Chapter 3

Fully Distributed Robust Synchronization of Undirected Lur’e Networks

This chapter deals with robust synchronization problems for unknown dynamical networks of diffusively interconnected identical Lur’e systems subject to incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively, in a fully distributed fashion. Whereas in stabilization of one single Lur’e system the conditions of passivity and sector boundedness for the unknown nonlinear function in its negative feedback loop are commonly assumed, in our context of networked Lur’e systems we adopt the stronger assumptions of incremental passivity and incremental sector boundedness. Throughout this chapter the interconnection topologies among these Lur’e agents are assumed to be undirected and connected. We design robustly synchronizing protocols and subsequently implement these protocols in a fully distributed way by means of an adaptive control law that adjusts the coupling weights between neighboring agents. Both for the case of incrementally passive as well as incrementally sector bounded nonlinearities we obtain sufficient conditions for the existence of fully distributed robustly synchronizing protocols. The state feedback gain matrices are computed by solving LMI’s in terms of the matrices defining the individual agent dynamics. Numerical simulation examples illustrate our theoretical results.

3.1 Introduction

As announced above, we consider undirected nonlinear multi-agent networks in which the dynamics of the individual agents is described by a Lur’e system, i.e. a nonlinear system consisting of the negative feedback interconnection of a nominal linear system with an unknown static nonlinearity around it, see e.g. [26]. We restrict ourselves to homogeneous networks, in the sense that we assume all the agents to have an identical nominal linear dynamics and also identical nonlinearity in the feedback loop. Such system model is used in many control system applications, e.g. Chua’s circuits, flexible robotic arms and aircrafts [32]. The feedback loop can represent different kinds of nonlinearities such as saturation and dead zone. Often, we can reconfigure a linear system with nonlinearities at its input/output as a Lur’e system [72]. In the present chapter we assume
these unknown nonlinearities to be incrementally passive or incrementally sector bound. In contrast with [65], the uncertainties we consider here are not additive perturbations and the networks are intrinsically nonlinear.

Conditions for (global) asymptotic stability of a single Lur'e system are of course well known [26, 32, 41]. Master-slave synchronization of two coupled Lur'e systems was studied for applications in secure communication, see e.g. [63]. In [29] pinning synchronization of a Lur'e dynamical network was converted into global asymptotic stability of a set of Lur'e systems, and frequency-domain and time-domain synchronization criteria were derived in virtue of the main result in [41]. We note that the time-domain criterion in [29] is a corollary of the frequency-domain one and can not be obtained directly. In contrast to [29], our setting does not start with any assumption on the existence of a global synchronization manifold (isolated agent, virtual leader or exosystem), a condition which is frequently employed in nonlinear multi-agent coordination. In [58], synchronization of incrementally passive oscillators described by a Lur'e system was discussed. We stress that in our work we do not make any assumption on incremental passivity of the agents.

In general, in multi-agent coordination the nonzero eigenvalues of the Laplacian matrix associated with the interconnection topology are assumed to be known, see e.g. [28, 29, 65, 70]. However, the Laplacian matrix depends on the entire interconnection topology, which is global information and impossible to be known if the network is large. Thus, numerous synchronization protocols can actually not be implemented by each agent in a fully distributed fashion. Such problems were discussed in [13, 14, 30, 31, 53] etc. The idea to solve this problem is to introduce adaptively updated coupling weights in the protocol, whose dynamics are driven by the local synchronization errors over each edge in the network. It is essential to adjust the coupling strength between neighboring agents to guarantee convergence. In [13, 30] this idea was applied to deal with adaptive pinning synchronization of networks of Lur'e systems and Lipschitz nonlinear systems, respectively. In this chapter we consider adaptive self-synchronization of Lur'e networks. In addition, the structure of the unknown nonlinearities we deal with is more general than the one used in [13].

In short, the main contribution of the present chapter is that we provide sufficient conditions for the existence of fully distributed robustly synchronizing protocols, both for incrementally passive and for incrementally sector bounded nonlinearities.

The remainder of this chapter is organized as follows. Section 3.2 introduces the individual agent dynamics we will consider in this chapter. Our main results are presented in Sections 3.3 and 3.4. In Section 3.3 we establish sufficient conditions for the existence of robustly synchronizing protocols and discuss how to compute these protocols. In Section 3.4 we explain how to modify these protocols in order
3.2 Problem formulation

In this chapter, we consider an undirected multi-agent network of \( N (\geq 2) \) nonlinear dynamical systems described by the following identical Lur’e systems (see Fig. 3.1),

\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i + Ez_i \\
y_i &=Cx_i \\
z_i &= -\phi(y_i, t)
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^n \), \( u_i(t) \in \mathbb{R}^m \) and \( y_i(t) \in \mathbb{R}^s \) are the state to be synchronized, the diffusive coupling input and the output of the \( i \)th agent, respectively. The equation \( z_i = -\phi(y_i, t) \) represents a time-varying, memoryless, nonlinear negative feedback loop. The function \( \phi(\cdot, t) \) from \( \mathbb{R}^s \times \mathbb{R}^+ \) to \( \mathbb{R}^s \) is unknown and can be any one from a set of functions to be specified later. \( A, B, C \) and \( E \) are known constant matrices of compatible dimensions. Without loss of generality, we assume that the number of control inputs \( m \) is strictly less than the state space dimension \( n \). In this case the rows of matrix \( B \) are linearly dependent and thus \( B^\perp \) exists. We also assume that \((A, B)\) is stabilizable. The interconnection topology among these agents is represented by a connected undirected graph \( G \).

3.3 Robust Synchronization

In this section, the agents (3.1) in the network are assumed to be interconnected by means of the following distributed static protocol

\[
u_i = F \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, 2, \cdots, N,
\]
where $F \in \mathbb{R}^{m \times n}$ is a common feedback gain matrix to be determined later, and $A = [a_{ij}]$ is the adjacency matrix of the graph $G$.

**Definition 3.1.** The network of agents (3.1) with the protocol (3.2) is robustly synchronized if $x_i(t) - x_j(t) \to 0$ as $t \to \infty$, $\forall i, j = 1, 2, \cdots, N$, for all initial conditions and all uncertainties $\phi(\cdot, t)$ from a specified function set.

By interconnecting (3.1) and (3.2) we get the Lur’e dynamical network

$$
\begin{cases}
\dot{x} = (I_N \otimes A + L \otimes BF)x - (I_N \otimes E)\Phi(y, t) \\
y = (I_N \otimes C)x
\end{cases},
$$

(3.3)

where $x = [x_1^T, x_2^T, \cdots, x_N^T]^T$, $y = [y_1^T, y_2^T, \cdots, y_N^T]^T$, $\Phi(y, t) = [\phi(y_1, t)^T, \phi(y_2, t)^T, \cdots, \phi(y_N, t)^T]^T$, and $L$ is the Laplacian matrix of the graph $G$.

