Chapter 4

Lorentz symmetry breaking in $\beta$ decay

Lorentz violation can be parametrized by adding a general tensor $\chi^{\mu\nu}$ to the Minkowski metric. In this chapter we further discuss the possibilities to improve the current bounds on $\chi^{\mu\nu}$, discussed in Sec. 2.6. We focus on the possibilities for $\beta$ decay and electron capture. Tests of Lorentz invariance in kaon and pion decay are discussed in Chapter 5.

4.1 Concurrent tests of Lorentz invariance in $\beta$-decay experiments

Modern experiments on neutron and allowed nuclear $\beta$ decay search for new semileptonic interactions, beyond the “left-handed” electroweak force. We show that ongoing and planned $\beta$-decay experiments, with isotopes at rest and in flight, can be exploited as sensitive tests of Lorentz invariance. The variety of correlations that involve the nuclear spin, the direction of the emitted $\beta$ particle, and the recoil direction of the daughter nucleus allow for relatively simple experiments that give direct bounds on Lorentz violation. The pertinent observables are decay-rate asymmetries and their dependence on sidereal time and the dependence of the lifetime on the speed and direction of the source’s motion. We discuss the potential of several asymmetries that together cover a large part of the parameter space for Lorentz violation in the gauge sector. High counting statistics is required.

4.1.1 Motivation

$\beta$ decay is a recognized probe of symmetry violation in the electroweak interaction. Because of the wide choice of $\beta$ emitters and the various observables that can be measured with high precision, one can select isotopes that are tailored to specific searches for particle physics beyond the Standard Model (SM) [14, 22, 95, 192]. Over the years, strong limits were put on scalar, right-handed vector and axial vector, and tensor contributions to the

semileptonic process $d \rightarrow u + e^- + \nu_e$. Recently, it was shown that $\beta$ decay is moreover a unique laboratory for testing Lorentz invariance in the weak gauge [17, 21, 172, 192] and neutrino [39, 186] sectors. Such studies are strongly motivated by ideas how to unify the SM and general relativity in a theory of “quantum gravity” [27, 210]. We demonstrate here that ongoing and planned $\beta$-decay experiments can, with moderate modifications in the setup and data analysis, be exploited to improve the existing limits on Lorentz violation.

We base our studies on the theoretical framework for Lorentz and CPT violation developed in Refs. [17, 21] for $\beta$ decay and in Ref. [172] for orbital electron capture. It covers effects from e.g. a modified low-energy $W$-boson propagator $\langle W^{\mu} + W^{\nu} \rangle = -i(g^{\mu\nu} + \chi^{\mu\nu})/M_W^2$. The tensor components $\chi^{\mu\nu}$ were limited with data on allowed [173, 38, 180, 174] and forbidden [21] $\beta$ decay, pion decay [175, 176], nonleptonic kaon decay [178], and muon decay [177]. The best upper bounds were derived from experiments on forbidden $\beta$ decays [21], while a first experiment on allowed $\beta$ decay with polarized nuclei gave additional, partly complementary information [38, 180]. These results were translated into bounds on Higgs- and $W$-boson parameters of the Standard Model Extension (SME) [29, 16, 37], the general effective field theory for Lorentz and CPT violation at low energies.

The allowed-$\beta$-decay rate with Lorentz violation was derived in Ref. [17] and given in Eq. (2.84). Compared to ordinary $\beta$ decay, it contains additional, frame-dependent correlations between the momenta and spins of the nuclei and leptons and the tensor $\chi$. The correlations involve linear combinations of the components $\chi^{\mu\nu}$, depending on the type of $\beta$ decay, Fermi, Gamow-Teller, or mixed. While many of these correlations are hard to measure, a few appear relatively straightforward. We discuss a number of experiments on neutron and allowed nuclear $\beta$ decay that can give competing bounds on Lorentz violation. The pertinent observables are all rather simple asymmetries recorded with sidereal-time stamps. We also consider the $\beta$ decay of nuclei in flight, e.g. at proposed $\beta$-beam facilities, as a way to increase the sensitivity. We end with recommendations how to further explore Lorentz violation in weak decays.

### 4.1.2 Decay rate

We assume that Lorentz violation comes from propagator corrections and neglect momentum-dependent terms in $\chi$, which are suppressed by powers of the $W$-boson mass. Hermiticity of the Lagrangian then implies that $\chi^{\mu\nu} = (\chi^*)^{\nu\mu}$. We also neglect here terms with only neutrino-momentum or neutrino-spin correlations, which are important in electron capture [172] (Sec. 4.2), but in $\beta$ decay do not contain more information than the easier to measure $\beta$-particle correlations. In addition, we ignore for the moment terms proportional to the spin factor $\hat{A}_{\mu\nu}$ [17], which is associated with higher-order spin correlations ($\hat{A}_{\mu\nu} = 0$ for unpolarized and spin-1/2 nuclei).

With these simplifications the $\beta$-decay rate [17]$^1$, in the rest frame of the parent nucleus, reduces to ($\hbar = c = 1$)

$^1$In Eq. (18) in Ref. [17] it should read $w^l_3 = -x \chi^l_3 + \bar{g}(\chi^l_3 + \chi^l_3)$. This also corrects Eq. (38) in Ref. [17].
\[ dW = dW_0 \left\{ 1 + 2a\chi_{r}^{00} + 2\left(-a\chi_{r}^{0l} + i\chi_{i}^{l}\right) \frac{p_{e}^{l}}{E_{e}} + \left([a + 2\bar{\chi}_{r}^{00}]\delta_{lm} - 4\bar{\chi}_{r}^{lm}\right) \frac{\vec{p}_{e}^{m}p_{\nu}^{n}}{E_{e}E_{\nu}} + 2\chi_{r}^{0k}(\vec{p}_{e}\times\vec{p}_{\nu})^{k} + \frac{\langle f^{k}\rangle}{J}(-2\bar{L}\chi_{r}^{k} + \left(A + B\chi_{r}^{00}\right)\delta_{kl} - B\chi_{r}^{kl}) \frac{p_{e}^{l}}{E_{e}}\right\} \tag{4.1} \]

