Multi-loop Hysteresis and Recursive Remnant Control

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Chapter 3

The Preisach hysteresis operator

I remember seeking advice from someone—who could it have been?—about whether this work was worth submitting for publication; the reasoning it uses is so very simple... Fortunately he advised me to go ahead...
—Joseph Kruskal

It has been observed from experiments and numerical simulations that a Preisach operator, whose weighting function is not restricted to be sign-definite, can describe butterfly hysteresis loops (see, for instance [16, 17]). In this chapter, we study and characterize Preisach hysteresis operators capable of exhibiting butterfly loops. We firstly present the analysis of a class of Preisach operators whose weighting function has one positive and one negative domain. We show that under mild assumptions over the distributions of these domains, the input-output behavior of Preisach operator can exhibits butterfly loops. Subsequently, we introduce a general class of Preisach operator whose weighting function can assume more than one positive and one negative domain. We show that the input-output behavior of these operators can exhibit hysteresis loops with two or more sub-loops.

3.1 The Preisach butterfly hysteresis operator

For the past decades, hysteresis operators have been widely studied and characterized (see, for instance, the exposition in [78, chap. 3]) [11, chap. 2] [43, chap. 1]. When a simple periodic input with a single maximum and a single minimum in its periodic interval is applied to a hysteresis operator, the input-output phase plot will undergo a periodic closed orbit which is commonly referred to as hysteresis loop. Similar to the periodic input-output map introduced in [51, Definition 2.2] for Duhem models, we can define a *hysteresis loop* as follows.

Definition 3.1. *Consider a hysteresis operator* Φ *and an input-output pair* (u, y) *with* $y = \Phi(u)$*. Let* u *be periodic with a period of* $T > 0$*, with one maximum* $u_{\max} \in \mathbb{R}$ *and with one minimum* $u_{\min} \in \mathbb{R}$ *in its periodic interval. Assume that there exists a constant* $t_p \geq 0$ *such that* y *is periodic in the interval* $[t_p, \infty)$*. The periodic orbit given by* $\mathscr{H}_{u,y} = \{(u(t), y(t)) \mid t \in [t_p, \infty)\}$ *is called a* hysteresis loop *if there exists a* $\upsilon \in \mathbb{R}$ *such that*

$$card\big[\{(\upsilon, \gamma) \in \mathscr{H}_{u,y} \mid \gamma \in \mathbb{R}\}\big] = 2,$$

where card denotes the cardinality of a set. △

Remark 3.2. *The hysteresis loop as defined above means that the curve defined by* $\mathscr{H}_{u,y}$ *can have at most two elements* (υ, γ_1) *and* (υ, γ_2) *for any admissible point* υ*. This is due to the fact that we consider inputs with only one maximum and one minimum in its periodic interval and consequently, we exclude the possibility of the curve* $\mathscr{H}_{u,y}$ *to have inner minor loops. Nevertheless, it is possible to have multiple input-output loops formed by self-intersections of the curve* $\mathscr{H}_{u,y}$ *that will be studied further in this chapter.*

As shown in [29, 52, 53, 68], the input-output behavior of hysteresis operators can be classified by the type of hysteresis loops they produce. Simple hysteresis loops can have a clockwise or counterclockwise orientation which is given in terms of the signed-area enclosed by its phase plot. Following from Green's theorem, the signed-area enclosed by an input-output pair (u, y) that forms a closed curve in an interval $[t_1, t_2]$ is given by

$$\mathbb{A} := \frac{1}{2}\int_{t_1}^{t_2} \big[u(\tau)\dot{y}(\tau) - y(\tau)\dot{u}(\tau)\big]\,\mathrm{d}\tau. \tag{3.1}$$

Hence, generalizing this notion we can say that a hysteresis loop $\mathscr{H}_{u,y}$ is clockwise (resp. counterclockwise) if its signed-area $\mathbb{A}$ given by (3.1) with $t_1 \geq t_p$ and $t_2 = t_1 + T$ satisfies $\mathbb{A} < 0$ (resp. $\mathbb{A} > 0$). In a similar manner, we say that a hysteresis operator Φ exhibits clockwise (resp. counterclockwise) input-output behavior if there exists at least one hysteresis loop $\mathscr{H}_{u,y}$ corresponding to an input-output pair (u, y) with $y = \Phi(u)$ which is clockwise (resp. counterclockwise).

Based on these concepts, we can study hysteresis operators Φ that give rise to butterfly loops using the enclosed signed-area as in (3.1) of the resulting hysteresis loops. However, as depicted in Fig. 1.2, we would like to note that the so-called butterfly loops are composed of two subloops connected by a self-intersection point where one of the loops is clockwise, and the other is counterclockwise. Thus, the total signed-area of the butterfly loop could be either positive or negative depending on the difference between the individual signed-area of each subloop. For this reason, we define a *butterfly hysteresis operator* as follows.

Definition 3.3. *A hysteresis operator* Φ *is called a* butterfly hysteresis operator *if there exists a hysteresis loop* $\mathscr{H}_{u,y}$ *with* $y = \Phi(u)$ *such that* $\mathbb{A} = 0$*, where* $\mathbb{A}$ *is defined as in* (3.1) *with* $t_1 \geq t_p$ *and* $t_2 = t_1 + T$*.* △

Remark 3.4. *The* butterfly hysteresis operator *as defined above only requires the existence of a single input-output pair whose total enclosed area is zero, and we do not impose restrictions on the rest of the input-output pairs to have zero enclosed area or to be symmetric. Moreover, this definition is equivalent to the one introduced in [26, Definition 2.2]. However, in this case the concept of* hysteresis loop *in Definition 3.1 has been used for defining the butterfly hysteresis operator with a particular input-output pair* (u,y) *that forms a periodic orbit, e.g. on the interval* $[t_1, t_1 + T]$*.*

Remark 3.5. *It is not straightforward to use the characterization presented in [2] to characterize a butterfly hysteresis operator since a butterfly hysteresis operator may exhibit hysteresis loops with negative enclosed area. For this reason, Definition 3.3 focuses on finding one input-output pair whose total enclosed area is zero.*

To present a Preisach hysteresis operator that can produce butterfly loops, we firstly introduce the following lemma that allows us to compute the enclosed area of the hysteresis loop of a single relay operator.

Lemma 3.6. *Consider the counterclockwise relay operator* $\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}$ *as in* (2.1) *with* $\alpha > \beta$*. For every periodic signal* u *with a period of* T *and with one maximum* $u_{\max} \in \mathbb{R}$ *and one minimum* $u_{\min} \in \mathbb{R}$ *in the periodic interval, the signed-area* $\mathbb{A}$ *corresponding to the input-output pair* (u,y) *with* $y = \mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u, r_0)$ *and* $r_0 \in \{-1,+1\}$ *is given by*

$$\mathbb{A} = \begin{cases} 2(\alpha - \beta) & \text{if } u_{\min} < \beta \text{ and } u_{\max} > \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It can be checked that the influence of relay initial condition to the output signal $y = \mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u, r_0)$ disappears after one period. In other words, the pair (u,y) will form a hysteresis loop $\mathscr{H}_{u,y}$ or a line in the time interval $[t_1, \infty)$ with $t_1 \geq t_p = T$. When the range of the input covers both switching points of the relay, i.e., $u_{\min} < \beta$ and $u_{\max} > \alpha$, then the relay will switch periodically forming a hysteresis loop $\mathscr{H}_{u,y}$ as in Fig. 2.1. Otherwise, it will be a line. If it is a hysteresis loop $\mathscr{H}_{u,y}$ then the signed-area that is produced is given by the area of the corresponding rectangle in the phase plot of $\{(u(\tau), y(\tau)) \mid \tau \in [t_1, t_1 + T]\}$ which is equal to $2(\alpha - \beta)$. If it is a line then the signed-area is equal to zero. □