In the next two subsections we will discuss robust synchronization of the Lur’e network (3.3). In the first subsection, the uncertainty in the negative feedback loop is modeled by assuming the unknown nonlinear function $\phi(\cdot, t)$ to be incrementally passive. In the second subsection, we model the uncertainty by assuming that the function $\phi(\cdot, t)$ satisfies an incremental sector boundedness condition.

### 3.3.1 Incrementally Passive Nonlinearities

In this subsection we assume that the unknown functions $\phi(\cdot, t)$ are incrementally passive. Incremental passivity for static systems of the form

$$
z = \phi(y, t)
$$

(3.4)

with input $y(t) \in \mathbb{R}^s$ and output $z(t) \in \mathbb{R}^s$ is defined as follows:

**Definition 3.2.** [42] The system (3.4) is called *incrementally passive* if the function $\phi(\cdot, t)$ satisfies

$$
(y_1 - y_2)^T(\phi(y_1, t) - \phi(y_2, t)) \geq 0
$$

(3.5)

for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$.

In general, incremental passivity is stronger than the property of passivity, which requires $y^T \phi(y, t) \geq 0$ for all $y \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$. Passivity implies incremental passivity for linear systems, and also for monotonically increasing static nonlinearities [58].

Recall that we use the standard assumption that the undirected graph $G$ is connected. Also recall that the nonzero eigenvalues of its Laplacian matrix are given by $\lambda_2, \lambda_3, \ldots, \lambda_N$. Then we have the following lemma:
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Lemma 3.1. Assume that the unknown functions \( \phi(\cdot, t) \) satisfy (3.5) for all \( y_1, y_2 \in \mathbb{R}^n \) and \( t \in \mathbb{R}^+ \). If there exist matrices \( P > 0 \) and \( F \) such that

\[
(A + \lambda_i BF)^T P + P(A + \lambda_i BF) < 0
\]

for all \( i = 2, \cdots, N \), and

\[
PE = C^T,
\]

then the network of Lur'e agents (3.1) with the protocol (3.2) is robustly synchronized, i.e. the Lur'e network (3.3) is synchronized for all incrementally passive \( \phi(\cdot, t) \).

Proof. Let \( U \) be an orthogonal matrix such that \( UT \mathcal{L} U = \Lambda \), where \( \Lambda = \text{diag}(0, \lambda_2, \cdots, \lambda_N) \). Let \( \bar{x} = (U^T \otimes I_n) x \) and \( \bar{x} = (U_2^T \otimes I_n) x \), where \( \bar{x} = [\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N]^T \) and \( \bar{x} = [\bar{x}_2, \cdots, \bar{x}_N]^T \). Then after the coordinate transformation the differential equation in (3.3) takes the form

\[
\dot{x} = (I_{N-1} \otimes A + \bar{\Lambda} \otimes BF) \bar{x} - (U_2^T \otimes E) \Phi(y, t),
\]

(3.8)

where \( \bar{\Lambda} = \text{diag}(0, \lambda_2, \cdots, \lambda_N) \). We know that \( x_i(t) - x_j(t) \to 0 \) as \( t \to \infty \), \( \forall \ i, j = 1, 2, \cdots, N \), if and only if \( \bar{x}(t) \to 0 \) as \( t \to \infty \), see Lemma 3.2 in [65]. Thus the robust synchronization of \( x \) is equivalent to the global asymptotical stability of \( \bar{x} \).

By Lemma 2.1, we have

\[
\begin{aligned}
&x^T (U_2 U_2^T \otimes C^T) \Phi(y, t) \\
= &x^T (I_N \otimes C^T) (U_2 U_2^T \otimes I_N) \Phi(y, t) \\
= &y^T (U_2 U_2^T \otimes I_N) \Phi(y, t) \\
= &\frac{1}{N} \sum_{1 \leq i < j \leq N} (y_i - y_j)^T (\phi(y_i, t) - \phi(y_j, t)) \geq 0.
\end{aligned}
\]

Choose a quadratic Lyapunov function candidate \( V_1(\bar{x}) = \frac{1}{2} \bar{x}^T (I_{N-1} \otimes P) \bar{x} \), where \( P > 0 \) together with the feedback gain matrix \( F \) satisfies (3.6) and (3.7). Obviously, \( V_1(\bar{x}) \) is positive definite and radially unbounded. The time derivative of \( V_1(\bar{x}) \) along the trajectories of the system (3.8) is given by

\[
\begin{aligned}
\dot{V}_1(\bar{x}) \\
= &\bar{x}^T (I_{N-1} \otimes P) \dot{\bar{x}} \\
= &\bar{x}^T (I_{N-1} \otimes P) \left[ (I_{N-1} \otimes A + \bar{\Lambda} \otimes BF) \bar{x} - (U_2^T \otimes E) \Phi(y, t) \right] \\
= &\bar{x}^T (I_{N-1} \otimes PA + \bar{\Lambda} \otimes PBF) \bar{x} - \bar{x}^T (U_2^T \otimes PE) \Phi(y, t) \\
= &\sum_{i=2}^{N} \bar{x}_i^T (PA + \lambda_i PBF) \bar{x}_i - x^T (U_2 \otimes I_n) (U_2^T \otimes C^T) \Phi(y, t)
\end{aligned}
\]
\[
\sum_{i=2}^{N} \tilde{x}_i^T (P A + \lambda_i P B F) \tilde{x}_i - x^T (U_2 U_2^T \otimes C^T) \Phi(y, t)
\]
\[
\leq \frac{1}{2} \sum_{i=2}^{N} \tilde{x}_i^T [(A + \lambda_i BF)^T P + P(A + \lambda_i BF)] \tilde{x}_i,
\]
which is negative definite. Thus the system (3.8) is globally asymptotically stable, i.e. the Lur’e network (3.3) is robustly synchronized. This completes the proof. □

Remark 3.2. It is well known that the nonzero eigenvalues of the graph Laplacian matrix in general play an important role in synchronization conditions, see e.g. [28], where it was shown that the protocol (3.2) synchronizes the linear network if and only if the feedback gain matrix \( F \) stabilizes \( A + \lambda_i BF \) for all \( i = 2, \cdots, N \). In our work, due to the strict passivity requirements (3.6) and (3.7), a common \( P > 0 \) is required as well as a common \( F \). Without the Lur’e-type nonlinearities, the Lur’e network (3.3) becomes a linear network, and the requirement of a common \( P \) and the condition (3.7) are removed.

