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Vinjamoor, Harsh; Schaft, Arjan van der

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Asymptotic achievability for linear time invariant state space systems.

Harsh Vinjamoor* and Arjan van der Schaft*

Abstract— **We consider here the problem of finding a controller such that when interconnected to the plant, we obtain a system which is asymptotically equivalent to a desired system. Here 'asymptotic equivalence' is formalized as 'asymptotic bisimilarity'. Intuitively speaking, two systems are asymptotically bisimilar if the difference between their outputs decays to zero with time. We give necessary and sufficient conditions for the existence of such a controller. These conditions can be verified computationally using standard algorithms in linear geometric control. The systems we consider are linear time invariant input-state-output systems.**
Keywords: Linear systems, bisimulations, achievability, canonical controller, interconnection.

I. Introduction and motivation

A basic question in systems and control theory is the following: given a plant system, by constructing another dynamical system called a controller and interconnecting this to the plant, what are the possibilities of modifying the input-output behavior of the plant? Now, to decide whether the interconnected system does indeed have the desired dynamics, we need some notion of equivalence between systems. Also, we must specify exactly what we mean by 'interconnection'. In this paper we shall deal with 'asymptotic bisimilarity', i.e., we require that the controlled system behaves like the desired system asymptotically. The notion of asymptotic achievability extends the notion of achievability studied in [VvdS09]. In the latter case one looks for a controller such that the controlled system behaves exactly as desired. Related problems for the exact bisimulation case have also been addressed in [Tab08] and [PvdSB05].

In the following section, Section II, we explain why we consider a more general class of interconnections than is usually considered. Section III elaborates on the notational issues and also states the problem statement precisely. The main result of the paper is stated and proved in Section IV. We explain how the conditions in the main result can be verified computationally in Section V and conclude with some remarks and future directions in Section VI.

Throughout this paper we will have to deal with three systems viz. the plant P, the desired system S and the controller system C. The goal is to find necessary and sufficient conditions under which there exists a controller such that when interconnected to P, the resulting system, called a *controlled system*, behaves *like* S. When we say 'behaves like S' we mean that the outputs become equal asymptotically with time. We shall make this precise later. If such a C exists we say that C *asymptotically achieves* S.

We first describe the class of systems that we wish to consider. All our systems shall have linear time invariant state space descriptions. All state spaces are finite dimensional real vector spaces. Let P denote the plant. It is a system which has an input vector u_P, two output vectors y_P and z_P and a state vector x_P in some state space $\mathcal{X}(P)$, given as

$$P: \quad \begin{aligned} \dot{x}_P &= A_P x_p + B_P^u u_P \\ \begin{bmatrix} z_P \\ y_P \end{bmatrix} &= \begin{bmatrix} C_P^z \\ C_P^y \end{bmatrix} x_P \end{aligned}$$

Next we have the desired system S. This is an autonomous system with state x_S in state space $\mathcal{X}(S)$ and an output vector z_S, given as

$$S: \quad \begin{aligned} \dot{x}_S &= A_S x_S \\ z_S &= C_S^z x_S \end{aligned}$$

The equations describing the controller C are

$$C: \quad \begin{aligned} \dot{x}_C &= A_C x_C + B_C^u u_C \\ y_C &= C_C^y x_C \end{aligned}$$

The aim is to make sure that the variable z_P behaves asymptotically as it behaves in S. We assume that only the variables (u_P, y_P) of the plant are available for interconnection with the controller. Practically this is quite a common situation; for instance, if the variable z_P is difficult to measure or if the sensor required is too expensive compared to the sensors for measuring y_P. We shall call the variable z_P *manifest* (denoted by m) as it is the variable whose behaviour we are interested in. For the variables (u_P, y_P) we shall use the term *control* variables, denoted by c, since they are available for control.

II. More general interconnections

Classical control theory deals with input/output controllers, i.e. controllers which accept the output of the plant as their input and produce an output which acts as an input to the plant. Thus a controller is looked at as a signal processing unit. These controllers have many advantages. For instance, in the case of linear time invariant state space systems without feed-through terms, an input/output interconnection is guaranteed to be well-posed, in the sense that *after* attaching the controller, the space of initial conditions of the

* The authors are with the Johann Bernoulli Institute of Mathematics and Computer Science, University of Groningen. The corresponding author for this paper is Harsh Vinjamoor. Email: h.g.vinjamoor@rug.nl, a.j.van.der.schaft@rug.nl

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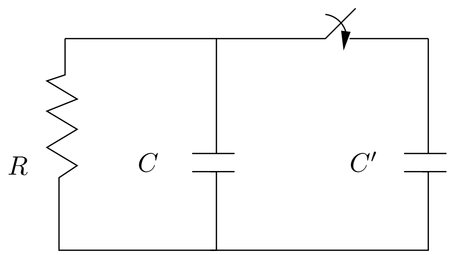
plant does not become a proper subspace of the plant state space; a property that is often desirable.

