi_{\nu]} \\
&\quad +4iT^{1/2}\bar{\epsilon}\gamma_{[\mu}\psi_{C\,\nu]} \,, && (1.3.66)\\
\tilde{q}_\alpha\tilde{\delta} A^\alpha_{\mu\nu} &= -\frac{i}{2}T^{-1/2}\bar{\epsilon}\gamma_{\mu\nu}\lambda + \frac{i}{2}T^{-1/2}\bar{\epsilon}_C\gamma_{\mu\nu}\lambda_C + 2T^{-1/2}\bar{\epsilon}_C\gamma_{[\mu}\psi_{\nu]} \\
&\quad -2T^{-1/2}\bar{\epsilon}\gamma_{[\mu}\psi_{C\,\nu]} + \frac{1}{2}\chi' q_\alpha\tilde{\delta} A^\alpha_{\mu\nu} \,, && (1.3.67)\\
\delta A_{\mu\nu\rho\sigma} &= \bar{\epsilon}\gamma_{[\mu\nu\rho}\psi_{\sigma]} - \bar{\epsilon}_C\gamma_{[\mu\nu\rho}\psi_{C\,\sigma]} \\
&\quad -\frac{3}{8}\,q_\alpha A^\alpha_{[\mu\nu}\,\tilde{q}_\beta\tilde{\delta} A^\beta_{\rho\sigma]} + \frac{3}{8}\,\tilde{q}_\alpha A^\alpha_{[\mu\nu}\,q_\beta\tilde{\delta} A^\beta_{\rho\sigma]} \,, && (1.3.68)\\
q_\alpha\tilde{\delta} A^\alpha_{\mu_1\ldots\mu_6} &= iT^{1/2}\bar{\epsilon}\gamma_{\mu_1\ldots\mu_6}\lambda - iT^{1/2}\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_6}\lambda_C \\
&\quad +12T^{1/2}\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} - 12T^{1/2}\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]} \\
&\quad +40A_{[\mu_1\ldots\mu_4}\,q_\alpha\tilde{\delta} A^\alpha_{\mu_5\mu_6]} - 20\tilde{\delta} A_{[\mu_1\ldots\mu_4}\,q_\alpha A^\alpha_{\mu_5\mu_6]} \\
&\quad -\frac{15}{2}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\left(q_\beta A^\beta_{\mu_3\mu_4}\,\tilde{q}_\gamma\tilde{\delta} A^\gamma_{\mu_5\mu_6]} - \tilde{q}_\beta A^\beta_{\mu_3\mu_4}\,q_\gamma\tilde{\delta} A^\gamma_{\mu_5\mu_6]}\right), && (1.3.69)\\
\tilde{q}_\alpha\tilde{\delta} A^\alpha_{\mu_1\ldots\mu_6} &= \frac{i}{2}T^{1/2}(\chi' - \frac{i}{T})\bar{\epsilon}\gamma_{\mu_1\ldots\mu_6}\lambda - \frac{i}{2}T^{1/2}(\chi' + \frac{i}{T})\bar{\epsilon}_C\gamma_{\mu_1\ldots\mu_6}\lambda_C \\
&\quad +6T^{1/2}(\chi' - \frac{i}{T})\bar{\epsilon}_C\gamma_{[\mu_1\ldots\mu_5}\psi_{\mu_6]} - T^{1/2}(\chi' + \frac{i}{T})\bar{\epsilon}\gamma_{[\mu_1\ldots\mu_5}\psi_{C\,\mu_6]}
\end{aligned}$$

$$
\begin{aligned}
& +40 A_{[\mu_1\ldots\mu_4}\,\tilde q_\alpha \tilde\delta A^\alpha_{\mu_5\mu_6]} - 20\tilde\delta A_{[\mu_1\ldots\mu_4}\,\tilde q_\alpha A^\alpha_{\mu_5\mu_6]} \\
& -\frac{15}{2}\,\tilde q_\alpha A^\alpha_{[\mu_1\mu_2}\left(q_\beta A^\beta_{\mu_3\mu_4}\,\tilde q_\gamma \tilde\delta A^\gamma_{\mu_5\mu_6]} - \tilde q_\beta A^\beta_{\mu_3\mu_4}\,q_\gamma \tilde\delta A^\gamma_{\mu_5\mu_6]}\right)\,, \qquad (1.3.70)
\end{aligned}
$$