By applying the Kalman-Yakubovich-Popov lemma, the existence of \( P > 0 \) and \( F \) such that (3.6) and (3.7) hold is equivalent to the existence of a common state feedback control law \( u_i = F x_i \) for the \( N - 1 \) systems

\[
\begin{align*}
\dot{x}_i &= A x_i + \lambda_i B u_i + E z_i, \\
y_i &= C x_i,
\end{align*}
\]
which renders all the resulting closed-loop systems strictly passive from \( z_i \) to \( y_i \) with the common storage function \( \frac{1}{2} x_i^T P x_i \).

We will now study the problem of finding necessary and sufficient conditions under which such common \( P > 0 \) and \( F \) exist for the set of systems (3.9). We first consider the case that we have a single system

\[
\begin{align*}
\dot{x} &= A x + B u + E z, \\
y &= C x,
\end{align*}
\]
and obtain necessary and sufficient conditions for the existence of a state feedback control law \( u = F x \) that renders this system strictly passive from \( z \) to \( y \).

**Lemma 3.3.** There exist matrices \( P > 0 \) and \( F \) such that

\[
(A + BF)^T P + P(A + BF) < 0,
\]
\[
PE = C^T,
\]
if and only if there exists a matrix $Q > 0$ such that
\[
B^\top (QA^T + AQ)(B^\top)^T < 0,
\]
(3.10)
\[
E = QC^T.
\]
(3.11)

In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by $F = \mu B^T Q^{-1}$, where $\mu$ is any real number satisfying
\[
QA^T + AQ + 2\mu BB^T < 0.
\]

**Proof.** For the ‘only if’ part, let $Q = P^{-1}$. Then we get (3.11) and $QA^T + AQ + QF^T B^T + BFQ < 0$ which yields (3.10). For the ‘if’ part, by Finsler’s lemma, there exists a real $\mu$ such that $QA^T + AQ + 2\mu BB^T < 0$. Let $P = Q^{-1}$ and $F := \mu B^T Q^{-1}$. Then we get $PE = C^T$ and
\[
(A + BF)^T P + P(A + BF)
= (A + \mu BB^T Q^{-1})^T Q^{-1} + Q^{-1} (A + \mu BB^T Q^{-1})
= A^T Q^{-1} + Q^{-1} A + 2\mu Q^{-1} BB^T Q^{-1} < 0.
\]

This completes the proof. 

**Remark 3.4.** Obviously, if $P > 0$ and $F$ satisfying the conditions in Lemma 3.3 exist, then the state feedback control law $u = Fx$ renders the system $\dot{x} = Ax + Bu + Ez$, $y = Cx$ strictly passive from $z$ to $y$. Equivalently, the resulting closed-loop system is robustly stabilized against passive unknown nonlinearities in its negative feedback loop $z = -\hat{\phi}(y, t)$, see also [26]. More on static output feedback passification can be found in [53].

**Remark 3.5.** If $\mu$ is a real number as in the formulation of Lemma 3.3, then by the fact that $BB^T$ is positive semi-definite, $QA^T + AQ - 2\hat{\mu}BB^T < 0$ holds for any $\hat{\mu}$ such that $\hat{\mu} \geq -\mu$. Clearly, such $\hat{\mu}$ can always be taken positive. In this case, $F' := -\hat{\mu} B^T Q^{-1}$ is also a suitable control law.

Next we will focus on conditions for the existence of a protocol (3.2) that robustly synchronizes the Lur’e network (3.3), i.e. on finding a common $P > 0$ and $F$ such that (3.6) and (3.7) in Lemma 3.1 hold. The following theorem establishes necessary and sufficient conditions for the existence of such common $P > 0$ and $F$:

**Theorem 3.6.** Let $\lambda_2$ be the smallest nonzero Laplacian eigenvalue. There exist matrices $P > 0$ and $F$ such that (3.6) and (3.7) hold for all $i = 2, \ldots, N$ if and only if there exists a matrix $Q > 0$ such that (3.10) and (3.11) hold. In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by $F = -kB^T Q^{-1}$,
where the positive real number \( k \) satisfies
\[
Q A^T + A Q - 2 k \lambda_2 B B^T < 0.
\] (3.12)

**Proof.** The ‘only if’ part is obvious. For the ‘if’ part, there exists \( k > 0 \) such that (3.12) holds. Let \( P = Q^{-1} \) and \( F := - k B^T Q^{-1} \). Then we get \( P E = C^T \) and
\[
(A + \lambda_i B F)^T P + P (A + \lambda_i B F)
\]
\[
= (A - k \lambda_i B B^T Q^{-1})^T Q^{-1} + Q^{-1} (A - k \lambda_i B B^T Q^{-1})
\]
\[
= A^T Q^{-1} + Q^{-1} (A - 2 k \lambda_i Q^{-1} B B^T Q^{-1})
\]
\[
\leq A^T Q^{-1} + Q^{-1} (A - 2 k \lambda_2 Q^{-1} B B^T Q^{-1}) < 0
\]
for all \( i = 2, \cdots, N \). This completes the proof. \( \square \)

**Remark 3.7.** We want to stress that our conditions in Theorem 3.6 are equivalent to the existence of \( P > 0 \) and \( F \) for a single agent, see Lemma 3.3. From the computational point of view this is advantageous since it reduces the computation of a synchronization protocol for a possibly large network (\( N \) large) to the computation of a state feedback controller that renders a single agent strictly passive.

### 3.3.2 Incrementally Sector Bounded Nonlinearities

In this subsection we consider unknown feedback nonlinearities \( \phi(\cdot, t) \) given by incrementally sector bounded functions within the sector \([S_1, S_2]\), where \( S_1, S_2 \in \mathbb{R}^{s \times s} \) are real symmetric matrices with \( 0 \leq S_1 < S_2 \). This property is defined as follows.

**Definition 3.3.** The system (3.4) is called **incrementally sector bounded** within the sector \([S_1, S_2]\) if the function \( \phi(\cdot, t) \) satisfies
\[
[z_1 - z_2 - S_1 (y_1 - y_2)]^T [z_1 - z_2 - S_2 (y_1 - y_2)] \leq 0
\] (3.13)
for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \), where \( z_1 = \phi(y_1, t) \) and \( z_2 = \phi(y_2, t) \).

Incremental sector boundedness was investigated before, for example in [9]. Note that any function \( \phi(\cdot, t) \) satisfying the incremental sector boundedness condition (3.13) also satisfies the ordinary sector boundedness condition, i.e.
\[
(\phi(y, t) - S_1 y)^T (\phi(y, t) - S_2 y) \leq 0
\]
for all \( y \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \). For the SISO case, i.e. the case that \( z, y \) are scalars and hence \( S_1 = \alpha, S_2 = \beta \) are nonnegative real numbers [73], the incremental sector boundedness condition with \( 0 \leq \alpha < \beta \) is equivalent to the slope-restrictedness
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condition that was used in [13, 29]. In contrast to [13, 29], the incremental sector boundedness condition allows us to explore Lur'e network synchronization via linear matrix inequalities immediately.

