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Minimality of Linear Switched Systems with known switching signal

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Minimal realization is discussed for linear switched systems with a given switching signal. We propose a consecutive forward and backward approach for the time-interval of interest. The forward approach refers to extending the reachable subspace at each switching time by taking into account the nonzero reachable space from the previous mode. Afterwards, the backward approach extends the observable subspace of the current mode by taking observability information from the next mode into account. This results in an overall reduced switched system which is minimal and has the same input-output behavior as original system. Some examples are provided to illustrate the approach.

1 Introduction

We consider linear switched system of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_q(t) = A_{\sigma(t)} x_q(t) + B_{\sigma(t)} u(t), \ t \in (t_q, t_{q+1}), \\ x_q(t_q^+) = J_{\sigma(t_q^+),\sigma(t_q^-)} x_{q-1}(t_q^-), \\ y(t) = C_{\sigma(t)} x_q(t), \quad t \in \mathbb{R}, \end{cases} \tag{1}$$

where $x_q : (t_q, t_{q+1}) \to \mathbb{R}^{n_q}$ is the absolutely continuous q-th piece of the state, $u : \mathbb{R} \to \mathbb{R}^u$ is the input and y is the measured output. The switching signal $\sigma : \mathbb{R} \to \mathcal{Q} = \{1, 2, \cdots, f\} \subset \mathbb{N}$ is a given piecewise constant function with finitely many switching times: $\{ \ t_q \ | \ q \in \mathcal{Q}, t_1 < t_2 < \cdots < t_f \ \}$ in the bounded interval (t_1, t_{f+1}) of interest. For each $q \in \mathcal{Q}$, the matrices A_q, B_q, C_q, are of appropriate q-dependent size. We need a jump map $J_{q^+,q^-} : \mathbb{R}^{n_{q^-}} \to \mathbb{R}^{n_{q^+}}$ to relate different state-space dimensions and simplify the notation $J_{\sigma(t_q^+),\sigma(t_q^-)} = J_{q,q-1} =: J_q$.

The general idea of minimal realization is to construct a state-space model from a given input-output behavior of the system. In particular, finding a minimal realization could be seen as the first step towards model reduction. In [1], we have presented a time-varying model reduction approach for linear switched system which was not a switched system anymore. Therefore, our aim is to gain insight into a more suitable model-reduction approach by studying the minimal realization problem for switched systems of the form (1) *within* this system class.

Several approaches have been discussed in the cases of arbitrary and constrained switching e.g. in [1–5] where switching signal are viewed as input to the switched systems. It can be seen that (minimal) realization in general depends on the specifically given switching signal, so in contrast to the existing literature, we view the switched system (1) as a piecewise-constant time-varying linear system. We begin with the formal definition of minimality.

Definition 1.1 For Σ_σ as in (1), the *total dimension* is defined by $\dim \Sigma_\sigma := \sum_{q \in \mathcal{Q}} n_q$. Furthermore, we define its input-output behaviour as follows

$$\mathfrak{B}_\sigma^{io} := \left\{ \ (u, y) \ \middle| \ \forall q \in \mathcal{Q} \ \exists x_q : (t_q, t_{q+1}) \to \mathbb{R}^{n_q} \text{ satisfying (1) and } x_1(t_1^+) = 0 \ \right\}.$$

A linear switched system $\widehat{\Sigma}_\sigma$ with corresponding input-output behavior $\widehat{\mathfrak{B}}_\sigma^{io}$ is said to be a minimal realization of switched system Σ_σ if 1) $\mathfrak{B}_\sigma^{io} = \widehat{\mathfrak{B}}_\sigma^{io}$ and 2) for any $\widetilde{\Sigma}_\sigma$ with $\mathfrak{B}_\sigma^{io} = \widetilde{\mathfrak{B}}_\sigma^{io}$ satisfies $\dim \widehat{\Sigma}_\sigma \leq \dim \widetilde{\Sigma}_\sigma$.

Remark 1.2 The above definition of minimality is not specifying any *method* to obtain a minimal realization from a given switched system as in (1). In general, a minimal realization can only be obtained by considering each mode individually (and by properly taking the effect on the other modes into account).