Remark 3.7. *Lemma 3.6 is formulated to consider the hysteresis loop obtained from any simple periodic input with one maximum and one minimum. However, since the relay operator has only two output values* -1 *and* $+1$*, applying any input whose maxima and minima are greater and lower than the switching values* α *and* β*, correspondingly, will produce an input-output phase plot with the same shape. Therefore, an immediate consequence of Lemma 3.6, is that the signed-area enclosed by the hysteresis loop of a clockwise relay operator* $\mathscr{R}^{\circlearrowright}_{\alpha,\beta}$ *defined by* (2.2) *is given by* $-2(\alpha-\beta)$ *when* $u_{\min} < \beta$ *and* $u_{\max} > \alpha$*.*

We introduce now a particular class of the Preisach hysteresis operator with two-sided weighting function μ whose input-ouput behavior can exhibit butterfly loops, and whose proof is the main result of this section. By two-sided we mean that there exists a simple curve B that divides the Preisach domain P into two disjoint subdomains B_+ and B_- such that $P = B_+ \cup B_- \cup B$ and where $\mu(\alpha,\beta) \geq 0$ for every $(\alpha,\beta) \in B_+$ and $\mu(\alpha,\beta) \leq 0$ for every $(\alpha,\beta) \in B_-$.

Theorem 3.8. *Consider a Preisach hysteresis operator* $\mathscr{P}$ *as in* (2.3) *with* μ *be a two-sided weighting function. Suppose that the first order lower and upper partial moments of* μ *satisfy*

$$\int_r^{\infty} \mu(\alpha,\beta)\beta \mathrm{d}\beta = \infty \tag{3.2}$$

$$\int_{-\infty}^{r} \mu(\alpha,\beta)\alpha \mathrm{d}\alpha = \infty, \tag{3.3}$$

for all $(\alpha,\beta) \in P$*. Assume that the boundary curve B is monotonically decreasing. Then* $\mathscr{P}$ *is a* butterfly hysteresis operator.

Proof. Let us take arbitrary $(\alpha_1,\beta_1) \in B$ with $\alpha_1 > \beta_1$ (i.e., the point (α_1,β_1) is not on the boundary of P). Consider a subset of Preisach domain $P_1 := \{(\alpha,\beta) \in P \mid \alpha < \alpha_1, \beta > \beta_1\}$ which is a solid triangle whose vertices are at (α_1,β_1), (α_1,α_1) and (β_1,β_1). Note that since B is monotonically decreasing, it separates P in two polar regions where the weighting function μ assigned to the domain above B has a different sign with that below B. Without loss of generality, we consider the case where B_{1-} is below B and B_{1+} is above B. The arguments below are still valid when we consider the reverse case.

Due to the monotonicity of B, if we consider the extended area on left of P_1, which is given by $P_{1-}^{\text{ext}} := \{(\alpha,\beta) \in P \mid \beta < \beta_1, \alpha < \alpha_1\}$, the weight μ in this area will have the same sign as that in B_{1-}. The same holds for the extended area above P_1 where the weight μ in $P_{1+}^{\text{ext}} := \{(\alpha,\beta) \in P \mid \beta > \beta_1, \alpha > \alpha_1\}$ has the same sign as that in B_{1+}.

Let us now analyze the input-output behavior when the input u of the Preisach hysteresis operator is a periodic signal with a period of T and with one maximum $u_{\max} = \alpha_1$ and one minimum $u_{\min} = \beta_1$. It is clear that for every $t \geq t_p = T$, the initial conditions of all relays in P_1 no longer affect output y and it becomes periodic. Therefore, the relays $\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}$ whose states are switching periodically correspond to the domain P_1 while the state of all the relays in $P \backslash P_1$ remains the same as given by the initial condition. Consequently, following from Lemma 3.6, the signed-area of the hysteresis loop $\mathscr{H}_{u,y}$ obtained from the input-output pair (u,y) is given by

$$\begin{aligned}\mathbb{A} &= 2 \iint\limits_{(\alpha,\beta)\in P_1} \mu(\alpha,\beta)\ \big[\alpha-\beta\big]\mathrm{d}\alpha\mathrm{d}\beta \\ &= \underbrace{2 \iint\limits_{(\alpha,\beta)\in B_{1+}} |\mu(\alpha,\beta)|\big[\alpha-\beta\big]\mathrm{d}\alpha\mathrm{d}\beta}_{\mathbb{A}_+} \\ &\quad \underbrace{-2 \iint\limits_{(\alpha,\beta)\in B_{1-}} |\mu(\alpha,\beta)|\big[\alpha-\beta\big]\mathrm{d}\alpha\mathrm{d}\beta}_{\mathbb{A}_-}\end{aligned}$$

When the variation of μ is such that $\mathbb{A}_+ = \mathbb{A}_-$, we have obtained the condition for $\mathscr{P}$ to be a *butterfly hysteresis operator* where the chosen periodic input signal u ensures that $\mathbb{A} = 0$. However, since in general the variation in μ can be asymmetric, the signed-area $\mathbb{A}_+$ may not be equal to $\mathbb{A}_-$.

Let us consider the case when $\mathbb{A}_- > \mathbb{A}_+$ (i.e., the negative weight is dominant in P_1). In this case, we modify the periodic input signal u such that its maximum $u_{\max}$ is parametrized by $\lambda > \alpha_1$. Similar as before, we have that the relays $\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}$ whose states are switching periodically correspond to the domains P_1 and $P_{1+}^{\text{ext},\lambda} := \{(\alpha,\beta) \in P \mid \beta > \beta_1, \lambda > \alpha > \alpha_1\}$ while the state of relays corresponding to $P \backslash (P_1 \cup P_{1+}^{\text{ext},\lambda})$ remains the same as given by the initial condition. Hence, using again Lemma 3.6, the signed-area of the hysteresis loop $\mathscr{H}_{u,y}$ corresponding to the input-output pair (u,y) with modified u is now given by

$$\mathbb{A} = 2(\mathbb{A}_+ - \mathbb{A}_-) + \underbrace{2 \iint\limits_{(\alpha,\beta)\in P_{1+}^{\text{ext},\lambda}} |\mu(\alpha,\beta)|\big[\alpha-\beta\big]\mathrm{d}\alpha\mathrm{d}\beta}_{h(\lambda)}.$$

Since μ is a piecewise continuous function, the function $h(\lambda)$ is also a continuous function, $h(\alpha_1) = 0$ and is strictly increasing (as $\mu > 0$ in P_{1+}^{ext}). Due to the unboundedness of the first-order upper partial moment of μ as in (3.2), it follows that $h(\lambda) \to \infty$ as $\lambda \to \infty$. This implies that there exists $\alpha_2 > \alpha_1$ such that $h(\alpha_2) = \mathbb{A}_- - \mathbb{A}_+$. In this case, by taking a periodic signal with its maximum $u_{\max} = \alpha_2$ and its minimum $u_{\min} = \alpha_1$, the signed-area of the corresponding hysteresis loop $\mathscr{H}_{u,y}$ is equal to zero as claimed.

On the other hand, when $\mathbb{A}_+ > \mathbb{A}_-$, we can use *vis-a-vis* similar arguments as above where $u_{\min}$ is now parametrized by $\lambda < \beta_1$, instead of parameterizing $u_{\max}$ as before. For this situation, the additional relays that are affected by the modified input correspond to the domain $P_{1-}^{\text{ext},\lambda} := \{(\alpha,\beta) \in P \mid \lambda < \beta < \beta_1, \alpha < \alpha_1\}$. The claim then follows similarly as above where the additional signed-area of the corresponding hysteresis loop $\mathscr{H}_{u,y}$ is a continuous function $h(\lambda)$ that is strictly increasing and approaches $-\infty$ as $\lambda \to -\infty$.