Lemma 3.8. Assume that the unknown functions \( \phi(\cdot, t) \) satisfy (3.13) for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \). If there exist matrices \( P > 0, F \) and a positive real number \( \tau \) such that the strict inequality

\[
\begin{bmatrix}
(A + \lambda_i BF)^T P & -PE \\
+P(A + \lambda_i BF) & +\tau C^T (S_1 + S_2)
\end{bmatrix}
- \tau C^T (S_1 S_2 + S_2 S_1) C
- E^T P
+ \tau (S_1 + S_2) C
- 2\tau I_s < 0
\quad (3.14)
\]

holds for all \( i = 2, \ldots, N \), then the network of Lur'e agents (3.1) with the protocol (3.2) is robustly synchronized for all incrementally sector bounded \( \phi(\cdot, t) \) within the sector \([S_1, S_2]\).

Proof. As in the proof of Lemma 3.1, choose the Lyapunov function candidate

\[
V_1(\bar{x}) = \frac{1}{2} \bar{x}^T(I_{N-1} \otimes P) \bar{x},
\]

where \( P > 0 \) together with \( F \) and \( \tau > 0 \) satisfies (3.14). Then the time derivative of \( V_1(\bar{x}) \) along the trajectories of the system (3.8) is equal to

\[
\dot{V}_1(\bar{x}) = \bar{x}^T \left[ (I_{N-1} \otimes PA + \bar{\Lambda} \otimes PBF) \bar{x} \right.
\left. - (I_{N-1} \otimes PE) (U_2^T \otimes I_s) \Phi(y, t) \right]
= \frac{1}{2} \left[ (U_2^T \otimes I_s) \Phi(y, t) \right]^T \left[ \begin{array}{cc}
I_{N-1} \otimes \left( A^T P + PA \right) + \bar{\Lambda} \otimes \left( F^T B^T P + PBF \right)
& -I_{N-1} \otimes PE \\
-I_{N-1} \otimes E^T P
\end{array} \right]
\left[ \begin{array}{c}
\bar{x} \\
(U_2^T \otimes I_s) \Phi(y, t)
\end{array} \right].
\]

Denote

\[
\Xi := \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_1) (U_2^T \otimes I_s) y \right]^T
\left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_2) (U_2^T \otimes I_s) y \right].
\]
By Lemma 2.1, we have

\[ \Xi = \left[ (U_2^T \otimes I_s) \Phi(y, t) - (U_2^T \otimes I_s) (I_N \otimes S_1)y \right]^T \]

\[ \left[ (U_2^T \otimes I_s) \Phi(y, t) - (U_2^T \otimes I_s) (I_N \otimes S_2)y \right] \]

\[ = \left[ \Phi(y, t) - (I_N \otimes S_1)y^T (U_2U_2^T \otimes I_s) \left[ \Phi(y, t) - (I_N \otimes S_2)y \right] \right] \]

\[ \frac{1}{N} \sum_{1 \leq i < j \leq N} \left[ \phi(y_i, t) - \phi(y_j, t) - S_1(y_i - y_j) \right]^T \]

\[ \left[ \phi(y_i, t) - \phi(y_j, t) - S_2(y_i - y_j) \right] \leq 0. \]

On the other hand, we have

\[ \frac{1}{2} \left[ (U_2^T \otimes I_s) \Phi(y, t) \right]^T \]

\[ \left[ I_{N-1} \otimes C^T(S_1S_2 + S_2S_1)C - I_{N-1} \otimes C^T(S_1 + S_2) \right] \]

\[ = \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_1 C) \bar{x} \right]^T \]

\[ \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_2 C) \bar{x} \right] \]

\[ = \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_1 C) (I_{N-1} \otimes C) (U_2 \otimes I_n) x \right]^T \]

\[ \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_2 C) (I_{N-1} \otimes C) (U_2 \otimes I_n) x \right] \]

\[ = \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_1) (U_2^T \otimes I_s) (I_N \otimes C)x \right]^T \]

\[ \left[ (U_2^T \otimes I_s) \Phi(y, t) - (I_{N-1} \otimes S_2) (U_2^T \otimes I_s) (I_N \otimes C)x \right] \]

\[ = \Xi \leq 0. \]

Now using (3.14), it can be seen that \( \dot{V}_1(\bar{x}) \) is negative definite for all \( \bar{x} \) satisfying (3.8) and for all incrementally sector bounded \( \phi(\cdot, t) \) within \([S_1, S_2]\). This completes the proof. \( \square \)

By taking the Schur complement, (3.14) is equivalent to

\[ \Theta_i := \left( A - \frac{1}{2} E(S_1 + S_2)C + \lambda_i BF \right)^T P + P \left( A - \frac{1}{2} E(S_1 + S_2)C + \lambda_i BF \right) \]

\[ + \frac{1}{2} \tau C^T(S_2 - S_1)^2 C + \frac{1}{2\tau} P E E \tau P < 0, \]
equivalently,

\[
\begin{bmatrix}
(A - \frac{1}{2}E(S_1 + S_2)C + \lambda_i BF)^T P \\
+ P(A - \frac{1}{2}E(S_1 + S_2)C + \lambda_i BF) \\
+ \frac{1}{2}\tau C^T(S_2 - S_1)^2 C \\
E^T P
\end{bmatrix}
\begin{bmatrix}
PE \\
E^T P
\end{bmatrix}
< 0
\]

for all \(i = 2, \cdots, N\). Consequently, by applying the bounded real lemma, the existence of \(P > 0\), \(F\) and \(\tau > 0\) such that (3.14) holds is equivalent to the existence of a common state feedback control law \(u_i = Fx_i\) for the \(N - 1\) systems

\[
\begin{align*}
\dot{x}_i &= (A - \frac{1}{2}E(S_1 + S_2)C) x_i + \lambda_i Bu_i + E z_i \\
y_i &= \sqrt{\frac{\tau}{2}}(S_2 - S_1)Cx_i, \quad i = 2, \cdots, N
\end{align*}
\]

such that the resulting closed-loop systems are dissipative with respect to the supply rate \(s(z_i, y_i) = \tau z_i^T z_i - y_i^T y_i\) with the common storage function \(x_i^T P x_i\). In particular the common state feedback control law \(u_i = Fx_i\) renders the \(H_\infty\) gains from \(z_i\) to \(y_i\) less than or equal to \(\sqrt{\frac{\tau}{2}}\).

Using the same idea as in Lemma 3.3, we now establish necessary and sufficient conditions for the existence of \(P > 0\), \(F\) and \(\tau > 0\) satisfying (3.14), thus obtain a distributed static protocol that robustly synchronizes the Lur’e network (3.3) against incrementally sector bounded nonlinearities.