$$
\begin{aligned}
q_\alpha q_\beta \tilde\delta A^{\alpha\beta}_{\mu_1\ldots\mu_8} \;=\; & -iT\bar\epsilon\gamma_{\mu_1\ldots\mu_8}\lambda_C + iT\bar\epsilon_C\gamma_{\mu_1\ldots\mu_8}\lambda \\
& +8T\,\bar\epsilon\gamma_{[\mu_1\ldots\mu_7}\psi_{\mu_8]} - 8T\,\bar\epsilon_C\gamma_{[\mu_1\ldots\mu_7}\psi_{C\,\mu_8]} \\
& +\frac{21}{4}\,q_\alpha A^\alpha_{[\mu_1\ldots\mu_6}\,q_\beta\tilde\delta A^\beta_{\mu_7\mu_8]} - \frac{7}{4}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta\tilde\delta A^\beta_{\mu_3\ldots\mu_8]} \\
& -35\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta A^\beta_{\mu_3\mu_4}\tilde\delta A_{\mu_5\ldots\mu_8]} + 70 A_{[\mu_1\ldots\mu_4}\,q_\alpha A^\alpha_{\mu_5\mu_6}\,q_\beta\tilde\delta A^\beta_{\mu_7\mu_8]} \\
& -\frac{105}{8}\,q_\alpha A^\alpha_{[\mu_1\mu_2}\,q_\beta A^\beta_{\mu_3\mu_4}\left(q_\gamma A^\gamma_{\mu_5\mu_6}\,\tilde q_\delta\tilde\delta A^\delta_{\mu_7\mu_8]} - \tilde q_\gamma A^\gamma_{\mu_5\mu_6}\,q_\delta\tilde\delta A^\delta_{\mu_7\mu_8]}\right)\,,
\end{aligned}
\qquad (1.3.71)
$$

$$
\tilde\delta\lambda = iP_\mu\gamma^\mu\epsilon_C - \frac{i}{24}G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon\,, \qquad (1.3.72)
$$

$$
\begin{aligned}
\tilde\delta\psi_\mu = & \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}Q_\mu\right)\epsilon + \frac{i}{480}F_{\mu\nu_1\ldots\nu_4}\gamma^{\nu_1\ldots\nu_4}\epsilon \\
& +\frac{1}{96}G^{\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}\epsilon_C - \frac{3}{32}G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon_C\,,
\end{aligned}
\qquad (1.3.73)
$$

with P, Q, G_3 and F_5 given by

$$
\begin{aligned}
P &= \frac{1}{2}\frac{dT}{T} + \frac{i}{2}T\,d\chi'\,, && (1.3.74)\\
Q &= \frac{T}{2}\,d\chi'\,, && (1.3.75)\\
G_3 &= iT^{1/2}\tilde q_\alpha F_3^\alpha - \frac{i}{2}T^{1/2}(\chi' + \frac{i}{T})q_\alpha F_3^\alpha\,, && (1.3.76)\\
F_5 &= dA_4 + \frac{1}{16}\left(q_\alpha A_2^\alpha\wedge\tilde q_\beta F_3^\beta - \tilde q_\alpha A_2^\alpha\wedge q_\beta F_3^\beta\right)\,. && (1.3.77)
\end{aligned}
$$

For the following pair of values of q_α and $\tilde q_\alpha$

$$
q_1 = q_2 = \frac{y}{2}\,, \qquad \tilde q_1 = -\tilde q_2 = -\frac{i}{y}\,, \qquad (1.3.78)
$$

it is customary to write

$$
\begin{aligned}
T^{1/2} &= \sqrt{p}\frac{1}{(\mathrm{Im}\,\tau)^{1/2}}\,, && (1.3.79)\\
\chi' &= \frac{1}{p}\,\chi\,, && (1.3.80)
\end{aligned}
$$

where p is an arbitrary real but positive number. In fact p is one of the three parameters p, q, r that parameterize Q, see eq. (1.3.38). The complex field τ is the axidilaton field: $\tau = \chi + ie^{-\phi}$ where χ is the RR axion and ϕ is the dilaton. For the choice (1.3.78) and with the definitions (1.3.79) and (1.3.80) the parametrization of $V^\alpha_\pm$, obtained via (1.2.12) and (1.2.13) is