Finally, we can follow the same reasoning as above for the case when B_{1-} is above B and B_{1+} is below B. □

In Theorem 3.8, we consider a general case where μ can be any two-sided function, as long as, its decay to zero, which is measured by its upper and lower partial moments, is not too fast. If this condition is not satisfied, we may not be able to find an extended subset in P parametrized by λ (as used in the proof of Theorem 3.8) such that the total signed-area $\mathbb{A}$ of the hysteresis loop $\mathscr{H}_{u,y}$ with the modified u is zero. Nevertheless, the conditions over the upper and lower partial moments of μ can be relaxed if we focus on a small region close to the meeting point of B and the line $\{(\alpha,\beta) \mid \alpha = \beta\}$. This is the case for the class of the Preisach hysteresis operator considered in the next result, which was also presented in [26].

Corollary 3.9. *Consider a Preisach hysteresis operator Φ as in* (2.3) *with a two-sided weighting function μ. Assume that the boundary curve of μ is given by $B = \{(\alpha,\beta) \in P \mid \alpha = -\beta + \kappa\}$ where $\kappa \in \mathbb{R}_+$ is an offset and μ is anti-symmetric with respect to B, i.e., $\mu(\alpha,\beta) = -\mu(-\alpha,-\beta)$ holds for all $(\alpha,\beta) \in P$. Then Φ is a butterfly hysteresis operator.*

The previous corollary follows directly from the proof of Theorem 3.8. In this case, it suffices to have a periodic input signal u whose maximum and minimum satisfy $u_{\max} = -u_{\min} + 2\kappa$. The relays $\mathscr{R}_{\alpha,\beta}$ whose state switches periodically will lie in a subset of P, which has the form of an isosceles and right triangle. Since the weighting function is anti-symmetric with respect to B then the signed-area of the hysteresis loop $\mathscr{H}_{u,y}$ will be zero.

3.1.1 Numerical example: Preisach butterfly operator with symmetrical two-sided weighting function

As a particular case of study, we analyze in the next example a Preisach butterfly operator belonging to the class of Proposition 3.9 with symmetrical two-sided weighting function. Let $B := \{(\alpha,\beta) \in P \mid \alpha = -\beta\}$ (with $\kappa = 0$) and consider a point $(-\beta_1, \beta_1) \in B$ such that $P_1 := \{(\alpha,\beta) \in P \mid \alpha < \beta_1, \beta > -\beta_1\}$. In this case, the subdomain of interest P_1 in the Preisach plane is an isosceles triangle with vertices in (α_1, α_1), $(\alpha_1, -\alpha_1)$, and $(-\alpha_1, -\alpha_1)$. Let us define the weighting function by

$$\mu(\alpha,\beta) := \begin{cases} -1 & \text{if } \alpha \leq -\beta,\ (\alpha,\beta) \in P_1 \\ 1 & \text{if } \alpha > -\beta,\ (\alpha,\beta) \in P_1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

An illustration of the weighting function (3.4) is included in Fig. 3.1. The Preisach

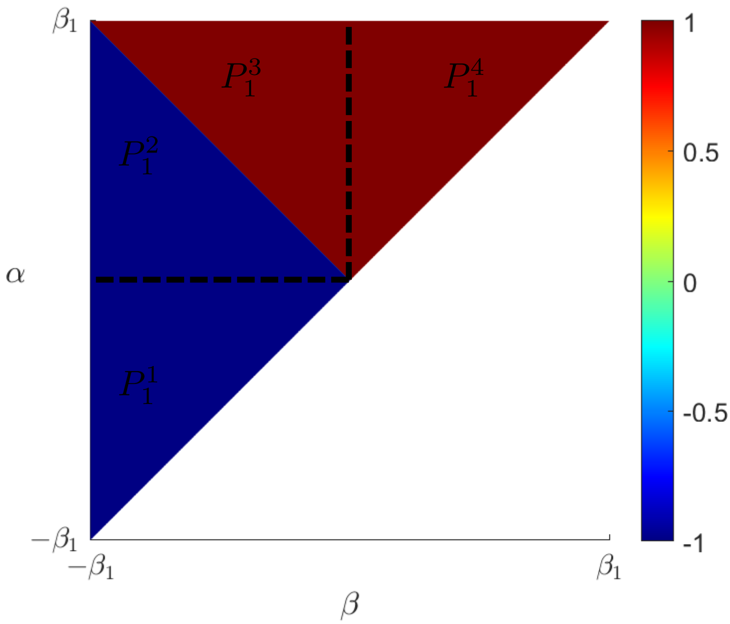
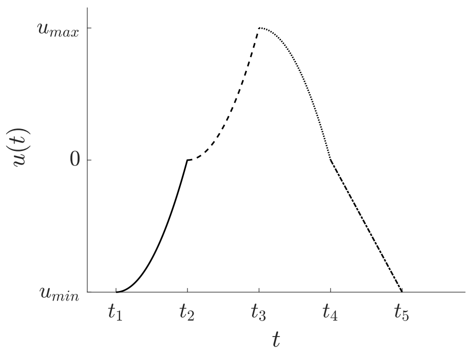


Figure 3.1: Weighting function $\mu(\alpha,\beta)$ as defined in (3.4).

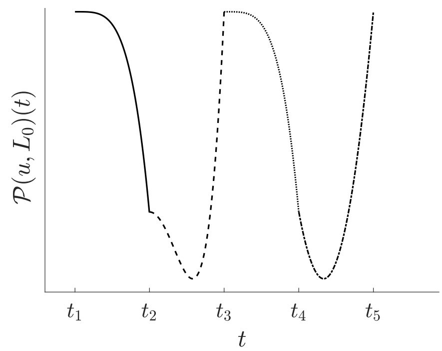
hysteresis operator with this weighting function clearly satisfies the conditions of Proposition 3.8 and consequently it is a *butterfly hysteresis operator*. It follows from this proposition that we no longer need the extended areas P_{1-}^{ext} and P_{1+}^{ext}. The subset of the

Preisach domain P_1 is now subdivided in four disjoint regions defined by

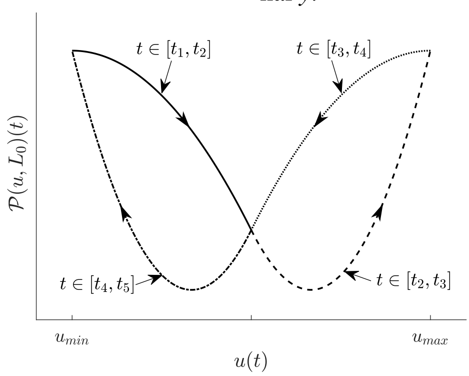
$$
\begin{aligned}
P_1^1 &:= \{(\alpha,\beta) \mid \alpha \leq 0,\ \beta \leq 0\},\\
P_1^2 &:= \{(\alpha,\beta) \mid \alpha > 0,\ \beta \leq 0,\ \alpha \leq -\beta\},\\
P_1^3 &:= \{(\alpha,\beta) \mid \beta \leq 0,\ \alpha > -\beta\},\\
P_1^4 &:= \{(\alpha,\beta) \mid \alpha > 0,\ \beta > 0\}.
\end{aligned}
$$



(a) The plot of input signal u in a periodic time interval $[t_1, t_5]$.



(b) The corresponding plot of output signal y.



(c) The input-output phase plot of input and output signal which shows a symmetric butterfly loop and whose signed-area is equal to zero.

Figure 3.2: Input-output phase plot using the Preisach butterfly hysteresis operator with symmetric two-sided weighting function.