**Theorem 3.9.** There exist \(P > 0\), \(F\) and \(\tau > 0\) such that (3.14) holds for all \(i = 2, \cdots, N\) if and only if there exist a matrix \(Q > 0\) and a positive real number \(\rho\) such that the following LMI holds:

\[
\begin{bmatrix}
B & \frac{Q(A - \frac{1}{2}E(S_1 + S_2)C)^T}{+(A - \frac{1}{2}E(S_1 + S_2)C)Q} \\
CQ & -2\rho(S_2 - S_1)^{-2}
\end{bmatrix}
\begin{bmatrix}
QC^T \\
Q C^T
\end{bmatrix}
< 0. \tag{3.15}
\]

In this case, a suitable \(P\) is given by \(P = Q^{-1}\), a suitable \(\tau\) is given by \(\tau = \frac{1}{\rho}\), and a suitable \(F\) is given by \(F = -kB^T Q^{-1}\), where the positive real number \(k\) is chosen to satisfy

\[
\begin{bmatrix}
Q(A - \frac{1}{2}E(S_1 + S_2)C)^T \\
+(A - \frac{1}{2}E(S_1 + S_2)C)Q \\
+ \frac{1}{2}\rho E E^T - 2k\lambda_2 BB^T
\end{bmatrix}
\begin{bmatrix}
QC^T \\
CQ
\end{bmatrix}
< 0. \tag{3.16}
\]

**Proof.** For the ‘only if’ part, by taking the Schur complement, (3.14) is also
equivalent to

\[
\begin{bmatrix}
(A - \frac{1}{2}E(S_1 + S_2)C)^T P & C^T \\
+ P(A - \frac{1}{2}E(S_1 + S_2)C) & + \lambda_i (F^TB^TP + PBF) + \frac{1}{2}\tau PEE^TP \\
\vdots & \vdots \\
+ \frac{1}{2}\tau PEE^TP & - \frac{2}{\tau}(S_2 - S_1)^{-2}
\end{bmatrix} < 0.
\]

Let \( Q = P^{-1} \) and \( \rho = \frac{1}{\tau} \). Then we get

\[
\begin{bmatrix}
Q(A - \frac{1}{2}E(S_1 + S_2)C)^T & QC^T \\
+ (A - \frac{1}{2}E(S_1 + S_2)C)Q & + \frac{1}{2}\rho EET + \frac{1}{2}\rho EET \\
\vdots & \vdots \\
+ \frac{1}{2}\rho EET & -2\rho(S_2 - S_1)^{-2}
\end{bmatrix} < 0.
\]

Without loss of generality, we have

\[
\begin{bmatrix}
B \\
0_{s \times m}
\end{bmatrix}^\perp = \begin{bmatrix}
B^\perp \\
0_{(n-r) \times s} \\
0_{s \times s} \\
I_{s \times s}
\end{bmatrix}.
\]

(3.17)

By premultiplying with (3.17) and postmultiplying with the transpose of (3.17), (3.15) is obtained.

For the ‘if’ part, again by taking the Schur complement, (3.15) implies

\[
B^\perp \left[ Q \left( A - \frac{1}{2}E(S_1 + S_2)C \right)^T + \left( A - \frac{1}{2}E(S_1 + S_2) \right) \right] Q + \frac{1}{2}\rho EET + \frac{1}{2}\rho QC^T(S_2 - S_1)^2CQ (B^\perp)^T < 0.
\]

By Finsler’s lemma, it follows that there exists \( k > 0 \) such that

\[
Q \left( A - \frac{1}{2}E(S_1 + S_2)C \right)^T + \left( A - \frac{1}{2}E(S_1 + S_2) \right) Q + \frac{1}{2}\rho EET + \frac{1}{2}\rho QC^T(S_1 - S_2)^2CQ - 2k\lambda_2 BBT < 0,
\]

i.e. (3.16). Let \( P = Q^{-1} \), \( \tau = \frac{1}{\rho} \) and \( F := -kB^TP \). Then we get

\[
\Theta_2 = \left( A - \frac{1}{2}E(S_1 + S_2)C \right)^T P + P \left( A - \frac{1}{2}E(S_1 + S_2)C \right)
\]
3.3. Robust Synchronization

\[ -2k\lambda_2 PBB^T P + \frac{1}{2}\tau C^T (S_2 - S_1)^2 C + \frac{1}{2\tau} PEE^T P < 0. \]

Thus \( \Theta_i \leq \Theta_2 < 0 \) for all \( i = 2, \cdots, N \). This completes the proof. \( \square \)

Remark 3.10. Obviously, if we take the sector \([0, kI]\) with \( k \) a positive real number, then by formally setting \( k = +\infty \) we derive incremental passivity. Thus the question arises whether we can obtain the conditions of Theorem 3.6 by letting \( k \to +\infty \) in the condition of Theorem 3.9. In the case of the sector \([0, kI]\) the LMI (3.15) in Theorem 3.9 becomes

\[
\begin{bmatrix}
B \\
0
\end{bmatrix}^\perp
\begin{bmatrix}
Q(A - \frac{1}{2}kEC)^T & \rho EE^T \\
(A - \frac{1}{2}kEC)Q & ky + \frac{1}{2} \rho EE^T + \frac{1}{2}k^2 QC^T CQ
\end{bmatrix}
\begin{bmatrix}
B \\
0
\end{bmatrix}^\perp < 0
\]

or, equivalently,

\[
B^\perp \left[ Q \left( A - \frac{1}{2}kEC \right)^T + \left( A - \frac{1}{2}kEC \right) Q + \frac{1}{2} \rho EE^T + \frac{1}{2}k^2 QC^T CQ \right] (B^\perp)^T < 0
\]

for some \( Q > 0 \) and \( \rho > 0 \). Note that, in fact, \( Q \) and \( \rho \) depend on \( k \), i.e., \( Q = Q(k) \) and \( \rho = \rho(k) \). Clearly, the strict inequality above is equivalent to

\[
B^\perp \left[ QA^T + AQ + \frac{1}{2} \rho \left( E - \frac{1}{\rho} kQC^T \right) \left( E - \frac{1}{\rho} kQC^T \right)^T \right] (B^\perp)^T < 0.
\]

From this we indeed immediately obtain

\[
B^\perp \left( QA^T + AQ \right) (B^\perp)^T < 0,
\]

which is exactly the first condition, (3.10), of Theorem 3.6. The question then of course arises whether also the second condition of Theorem 3.6, i.e. \( QC^T = E \), can be obtained. Here the idea would be to let \( k \) run off to infinity. If we would know that \( \frac{k}{\rho(k)} \to 1 \) and \( Q(k) \to \bar{Q} \) for \( k \to +\infty \) (for some limited \( \bar{Q} \)), then from the latter inequality we would indeed obtain \( QC^T = E \). However, it is unclear how \( Q(k) \) and \( \rho(k) \) depend on the sector bound \( k \), and what happens if \( k \) goes to infinity. Summarizing, the condition of Theorem 3.9 indeed immediately implies the first condition of Theorem 3.6, but it is unclear how the second condition of Theorem 3.6 follows from the condition of Theorem 3.9.
3.4 Fully distributed robust synchronization