$$V^1_+ = \frac{1+i\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.81}$$

$$V^2_+ = \frac{1-i\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.82}$$

$$V^1_- = \frac{1+i\bar\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,, \tag{1.3.83}$$

$$V^2_- = \frac{1-i\bar\tau}{2(\mathrm{Im}\,\tau)^{1/2}}\,. \tag{1.3.84}$$

Further using the definitions (1.3.44) to (1.3.48) it is clear that one has

$$q_\alpha A^\alpha_2 = 2\sqrt{p}\,B_2\,, \tag{1.3.85}$$

$$\tilde q_\alpha A^\alpha_2 = -\frac{1}{\sqrt{p}}C_2\,, \tag{1.3.86}$$

$$q_\alpha A^\alpha_6 = 2\sqrt{p}\,C_6\,, \tag{1.3.87}$$

$$\tilde q_\alpha A^\alpha_6 = -\frac{1}{\sqrt{p}}B_6\,, \tag{1.3.88}$$

$$q_\alpha q_\beta A^{\alpha\beta}_8 = pC_8\,, \tag{1.3.89}$$

The supersymmetry transformation rules expressed in terms of the fields τ, B_2, C_2, B_6, C_6 and C_8 can be read off directly from eqs. (1.3.63) to (1.3.73).

In the formulation containing both the RR scalar as well as the RR 8-form the following symmetry is manifest:

$$\phi \to \phi\,, \tag{1.3.90}$$

$$\chi \to \chi + b \tag{1.3.91}$$

$$B_2 \to B_2\,, \tag{1.3.92}$$

$$C_2 \to C_2 + bB_2\,, \tag{1.3.93}$$

$$B_6 \to B_6 + bC_6\,, \tag{1.3.94}$$

$$C_6 \to C_6\,, \tag{1.3.95}$$

$$C_8 \to C_8\,, \tag{1.3.96}$$

$$\lambda \to \lambda\,, \tag{1.3.97}$$

$$\psi_\mu \to \psi_\mu\,, \qquad \epsilon \to \epsilon\,. \tag{1.3.98}$$

This set of transformation forms the $\mathbb{R}$ subgroup of the classical $SL(2,\mathbb{R})$ duality group.

Fields	Group	Order of S
τ	$PSL(2,\mathbb{R})$	2
B_2, C_2	$SL(2,\mathbb{R})$	4
$\lambda, \psi_\mu, \epsilon$	double cover of $SL(2,\mathbb{R})$	8

Table 1.3.1: Transformation groups for the IIB fields that transform non-trivially under duality.

The spinor transformation rules under $SL(2,\mathbb{R})$ in terms of τ can be obtained from the spinor supersymmetry rules (1.3.72) and (1.3.73) in which

$$P = -\frac{d\tau}{\tau - \bar{\tau}}\,, \qquad (1.3.99)$$

$$Q = i\frac{d(\tau + \bar{\tau})}{2(\tau - \bar{\tau})}\,, \qquad (1.3.100)$$

$$G_3 = \frac{-i}{(\mathrm{Im}\tau)^{1/2}}\left(-dC_2 + \tau dB_2\right)\,, \qquad (1.3.101)$$

$$F_5 = dA_4 + \frac{1}{8}\left(C_2 \wedge dB_2 - B_2 \wedge dC_2\right)\,. \qquad (1.3.102)$$

Under the full $SL(2,\mathbb{R})$ duality group the spinors λ, ψ_μ and ϵ transform as:

$$\lambda \to e^{3i\beta}\lambda\,, \quad \psi_\mu \to e^{i\beta}\psi_\mu\,, \quad \epsilon \to e^{i\beta}\epsilon\,, \qquad (1.3.103)$$

where

$$e^{2i\beta} = \frac{|c\tau + d|}{c\tau + d}\,. \qquad (1.3.104)$$

Using (1.3.103) it can concluded that one needs to perform the S-duality transformation, $S : \tau \to -\frac{1}{\tau}$, eight times on the spinors in order to return to the same configuration, that is, $S^8 = 1$. On the other hand the S-duality acting on the 2-forms B_2 and C_2 and on τ transforming as in (1.3.51) and (1.3.39) is of fourth, respectively, second order. The transformation groups for the various IIB fields and the order of S are summarized in table 1.3.1.