where \( dW_0 = |\vec{p}_{e}|E_{e}(E_{e} - E_{0})dE_{e}d\Omega_{e}d\Omega_{\nu}F(E_{e}, \pm Z)\bar{\xi}/(2\pi)^{5} \), \( \vec{p}_{e(\nu)}, E_{e(\nu)} \) are the momentum and energy of the \( \beta \) particle (electron or positron) and neutrino, and \( \langle f^{k}\rangle \) is the expectation value of the spin of the parent nucleus. \( F(E_{e}, \pm Z) \) is the usual Fermi function, with \( Z \) the atomic number of the daughter nucleus, and the upper (lower) sign holds for \( \beta^{-(+)} \) decays; \( \xi = 2g_{\nu}^{2}(1)^{2} + 2g_{A}^{2}(\sigma)^{2} \). The subscripts \( r \) and \( i \) denote the real and imaginary parts of \( \chi = \chi_{r} + i\chi_{i}, \bar{\chi}_{r}^{l} = \chi_{l}^{k}, k,l,m \) are spatial directions. The coefficients \( a, A, B \) are standard in \( \beta \) decay [24, 14], while \( \bar{a}, \bar{g}, \bar{L} \) multiply correlations that are Lorentz violating [17]. They are defined by

\[ a = \left(1 - \frac{1}{2}\rho^{2}\right) \left(1 + \rho^{2}\right), \tag{4.2a} \]
\[ A = \left(\mp\lambda_{J,J'}\rho^{2} + 2\delta_{J,J'}\sqrt{J/(J+1)}\rho\right) \left(1 + \rho^{2}\right), \tag{4.2b} \]
\[ B = \left(\pm\lambda_{J,J'}\rho^{2} + 2\delta_{J,J'}\sqrt{J/(J+1)}\rho\right) \left(1 + \rho^{2}\right), \tag{4.2c} \]
\[ \bar{a} = \left(1 + \frac{1}{2}\rho^{2}\right) \left(1 + \rho^{2}\right), \tag{4.2d} \]
\[ \bar{g} = \frac{1}{2}\rho^{2} \left(1 + \rho^{2}\right), \tag{4.2e} \]
\[ \bar{L} = \pm\frac{1}{2}\lambda_{J,J'}\rho^{2} \left(1 + \rho^{2}\right), \tag{4.2f} \]

where \( \rho = |g_{A}|/|g_{\nu}| |M_{GT}|/(g_{\nu}|M_{F}|) \) is the ratio between the Gamow-Teller and Fermi matrix elements. The value of the spin factor \( \lambda_{J,J'} \), where \( J (J') \) is the initial (final) nuclear spin, is \( \lambda_{J,J'} = 1 \) for \( J' = J - 1, 1/(J+1) \) for \( J' = J \), and \( -J/(J+1) \) for \( J' = J + 1 \).

### 4.1.3 Observables

There are 15 independent tensor components \( \chi^{\mu\nu} \). It is standard to translate the tensor \( \chi \) to the Sun-centered reference frame, in which it is denoted by \( X \), and report limits for

<table>
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<tr>
<th>( X_{r}^{00} )</th>
<th>( X_{r}^{0l} )</th>
<th>( X_{r}^{kl} )</th>
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<td>( 10^{-6} )</td>
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Table 4.1: Statistical precision for the components \( X^{\mu\nu} \) required to compete with the existing upper bounds from forbidden \( \beta \) decay [21]. The components \( X_{r}^{0l} \) are unconstrained at present.
the components $X^{\mu\nu}$ [37]. The best upper bounds on (linear combinations of) $X^{\mu\nu}$ are $\mathcal{O}(10^{-6})$, derived [21] from pioneering forbidden-$\beta$-decay experiments [181, 182] that used strong sources. In case there are accidental cancellations, the bounds on the individual components could be significantly weaker and range from $\mathcal{O}(10^{-4})$ to $\mathcal{O}(10^{-8})$ [211]. The order-of-magnitude precision required to improve the existing bounds on the various components $X^{\mu\nu}$ is summarized in Table 4.1. A statistical precision of $10^{-n}$ requires at least $\mathcal{O}(10^{2n})$ events. This would require one year of data taking with a source of 1 Curie for an experiment of the type performed in Ref. [181]. An alternative option is electron capture, which allows experiments at high rates and low dose [172]. We focus here on the possibilities to improve the existing bounds in allowed $\beta$ decay.

From Eq. (5.9) we derive asymmetries that are proportional to specific components $\chi^{\mu\nu}$. Asymmetries are practical to measure and ideal to control systematic errors. Expressed in terms of $X^{\mu\nu}$, they oscillate in time with the sidereal rotation frequency $\Omega = 2\pi/(23h56m)$ of Earth and depend on the colatitude $\zeta$ of the site of the experiment. These sidereal-time variations of the observables are a unique feature of Lorentz violation, and help to separate the desired signal from systematic errors. They also distinguish Lorentz violation from effects due to e.g. scalar or tensor interactions, which would produce deviations from SM predictions that are independent of Earth’s orientation.

(i) The simplest way to study Lorentz violation is to integrate over the neutrino direction and measure the dependence of the decay rate on the direction of the $\beta$ particle. The highest sensitivity can be reached in pure Fermi or Gamow-Teller decays. For Fermi decays, the experimental observable is the asymmetry

$$A_F = \frac{W_F^+ - W_F^-}{W_F^+ + W_F^-} = -2\chi_{r l}^0 \beta \hat{p}_e^l,$$  

(4.3)

where $\beta = |\vec{p}_e|/E_e$ and $W_F^\pm$ is the rate of $\beta$ particles measured in the $\pm\hat{p}_e$-direction. For Gamow-Teller decays of unpolarized nuclei, the analogous asymmetry is

$$A_{GT} = \frac{W_{GT}^+ - W_{GT}^-}{W_{GT}^+ + W_{GT}^-} = \frac{2}{3} \left(\chi_{r l}^0 + \bar{\chi}_{l i}^1\right) \beta \hat{p}_e^l.$$  

(4.4)

These two asymmetries are complementary and give direct bounds on $\chi_{r l}^0$ and $\bar{\chi}_{l i}^1$. Mixed decays are slightly less sensitive, e.g. for neutron $\beta$ decay, with $\rho = \sqrt{3}|g_A|$, where $g_A \approx -1.275$ [69, 68], the asymmetry is $A_n = (0.21\chi_{r l}^0 + 0.55\bar{\chi}_{l i}^1) \beta \hat{p}_e^l$.