As shown in the previous section, in order to compute the feedback gain matrix $F$ in the synchronization protocol (3.2), a positive real number $k$ must be computed that satisfies (3.12) or (3.16), respectively. Computation of such $k$ involves the second smallest eigenvalue $\lambda_2$ of the Laplacian matrix $L$, which is called the algebraic connectivity of the undirected graph $G$. However, knowledge of the exact value of $\lambda_2$ involves global information in the sense that each agent has to know $L$ and consequently the entire interconnection topology $G$ to compute it. Thus, in effect, the protocol (3.2) cannot be implemented by each agent in a fully distributed fashion.

In order to overcome this drawback, we adopt the following fully distributed synchronization protocol with an adaptive control law that adjusts the coupling weights in real time (see also [13, 30, 31]):

$$u_i = F \sum_{j=1}^{N} a_{ij} c_{ij} (x_i - x_j),$$

$$\dot{c}_{ij} = a_{ij} (x_i - x_j)^T H (x_i - x_j), \quad i,j = 1,2,\ldots,N,$$

where $c_{ij}(t)$ is a time-varying coupling weight between agents $i$ and $j$ with $c_{ij}(0) = c_{ji}(0)$, $H \in \mathbb{R}^{s \times s}$ is a common coupling gain matrix to be determined later. By interconnecting the agents (3.1) with the adaptive protocol (3.18), we get that the network is represented by

$$\dot{x}_i = Ax_i + BF \sum_{j=1}^{N} a_{ij} c_{ij} (x_i - x_j) - E\phi(Cx_i, t),$$

$$\dot{c}_{ij} = a_{ij} (x_i - x_j)^T H (x_i - x_j), \quad i,j = 1,2,\ldots,N.$$

We will now design $F$ and $H$ in (3.18) such that the Lur'e network (3.19) is robustly synchronized against uncertainties $\phi(\cdot, t)$. As in Section 3.3, we first consider the case of incrementally passive nonlinearities, and subsequently the case that the nonlinearities are incrementally sector bounded.

3.4.1 Incrementally Passive Nonlinearities

In this subsection we will study the incrementally passive nonlinearity case.

**Theorem 3.11.** Assume that the unknown functions $\phi(\cdot, t)$ satisfy (3.5) for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$. If there exists a matrix $Q > 0$ such that (3.10) and (3.11) hold, then the network of agents (3.1) with the adaptive protocol (3.18), where
3.4. Fully distributed robust synchronization

\( F = -B^T Q^{-1} \) and \( H = Q^{-1} B B^T Q^{-1} \), is robustly synchronized, i.e. the Lur’\'e network (3.19) is synchronized for all incrementally passive \( \phi(\cdot, t) \).

**Proof.** Let \( e_i = x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \), \( i = 1, 2, \ldots, N \), and denote \( e = [e_1^T, e_2^T, \ldots, e_N^T]^T \). Then we get

\[
\dot{e}_i = A e_i + B F \sum_{j=1}^{N} a_{ij} c_{ij} (e_i - e_j) - E \phi(C x_i, t) + \frac{1}{N} \sum_{j=1}^{N} E \phi(C x_j, t),
\]

(3.20)

\[
\dot{c}_{ij} = a_{ij} (e_i - e_j)^T H (e_i - e_j), \quad i, j = 1, 2, \ldots, N.
\]

It is obvious that \( x_i - x_j = 0 \) for all \( i, j = 1, 2, \ldots, N \) if and only if \( e = 0 \). Therefore, the synchronization of \( x \) is equivalent to the global asymptotical stability of \( e \).

Denote \( e = [c_{11}, \ldots, c_{1N}, c_{21}, \ldots, c_{NN}]^T \). Choose the Lyapunov function candidate

\[
V_2(e, c) = \sum_{i=1}^{N} e_i^T Q^{-1} e_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c})^2,
\]

where \( Q > 0 \) satisfies (3.10) and (3.11), and \( \bar{c} \) is a positive real number to be determined later. The time derivative of \( V_2(e, c) \) along the trajectories of (3.20) is given by

\[
\dot{V}_2(e, c)
\]

\[
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} \dot{e}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c}) \dot{c}_{ij}
\]

\[
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} \left[ A e_i + B F \sum_{j=1}^{N} a_{ij} c_{ij} (e_i - e_j) - E \phi(C x_i, t) + \frac{1}{N} \sum_{j=1}^{N} E \phi(C x_j, t) \right]
\]

\[
+ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) (e_i - e_j)^T H (e_i - e_j).
\]

Since \( a_{ij} = a_{ji}, c_{ij}(0) = c_{ji}(0) \) and \( \dot{c}_{ij} = \dot{c}_{ji}, \forall i, j = 1, 2, \ldots, N \), we know that \( c_{ij}(t) = c_{ji}(t), \forall t \geq 0, i, j = 1, 2, \ldots, N \). Then

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) (e_i - e_j)^T H (e_i - e_j)
\]
= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (e_{ij} - \bar{e}) e_i^T H (e_i - e_j).

Thus

\dot{V}_2(e, c)

= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i

- 2 \sum_{i=1}^{N} e_i^T Q^{-1} B B^T Q^{-1} \sum_{j=1}^{N} a_{ij} c_{ij} (e_i - e_j)

- \frac{2}{N} \sum_{i=1}^{N} e_i^T Q^{-1} E \sum_{j=1}^{N} (\phi(C x_i, t) - \phi(C x_j, t))

+ 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) e_i^T Q^{-1} B B^T Q^{-1} (e_i - e_j)

\leq 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i

- \frac{2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (C e_i - C e_j)^T (\phi(C x_i, t) - \phi(C x_i, t))

- 2 \bar{e} \sum_{i=1}^{N} \sum_{j=1}^{N} l_{ij} e_i^T Q^{-1} B B^T Q^{-1} e_j

= 2 e^T (I_N \otimes Q^{-1} A - \bar{c} \mathcal{L} \otimes Q^{-1} B B^T Q^{-1}) e.