However, there are desired systems S which can be achieved but not by this class of interconnections. These considerations are not new and have already been addressed; see for instance the example of the 'door closing mechanism' in [Kui95], [Wil97]. Consider also the example of an $RC-$circuit, see Figure II, in which we can attach another capacitance C' in parallel to the first capacitance C and thus 'shape' the capacitance of the circuit. This interconnection too is not a standard feedback interconnection since the voltages of the two capacitors (the outputs in this case) are equated. We now show an explicit mathematical example which illustrates the need for interconnections other than the standard feedback interconnection.



Suppose we have a linear time invariant plant described by the differential equation

$$\begin{aligned} \dot{x}_P &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_P + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z_P &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_P \\ y_P &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_P \end{aligned} \tag{1}$$

Suppose the desired system S is just the zero system; hence the aim is to design a controller so that the output z_P of the plant decays to zero. In this case, if we were to use the standard (well-posed) feedback configuration, the space of allowed initial conditions of the plant would continue to be the whole state space; as a result, we cannot impose the condition $x_P^1(0) = 0$. Since the first equation is $\dot{x}_P^1 = x_P^1$, if $x_P^1(0) \neq 0$ then the output z_P will not decay to zero with time. Note that due to the structures of the input matrix and the $A-$matrix i.e. it has a zero as its first entry, no feedback can influence the state x_P^1. Thus the standard feedback configuration cannot ensure that the output z_P will decay to zero. Now, if we set $y_P = 0$ then the output z_P will be given by $z_P = x_P^2(t)$. By choosing our controller $u_P = 0$ and $y_P = 0$ we can thus ensure that $z_P(t)$ decays zero. So $y_P = 0,\ u_P = 0$ is a controller which asymptotically achieves the desired behaviour and is *not* in the standard feedback configuration.

III. Definitions and notation

We shall use the notation[1] $(x_P(0), u_P, y_P, z_P) \in P$ to indicate that starting with an initial condition $x_P(0)$, if we apply the input function u_P to the system P, then (y_P, z_P) will be the resulting output functions. Similarly, $(x_S(0), z_S) \in S$ if for initial condition $x_S(0)$ the output is z_S. The interconnection of two systems will always be either with respect to the manifest variables or the control variables; we shall indicate this by subscripts m and c, respectively. Also, an interconnection means that we set some variables of one system equal to some variables of the other system. We now make this precise. Assume that $z_P(t)$ and $z_S(t)$ are vectors of the same size. Similarly, the number of variables in (u_P, y_P) and in (u_C, y_C) are the same. Then we define an interconnection of P and a controller C by the equations $(u_P, y_P) = \Pi(u_C, y_C)$ where Π is a *permutation* matrix; i.e., a matrix obtained by permuting the columns of an identity matrix. We shall refer to Π as the *interconnection matrix*. More precisely, $P \overset{\Pi}{\underset{c}{\|}} C := \{(x_P(0), u_P, y_P, z_P) \times (x_C(0), u_C, y_C) \mid (x_P(t), x_C(t)) \in \mathcal{X}(P) \times \mathcal{X}(C)$ and $\begin{bmatrix} u_P(t) \\ y_P(t) \end{bmatrix} = \Pi \begin{bmatrix} u_C(t) \\ y_C(t) \end{bmatrix}\ \forall t \geq 0\}$. The state space of this interconnected system is a subspace of $\mathcal{X}(P) \times \mathcal{X}(C)$ and we denote it by $\mathcal{X}(P \overset{\Pi}{\underset{c}{\|}} C)$. Note that this can be strictly smaller than $\mathcal{X}(P) \times \mathcal{X}(C)$. For the case when $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ we recover the standard feedback interconnection, i.e., $u_C = y_P$ and $y_C = u_P$.