Note that the output can be determined by the individual behavior of each region in the form

$$\begin{aligned}
&\big[\mathscr{P}(u,L_0)\big](t) := \\
&-\iint\limits_{(\alpha,\beta)\in P_1^1} \big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_{\alpha,\beta}(L_0))\big](t)\,\mathrm{d}\alpha\mathrm{d}\beta \;-\iint\limits_{(\alpha,\beta)\in P_1^2} \big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_{\alpha,\beta}(L_0))\big](t)\,\mathrm{d}\alpha\mathrm{d}\beta \\
&+\iint\limits_{(\alpha,\beta)\in P_1^3} \big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_{\alpha,\beta}(L_0))\big](t)\,\mathrm{d}\alpha\mathrm{d}\beta \;+\iint\limits_{(\alpha,\beta)\in P_1^4} \big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u,r_{\alpha,\beta}(L_0))\big](t)\,\mathrm{d}\alpha\mathrm{d}\beta
\end{aligned} \tag{3.5}$$

Let us analyze the input-output behavior when a periodic input u with period T and with one maximum $u_{\max} = \beta_1$ and one minimum $u_{\min} = -\beta_1$ is applied to this operator. Consider five time instances $t_1 < t_2 < t_3 < t_4 < t_5$ with $t_1 \geq T$ and such that $u(t_1) = u_{\min} = -\beta_1$, $u(t_2) = 0$, $u(t_3) = u_{\max} = \beta_1$, $u(t_4) = 0$, and $u(t_5) = u_{\min} = -\beta_1$. It is clear that $u(t)$ is monotonically increasing in the interval $t_1 \leq t < t_3$ and monotonically decreasing in the interval $t_3 \leq t < t_5$. An example of an input signal u satisfying these conditions is illustrated in Fig. 3.2(a). Using such input signal, the output signal $y(t)$ can be computed analytically for every $t \in [t_1, t_5]$ (i.e. for one periodic interval when the phase plot forms a hysteresis loop $\mathscr{H}_{u,y}$) based on (3.5) and is given by

$$\begin{aligned}
&\big[\mathscr{P}(u,L_0)\big](t) = \\
&\begin{cases}
-\big[u_{\max}+u(t)\big]^2 & \text{if } t_1 \leq t < t_2 \\
-\big[u_{\max}-u(t)\big]\big[u_{\max}+3u(t)\big] & \text{if } t_2 \leq t < t_3 \\
-\big[u_{\max}-u(t)\big]^2 & \text{if } t_3 \leq t < t_4 \\
-\big[u_{\max}+u(t)\big]\big[u_{\max}-3u(t)\big] & \text{if } t_4 \leq t < t_5
\end{cases}
\end{aligned} \tag{3.6}$$

Fig. 3.2(b) shows the corresponding output signal. By plotting the phase plot as in Fig. 3.2(c), we can see immediately that the resulting butterfly loop is symmetric as expected. Furthermore, using (3.1) it can be validated that the signed-area enclosed by this curve is equal to zero.

Remark 3.10. *Note that the assumption of the boundary B that separates the polar regions of the weighting function being monotonically decreasing is made to simplify the analysis in Proposition 3.8. Such assumption guarantees that it is always possible to find the extended domains* $P_{1+}^{ext,\lambda}$ *or* $P_{1-}^{ext,\lambda}$ *where the weighting function is sign-definite. However, according to Definition 3.3 and Lemma 3.6, we have that* $\mathscr{P}$ *is a* Preisach butterfly operator *as long as we can find a* hysteresis loop $\mathscr{H}_{u,y}$ *with an input u whose minimum and maximum can parameterize a subdomain* P_1 *of the form*

$P_1 := \{(\alpha,\beta) \in P \mid u_{\min} \leq \beta \leq \alpha,\, u_{\min} \leq \alpha \leq u_{\max}\}$ *which satisfies*

$$\iint_{(\alpha,\beta)\in P_1} \mu(\alpha,\beta)\ \big[\alpha-\beta\big]\mathrm{d}\alpha\mathrm{d}\beta = 0.$$

It follows that the monotonically decreasing property of the boundary B is sufficient but not necessary to obtain a Preisach butterfly operator.

Remark 3.11. *Constructing the weighting function to describe a specific butterfly hysteresis loop can be addressed by the classical identification scheme presented in [66], as it has been done in [26] for the butterfly hysteresis loop exhibited in the relation between strain and electric field of a Preisach hysteresis operator.*

3.2 Preisach multi-loop hysteresis operator

In the previous section, we have shown that a *butterfly hysteresis operator* can be obtained from a Preisach hysteresis operator with a two-sided weighting function. Following from the condition of zero total signed-area in Definition 3.3, the previous analysis is based on finding an input u such that each subloop contribution to the total signed-area of the hysteresis loop cancels each other. This analysis exploits the particular two-sided structure of the weighting function μ. However, imposing this structure to the weighting function μ is only a sufficient condition to obtain a *butterfly hysteresis operator* which, in addition, restricts all the hysteresis loops obtained from the Preisach hysteresis operator to have at most two subloops.

In this section, we study a larger class of Preisach hysteresis operators with more complex weighting functions that are not necessarily two-sided. The Preisach hysteresis operators in this class can produce hysteresis loops with more than two subloops. Therefore, using an analysis based only on the total enclosed signed-area of the hysteresis loops is no longer applicable for studying this class of hysteresis operators since the signed-area does not directly determine the number of subloops in a given hysteresis loop. In this case, we must note that if a hysteresis loop has two or more subloops each one of the subloops is connected to another subloop by at least one self-intersection point of the hysteresis loop. We will call these points the crossover points of a hysteresis loop and characterize them as follows. Consider a hysteresis loop $\mathscr{H}_{u,y}$ obtained from an input-output pair (u,y) of a hysteresis operator Φ with $y = \Phi(u)$. Let us select $t_1 \geq t_p$ such that $t_1 < t_2 < t_1 + T$ is a monotone partition of one periodic interval of u and $u(t)$ is monotonically increasing when $t \in [t_1,t_2]$ and monotonically decreasing when $t \in [t_2,t_1+T]$. We can split the hysteresis loop $\mathscr{H}_{u,y}$ into two segments that correspond to the subintervals of the monotone partition given by

$$\mathscr{H}_{u,y}^{+} := \{(u(t), y(t)) \mid t \in [t_1, t_2]\}, \tag{3.7}$$

$$\mathscr{H}_{u,y}^{-} := \{(u(t), y(t)) \mid t \in [t_2, t_1 + T]\}. \tag{3.8}$$

We define formally a crossover point as follows.

Definition 3.12. *Consider a hysteresis loop $\mathscr{H}_{u,y}$. A point $(u_c, y_c) \in \mathscr{H}_{u,y}$ is called a* crossover point *if $(u_c, y_c) \in \mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$.* △

Remark 3.13. *A hysteresis loop $\mathscr{H}_{u,y}$ will always have at least two crossover points corresponding to the points where the input u achieves its extrema $(u_{\min}, y_1), (u_{\max}, y_2) \in \mathscr{H}_{u,y}$ with $(u_{\min}, y_1) = (u(t_1), y(t_1)) = (u(t_1+T), y(t_1+T))$ and $(u_{\max}, y_2) = (u(t_2), y(t_2))$. Moreover, it is possible for a hysteresis loop $\mathscr{H}_{u,y}$ to have an infinite number of crossover points if, for instance, there exists a segment of intersection between $\mathscr{H}_{u,y}^{+}$ and $\mathscr{H}_{u,y}^{-}$, i.e. there exists a time subinterval $[t_3, t_4] \subset [t_1, t_1 + T]$ with $t_4 > t_3$ such that $(y(t), u(t)) \in \mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$ for every $t \in [t_3, t_4]$.*

Note that the numbers of subloops in a hysteresis loop $\mathscr{H}_{u,y}$ is not determined by the number of crossover points but by the number of maximal connected subsets in $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$, where each maximal connected subset can be a singleton in the case that the corresponding crossover point does not belong to a segment of intersection between $\mathscr{H}_{u,y}^{+}$ and $\mathscr{H}_{u,y}^{-}$. By a maximal connected subset in A we mean a connected subset $B \subset A$ with the property that there does not exist other connected subset $C \subset A$ such that $B \subset C$.

Using these notions we introduce the definition of *multi-loop hysteresis operator* as follows.