Let \mathcal{U} be an orthogonal matrix such that \mathcal{U}^T \mathcal{L} \mathcal{U} = \Lambda as defined in Chapter 2.
Let \bar{e} = (\mathcal{U}^T \otimes I_n) e and \tilde{e} = (\mathcal{U}_2^T \otimes I_n) e, where \tilde{e} = [\tilde{e}_1^T, \tilde{e}_2^T, \ldots, \tilde{e}_N^T]^T and \bar{e} =
3.4. Fully distributed robust synchronization

\[ [\tilde{e}_2^T, \cdots, \tilde{e}_N^T]^T \]. It is obvious that \( \tilde{e}_1 = \frac{1}{\sqrt{N}} 1_N^T e = 0 \). Then for all trajectories of \( e \) and \( e \) we get

\[
\dot{V}_2(e, c) \\
\leq 2\tilde{e}^T (I_{N-1} \otimes Q^{-1} A - \tilde{e} \Lambda \otimes Q^{-1} BB^T Q^{-1}) \tilde{e} \\
= \sum_{i=2}^{N} \tilde{e}_i^T (A^T Q^{-1} + Q^{-1} A - 2\tilde{e}_i Q^{-1} BB^T Q^{-1}) \tilde{e}_i.
\]

By Finsler’s lemma, (3.10) implies that there exists \( \tilde{e} > 0 \) such that

\[
AQ + QA^T - 2\tilde{e} \lambda_2 BB^T < 0.
\]

Obviously,

\[
AQ + QA^T - 2\tilde{e} \lambda_1 BB^T \\
\leq AQ + QA^T - 2\tilde{e} \lambda_2 BB^T < 0
\]

for all \( i = 2, \cdots, N \). Thus \( \dot{V}_2(e, c) \leq 0 \). Hence \( V_2(e, c) \) is bounded and so is each \( c_{ij} \). Note that \( H \geq 0 \). This implies that \( c_{ij} \) are nondecreasing. It follows that \( c_{ij} \) converge to certain finite values. Let \( S := \{(e(t), c(t)) \mid \dot{V}_2(e, c) \equiv 0\} \). Note that \( V_2(e, c) \equiv 0 \) implies \( \tilde{e} \equiv 0 \). We also know that \( \tilde{e}_1 \equiv 0 \). Therefore, by LaSalle’s invariance principle, \( \tilde{e}(t) \to 0 \) and \( e(t) \to 0 \) as \( t \to \infty \). This completes the proof. □

3.4.2 Incrementally Sector Bounded Nonlinearities

In this subsection we consider the incrementally sector bounded nonlinearity case.

**Theorem 3.12.** Assume that the unknown functions \( \phi(\cdot, t) \) satisfy (3.13) for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \). If there exist a matrix \( Q > 0 \) and a positive real number \( \rho \) such that (3.15) holds, then the network of agents (3.1) with the adaptive protocol (3.18), where \( F = -B^T Q^{-1} \) and \( H = Q^{-1} BB^T Q^{-1} \), is robustly synchronized, i.e. the Lur’e network (3.19) is synchronized for all incrementally sector bounded \( \phi(\cdot, t) \) within \( [S_1, S_2] \).

**Proof.** As in the proof of Theorem 3.11, let \( e_i = x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \) and denote \( e = [e_1^T, e_2^T, \cdots, e_N^T]^T \). Then again we obtain (3.20). Choose the same Lyapunov function candidate \( V_2(e, c) \). The time derivative of \( V_2(e, c) \) along the trajectories of (3.20) is given by

\[
\dot{V}_2(e, c) \\
= 2 \sum_{i=1}^{N} e_i Q^{-1} A e_i - 2 \sum_{i=1}^{N} e_i^T Q^{-1} BB^T Q^{-1} \sum_{j=1}^{N} a_{ij} c_{ij}(e_i - e_j)
\]
3. Fully Distributed Robust Synchronization of Undirected Lur’e Networks

\[-2 \sum_{i=1}^{N} e_i^T Q^{-1} E \phi(C x_i, t) + \frac{2}{N} \left( \sum_{i=1}^{N} e_i^T \right) Q^{-1} \sum_{j=1}^{N} E \phi(C x_i, t)\]

\[+ 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) e_i^T Q^{-1} B B^T Q^{-1} (e_i - e_j).\]

Since

\[\sum_{i=1}^{N} e_i = \sum_{i=1}^{N} \left( x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \right) = 0,\]

we obtain

\[\dot{V}_2(e, c) = 2 \sum_{i=1}^{N} e_i Q^{-1} A e_i - 2 \sum_{i=1}^{N} e_i^T Q^{-1} E \phi(C x_i, t)\]

\[- 2 \bar{c} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1} (e_i - e_j)\]

\[= e^T \left[ I_N \otimes \left( Q^{-1} A + A^T Q^{-1} \right) \right] e - 2 e^T \left( I_N \otimes Q^{-1} E \right) \Phi(y, t)\]

\[- 2 \bar{c} e^T \left( \mathcal{L} \otimes Q^{-1} B B^T Q^{-1} \right) e\]

\[= \begin{bmatrix} e^T \\ \Phi(y, t) \end{bmatrix}^T \begin{bmatrix} I_N \otimes \left( Q^{-1} A + A^T Q^{-1} \right) & -I_N \otimes Q^{-1} E \\ -2\bar{c} \mathcal{L} \otimes Q^{-1} B B^T Q^{-1} & 0 \end{bmatrix} \begin{bmatrix} e \\ \Phi(y, t) \end{bmatrix}.\]

Let \(\bar{e} = (U^T \otimes I_n) e\) and \(\bar{\bar{e}} = (U^T_2 \otimes I_n) e\), where \(\bar{e} = [\bar{e}_1^T, \bar{e}_2^T, \ldots, \bar{e}_N^T]^T\) and \(\bar{\bar{e}} = [\bar{\bar{e}}_2^T, \ldots, \bar{\bar{e}}_N^T]^T\). Then we get

\[\dot{V}_2(e, c) = \begin{bmatrix} \bar{e} \\ (U^T_2 \otimes I_s) \Phi(y, t) \end{bmatrix}^T \begin{bmatrix} I_{N-1} \otimes \left( Q^{-1} A + A^T Q^{-1} \right) & -I_{N-1} \otimes Q^{-1} E \\ -2\bar{c} \mathcal{L} \otimes Q^{-1} B B^T Q^{-1} & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ (U^T_2 \otimes I_s) \Phi(y, t) \end{bmatrix}.\]
It is easily seen that
\[ e = \left[ \left( I_N - \frac{1}{N} 1_N 1_N^T \right) \otimes I_n \right] x = (U_2 U_2^T \otimes I_n) x. \]

As \( \bar{e} = (U_2^T \otimes I_n) e \), we have \( \bar{e} = (U_2^T \otimes I_n) x = \bar{x} \). Thus, from the proof of Lemma 3.8, we have the following
\[ \bar{e} (U_2^T \otimes I_s) \Phi(y, t) \leq 0. \]