1.4 The classical moduli space

The classical moduli space of type IIB supergravity is the space of allowed values for the scalars. This space is the coset $SL(2,\mathbb{R})/SO(2)$ - also referred to as the upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$. The coordinates $(\tau, \bar{\tau})$ are called Poincaré coordinates whereas the coordinates (η, φ) are referred to as geodesic polar coordinates [28] for reasons that will become clear later. The coordinate ranges for η and φ are given by $0 \leq \eta < \infty$ and $0 \leq \varphi < 2\pi$.

The kinetic terms for the scalars can be thought of as defining a metric on the moduli space. The scalar kinetic terms in the action (1.1.17) are determined by $2 \star P \wedge \bar{P}$. Using for P the expression (1.3.99) one finds

$$2 \star P \wedge \bar{P} = \frac{1}{2} \frac{\star d\tau \wedge d\bar{\tau}}{(\mathrm{Im}\,\tau)^2} \,. \tag{1.4.1}$$

If instead one were to use the expression (1.3.34) for P the result would be

$$2 \star P \wedge \bar{P} = \frac{1}{2} \star d\eta \wedge d\eta + \frac{1}{2} \sinh^2 \eta \star d\varphi \wedge d\varphi \,. \tag{1.4.2}$$

The kinetic terms (1.4.1) are invariant under the transformations (1.3.39). These are fractional linear transformations and they form the group $PSL(2,\mathbb{R}) \equiv SL(2,\mathbb{R})/\{\pm \mathbb{1}\}$.

The relation between $(\tau, \bar{\tau})$ and (η, φ) can be obtained by starting with eq. (1.3.42), substituting in it eqs. (1.2.23) and (1.2.25) and subsequently use eqs. (1.3.23) and (1.3.24). The result is

$$\frac{\tau - \tau_0}{\tau - \bar{\tau}_0} = e^{i\varphi} \tanh \tfrac{\eta}{2} \,. \tag{1.4.3}$$

The point τ_0 is defined in eq. (1.3.41) to be a fixed point of the transformation $\tau \to e^Q \tau$.

There are two types of geodesics on the upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0\}$. The first type consists of straight semi-infinite vertical lines: $\mathrm{Re}\,\tau = \mathrm{cst}$. The second type is formed by semi-circles of radius r whose center a is on the real axis: $\mathrm{Im}\,\tau = \big(r^2 - (\mathrm{Re}\,\tau - a)^2\big)^{1/2}$.

The Poincaré coordinate axes are $\mathrm{Im}\,\tau = \mathrm{cst}$ and $\mathrm{Re}\,\tau = \mathrm{cst}$ of which the latter are geodesics. The coordinate axes in geodesic polar coordinates are formed by $\varphi = \varphi_0$ and $\eta = \eta_0$. These are the respective circles:

$$\varphi = \varphi_0 \;: \quad \left(\mathrm{Re}\,\tau - \mathrm{Re}\,\tau_0 + \cot\varphi_0 \,\mathrm{Im}\,\tau_0\right)^2 + (\mathrm{Im}\,\tau)^2 = \left(\frac{\mathrm{Im}\,\tau_0}{\cos\varphi_0}\right)^2 \,, \tag{1.4.4}$$

$$\eta = \eta_0 \;: \quad \left(\mathrm{Re}\,\tau - \mathrm{Re}\,\tau_0\right)^2 + \left(\mathrm{Im}\,\tau - \cosh\eta_0 \,\mathrm{Im}\,\tau_0\right)^2 = \left(\sinh\eta_0 \,\mathrm{Im}\,\tau_0\right)^2 \,. \tag{1.4.5}$$

The lines $\varphi = \varphi_0$ all go through the point τ_0 with $0 < \mathrm{Im}\,\tau_0 < \infty$ and form semi-circles with their center on the real axis (geodesics), see figure 1.4.1.