Along the same lines, $S \overset{I}{\underset{m}{\|}} P := \{(x_S(0), z_S) \times (x_P(0), u_P, y_P, z_P) \mid (x_S(t), x_P(t)) \in \mathcal{X}(S) \times \mathcal{X}(P)$ and $z_S(t) = z_P(t) \forall t \geq 0\}$. As earlier we denote the state space by $\mathcal{X}(S \overset{I}{\underset{m}{\|}} P)$; here the interconnection matrix is the identity matrix so that the interconnection equations become $z_P = z_S$. Interconnections of more than two systems are defined in a similar way. For example $S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P$ is defined as $(S \overset{I}{\underset{m}{\|}} P) \overset{I}{\underset{c}{\|}} P$. Before proceeding, we explain the state space of an interconnected system for one case. The explicit equations defining the interconnected system $S \overset{I}{\underset{m}{\|}} P$ are

$$\begin{aligned} \begin{bmatrix} \dot{x}_S \\ \dot{x}_P \end{bmatrix} &= \begin{bmatrix} A_S & 0 \\ 0 & A_P \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} + \begin{bmatrix} 0 \\ B_P^u \end{bmatrix} u_P \\ 0 &= \begin{bmatrix} C_S^z & -C_P^z \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} = z_S - z_P =: l \\ y &= \begin{bmatrix} 0 & C_P^y \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} \end{aligned} \tag{2}$$

The state space of this system is the largest controlled invariant subspace contained in the kernel of the output map for the output l (see [Won85] for details about controlled invariance). Also, computing the state space of the above interconnection is in this case equivalent to finding the largest simulation relation as explained in [vdS04]. Note that for S

[1]Note that the first entry $x_P(0)$ is a state vector while the other entries are functions.

we consider autonomous systems only; if the desired system S has an input then simulation relations and controlled invariant subspaces need not be equal.

We need to address one more question before we can state the problem precisely viz. the notion of equivalence. Given a controller C, when do we say that $P \overset{\Pi}{\underset{c}{\|}} C$ behaves like S? One intuitive idea is that for every initial condition in S there should exist an initial condition in $P \overset{\Pi}{\underset{c}{\|}} C$ such that the difference of the outputs $z_P - z_S$ decays to zero with time. The definitions that follow are very much in the spirit of the definitions of (exact) bisimulation as introduced in [Pap03], [vdS04]. A somewhat analogous notion is that of *approximate* bisimulation (see [GP07], [GP09]); note that this is *different* from the notion of asymptotic bisimulation that we shall now introduce.

Definition 1: A relation $\mathrm{R} \subseteq \mathcal{X}(P) \times \mathcal{X}(C) \times \mathcal{X}(S)$ is called an *asymptotic bisimulation* relation between $P \overset{\Pi}{\underset{c}{\|}} C$ and S if for all $(x_P(0), x_C(0), x_S(0)) \in \mathrm{R}$, $(x_P(t), x_C(t), x_S(t)) \in \mathrm{R}\ \forall\ t\ \geq 0$ and $\|z_P - z_S\| \to 0$ as $t \to \infty$.

Definition 2: We say that $P \overset{\Pi}{\underset{c}{\|}} C$ is *asymptotically bisimilar* to S, denoted by $P \overset{\Pi}{\underset{c}{\|}} C \rightleftharpoons S$, if there exists an asymptotic bisimulation relation R which projects *onto* the state space of $P \overset{\Pi}{\underset{c}{\|}} C$ and S.

Problem statement: Give P and S, find necessary and sufficient conditions for the existence of a controller C and an interconnection matrix Π such that $P \overset{\Pi}{\underset{c}{\|}} C \rightleftharpoons S$.

IV. Main result

We will need to talk about states of the system $S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P$; to make things easier to write we shall often write these systems as $S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$ and refer to states and trajectories in various components by using different primes whenever necessary. For example $(x_S(0), z) \times (x_P(0), u, y, z) \times (x_{P'}(0), u, y, z') \in S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$ has a state with $x_S(0) \in \mathcal{X}(S)$, $x_P(0) \in \mathcal{X}(P)$ and $x_{P'}(0) \in \mathcal{X}(P)$. Similarly it has output $z = z_S = z_P$, input $u = u_P = u_{P'}$, output $y = y_P = y_{P'}$ and output $z' = z_{P'}$. Note that P' is merely a copy of the system P, i.e., the equations defining P, P' are the same. However, the initial conditions need not be the same, i.e., there exist trajectories $(x_S(0), z) \times (x_P(0), u, y, z) \times (x_{P'}(0), u, y, z') \in S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$ where $x_P(0) \neq x_{P'}(0)$.