Figure 4.1 illustrates the sidereal-time dependence of the asymmetry $A_F$ for three different observation directions. When the $\beta$ particles are detected parallel to Earth’s rotation axis, no oscillation is observed. Observation of the $\beta$ particles perpendicular to the rotation axis, i.e. east-west, gives a sidereal-time variation. When the $\beta$ particles are observed in the up-down (↑↓) direction, this oscillation has a constant offset. Systematic errors can result in a finite offset, and therefore observation in the direction perpendicular to the rotation axis is favored. The asymmetries should preferably be measured in a rotating setup [181] to reduce systematic errors. Alternatively, a multi-detector setup with appropriate symmetry can exploit the full polar and azimuthal dependence as shown in Fig. 4.1, while reducing the counting rates of the individual detectors. An experiment with a duration of one year can use diurnal variations to reduce systematic errors.
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Figure 4.1: The sidereal time dependence of the asymmetry $A_F$ in Eq. (4.3), for $X_{0x}^{r} = 0.1$, $X_{0y}^{r} = 0.2$, $X_{0z}^{r} = 0.3$, and colatitude $\zeta = 45^\circ$. For $\beta$ particles observed parallel (||) to Earth’s rotation axis $A_F$ is constant. Observation in the ↑↓ (up-down) direction or perpendicular (⊥) to the rotation axis results in an oscillation of $A_F$ with sidereal time.

There are ongoing efforts to improve the bounds on tensor currents in $\beta$ decay. A promising observable for this purpose is the energy spectrum of the $\beta$ particles [111]. The Gamow-Teller decays of $^6$He [106, 212] and $^{45}$Ca [111] are currently under consideration. Such experiments require high statistics and accuracy. The $^6$He facility promises to produce $10^{16}$ particles/s, but it remains to be seen how such a beam can be used for Lorentz-violation measurements [212]. Isotopes such as $^{32}$P, $^{33}$P, $^{34}$S, and $^{63}$Ni are also of interest, because they have clean ground-state-to-ground-state $\beta^-$-transitions and low $Q$-values. For example, conveniently-shaped $^{63}$Ni sources of 1GBq are commercially available. Such sources have minimal contributions of secondary radiation that can complicate the measurements. Moreover, strong sources can be produced in reactors. For the Fermi asymmetry $A_F$, any of the superallowed $0^+ \rightarrow 0^+$ decays [43, 213] can be considered. We recommend that in these experiments the asymmetries $A_F$ of Eq. (4.3) and $A_{GT}$ of Eq. (4.4) are measured concurrently, with sidereal-time stamps.

(ii) With polarized nuclei one can measure the correlations that involve the nuclear spin. The simplest of these is the spin asymmetry

$$ A_J = \frac{W^\uparrow - W^\downarrow}{W^\uparrow + W^\downarrow} = -2\bar{\chi}\vec{J} \cdot \vec{P}, \quad (4.5) $$

where $\vec{J}$ is the unit vector in the direction of the parent spin, $P$ is the degree of nuclear polarization, and $W^\uparrow(\downarrow)$ is the integrated decay rate in the $\pm \vec{J}$-directions. For pure Gamow-Teller decays, $\bar{L} = \frac{1}{2} B = -\frac{1}{2} A$. Isotopes for which $\lambda J_{J'} = 1$ are optimal.

The first dedicated experiment to search for Lorentz violation in allowed $\beta$ decay measured $A_J$ in the $\beta^+$ decay of $^{20}$Na [38]. The result of the most recent measurement is $|\vec{\chi}| < 5 \times 10^{-4}$ with 90% confidence [180]. Data for polarized-neutron decay are currently being analyzed [174, 214]. When the sidereal-time dependence of $A_J$ is measured, it is not necessary to know $A$ and $P$ with high precision. If the polarization is not exactly equal in the two directions, $A_J$ will show an offset, which is independent of the sidereal frequency.
as long as the polarization can be kept independent of $\Omega$. Still, a measurement of the $\beta$ asymmetry $A_{GT}$, as discussed above, is probably preferable for improving the bounds on $\chi^k_i$.

(iii) The components $\chi^{ik}_i$ for which there are no bounds available yet, can be accessed through the correlations of $\hat{J} \times \hat{p}_e$ or $\hat{p}_e \times \hat{p}_\nu$ and a component of $\chi$. The first correlation can be measured with the asymmetry

$$A_{\beta\nu} = \frac{W_L^p W_R^k - W_R^p W_L^k}{W_R^k W_R^k + W_L^k W_L^k} = 4 \left( a \chi^{ik}_0 \epsilon_k l m - 2 \hat{\epsilon} \chi^{lm}_r \right) \beta_\nu \beta_\nu^l,$$

(4.6)

where $W_{L,R}$ is obtained by measuring the $\beta$ particles in the opposite left ($L$) and right ($R$) $\hat{p}_e$-directions, while the recoiling nucleus is detected in the perpendicular $\uparrow$ ($\downarrow$) direction. For this asymmetry, pure Fermi decays, with $a = 1$ and $\hat{\epsilon} = 0$, are preferred. Experiments that measure both the $\beta$ and the neutrino direction are thus of interest. Ref. [59] e.g. reports a search for a deviation from the SM prediction $a = 1$ for the $\beta$-$\nu$ correlation in $^{38}\text{mK}$, with an error on $a$ of order $\mathcal{O}(10^{-3})$, which would be the corresponding limit for $\chi^{ik}_0$.

With polarized nuclei, $\chi^{ik}_i$ can be measured from the asymmetry between the nuclear spin and the $\beta$ particle,

$$A_{J\beta} = \frac{W_L^p W_R^k - W_R^p W_L^k}{W_R^k W_R^k + W_L^k W_L^k} = -2 \left( A \chi^{im}_r \epsilon_m l k + B \chi^{kl}_r \right) J \beta^l \beta^l e,$$

(4.7)

where now $W_{L,R}$ is the rate with the $\beta$ particles in the opposite left ($L$) and right ($R$) $\hat{p}_e$-directions and the nuclei polarized in the perpendicular $\uparrow$ ($\downarrow$) $\hat{J}$-direction. Equation (4.7) holds for Gamow-Teller and mixed decays. Gamow-Teller decays with $\chi_{J,\nu} = 1$ are preferred.