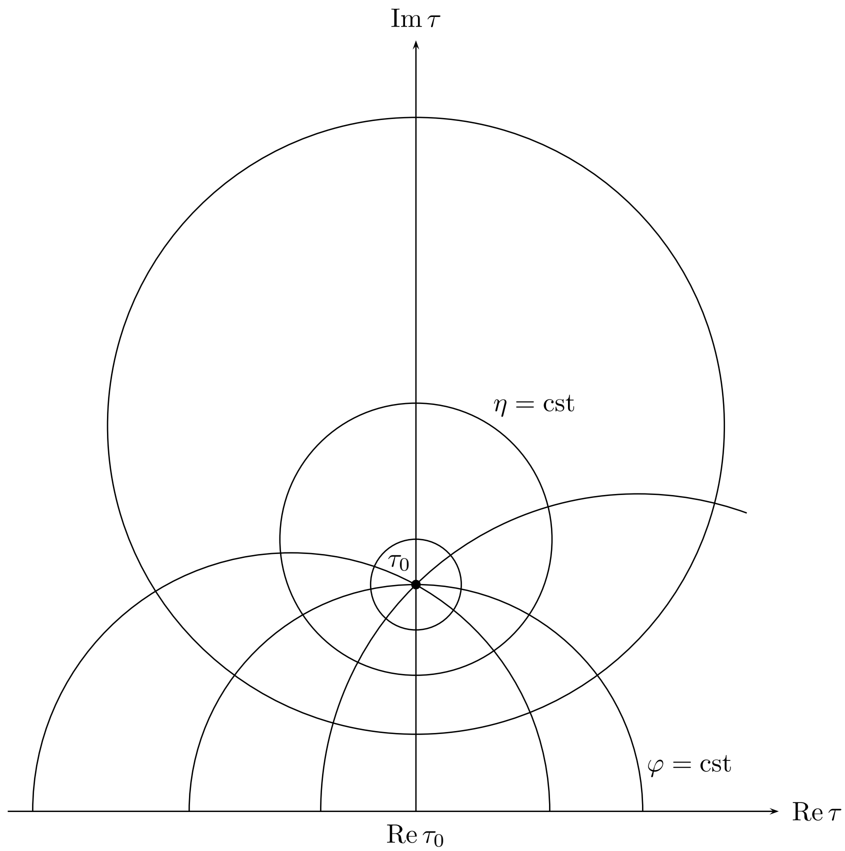

Figure 1.4.1: Geodesic polar coordinate system: the lines $\varphi = \text{cst}$ are geodesics of the Poincaré metric and η measures the Poincaré distance from τ to τ_0 along $\varphi = \text{cst}$.

The Poincaré distance measured along all the straight semi-infinite vertical lines that go through the point τ_0 with $\operatorname{Im}\tau_0 = 0$ is given by $\int d\phi$ where $\operatorname{Im}\tau = e^{-\phi}$. The Poincaré distance measured along all the semi-circles that go through the point τ_0 with $0 < \operatorname{Im}\tau_0 < \infty$ is given by $\int d\eta$. Hence, the scalars ϕ and η measure the Poincaré distance along a geodesic that goes through the point τ_0. This is the origin of the name geodesic polar coordinates for the coordinates (η, φ).

Fixed points τ_0 of $\tau \to e^Q\tau$ may be thought of as defining the origin of a coordinate system. For the fixed points with $\det Q = 0$ the origin is somewhere on the real axis $\mathrm{Im}\,\tau = 0$ (or after performing a certain $PSL(2,\mathbb{R})$ transformation at $\mathrm{Im}\,\tau = \infty$) and the coordinate axes are the semi-infinite vertical lines (geodesics) $\mathrm{Re}\,\tau = \mathrm{cst}$ together with the real line $\mathrm{Im}\,\tau = 0$. For the fixed points with $\det Q > 0$ the origin is located at a point τ_0 with $0 < \mathrm{Im}\,\tau_0 < \infty$ and the coordinate axes are formed by the lines $\eta = \mathrm{cst}$ together with all semi-circles that go through the point τ_0 that are the lines $\varphi = \mathrm{cst}$.

The matrix e^Q is equal to

$$e^Q = \cos(\sqrt{\det Q})\mathbb{1} + \frac{\sin(\sqrt{\det Q})}{\sqrt{\det Q}} Q\,. \tag{1.4.6}$$

From this expression it is clear that the set of transformations with $\det Q = 0$ forms the $\mathbb{R}$ subgroup of $SL(2,\mathbb{R})$, whereas the set of transformations with $\det Q > 0$ forms the $SO(2)$ subgroup of $SL(2,\mathbb{R})$. Any element Λ of $PSL(2,\mathbb{R})$ with $\mathrm{Tr}\,\Lambda \leq 2$ can be written as $\Lambda = e^Q$ where Q is given by (1.2.7) in which $p \geq 0$ and $q \geq 0$. The trace of Λ constitutes an $SL(2,\mathbb{R})$ conjugacy class.