By taking the Schur complement and applying Finsler's lemma, (3.15) implies that there exists \( \bar{c} > 0 \) such that
\[
\begin{bmatrix}
Q^{-1} A + A^T Q^{-1} & -Q^{-1} E \\
-2\bar{c}\lambda Q^{-1} BB^T Q^{-1} & +\tau C^T (S_1 + S_2) \\
-\tau C^T (S_1 S_2 + S_2 S_1) C & -2\tau I_s \\
-E^T Q^{-1} +\tau (S_1 + S_2) C & \end{bmatrix} < 0,
\]
where \( \tau = \frac{1}{\rho} \). Therefore
\[
\begin{bmatrix}
Q^{-1} A + A^T Q^{-1} & -Q^{-1} E \\
-2\bar{c}\lambda Q^{-1} BB^T Q^{-1} & +\tau C^T (S_1 + S_2) \\
-\tau C^T (S_1 S_2 + S_2 S_1) C & -2\tau I_s \\
-E^T Q^{-1} +\tau (S_1 + S_2) C & \end{bmatrix} < 0
\]
for all \( i = 2, \cdots, N \). It follows that \( \dot{V}_2(e, c) \leq 0 \) for all incrementally sector bounded \( \phi(\cdot, t) \) within \([S_1, S_2]\). Following a similar analysis as in the proof of Theorem 3.11, the proof is completed. \( \square \)
3.5 Simulation examples

In this section, we present some numerical simulations to illustrate the theoretical results obtained in this chapter. We consider Chua’s circuit, which is described by the following system of nonlinear ordinary differential equations [32]:

\[
\begin{align*}
\dot{x}_1 &= 10.0(-x_1 + x_2 - f(x_1)) \\
\dot{x}_2 &= x_1 - x_2 + x_3 \\
\dot{x}_3 &= -14.87x_2
\end{align*}
\]  
(3.21)

where \( x_1(t), x_2(t), x_3(t) \in \mathbb{R} \). \( f(x_1) \) is a piecewise linear function that represents the change in resistance vs. current across Chua’s diode, which is given by

\[
f(x_1) = -0.68x_1 - 0.295(|x_1 + 1| - |x_1 - 1|).
\]

It is possible to rewrite (3.21) in the form of a Lur’e system with control input \( u \in \mathbb{R} \):

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ez \\
y &= Cx \\
z &= -\phi_1(y)
\end{align*}
\]  
(3.22)

where \( x = [x_1, x_2, x_3]^T \), \( A = [-3.2, 10, 0; 1, -1, 1; 0, -14.87, 0] \), \( B = [1; 1; 0] \), \( C = [1, 0, 0] \), \( E = [-2.95; 0; 0] \) and \( \phi_1(x_1) = |x_1 + 1| - |x_1 - 1| \). \( B^\perp \) exists, and the pair \( (A, B) \) is controllable. It is easily seen that \( \phi_1(\cdot) \) is incrementally passive, but also incrementally sector bounded with \( S_1 = 0 \) and \( S_2 = 2 \), see Fig. 3.2. Taking (3.22) as the individual agent dynamics, a network of 9 such agents is shown in Fig. 3.3, in which the interconnection topology is undirected and connected. Let the graph be unweighted, i.e. \( a_{ij} = 1 \) when the edge \((i,j)\) (or \((j,i)\)) exists. We compute the

![Figure 3.2: Plots of \( \phi_1(x_1) \) and \( \phi_2(x_1) \)](image-url)
Figure 3.3: A network of 9 agents

second smallest Laplacian eigenvalue of the graph to be $\lambda_2 = 0.4822$.

**Example 1.** Since $\phi_1(\cdot)$ is incrementally passive, we first attempt to apply Theorem 3.6 to find a distributed static protocol that robustly synchronizes the network. Unfortunately, the conditions in Theorem 3.6 are not satisfied since there does not exist a positive definite matrix $Q$ such that $CQ = E^T$. We have the dynamical system $\dot{x}_1 = -x_1^3 + x_2 + u$, $\dot{x}_2 = -x_2 + u$ to be an available example for Theorem 3.6. A suitable positive definite matrix $Q$ could be $Q = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}$, where $q$ is any positive real number. Due to space limitations, the simulation plots for this example are omitted.

**Example 2.** As we have noted before, $\phi_1(\cdot)$ is incrementally sector bounded with $S_1 = 0$ and $S_2 = 2$. Therefore we proceed by trying to find an available synchronization protocol by means of Theorem 3.9. Using the LMI Control Toolbox in Matlab, we find that the positive definite matrix

$$Q = \begin{bmatrix} 540.7 & 248.5 & 56.1 \\ 248.5 & 413.3 & 23.0 \\ 56.1 & 23.0 & 2457.7 \end{bmatrix},$$

and the positive real number $\rho = 689.9865$ satisfy condition (3.15) in Theorem 3.9, and $k = 41893$ satisfies condition (3.16). The corresponding feedback gain matrix is computed as

$$F = -kB^TQ^{-1} = \begin{bmatrix} -42.8694 & -75.6846 & 1.6868 \end{bmatrix}.$$
we use different initial states in order to get a bounded synchronization trajectory.

**Example 3.** In this example we apply the results of Section 4 to Chua’s circuit as we did in Example 2. We provide simulation results for the incremental sector boundedness case. As the positive definite matrix $Q$ in Theorem 3.12 is equal to $Q$ in Theorem 3.9, we obtain

$$F = -B^T Q^{-1} = \begin{bmatrix} -0.0010 & -0.0018 & 0.0000 \end{bmatrix},$$

$$H = Q^{-1} BB^T Q^{-1} = \begin{bmatrix} 10470 & 18490 & -410 \\ 18490 & 32640 & -730 \\ -410 & -730 & 20 \end{bmatrix}.$$ 

We choose the same initial states as in Example 2. Similarly, the first component of the trajectories of the network (3.19), i.e. $X_1$, is plotted in Fig. 3.6. The network
3.6. Conclusions

Indeed reaches synchronization. The time-varying coupling weights $c_{ij}$ are shown in Fig. 3.7. These converge to finite steady-state values. Similar to the results of Example 2, we have tested the protocol for the nonlinearity $\phi_2 = \arctan(\cdot)$, and have found that the network is synchronized as well. Due to space limitations the simulation results are omitted.

3.6 Conclusions

In this chapter we have discussed the roles of incremental passivity and incremental sector boundedness conditions in robust synchronization of homogeneous Lur’e networks. Sufficient conditions for the existence of distributed static protocols to robustly synchronize Lur’e networks have been given. The protocols can be implemented by each agent in a fully distributed fashion. The required feedback
gain matrices are computed by solving LMI's, which can be easily done using the LMI Control Toolbox in Matlab. However, the assumption on the interconnection topologies that the graphs are undirected is not that applicable to practical applications. Therefore, the case that the graphs are directed will be discussed in the next chapter.