The bounds from forbidden $\beta$ decay give $|\chi^{kl}_r| < \mathcal{O}(10^{-6})$ [21]. A measurement of $A_{J\beta}$ or $A_{\beta\nu}$ with a precision lower than $10^{-6}$, therefore, translates to a bound on $\chi^{ik}_i$. The sidereal-time variation of $A_{J\beta}$ and $A_{\beta\nu}$ is similar to that shown in Fig. 4.1. To reduce systematic errors $\hat{J} \times \hat{p}_e$ or $\hat{p}_e \times \hat{p}_\nu$ should point perpendicular to Earth’s rotation axis. $A_{J\beta}$ can possibly be obtained in polarized-neutron decay by reanalyzing the data of Ref. [174].

Measuring the asymmetries better than $10^{-6}$ requires coincident event rates exceeding $3 \times 10^4$/s for a year, but will then also improve the bounds on $\chi^{kl}_r$.

### 4.1.4 Exploiting Lorentz boosts

So far we discussed $\beta$ decay of nuclei at rest. The required event rate in these measurements is a challenge. In forbidden $\beta$ decays one can benefit from an enhancement of Lorentz violation of one order of magnitude [21]. A much larger enhancement can be obtained when the decaying particle is in flight. Consider specifically the total decay rate, which in the rest frame depends only on the isotropic term in Eq. (4.1),

$$W/W_0 = 1 + 2a \chi^{00}_r,$$

(4.8)

where $W_0$ is the SM decay rate and $a = 1 (-1/3)$ for Fermi (Gamow-Teller) decays. The component $\chi^{00}_r$ can e.g. be measured from the ratio between the longitudinal $\beta$
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polarization, $P_3 = (1 + 2a\chi_r^{(0)})G\beta$, for Fermi and Gamow-Teller decays, where $G = \mp 1$ [24, 14]. Comparing the best value $P_F/P_{GT} = 1.0010(27)$ [63, 64] to the SM prediction $P_F/P_{GT} = 1$ gives $-1.3 \cdot 10^{-3} < X_r^{(0)} < 2.0 \cdot 10^{-3}$ with 90% confidence, which is a much weaker bound than the one obtained for forbidden $\beta$ decay [21], and hard to improve with nuclei at rest.

The decay rate in flight depends on the velocity $\vec{v} = v\hat{\vec{v}}$ of the nucleus that results from a Lorentz boost. In terms of the components $X_{\mu\nu}$ in the Sun-centered frame one has $\chi_r^{(0)} = \gamma_r^2 \left( X_{r\mu}^{(0)} - 2v X_{r\mu}^{(l)} \hat{v}_l + X_{r\mu}^{(kl)} v^2 \hat{v}_k \hat{v}_l \right)$, where $\gamma_r = 1/\sqrt{1 - \vec{v}^2}$ is the Lorentz factor. When the velocity $\vec{v}$ is perpendicular to Earth’s rotation axis (east-west) one finds

$$
\chi_r^{(0)} = \gamma_r^2 \left( X_{r0}^{(0)} + \frac{1}{2} v^2 \left[ X_{rxx} + X_{ryy} \right] + 2v X_{r0x} \sin \Omega t - 2v X_{r0y} \cos \Omega t - v^2 X_{ryy} \sin 2\Omega t - \frac{1}{2} v^2 \left[ X_{rxx} - X_{ryy} \right] \cos 2\Omega t \right),
$$

(4.9)

which is enhanced by a factor $\gamma_r^2$. The components $X_{r\mu
u}$ can be fitted to the sidereal-time dependence of the measured decay rate. Alternatively, one can measure the decay rate at time $t$ and 12 hours later, and isolate $X_{r0l}$ via the “asymmetry”

$$
A_t = \frac{W(\Omega t) - W(\Omega t + \pi)}{W(\Omega t) + W(\Omega t + \pi)} = 4a v^2 \gamma_r^2 \left( X_{r0x} \sin \Omega t - X_{r0y} \cos \Omega t \right),
$$

(4.10)

while $X_{r0l}$ can be obtained by measuring at intervals of 6 hours, with

$$
A_{2t} = \frac{W(\Omega t) - W(\Omega t + \frac{3}{2} \pi) + W(\Omega t + \pi) - W(\Omega t + \frac{5}{2} \pi)}{W(\Omega t) + W(\Omega t + \frac{3}{2} \pi) + W(\Omega t + \pi) + W(\Omega t + \frac{5}{2} \pi)} = -a v^2 \gamma_r^2 \left( [X_{rxx} - X_{ryy}] \cos 2\Omega t + 2X_{ryy} \sin 2\Omega t \right),
$$

(4.11)

which oscillates only with the double frequency $2\Omega$.

The $\gamma_r^2$ enhancement in Eqs. (4.9), (4.10), and (4.11) can be exploited at a $\beta$-beam facility planned for neutrino physics [183]. A good nucleus for such a facility is $^6$He, for which the production rates are estimated at $10^{12}$/s with $\gamma_r = 100$ [215]. A possible setup for a $\beta$-beam facility that uses the proton synchrotrons at CERN is discussed in Refs. [215, 183].

Of course, any weakly-decaying particle in flight can be used, provided the coefficient $a$ in Eq. (4.8) can be calculated reliably. Nonleptonic decays of strange hadrons such as kaons are problematic [178], but decays of heavy quarks do not have this drawback. Leptonic and semileptonic decays are clearly preferable. For fast-moving pions [216] bounds of $O(10^{-4})$ on $\chi_{\mu\nu}$ were obtained [175]. Semileptonic kaon decays have been studied at the SPS at CERN [217] with $\gamma_r \approx 150$ and will be part of the background in the NA62 experiment. LHCb, designed to observe decays at $\gamma_r \gtrsim 10$, is serendipitously oriented perpendicular to Earth’s rotation axis. For all accelerator studies, the precise normalization of the decay rate as function of sidereal time is necessary for a concurrent test of Lorentz invariance.

4.1.5 $\beta-\gamma$ correlations

We have only considered cases where the anisotropic decay rate is observed in the emission direction of the $\beta$ particles and/or is associated with the polarization direction of
the parent nucleus. The anisotropy can also be observed from γ rays when an excited state in the daughter nucleus is populated. In Gamow-Teller transitions the daughter nucleus is left in a polarized state that reflects the degree of anisotropy of the emission. When measuring the γ-decay angular distribution this anisotropy can be observed as a residual alignment. Inspection of Eq. (5.9) shows that this will be the case for the term $-2\hat{L} \tilde{\chi}^{k}_{i} \hat{J}^{k}$. Clearly, such a measurement will have lower sensitivity compared with the direct measurements discussed above. The last line of Eq. (5.9) can also be accessed by measuring β-γ correlations. The last term is relevant because it contains the “missing” components $\chi_{0k}^{l}$. In this case the lower sensitivity may be compensated by an efficient setup. To obtain the actual expressions and the corresponding asymmetries the terms proportional to $\tilde{\Lambda}_{J',J}$ [17] have to be added to Eq. (5.9). The evaluation depends on the particular details of detection method and will be considered when the need arises.