Definition 3.14. *A hysteresis operator Φ is called a* multi-loop hysteresis operator *if there exists a hysteresis loop $\mathscr{H}_{u,y}$, where $y = \Phi(u)$, with at least one maximal connected subset $C \subset \mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$ such that $u_{\min} \neq u_c \neq u_{\max}$ for every $(u_c, y_c) \in C$.* △

Definition 3.14 is asking for the existence of at least one maximal connected subset of crossover points besides the ones that contain the crossover points corresponding to the maximum and minimum values of the input. The existence of this maximal connected subset guarantees that the hysteresis loop will be composed of at least two subloops. Moreover, it is clear that every *butterfly hysteresis operator* is then a *multi-loop hysteresis operator*. Before characterizing the class of Preisach hysteresis operators that satisfy the conditions to be *multi-loop hysteresis operators*, we introduce the next lemma that allows us to relate the existence of a crossover point in a hysteresis loop obtained from a Preisach hysteresis operator with the integration of its weighting function μ over a rectangular region of P delimited by the maximum and minimum values of the input.

Lemma 3.15. *Consider a hysteresis loop $\mathscr{H}_{u,y}$ obtained from an input-output pair (u,y) of a Preisach hysteresis operator $\mathscr{P}$ with a weighting function μ. A point $(u_c,y_c) \in \mathscr{H}_{u,y}$ is a* crossover point *if and only if*

$$\iint_{(\alpha,\beta)\in\Omega_c} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta = 0 \tag{3.9}$$

where the region Ω_c is defined by $\Omega_c := \{(\alpha,\beta) \in P \mid u_c < \alpha < u_{\max},\ u_{\min} < \beta < u_c\}$.

Proof. (*Sufficiency*) Let (u_c,y_c) be the *crossover point* of a hysteresis loop $\mathscr{H}_{u,y}$ and consider the corresponding subsets $\mathscr{H}^{+}_{u,y}$ and $\mathscr{H}^{-}_{u,y}$ of $\mathscr{H}_{u,y}$ as defined in (3.7) and (3.8) with a monotone partition $t_1 < t_2 < t_1 + T$ of the periodic input u. When $u_c = u_{\min} = u(t_1) = u(t_1+T)$ or $u_c = u_{\max} = u(t_2)$, the region Ω_c is empty and (3.9) holds trivially. Therefore, let us consider the case when there exist two time instants $\tau_1 \in (t_1,t_2)$ and $\tau_2 \in (t_2,t_1+T)$ such that $(u_c,y_c) = (u(\tau_1),y(\tau_1)) = (u(\tau_2),y(\tau_2))$ with $u_{\min} \neq u_c \neq u_{\max}$.

Let us analyze the input-output behavior of the Preisach hysteresis operator in the intervals (t_1,t_2) and (t_1,t_2+T). For this, consider a subdomain of the Preisach plane given by $P_1 := \{(\alpha,\beta) \in P \mid \alpha < u_{\max},\ \beta > u_{\min}\}$ which is a triangle whose vertices are at $(u_{\max},u_{\min})$, $(u_{\max},u_{\max})$ and $(u_{\min},u_{\min})$. It is clear that at every time instance $t \geq t_p = T$ the state of relays in $P\backslash P_1$ remains the same as given by the initial condition. We define three time varying disjoint regions of P_1 whose boundaries depend on the instantaneous value of the input $u(t)$ and which are given by

$$\Omega_1(t) := \{(\alpha,\beta) \in P_1 \mid u(t) < \beta\}, \tag{3.10}$$

$$\Omega_2(t) := \{(\alpha,\beta) \in P_1 \mid \beta < u(t) < \alpha\}, \tag{3.11}$$

$$\Omega_3(t) := \{(\alpha,\beta) \in P_1 \mid \alpha < u(t)\}. \tag{3.12}$$

The region $\Omega_1(t)$ is a triangle whose vertices are at $(u(t),u(t))$, $(u_{\max},u(t))$ and $(u_{\max},u_{\max})$, the region $\Omega_3(t)$ is a triangle whose vertices are at $(u(t),u(t))$, $(u(t),u_{\min})$ and $(u_{\min},u_{\min})$, and the region $\Omega_2(t)$ is a rectangle whose vertices are at $(u(t),u(t))$, $(u_{\max},u_{\min})$, $(u(t),u_{\min})$ and $(u_{\max},u(t))$. It can be checked that for every time instance $t \in [t_1,t_1+T]$, all relays corresponding to the regions $\Omega_1(t)$ and $\Omega_3(t)$ are in state -1 and $+1$, respectively. Moreover, at time instances τ_1 and τ_2 we have that $\Omega_2(\tau_1) = \Omega_2(\tau_2) = \Omega_c$.

The required condition (3.9) is obtained computing the output of the Preisach hysteresis operator at both time instances τ_1 and τ_2 using the regions $\Omega_1(t)$, $\Omega_2(t)$ and $\Omega_3(t)$

as follows. At time instance t_1 the input u reaches its minimum value. Thus we have $u(t_1) = u_{\min}$ which implies that all relays in the subdomain P_1 are in -1 state. As the input increases, at every time instance $t \in (t_1, t_2)$ the region $\Omega_3(t)$ indicates the relays whose state have changed from -1 to $+1$ while in the regions $\Omega_2(t)$ and $\Omega_1(t)$ all relays remain in -1 state. Therefore, at time instance τ_1 the output of the Preisach hysteresis operator is given by

$$\begin{aligned} y(\tau_1) &= \big[\mathscr{P}(u, L_0)\big](\tau_1) \\ &- \iint_{(\alpha,\beta)\in\Omega_1(\tau_1)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \;-\; \iint_{(\alpha,\beta)\in\Omega_2(\tau_1)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint_{(\alpha,\beta)\in\Omega_3(\tau_1)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint_{(\alpha,\beta)\in P\setminus P_1} \mu(\alpha,\beta)\big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u, r_{\alpha,\beta}(L_0))\big](\tau_1)\,\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned} \tag{3.13}$$

At time instance t_2 the input u reaches now its maximum value. Thus, in this case we have $u(t_2) = u_{\max}$ which implies that all relays in the subdomain P_1 are in $+1$ state. Subsequently, as the input decreases, at every time instance $t \in (t_2, t_1 + T)$ the region $\Omega_1(t)$ indicates the relays whose state have changed from $+1$ to -1 while in the regions $\Omega_2(t)$ and $\Omega_3(t)$ all relays remain in $+1$ state. Therefore, at time instance τ_2 the output of the Preisach hysteresis operator is given by

$$\begin{aligned} y(\tau_2) &= \big[\mathscr{P}(u, L_0)\big](\tau_2) \\ &- \iint_{(\alpha,\beta)\in\Omega_1(\tau_2)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \;+\; \iint_{(\alpha,\beta)\in\Omega_2(\tau_2)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint_{(\alpha,\beta)\in\Omega_3(\tau_2)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &+ \iint_{(\alpha,\beta)\in P\setminus P_1} \mu(\alpha,\beta)\big[\mathscr{R}^{\circlearrowleft}_{\alpha,\beta}(u, r_{\alpha,\beta}(L_0))\big](\tau_2)\,\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned} \tag{3.14}$$

Subtracting (3.14) and (3.13) we have

$$\begin{aligned} 0 &= y(\tau_2) - y(\tau_1) \\ &= \iint_{(\alpha,\beta)\in\Omega_2(\tau_2)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta + \iint_{(\alpha,\beta)\in\Omega_2(\tau_1)} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ &= 2\iint_{(\alpha,\beta)\in\Omega_c} \mu(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned} \tag{3.15}$$

(*Necessity*) Assume that (3.9) holds for some value $u_c \in [u_{\min}, u_{\max}]$. Consider the time instance $\tau_1 \in [t_1, t_2]$ when $u(\tau_1) = u_c$. At this time instance, the output $y(\tau_1)$ is given as in (3.13) and $(u(\tau_1), y(\tau_1)) \in \mathscr{H}_{u,y}^{+}$. Similarly, let $\tau_2 \in [t_2, t_1 + T]$ be the time instance when $u(\tau_2) = u_c$. At this time instance the output $y(\tau_2)$ is given as in (3.14) and $(u(\tau_2), y(\tau_2)) \in \mathscr{H}_{u,y}^{-}$. Since (3.9) holds, by subtracting (3.14) and (3.13) we obtain (3.15) again. It follows that $y(\tau_2) = y(\tau_1) = y_c$, and consequently $(u_c, y_c) \in \mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$. □

It is clear that Lemma 3.15 can be used to find the crossover points of a hysteresis loop obtained from a Preisach hysteresis operator. However, noting that the region Ω_c in (3.9) depends explicitly on the maximum and minimum of the input u applied to the Preisach hysteresis operator, we could also use Lemma 3.15 to estimate an input u that produces a hysteresis loop with crossover points additional to the trivial ones corresponding to the maximum and minimum value of the input.