2 Minimal realization of single switch switched system

We propose a method to find a minimal realization of linear switched system of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u, \quad \text{on } (t_1, t_2), \quad x_1(t_1^+) = 0, \\ \dot{x}_2 = A_2 x_2 + B_2 u, \quad \text{on } (t_2, t_f), \quad x_2(t_2^+) = J_2 x_1(t_2^-). \end{cases} \tag{2}$$

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The Kalman decomposition (KD), [6], is a well known method to find a minimal realization of a system, however, this method is based on the assumption that the initial value is zero. In system (2), we have seen that second mode starts with nonzero initial values which are not completely arbitrary, but are constraint to the reachable space of the first mode. By taking into account the reachable subspace of the first mode, we construct an *input-extended* system which is input-output equivalent to the second mode (cf. [7] in the context of model reduction). Then we extend the first mode by taking into account the observable states of the second mode. Finally we define the jump map from mode 1 to mode 2.

Overall, the algorithm of the proposed method is summarized as follows.

Step 1a. Compute the reachable subspace $\mathcal{R}_1 = \operatorname{im} R_1$ of first subsystem (A_1, B_1, C_1) and extend the input matrix of the second mode to

$$B_{2,e} := \operatorname{im}[B_2, J_2 R_1].$$

Step 1b. Calculate the KD of $(A_2, B_{2,e}, C_2)$ with corresponding transformation matrix $\mathbb{V}_2$ and left- and right-projectors W_2, V_2 (i.e. the corresponding rows and columns of $\mathbb{V}_2^{-1}$ and $\mathbb{V}_2$) and let

$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = (W_2 A_2 V_2, W_2 B_2, C_2 V_2).$$

Step 2a. Calculate the space $\mathcal{L}_2 = \mathcal{R}_1 \cap \mathcal{K}_2 =: \operatorname{im} L_2$ of additional observable states, where $\mathcal{K}_2 = \operatorname{im} K_2$ for some full column rank matrix $K_2 \in \mathbb{R}^{n_1 \times n_2^J}$ such that $J_2 K_2 = V_2^J$ for a full column rank matrix $V_2^J \in \mathbb{R}^{n_2 \times n_2^J}$ with $\operatorname{im} V_2^J := \operatorname{im} V_2 \cap \operatorname{im} J_2$. Then extend the output matrix of the first mode as

$$C_{1,e} := \operatorname{im} \begin{bmatrix} C_1 \\ L_2^\top \end{bmatrix}.$$

Step 2b. Calculate the KD of $(A_1, B_1, C_{1,e})$ with corresponding transformation matrix $\mathbb{V}_1$ and left- and right-projectors W_1, V_1 (i.e. the corresponding rows and columns of $\mathbb{V}_1^{-1}$ and $\mathbb{V}_1$) and let

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (W_1 A_1 V_1, W_1 B_1, C_1 V_1).$$

Step 3. The reduced jump $\widehat{J}_2 : \mathbb{R}^{\widehat{n}_1} \to \mathbb{R}^{\widehat{n_2}}$ is calculated as $\widehat{J}_2 := W_2 J_2 V_1$.

The overall reduced switched system is then given by

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_1 = \widehat{A}_1 \widehat{x}_1 + \widehat{B}_1 u, & \text{on } (t_1, t_2), \quad \widehat{x}_1(t_1^+) = 0, \\ \dot{\widehat{x}}_2 = \widehat{A}_2 \widehat{x}_2 + \widehat{B}_2 u, & \text{on } (t_2, t_f), \quad \widehat{x}_2(t_2^+) = \widehat{J}_2 \widehat{x}_1(t_2^-). \end{cases} \tag{3}$$

The above algorithm ensures following observations. Due to page limitation, we ignore details.

Theorem 2.1 *Consider the switched system Σ_σ and the reduced system $\widehat{\Sigma}_\sigma$ obtained via the above algorithm. Then both systems are input-output equivalent in the sense of Definition 1.1. Also, $\widehat{\Sigma}_\sigma$ has minimal total dimension under all possible input-output equivalent system of Σ_σ.*

The proposed approach is illustrated by the following example.

Example 2.2 Consider a switched system as in (2) with modes

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right) \text{ and } (A_2, B_2, C_2, J_2) = \left(\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right).$$

It is easily seen, that each modes is unreachable and unobservable, however, the switched system is reachable and observable. We apply the proposed method. Via the KD of the extended 2nd mode $(A_2, [B_2, J_2 R_1], C_2]$ and the extended 1st mode $(A_1, B_1, [C_1^\top, L_2]^\top)$ respectively, we obtain the left- and right-projectors $W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $W_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ with $R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. The corresponding input-output equivalent minimal switched system is given by

$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = (W_1 A_1 V_1, W_1 B_1, C_1 V_1) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right),$$
$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = (W_2 A_2 V_2, W_2 B_2, C_2 V_2) = \left(\begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right) \text{ and } \widehat{J}_2 = W_2 J_2 V_1 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

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