4.1.6 Conclusion

The breaking of Lorentz invariance in the weak interaction can be probed in relatively simple allowed-β-decay experiments. We propose to measure a number of decay-rate asymmetries as function of sidereal time, which together can constrain all Lorentz-violating gauge components. Measurements of the β-decay asymmetry in Fermi and Gamow-Teller decays, Eq. (4.3) and Eq. (4.4), give direct bounds on $\chi_{0l}^{l}$ and $\tilde{\chi}^{k}_{i}$. The most complicated experiments require the measurement of a correlation between two observables, as in Eq. (4.6) or Eq. (4.7). The components $\chi_{0k}^{l}$ are still unconstrained and these measurements will give the first bounds. In addition, we point out the potential of β beams and LHCb for tests of Lorentz invariance. Ultimately, the experiments should aim to improve the existing forbidden-β-decay limits starting at $\mathcal{O}(10^{-6})$, which requires high-intensity sources and excellent control of systematic uncertainties. As we have shown, this can go hand-in-hand with high-precision allowed-β-decay experiments that search for new semileptonic physics. Such efforts are, therefore, of considerable general interest.
4.2 Testing Lorentz invariance in orbital electron capture

Searches for Lorentz violation were recently extended to the weak sector, in particular neutron and nuclear $\beta$ decay [17]. From experiments on forbidden $\beta$-decay transitions strong limits in the range of $10^{-6}-10^{-8}$ were obtained on Lorentz-violating components of the $W$-boson propagator [21]. In order to improve on these limits strong sources have to be considered. In this section we study isotopes that undergo orbital electron capture and allow experiments at high decay rates and low dose. We derive the expressions for the Lorentz-violating differential decay rate and discuss the options for competitive experiments and their required precision.

4.2.1 Introduction

Motivated by insights that Lorentz and CPT invariance can be violated in unifying theories of particle physics and quantum gravity, a theoretical framework was developed in Refs. [17, 21] to study Lorentz violation in the weak gauge sector in neutron and (allowed and forbidden) nuclear $\beta$ decay. This approach, which parametrizes Lorentz violation by adding a complex tensor $\chi^{\mu\nu}$ to the Minkowski metric, includes a wide class of Lorentz-violating effects, in particular contributions from a modified low-energy $W$-boson propagator $\left\langle W^{\mu+}W^{\nu-} \right\rangle = -i(g^{\mu\nu} + \chi^{\mu\nu})/M_W^2$ or from a modified vertex $\Gamma^\mu = (g^{\mu\nu} + \chi^{\mu\nu})\gamma_\nu$. Limits on Lorentz violation were subsequently extracted from experiments on allowed [38, 173, 174] and forbidden [21] $\beta$ decay (see Sec. 2.6), pion [175, 176] (Sec. 5.2), kaon [178] (Sec. 5.1), and muon decay [177].

The strongest bounds on components $\chi^{\mu\nu}$ were obtained [21] from forbidden-$\beta$-decay experiments [181, 182] and range from $10^{-6}-10^{-8}$ on different linear combinations. We also discussed these bounds in Section 2.6. These bounds were translated in limits on parameters of the Standard Model Extension [16], which is the most general effective field theory for Lorentz and CPT violation at low energy. Specifically, $\chi^{\mu\nu} = -k^{\mu\nu}_{\phi\phi} - i k^{\mu\nu}_{\phi W}/2g$ in terms of parameters in the Higgs and $W$-boson sector, where $g$ is the SU(2) electroweak coupling constant [17]. The resulting bounds on linear combinations of $k^{\mu\nu}_{\phi\phi}$ and $k^{\mu\nu}_{\phi W}$ can be found in Ref. [21] and in the 2014 Data Tables in Ref. [37]. The best bounds from allowed $\beta$ decays are $\mathcal{O}(10^{-2})$ [38, 174] and from pion decay $\mathcal{O}(10^{-4})$ [175].

When seeking further improvement, one should realize that the bounds from forbidden $\beta$ decay benefited from the use of high-intensity sources. Such strong $\beta$-decay sources, however, are hazardous because they have high disintegration rates (Bq) and high doses (Sv). In this section we consider orbital electron capture [218], because the pertinent sources can give high decay rates at a low dose. We first derive the theoretical expression for the differential decay rate including Lorentz violation. Next, we discuss the experimental possibilities to constrain the various components $\chi^{\mu\nu}$. Finally, we explore which isotopes are suitable for a competitive measurement. We end with our conclusions.

### 4.2.2 Decay rate

We consider allowed $K$-orbital electron capture [219] mediated by $W$-boson exchange with a propagator that includes $\chi^{\mu\nu}$. We follow the notation and conventions of Ref. [17] ($\hbar = c = 1$). The derivation of the two-body capture decay rate is similar to the calculation of allowed $\beta$ decay [17], but with the electron in a bound state with binding energy $|E_K|$. Since Lorentz violation results in unique experimental signals, we restrict ourselves to the allowed approximation with a nonrelativistic electron wave function with $\psi_e(\vec{r} = 0) = \sqrt{Z^3/(\pi a_0^3)} \chi_\alpha$, where $Z$ is the atomic number of the parent nucleus, $a_0 = 1/(\alpha m_e)$ is the Bohr radius, and $\chi_\alpha$ is a Pauli spinor. The neutrino is emitted with momentum $\vec{p}_\nu$ with $|\vec{p}_\nu| = E_\nu$ and the recoiling daughter nucleus has momentum $\vec{p}_r$ and kinetic energy $T_r$. Because $E_\nu = Q - |E_K| - T_r \simeq Q$, the $Q$-value of the reaction, the recoil energy is $T_r \simeq Q^2/(2M_r)$, which is typically 1-10 eV.