3.2.1 Domain Ω_c in weighting function of Example 3.1.1

Let us recall the Preisach hysteresis operator from Example 3.1.1 whose weighting function μ is defined in (3.4). We can check that a non-empty region Ω_c satisfying (3.9) is given by

$$\Omega_c = \{(\alpha,\beta) \in P \mid 0 < \alpha < \beta_1,\ -\beta_1 < \beta < 0\}, \tag{3.16}$$

which is illustrated in Fig. 3.3. Therefore, noting the limits of region Ω_c in (3.16), it follows that applying to this Preisach hysteresis operator an input u with one maximum $u_{\max} = \beta_1$ and one minimum $u_{\min} = -\beta_1$ yields a crossover point (u_c, y_c) with $u_c = 0$ as it has been shown in the phase plot of Fig 3.2. Furthermore, using (3.6) we can find that $y_c = -u_{\max}^2$.

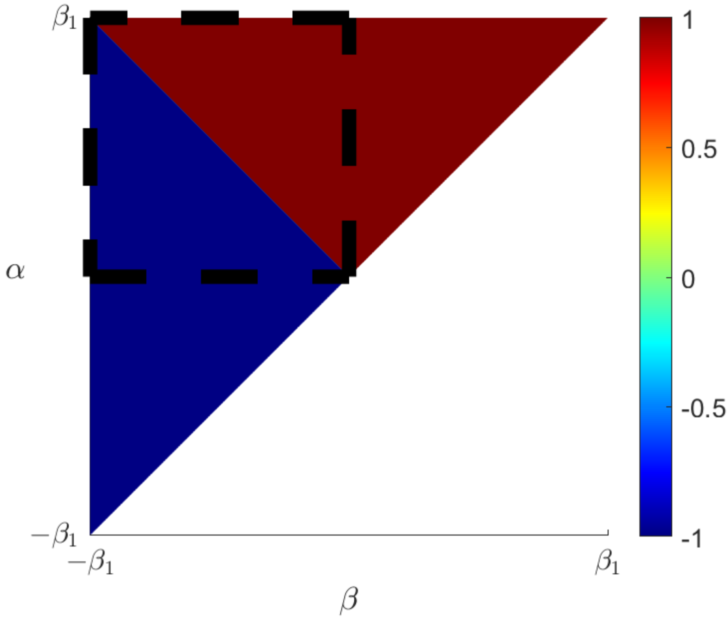


Figure 3.3: Weighting function $\mu(\alpha,\beta)$ of Example 3.1.1 defined in (3.4) where the region Ω_c given by (3.4) and that satisfies (3.9) is indicated by the dashed line.

3.2.2 Numerical example: crossover points in single-oriented hysteresis loop

We illustrate with the next example that only the existence of crossover points does not guarantee that the hysteresis loop is composed of subloops with different orientation. Consider a subdomain of the Preisach plane $P_1 = \{(\alpha,\beta) \in P \mid \alpha < \beta_1, \beta > -\beta_1\}$ with $\beta_1 > 0$ and define a weighting function μ given by

$$\mu(\alpha,\beta) := \begin{cases} -1, & \text{if } (\alpha,\beta) \in P_{1-}, \\ 1, & \text{if } (\alpha,\beta) \in P_1 \backslash P_{1-}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.17}$$

where $P_{1-} = \{(\alpha,\beta) \in P_1 \mid -\beta_1 < \beta < 0, 0 < \alpha < \beta + \beta_1\}$. It can be checked that the same region Ω_c given as in (3.16) satisfies condition (3.9) with the weighting function μ defined by (3.17). Fig. 3.4 illustrates this weighting function with the region Ω_c indicated by a dashed line. It follows that the hysteresis loop obtained from a Preisach hysteresis operator with a weighting function μ defined by (3.17) and whose input u has one maximum $u_{\max} = \beta_1$ and one minimum $u_{\min} = -\beta_1$ has a crossover point with coordinates $(u_c, y_c) = (0,0)$, where y_c is computed using (3.13) or (3.14). Nevertheless, by simple inspection of the phase plot included in Fig. 3.5, we can check that the hysteresis loop is composed of two subloops with the same orientation.

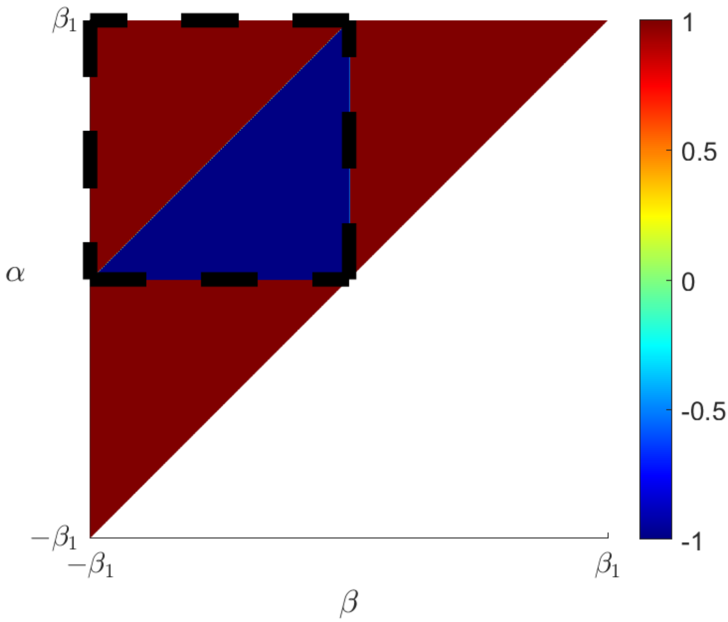


Figure 3.4: Weighting function $\mu(\alpha,\beta)$ of a Preisach hysteresis operator which can produce a hysteresis loop whose subloops have the same orientation. The region Ω_c that satisfies (3.9) is indicated by the dashed line.

We now introduce a theorem that shows how a *multi-loop hysteresis operator* can be obtained from a Preisach hysteresis operator.