Before proceeding we observe that since our goal is to achieve S asymptotically, we can assume without loss of generality that S is anti-stable, i.e., the system matrix for S has all its eigen values in the closed right half of the complex plane. For if S has a stable part, then we can write it as $\begin{bmatrix} \dot{x}_- \\ \dot{x}_+ \end{bmatrix} = \begin{bmatrix} A_S^- & 0 \\ 0 & A_S^+ \end{bmatrix} \begin{bmatrix} x_- \\ x_+ \end{bmatrix}$ where A_S^- is Hurwitz. Consequently, for any initial condition with $x_-(0) \neq 0$ and $x_+(0) = 0$ we can choose the zero state in *any* controlled system; so trivially also the plant itself. Stated differently, a stable system is asymptotically bisimilar to the zero system. We shall henceforth assume that S is anti-stable.

We need the following one-sided version of asymptotic bisimulation.

Definition 3: $\mathrm{R} \subseteq \mathcal{X}(S) \times \mathcal{X}(P)$ is said to be an *asymptotic simulation* relation of S by P if for all $(x_S(0), x_P(0)) \in \mathrm{R}$ there exists an input function u_P such that $(x_S(t), x_P(t)) \in \mathrm{R}\ \forall\, t\ \geq 0$ and $\|z_S - z_P\| \to 0$.

Definition 4: S is said to be asymptotically simulated by P if there exists a simulation relation R which projects *onto* the state space of S.

We define N as the subsystem of the plant which is 'hidden' from the controller: it is the system obtained by setting the control variables of the plant to zero, i.e., $u_P = 0$ and $y_P = 0$. Its state space $\mathcal{X}(N)$ is the maximal controlled invariant subspace of the plant contained in the kernel of the output map C_P^y. Since N is 'hidden' from the controller, it is to be expected to play a role in the necessary and sufficient conditions for asymptotic achievability. For stating these conditions we need one more definition.

Definition 5: $\mathrm{R} \subseteq \mathcal{X}(N) \times \mathcal{X}(S)$ is said to be an *asymptotic simulation* relation of N by S if for all $(x_N(0), x_S(0)) \in \mathrm{R}$, $\|z_S - z_P\| \to 0$. We say that N is asymptotically simulated by S if R projects *onto* $\mathcal{X}(N)$.

As we shall see in Section V, both, simulation and bisimulation relations are controlled invariant subspaces and hence we can also compute the largest such asymptotic simulation and asymptotic bisimulation relations between two systems. We shall denote the maximal asymptotic simulation relations of S by P and that of N by S with the symbols R_{SP} and R_{NS} respectively.

Theorem 6: There exists a controller C which asymptotically achieves S if and only if S is asymptotically simulated by P and N is asymptotically simulated by S.

In the proof of the main result we need the following propositions.

Proposition 7: Assume S is anti-stable and that S is asymptotically simulated by P. Then the state space of $S \overset{I}{\underset{m}{\|}} P$ projects *onto* the state space of S.

The above proposition says that if an anti-stable S is asymptotically simulated by P, then it is in fact exactly simulated by P, i.e., for all $x_S(0)$ there exists an $x_P(0)$ and an input function u_P such that $z_S(t) = z_P(t)\,\forall\, t\ \geq 0$. Note that this is akin to the internal model principle. We skip the proof of this proposition due to lack of space.

Proposition 8: [VvdS09] The trajectories $(x_P(0), u, y, z)$ and $(x'_P(0), u, y, z')$ are *both* trajectories in P if and only if $x'_P(0) - x_P(0) \in \mathcal{X}(N)$.

We shall now prove Theorem 6.

Proof :

Only if: Since C asymptotically achieves S, i.e., $P \overset{\Pi}{\underset{c}{\|}}$

$C \rightleftharpoons S$, there exists an asymptotic bisimulation relation R which projects *onto* the state space of S and $P \overset{\Pi}{\underset{c}{\|}} C$. Hence for every state $x_S(0) \in \mathcal{X}(S)$ there exists a state $x_P(0) \in \mathcal{X}(P)$ and a function u_P such that $\|z_S - z_P\| \to 0$. Hence S is asymptotically simulated by P. Similarly, let $(x_P(0), x_C(0)) \in \mathcal{X}(P) \times \mathcal{X}(C)$ with $x_P(0) \in \mathcal{X}(N)$ and hence $(x_P(0), u_P = 0, y_P = 0, z_P) \times (x_C(0), u_C = 0, y_C = 0) \in P \overset{\Pi}{\underset{c}{\|}} C$. Since $P \overset{\Pi}{\underset{c}{\|}} C \rightleftharpoons S$ there exists a state $x_S(0) \in \mathcal{X}(S)$ and $(x_S(0), z_S) \in S$ such that $\|z_P - z_S\| \to 0$. Thus N is asymptotically simulated by S.