The differential decay rate is given by

$$dW = \delta(E_\nu - Q)N_K \frac{1}{2} \sum_{s_e,s_\nu} |M|^2 d^3p_\nu,$$

(4.12)

with $N_K = 2$ the number of $K$-shell electrons. We define $\xi = 2C_V^2 \langle 1 \rangle^2 + 2C_A^2 \langle \sigma \rangle^2$, $x = 2C_V^2 \langle 1 \rangle^2/\xi$, $y = 2C_V C_A \langle 1 \rangle \langle \sigma \rangle/\xi$, and $z = 1 - x = 2C_A^2 \langle \sigma \rangle^2/\xi$, where $C_V = G_F \cos \theta_C/\sqrt{2}$ and $C_A \simeq -1.27 C_V$ are the vector and axial-vector coupling constants; $M_F = \langle 1 \rangle$ and $M_{GT} = \langle \sigma \rangle$ are the Fermi and Gamow-Teller reduced nuclear matrix elements. For a polarized source we find for the Lorentz-violating decay rate

$$dW = dW^0 \left[ (1 + B \hat{p}_\nu \cdot \hat{J})/2 + t + \tilde{w}_1 \cdot \hat{p}_\nu + \tilde{w}_2 \cdot \hat{J} + T_{1km}^j \hat{J}^k \hat{j}^m + T_{2kj}^j \hat{P}_\nu^r + S_{kmj}^{l} \hat{J}^k \hat{j}^m \hat{P}_\nu^j \right],$$

(4.13)

where $dW^0 = (Z/a_0)^3 E_\nu^2 d\Omega_\nu \xi/(2\pi^3)$, $\hat{p}_\nu = |\vec{p}_\nu|/E_\nu$, and $\hat{J}$ is the nuclear polarization axis. Latin indices run over the three spatial directions, with summation over repeated indices implied. The Lorentz-violating tensors for electron capture read, in terms of the components $\chi^{\mu\nu}$,

$$t = (a - c/2) \chi^{00}_r,$$

$$w_i^j = -x \chi^{0j}_r - z(1 + 3\Lambda^{(2)}/2)(\chi^{ij}_r - \chi^{0j}_r)/3,$$

$$w_i^k = -y \Lambda_z(\chi^{0k}_r - \chi^{00}_r) + z\Lambda^{(1)} \chi^{i}_r/2,$$

$$T_{1km}^{l} = 3c \chi^{kmj}/2,$$

$$T_{2kj}^{l} = A \chi^{00}_r \chi^{ij}_r / 2 - z \Lambda^{(1)} (\chi^{ik}_r + \chi^{0i}_r \epsilon^{sjj}) / 2 + y \Lambda_z(\chi^{kj}_r + \chi^{0s}_r \epsilon^{sjj}),$$

$$S_{kmj}^{l} = -3c(\chi^{00}_r \delta^{mj} - \chi^{0s}_r \epsilon^{sjj}) / 2.$$

(4.14a-4.14f)

The subscripts $r$ and $i$ denote the real and imaginary parts, respectively, of $\chi^{\mu\nu} = \chi^{\mu\nu}_r + i\chi^{\mu\nu}_i$, and $\chi^l = e^{imk} \chi^{mk}$. The $V - A$ correlation coefficients [14, 24, 132] that appear are

$$a = (4x - 1)/3, \quad c = z \Lambda^{(2)}, \quad A = z \Lambda^{(1)} - 2y \Lambda_z, \quad B = -z \Lambda^{(1)} - 2y \Lambda_z.$$  

(4.15)

The angular-momentum coefficients $\Lambda^{(1)}$, $\Lambda^{(2)}$, and $\Lambda_z$ are given in the Appendix 4.A. We absorbed a factor $\Lambda^{(2)}/3$ in $c$ and a factor $\langle m \rangle / j$ in $A$ and $B$ [17].
Eq. (4.13) reduces to the simple $V - A$ expression for the electron-capture decay rate when the Lorentz-violating parameters are set to zero. In particular, the $B$ term in the first line of Eq. (4.13) is the correlation between the spin of the parent nucleus and the recoil direction of the daughter nucleus discussed in Refs. [220, 221]. The second line of Eq. (4.13) gives Lorentz-violating, frame-dependent contributions to the decay rate.

### 4.2.3 Observables

From Eq. (4.13) we see that the possibilities to test Lorentz invariance in electron capture lie in measuring the decay rate as function of either the nuclear polarization or the recoil momentum, or both. We restrict ourselves to dimension-four propagator corrections, for which $\chi^{\mu\nu}(p) = \chi^{\mu\nu}(-p)$ holds [17]. (The tensor $\chi^{\mu\nu}$ may contain higher-dimensional, momentum-dependent terms, but such terms are suppressed by at least one power of the $W$-boson mass.) Since $\chi$ is traceless, this gives a total of 15 independent parameters, of which at present only $\chi_{00}^\nu\nu$ are unconstrained. The $\chi_{00}^\nu\nu$ term will not be considered, because it can only be accessed when comparing capture or $\beta$-decay rates between particles at rest and with a large Lorentz boost factor $\gamma \gg 1$. In addition, we specialize to the suitable isotopes (identified below), which decay by Gamow-Teller transitions, and for simplicity we assume that the source has vector polarization. This leaves

\[
dW = \frac{1}{2} dW^0 \left[ (1 + B \hat{p}_\nu \cdot \hat{J}) - \left( \frac{2}{3} + A^{(j)} \right) (\hat{\chi}_i^j - \hat{\chi}_r^{j0}) \hat{p}_\nu^i + A \hat{\chi}_i^k \hat{J}^k - A \hat{\chi}_i^{jk} \hat{p}_\nu^i \hat{J}^k - A \hat{\chi}_s^{0i} (\hat{p}_\nu \times \hat{J})^s \right] \tag{4.16}
\]

where for pure Gamow-Teller decays $A = -B = A^{(1)}$. The different components $\chi^{\mu\nu}$ can be accessed by measuring asymmetries. We give three examples.