Theorem 3.16. *Consider a Preisach hysteresis operator $\mathscr{P}$ as in* (2.3) *with a weighting function μ. Assume that there exists a point $(\alpha_0,\beta_0) \in P$ with $\alpha_0 > \beta_0$ and three values $\alpha_{1^-} < \alpha_1 < \alpha_{1^+}$ such that $\beta_0 < \alpha_{1^-}$ and $\alpha_{1^+} < \alpha_0$, and* (3.9) *holds for the region*

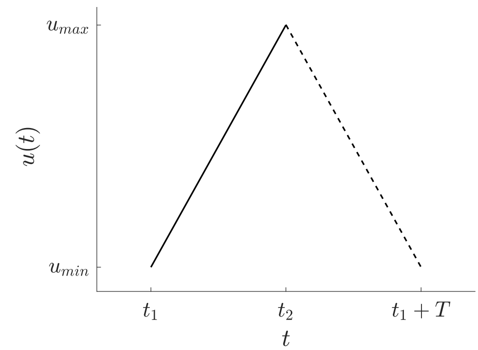
$$\Omega_{\alpha_1} = \{(\alpha,\beta) \in P \mid \alpha_1 < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_1\} \tag{3.18}$$

but does not hold for the regions

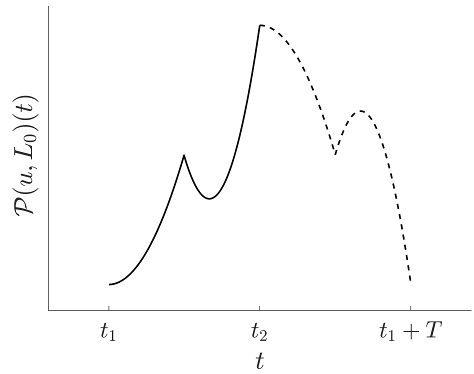
$$\Omega_{\alpha_{1^-}} = \{(\alpha,\beta) \in P \mid \alpha_{1^-} < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_{1^-}\}, \tag{3.19}$$

$$\Omega_{\alpha_{1^+}} = \{(\alpha,\beta) \in P \mid \alpha_{1^+} < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_{1^+}\}. \tag{3.20}$$

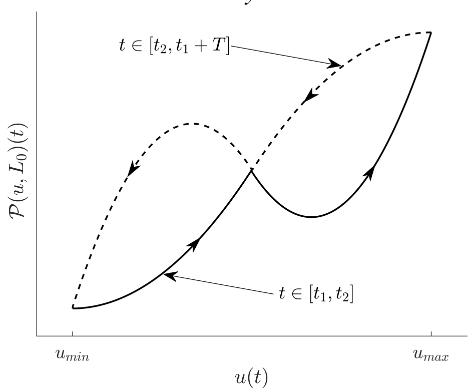
Then $\mathscr{P}$ is a multi-loop hysteresis operator.



(a) The plot of input signal u in a periodic time interval $[t_1, t_1+T]$.



(b) The corresponding plot of output signal y.



(c) The input-output phase plot of input and output signal which shows a hysteresis loop consisting of two subloops with the same orientation.

Figure 3.5: Input-output hysteresis response using the multiple loops Preisach multi-loop operator.

Proof. Consider the hysteresis loop $\mathscr{H}_{u,y}$ obtained from the input-output pair (u,y) with the input u being periodic with one maximum $u_{\max} = \alpha_0$ and one minimum $u_{\min} = \beta_0$, and $y = \mathscr{P}(u, L_0)$. Let $t_1 < t_2 < t_1+T$ be the monotonic partition of the input that divides $\mathscr{H}_{u,y}$ into $\mathscr{H}_{u,y}^-$ and $\mathscr{H}_{u,y}^+$ with $u(t_1) = u(t_1+T) = u_{\min}$ and $u(t_2) = u_{\max}$, and consider six time instances $\tau_{1^-}, \tau_1, \tau_{1^+} \in [t_1, t_2]$ and $\tau_{2^-}, \tau_2, \tau_{2^+} \in [t_2, t_1+T]$ with $\tau_{1^-} < \tau_1 < \tau_{1^+}$ and $\tau_{2^-} > \tau_2 > \tau_{2^+}$ such that $u(\tau_{1^-}) = u(\tau_{2^-}) = \alpha_{1^-}$, $u(\tau_1) = u(\tau_2) = \alpha_1$ and $u(\tau_{1^+}) = u(\tau_{2^+}) = \alpha_{1^+}$.

Using Lemma 3.15 with the region Ω_{α_1} defined in (3.18), the hysteresis loop $\mathscr{H}_{u,y}$ has a crossover point (u_c, y_c) with $u_c = u(\tau_1) = u(\tau_2) = \alpha_1$ and where $y_c = y(\tau_1) = y(\tau_2)$ is given by (3.13) or (3.14).

Without loss of generality, let C be the maximal connected subset of $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$ that contains (u_c, y_c). To check that C does not contain a crossover point of the form $(u_{\min}, y_1)$ observe that since (3.9) does not hold for the region $\Omega_{\alpha_{1^-}}$ then using again Lemma 3.15 we have that $(u(\tau_{1^-}), y(\tau_{1^-}))$ and $(u(\tau_{2^-}), y(\tau_{2^-}))$ are not crossover points and are not in $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$. Consequently, there does not exist a connected subset of $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$ that could contain both (u_c, y_c) and $(u_{\min}, y_1)$. Similarly, we can check that C does not contain a crossover point of the form $(u_{\max}, y_2)$ by noting that (3.9) does not hold for the region $\Omega_{\alpha_{1^+}}$ which by Lemma 3.15 implies that $(u(\tau_{1^+}), y(\tau_{1^+}))$ and $(u(\tau_{2^+}), y(\tau_{2^+}))$ are not crossover points and are not in $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$. It follows again that there does not exist connected subset of $\mathscr{H}_{u,y}^{+} \cap \mathscr{H}_{u,y}^{-}$ that could contain both (u_c, y_c) and $(u_{\max}, y_2)$. □

Remark 3.17. *It is possible to consider more than one region Ω_α where* (3.9) *holds. Let μ be a weighting function and consider values*

$$\alpha_{i^-} < \alpha_i < \alpha_{i^+}, \quad \text{where } i \in \{1, \dots, m\} \text{ and } m \in \mathbb{Z}_+,$$

with $\beta_0 < \alpha_{1^-}$ and $\alpha_{m^+} < \alpha_0$, and such that for every $j \in \{1, \dots, m-1\}$ we have that $\alpha_{j^+} < \alpha_{(j+1)^-}$. Assume that using these values, we can construct regions given by

$$\Omega_{\alpha_i} = \{(\alpha, \beta) \in P \mid \alpha_i < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_i\},$$

such that (3.9) *holds but does not hold for regions given by*

$$\begin{aligned} \Omega_{\alpha_{i^-}} &= \{(\alpha, \beta) \in P \mid \alpha_{i^-} < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_{i^-}\}, \\ \Omega_{\alpha_{i^+}} &= \{(\alpha, \beta) \in P \mid \alpha_{i^+} < \alpha < \alpha_0,\ \beta_0 < \beta < \alpha_{i^+}\} \end{aligned}$$

for every $i \in \{1, \dots, m\}$. It follows immediately from Theorem 3.16 that a Preisach hysteresis operator with this weighting function is a multi-loop hysteresis operator. *Moreover, it can be checked that in this case the hysteresis loop $\mathscr{H}_{u,y}$ obtained from such Preisach hysteresis operator with an input whose maximum is $u_{\max} = \alpha_0$ and minimum is $u_{\min} = \beta_0$ will be composed of $m+1$ subloops.*

3.2.3 Numerical example: Preisach multi-loop hysteresis operator

The final example of this section illustrates a Preisach hysteresis operator with a weighting function that has a complex distribution of positive and negative domains and whose hysteresis loops has four subloops. Consider a subset of the Preisach domain given by

$$P_1 := \{(\alpha, \beta) \mid -1 < \beta < 1,\ \beta < \alpha < 1\},$$

and a weighting function defined by

$$\mu(\alpha,\beta) := \begin{cases} \sin(2\pi(\alpha-\beta)) + \sin(2\pi(\alpha+\beta)), & \text{if } (\alpha,\beta) \in P_1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.21}$$

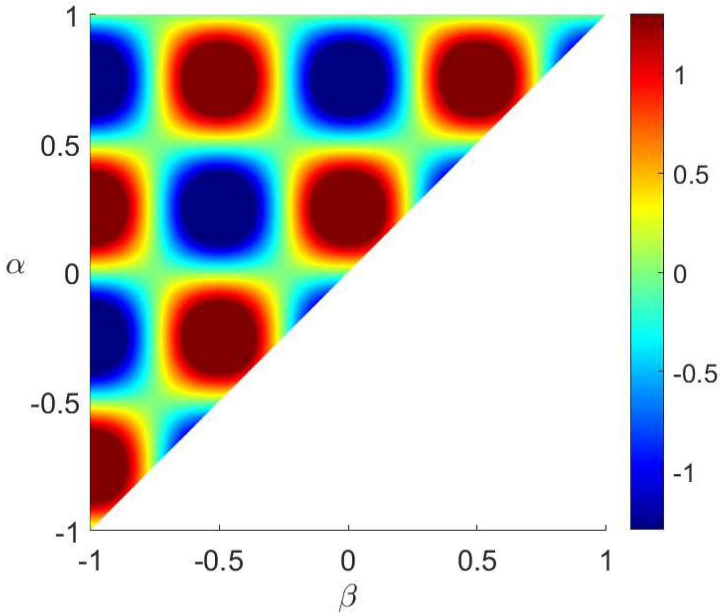


Figure 3.6: Weighting function $\mu(\alpha,\beta)$ defined in (3.21) corresponding to a *Preisach multi-loop operator*.