If: Since S is asymptotically simulated by P and is anti-stable, we can conclude from proposition 7 that the $S \overset{I}{\underset{m}{\|}} P$ has a state space which projects *onto* $\mathcal{X}(S)$. We shall show that $(S \overset{I}{\underset{m}{\|}} P) \overset{I}{\underset{c}{\|}} P \rightleftharpoons S$ so that the term in brackets viz. $S \overset{I}{\underset{m}{\|}} P$ is our controller. For convenience we shall denote this second S by S' and the second P by P' i.e., we will show that $(S \overset{I}{\underset{m}{\|}} P) \overset{I}{\underset{c}{\|}} P' \rightleftharpoons S'$. Let $(x_S(0), x_P(0), x'_P(0)) \in \mathcal{X}(S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P')$ with $(x_S(0), z = z_S) \times (x_P(0), u, y, z = z_P) \times (x'_P(0), u, y, z_{P'}) \in S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$. Then by Proposition 8 we can write $x'_P(0) = x_P(0) + x_N(0)$ where $x_N(0) \in \mathcal{X}(N)$. We can accordingly write the above trajectory as
$(x_S(0), z) \times (x_P(0), u, y, z) \times (x_P(0), u, y, z)$
+
$(0, 0) \times (0, 0, 0, 0) \times (x_N(0), 0, 0, z_N) \in S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$ where $z_N := z_{P'} - z_P$. Since N is asymptotically simulated by S there exists $x_S^N(0)$ such that $(x_N(0), x_S^N(0)) \in \mathrm{R_{NS}}$. Choose a state $x_S(0) + x_S^N(0)$ in $\mathcal{X}(S')$ with $(x_S(0), z) + (x_S^N(0), z_S^N) \in S'$; denote $z_{S'} := z + z_S^N$. Since N is asymptotically simulated by S we have that $\|z_N - z_S^N\| \to 0$ as $t \to \infty$. Thus, $\|z_{S'} - z_{P'}\| = \|z + z_S^N - z - z_N\| = \|z_S^N - z_N\| \to 0$.

We can hence conclude that the bisimulation relation between $S \overset{I}{\underset{m}{\|}} P \overset{I}{\underset{c}{\|}} P'$ and S' is given by

$$\{(a, b, c, d) \mid (a, b) \in \mathrm{R_{SP}}, c - b \in \mathcal{X}(N),$$

$$(c - b, d - a) \in \mathrm{R_{NS}}\}.$$

□

V. Computational aspects

In this section we illustrate how the conditions in Theorem 6 can be checked computationally. Observe that for both the conditions, viz. S being asymptotically simulated by P and N being asymptotically simulated by S we have to check whether an autonomous system is being simulated by another system. We shall show how this is done for checking whether S is simulated by P; N being simulated by S can be checked along the same lines.

We first prove a simple result about observable autonomous linear systems.

Lemma 9: Consider the system $\dot{x} = Ax$, $y = Cx$ with (C, A) observable and A anti-stable. Then for all $x(0) \neq 0$ there exists $\epsilon > 0$ and $T \geq 0$ such that $\|y(t)\| > \epsilon$ for all $t \geq T$.

Proof : By observability, $x(0) \neq 0$ implies $y(.) \neq 0$. $y(.)$ is a linear combination of unstable exponentials we conclude that $\|y(t)\| \to \infty$ as $t \to \infty$. Hence there exists $\epsilon > 0$ and $T \geq 0$ such that $\|y(t)\| > \epsilon$ for all $t \geq T$. □

Consider the equations for $S \overset{I}{\underset{m}{\|}} P$:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_S \\ \dot{x}_P \end{bmatrix} &= \begin{bmatrix} A_S & 0 \\ 0 & A_P \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} + \begin{bmatrix} 0 \\ B_P^u \end{bmatrix} u \\
l &:= \begin{bmatrix} C_S^z & -C_P^z \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix} = z_S - z_P \\
y &= \begin{bmatrix} 0 & C_P^y \end{bmatrix} \begin{bmatrix} x_S \\ x_P \end{bmatrix}
\end{aligned} \tag{3}$$

The equations for y are not relavant for the following computations and we ignore them. Suppose the largest controlled invariant subspace contained in the kernel of the output map corresponding to output l is $\mathcal{V}$. Let the corresponding input which renders $\mathcal{V}$ invariant be $u = F \begin{bmatrix} x_S^T & x_P^T \end{bmatrix}^T + v$. On adapting the basis to $\mathcal{V}$ the above equations take the following form:

