Do Diophantine vectors form a Cantor bouquet?

Henk Broer*

Institute for Mathematics and Computing Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands

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Dedicated to Robert Devaney on the occasion of his 60th birthday

We introduce the set of Diophantine vectors in $\mathbb{R}^n$, which is a standard notion in KAM Theory. The problem is whether this set forms a Cantor bouquet.

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1. Definition and elementary properties

Let $n \geq 2$ and for $\tau > 0$ and $\gamma > 0$ consider the set of $(\tau, \gamma)$-Diophantine frequency vectors by

$$\mathbb{R}^n_{\tau, \gamma} = \{ \omega \in \mathbb{R}^n | \forall k \in \mathbb{Z}^n \setminus \{0\} : |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau} \}.$$ 

Here, $\langle \omega, k \rangle = \sum_{j=1}^n \omega_j k_j$ is the usual inner product and $|k| = \sum_{j=1}^n |k_j|$ is the length of the integer vector $k$.

We summarize a few properties of $\mathbb{R}^n_{\tau, \gamma}$ (for details see [3,4]). First of all, the set $\mathbb{R}^n_{\tau, \gamma}$ is a union of closed half lines in the sense that for $\omega \in \mathbb{R}^n_{\tau, \gamma}$, then $s\omega \in \mathbb{R}^n_{\tau, \gamma}$ for all $s \geq 1$. Moreover, if $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, we consider the intersection $\mathbb{R}^n_{\tau, \gamma} \cap S^{n-1}$. This is a closed (and hence compact) set that by the Cantor–Bendixson Theorem is the union of a perfect and a discrete set. Due to the fact that for any $k \in \mathbb{Z}$ the resonant hyperplane

$$\{ \omega \in \mathbb{R}^n | \langle \omega, k \rangle = 0 \},$$

is in its complement, the perfect set is totally disconnected and hence is a Cantor set.

For $\tau > n - 1$, the intersection $\mathbb{R}^n_{\tau, \gamma} \cap S^{n-1}$ has positive measure in $S^{n-1}$ for sufficiently small $\gamma > 0$. Indeed, one even has that the measure of the complement $S^{n-1} \setminus \mathbb{R}^n_{\tau, \gamma}$ is of order $O(\gamma)$ as $\gamma \downarrow 0$. For a sketch of the planar case $\mathbb{R}^2_{\tau, \gamma} \subset \mathbb{R}^2$, see Figure 1.

For $n = 2$, by taking the ratio $\varrho = \omega_1 / \omega_2$, one finds a relationship with the Diophantine numbers $\varrho \in \mathbb{R}$ satisfying

$$\left| \frac{p}{q} - \varrho \right| \leq \gamma q^{-(\tau+1)},$$

for all rationals $p/q \in \mathbb{Q}$, which is a subset of the Bruno numbers [7]. Compare with the horizontal line in Figure 1. Generally, for $n \geq 3$, a similar relationship is established.

*Email: h.w.broer@rug.nl

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between $\omega \in \mathbb{R}^n$ and $[\omega_1 : \omega_2, \ldots, \omega_n] \in \mathbb{P}^{n-1}(\mathbb{R})$, the $(n - 1)$-dimensional projective space [2].

2. The problem

The problem is to show that $\mathbb{R}^n_{\tau, \gamma}$ is a Cantor bouquet [1,5]. For this, it is needed to know more about the endpoints of the closed half lines in $\mathbb{R}^n_{\tau, \gamma}$, in particular, one has to prove that the set of endpoints accumulates on every point of $\mathbb{R}^n_{\tau, \gamma}$.

For $n = 2$, the answer may be related to a similar result that concerns the graph of the Bruno numbers, which uses continued fraction expansions [6]. For $n \geq 3$, the number theoretic aspects of the sets $\mathbb{R}^n_{\tau, \gamma}$ and $\mathbb{R}^n_{\tau, \gamma} \cap \mathbb{S}^{n-1}$ may be a lot more involved.

References