The weighting function μ defined in (3.21) is illustrated in Fig 3.6. For this weighting function there exist three non-empty regions Ω_{α_1}, Ω_{α_2} and Ω_{α_3} that satisfy (3.9) and which are given by

$$\begin{aligned} \Omega_{\alpha_1} &= \{(\alpha,\beta) \in P_1 \mid -0.5 < \alpha < 1,\ -1 < \beta < -0.5\}, \\ \Omega_{\alpha_2} &= \{(\alpha,\beta) \in P_1 \mid 0 < \alpha < 1,\ -1 < \beta < 0\}, \\ \Omega_{\alpha_3} &= \{(\alpha,\beta) \in P_1 \mid 0.5 < \alpha < 1,\ -1 < \beta < 0.5\}. \end{aligned} \tag{3.22}$$

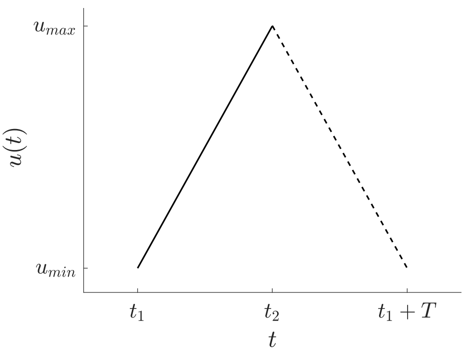
Moreover, it can be verified that (3.9) does not hold for every of the next regions

$$\begin{aligned}
\Omega_{\alpha_{1-}} &= \{(\alpha,\beta) \in P_1 \,|\, -0.75 < \alpha < 1, -1 < \beta < -0.75\},\\
\Omega_{\alpha_{1+}} &= \{(\alpha,\beta) \in P_1 \,|\, -0.25 < \alpha < 1, -1 < \beta < -0.25\},\\
\Omega_{\alpha_{2-}} &= \{(\alpha,\beta) \in P_1 \,|\, -0.25 < \alpha < 1, -1 < \beta < -0.25\},\\
\Omega_{\alpha_{2+}} &= \{(\alpha,\beta) \in P_1 \,|\, 0.25 < \alpha < 1, -1 < \beta < 0.25\},\\
\Omega_{\alpha_{3-}} &= \{(\alpha,\beta) \in P_1 \,|\, 0.25 < \alpha < 1, -1 < \beta < 0.25\},\\
\Omega_{\alpha_{3+}} &= \{(\alpha,\beta) \in P_1 \,|\, 0.75 < \alpha < 1, -1 < \beta < 0.75\}.
\end{aligned}$$

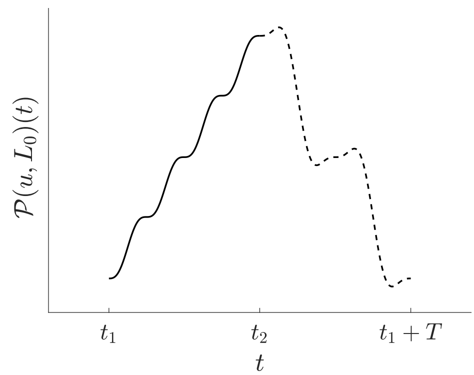
Fig. 3.8 illustrates the three regions Ω_{α_1}, Ω_{α_2} and Ω_{α_3} with a dashed line and Fig. 3.7 shows the input-output phase plot of the Preisach hysteresis operator with the weighting function (3.21) with a periodic input whose maximum and minimum are $u_{\min} = -1$ and $u_{\max} = 1$. It can be verified that the hysteresis loop is composed of four subloops and that there exist three crossover points additional to the trivial ones corresponding to the maximum and minimum of the input.

3.3 Conclusions

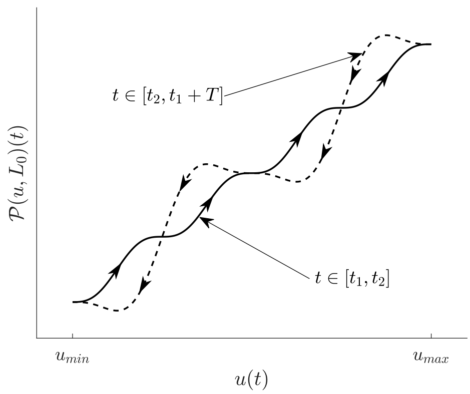
In this chapter, we introduced the concepts of the butterfly hysteresis operator based on the characterization of the enclosed signed-area of its hysteresis loops and multi-loop hysteresis operator based on the self-intersections of its hysteresis loops. We have studied the Preisach operator and provided conditions over its weighting function such that a butterfly or a multi-loop hysteresis operator can be obtained. The characterizations presented in this chapter will be used in the subsequent chapters.



(a) Plot of input signal u in a periodic time interval $[t_1, t_1 + T]$.

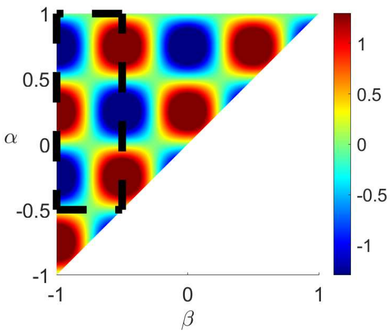


(b) Plot of the corresponding output signal $y = \mathscr{P}(u, L_0)$.

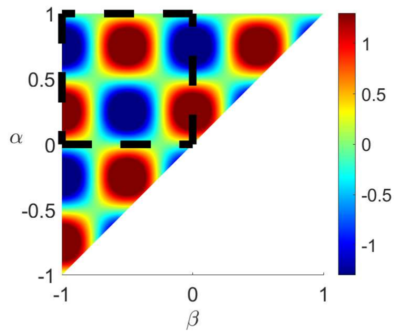


(c) The input-output phase plot of input and output signal which shows the hysteresis loop consisting of more than two subloops with different orientations.

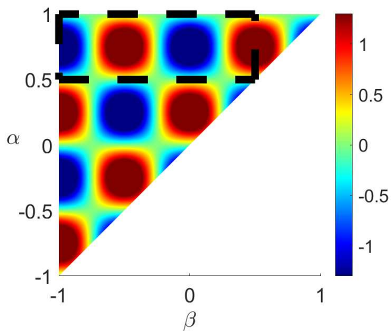
Figure 3.7: Input-output response using the multiple loops Preisach multi-loop operator.



(a) Region Ω_{α_1}.



(b) Region Ω_{α_2}.



(c) Region Ω_{α_3}.

Figure 3.8: Weighting function $\mu(\alpha,\beta)$ defined in (3.21) corresponding to a *Preisach multi-loop operator*. The regions defined in (3.22) that satisfy (3.9) are indicated by the dashed line.