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} A_F^{11} & A_F^{12} \\ 0 & A_F^{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_L \\ 0 \end{bmatrix} w + Gv \\
l &= C_2 z_2
\end{aligned} \tag{4}$$

where T is the associated state space transformation matrix. Here the component $z_1 \in \mathcal{V}$ and $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = TB$. Consider the equation $\dot{z}_2 = A_F^{22} z_2 + G_2 v$. The largest stabilizability subspace of this system is $(A_F^{22}, G_2)-$invariant. We adapt the basis of this subsystem so that the overall system is described by the equations

$$\begin{aligned}
\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2^1 \\ \dot{z}_2^2 \end{bmatrix} &= \begin{bmatrix} A_F^{11} & A_F^{12} & * \\ 0 & A_F^{22^{11}} & * \\ 0 & 0 & A_F^{22^{22}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2^1 \\ z_2^2 \end{bmatrix} \\
&\quad + \begin{bmatrix} B_L \\ 0 \\ 0 \end{bmatrix} w + \begin{bmatrix} H_1 \\ H_2 \\ 0 \end{bmatrix} v \\
z_S - z_P &= C_2 z_2^1 + C_3 z_2^2
\end{aligned} \tag{5}$$

where $*$ indicates blocks of appropriate sizes whose precise entries are not of interest for our analysis. Note that $A_F^{22^{22}}$ is anti-stable and the pair $(C_3, A_F^{22^{22}})$ is observable, for otherwise the maximality of $\mathcal{V}$ would be contradicted. By Lemma 9 we conclude that the largest subspace of $\mathcal{X}(S) \times \mathcal{X}(P)$ such that it is controlled invariant and $\|z_S - z_P\| \to 0$ is given by

$\mathcal{V} + \mathcal{X}_{\text{stab}}$ where $\mathcal{X}_{\text{stab}}$ is the largest stabilizability subspace of $S \overset{I}{\underset{m}{\|}} P$; this is the maximal asymptotic simulation relation of S by P. In the above equations $\mathcal{V} + \mathcal{X}_{\text{stab}}$ is the subspace corresponding to the vectors with the z_2^2 component zero.

VI. Conclusions

We presented necessary and sufficient conditions for the existence of a controller C which asymptotically achieves S asymptotically. The conditions can be verified using standard algorithms from the theory of linear geometric control. The results we presented were for the case of an autonomous S which is anti-stable. The results also hold true for a general autonomous S, the difference being that our controller $S \overset{I}{\underset{m}{\|}} P$ has a state space which need not project onto the state space of S. Extending the results for the case of non-autonomous desired systems is being investigated. Also observe that our results bear relations with the problem of output regulation. The difference is that we have *partial* interconnection and no state feedback. The precise relations between our results and the classical output regulation are currently being studied.

References

[GP07] A. Girard and G.J. Pappas. Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5):782–798, May 2007.

[GP09] A. Girard and G.J. Pappas. Hierarchical control system design using approximate simulation. *Automatica*, 45:566–571, 2009.

[Kui95] Margreet Kuijper. Why do stabilizing controllers stabilize? *Automatica*, 31(4):621–625, 1995.

[Pap03] G.J. Pappas. Bisimilar linear systems. *Automatica*, 39(12):2035–2047, 2003.

[PvdSB05] G. Pola, A.J. van der Schaft, and M.D. Di Benedetto. Achievable bisimilar behavior of abstract state systems. pages 1535–1540. in Proc. 44th IEEE Conference on Decision and Control and the European Control Conference, Seville, Spain, December 2005.

[Tab08] P. Tabuada. Controller synthesis for bisimulation equivalence. *Systems & Control letters.*, 57:443–452, 2008.

[vdS04] A.J. van der Schaft. Equivalence of dynamical systems by bisimulation. *IEEE Transaction on Automatic Control*, 49(12):2160–2172, December 2004.

[VvdS09] Harsh Vinjamoor and Arjan van der Schaft. On achievable bisimulations for linear time invariant systems. in Proc. European Control Conference, Budapest, Hungary, August 23-26 2009.

[Wil97] J.C. Willems. On interconnections, control and feedback. *IEEE Trasactions on Automatic Control*, 42:326–339, 1997.

[Won85] W.M. Wonham. *Linear multivariable control: a geometric approach.* Springer, New York, Third Edition, 1985.