(i) $\chi_{ik}^j$ can be obtained from $\hat{\chi}_i$, which can be measured from an asymmetry that depends on the nuclear polarization, viz.

\[
A_J = \frac{\tau^+ - \tau^-}{\tau^+ + \tau^-} = \frac{W^- - W^+}{W^+ + W^-} = -A \hat{\chi}_i^k \hat{J}^k , \tag{4.17}
\]

where $A$ contains the degree of polarization of the source and $\tau^\pm$ and $W^\pm$ are the lifetime and decay rate, respectively, in two opposite polarization directions $\pm$. Such an experiment only requires to flip the spin of the sample and observe the change in decay rate. For a discussion on using the direction of polarization to reduce systematic errors, see Ref. [173]. In general, the observables must be expressed in a standard inertial frame, for example the Sun-centered frame [37]. In the laboratory frame $A_J$ will vary with $\Omega$, the angular rotation frequency of Earth, and the results depend on the colatitude $\zeta$ of the site of the experiment [17], cf. Fig. 4.2. In practice one searches for these variations as function of sidereal time in order to isolate the Lorentz-violating signal and to reduce systematical errors.

(ii) The asymmetry of the recoil emission direction in an unpolarized sample is given by

\[
A_r = \frac{W(-\hat{p}_\nu) - W(\hat{p}_\nu)}{W(-\hat{p}_\nu) + W(\hat{p}_\nu)} = \frac{2}{3} (\hat{\chi}_i^j - \hat{\chi}_r^{j0}) \hat{p}_\nu^i , \tag{4.18}
\]
LORENTZ SYMMETRY BREAKING IN $\beta$ DECREMENT

Figure 4.2: The oscillation of the asymmetries in Eqs. (4.17) and (4.19) as function of sidereal time, for $|\chi^{\mu\nu}| = 0.1$, $A = 1$, and colatitude $\zeta = 45^\circ$. To avoid a constant offset of the signal, we assumed polarization in the east-west ($\hat{y}$) direction. For $A_{Jr}$, $\hat{p}_\nu$ was taken in the laboratory $\hat{z}$ direction, i.e. perpendicular to Earth's surface.

which requires measuring the recoil direction $\hat{p}_r$ of the daughter nucleus. In this experiment it is necessary to rotate the setup as a whole to isolate the Lorentz-violating signal and reduce systematic errors, as was done in Refs. [181, 182]. $A_J$ and $A_r$ have the same sidereal frequency, but they can differ in phase.

(iii) When measurements of the recoil direction and polarization are combined, electron capture also offers the possibility to constrain the parameters $\chi^{ij}_s$, for which no bounds have been set so far. Such an experiment should measure $\chi^{ij}_s(\hat{p}_\nu \times \hat{J})^s$, similar to a triple-correlation experiment to measure time-reversal violation in $\beta$ decay. For example, the asymmetry

$$A_{Jr} = \frac{W(-\hat{p}_\nu)^+ W(\hat{p}_\nu)^- - W(-\hat{p}_\nu)^- W(\hat{p}_\nu)^+}{W(-\hat{p}_\nu)^+ W(\hat{p}_\nu)^- + W(-\hat{p}_\nu)^- W(\hat{p}_\nu)^+} = 2A(\chi^{jk}_r + \chi^{s0}_i \epsilon^{ijk})\hat{p}_r \hat{J}^k,$$

(4.19)

where $\hat{p}_r$ is measured perpendicular to $\hat{J}$, contains both $\chi^{jk}_r$ and $\chi^{s0}_i$. The first term also produces sidereal oscillations with frequency $2\Omega$. The difference for the asymmetries in Eqs. (4.17) and (4.19) is illustrated in Fig. 4.2.

4.2.4 Isotopes

The most stringent bounds found for a single component $\chi^{\mu\nu}$ so far are at a level of $O(10^{-8})$, other components are at least as small as $O(10^{-6})$ [21]. Most of the existing bounds concern linear combinations of several components $\chi^{\mu\nu}$, so that cancellations are in principle possible. Assuming maximal fine-tuning, the best bound for a real component is $O(10^{-6})$ and for an imaginary term $O(10^{-4})$ [211]. To achieve the highest statistical relevance very strong sources should be considered. In order to reach $10^{-9}$ statistical accuracy a source with a strength in the order of Curies (1 Curie-year $\simeq 10^{18}$ disintegrations) is required. For a high-statistics experiment a source that decays exclusively by electron capture is attractive, because the emission of ionizing radiation is strongly reduced: only X-ray emission and Auger electrons are involved. The most energetic radiation is due to internal bremsstrahlung, which is suppressed by at least the fine-structure constant.
4.2 TESTING LORENTZ INvariance in orbital electron capture

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$t_{1/2}$ [s]</th>
<th>$Q$ [keV]</th>
<th>$j^{pi} \rightarrow j'^{pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{37}$Ar</td>
<td>$3.0 \times 10^6$</td>
<td>814</td>
<td>$\frac{3}{2}^+ \rightarrow \frac{3}{2}^+$</td>
</tr>
<tr>
<td>$^{40}$V</td>
<td>$2.9 \times 10^7$</td>
<td>602</td>
<td>$\frac{7}{2}^- \rightarrow \frac{7}{2}^-$</td>
</tr>
<tr>
<td>$^{55}$Fe</td>
<td>$8.6 \times 10^7$</td>
<td>231</td>
<td>$\frac{3}{2}^- \rightarrow \frac{5}{2}^-$</td>
</tr>
<tr>
<td>$^{71}$Ge</td>
<td>$9.9 \times 10^5$</td>
<td>232</td>
<td>$\frac{1}{2}^- \rightarrow \frac{3}{2}^-$</td>
</tr>
<tr>
<td>$^{131}$Cs</td>
<td>$8.4 \times 10^5$</td>
<td>355</td>
<td>$\frac{5}{2}^+ \rightarrow \frac{3}{2}^+$</td>
</tr>
<tr>
<td>$^{163}$Ho</td>
<td>$1.4 \times 10^{11}$</td>
<td>2.6</td>
<td>$\frac{7}{2}^- \rightarrow \frac{5}{2}^-$</td>
</tr>
<tr>
<td>$^{165}$Er</td>
<td>$3.7 \times 10^4$</td>
<td>376</td>
<td>$\frac{5}{2}^- \rightarrow \frac{3}{2}^-$</td>
</tr>
<tr>
<td>$^{179}$Ta</td>
<td>$5.7 \times 10^7$</td>
<td>106</td>
<td>$\frac{7}{2}^+ \rightarrow \frac{9}{2}^+$</td>
</tr>
<tr>
<td>$^{53}$Mn</td>
<td>$1.2 \times 10^{14}$</td>
<td>597</td>
<td>$\frac{7}{2}^- \rightarrow \frac{3}{2}^-$</td>
</tr>
<tr>
<td>$^{97}$Tc</td>
<td>$1.3 \times 10^{14}$</td>
<td>320</td>
<td>$\frac{9}{2}^+ \rightarrow \frac{3}{2}^+$</td>
</tr>
<tr>
<td>$^{137}$La</td>
<td>$1.9 \times 10^{12}$</td>
<td>621</td>
<td>$\frac{7}{2}^+ \rightarrow \frac{3}{2}^+$</td>
</tr>
<tr>
<td>$^{205}$Pb</td>
<td>$5.5 \times 10^{14}$</td>
<td>51</td>
<td>$\frac{5}{2}^- \rightarrow \frac{1}{2}^+$</td>
</tr>
</tbody>
</table>

Table 4.2: Isotopes that decay exclusively by orbital electron capture to a stable ground state. The top eight are relatively short-lived species that decay via allowed transitions, the bottom four are long-lived isotopes that undergo forbidden transitions. $^{163}$Ho is long-lived because of the very low $Q$-value. The isotopes check-marked in the last column can be polarized directly by optical pumping, or possibly also via an optically-pumped buffer gas.

A list of possible isotopes is given in Table 4.2. Which isotope is the most suitable depends on the detection and production method. The decay rate can be measured from the ionization current due to Auger processes and the shake-off of electrons that follows capture. This requires that the radioactive isotopes are available as atoms, possibly in a buffer gas. In this way one can polarize nuclei via optical pumping. The four isotopes for which this strategy is feasible are indicated in Table 4.2. To observe the nuclear polarization, internal bremsstrahlung can be used, which is anisotropic with respect to the spin direction [222, 223].

Because there are only four options we discuss the production of these isotopes separately:

- $^{37}$Ar can be produced in a reactor via the reaction $^{40}$Ca($n$,α)$^{37}$Ar. A source of 35 mCi was produced from 0.4 g of CaCO$_3$ for a transient NMR experiment [224] to test the linearity of quantum mechanics [225, 226]. An alternative method would be proton activation. A cyclotron beam of 25 MeV protons on $^{37}$Cl allows for a production of $10^7$ Bq/$\mu$Ah [227], so that a source of one Curie can be produced well within a week.
• The production of $^{131}$Cs was developed for brachytherapy. Neutron and proton activation are both options. Neutron activation is possible by using $^{130}$Ba [228], and proton activation by using Xe or Ba isotopes, which gives a yield of $> 10^7$ Bq/µAh [229]. Commercial sources are available. Cs can be separated well from other radioactive by-products.

• $^{163}$Ho is an isotope of interest for measuring the neutrino mass, and is studied by for instance the ECHo collaboration [230]. The production of Ho has been considered in detail [209]. The maximal production rate is projected to be about $10^4$ Bq/h, which is insufficient for a competitive measurement to test Lorentz invariance.

• $^{165}$Er can be produced with a proton beam on a Ho target [231] with a yield of $10^8$ Bq/µAh. In view of its short half-life of 10 h, this is the only practical method. Although the production is sufficient, radioactive Ho is a by-product and Er cannot be separated effectively from Ho.

We conclude that $^{37}$Ar and $^{131}$Cs are the only viable isotopes to obtain competitive values for $\chi_{\mu\nu}$. $^{37}$Ar has the lowest ionizing yield and in this respect may be preferred. It can be polarized via a buffer gas, or by first exciting the atom into the metastable state. In Ref. [224] the $^{37}$Ar nuclei were polarized by spin exchange with optically-pumped K atoms and a nuclear polarization of 56% was achieved.

The experimental apparatus for a measurement of $A_{Jr}$ in Eq. (4.19) could be based on that used to measure the recoil in electron capture of $^{37}$Ar, first used to verify the existence of neutrinos [232]. In particular, the crossed-field spectrometer developed at that time [233] can be read with modern electronics and adapted to include polarization of $^{37}$Ar. It is necessary to detect ionization currents instead of counting the recoils in order to accommodate the high event rate if one wants to aim for an accuracy of $10^{-9}$. However, because there are no limits yet on $\chi_{\nu}$, such an experiment would immediately produce new results with a much more modest effort, while allowing to investigate the systematic errors that will limit the ultimate high-statistics and high-precision experiments.

4.2.5 Conclusions

We have explored the potential of orbital electron capture to put limits on Lorentz violation in $\beta$ decay. The limits set in earlier work [21] are already so strong that high-intensity sources are required. A source with a strength of at least one Curie that decays solely by electron capture may allow such experiments. Our survey limits the choice to $^{37}$Ar and possibly $^{131}$Cs. The theoretical formalism for such experiments was developed, following Ref. [17], in a form applicable to any allowed electron-capture process. For one set of parameters quantifying Lorentz violation, no bound has been obtained as yet. These can be accessed in an experiment that measures the recoil from the neutrino emitted from a polarized nucleus, thus producing a new result while testing the viability of the suggested experimental program.
Appendix 4.A  Angular-momentum coefficients

The angular-momentum coefficients in Eqs. (4.14b), (4.14c), (4.14e), and (4.15) are

\[ \Lambda^{(1)} = \begin{cases} \frac{\langle m \rangle}{j} & (j' = j - 1) \\ \frac{\langle m \rangle}{j(j+1)} & (j' = j) \\ -\frac{\langle m \rangle}{j+1} & (j' = j + 1) \end{cases} \]

\[ \Lambda^{(2)} = \begin{cases} \frac{\langle m^2 \rangle - \frac{1}{2}j(j+1)}{j(2j-1)} & (j' = j - 1) \\ \frac{\langle m^2 \rangle + \frac{1}{2}j(j+1)}{j(j+1)} & (j' = j) \\ \frac{\langle m^2 \rangle - \frac{1}{2}j(j+1)}{(j+1)(2j+3)} & (j' = j + 1) \end{cases} \]  

(4.20)

\[ \Lambda_z = \frac{\langle m \rangle}{j} \sqrt{\frac{j}{j + 1}} \delta_{jj'} \]  

(4.21)

where \( j \) and \( j' \) denote the initial and final nuclear spin, respectively, and \( \langle m \rangle \) and \( \langle m^2 \rangle \) denote the incoherent average of \( m \) and \( m^2 \) over the populations of the states \( m = -j, \ldots, j \). \( \Lambda^{(2)} \) vanishes for unpolarized sources and for decays with \( j = j' = \frac{1}{2} \).