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van Waarde, Henk; Camlibel, Kanat; Mesbahi, Mehran

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From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma

Henk J. van Waarde , M. Kanat Camlibel , *Member, IEEE*, and Mehran Mesbahi , *Fellow, IEEE*

Abstract—In this article, we propose a new method to obtain feedback controllers of an unknown dynamical system directly from noisy input/state data. The key ingredient of our design is a new matrix S-lemma that will be proven in this article. We provide both strict and nonstrict versions of this S-lemma, which are of interest in their own right. Thereafter, we will apply these results to data-driven control. In particular, we will derive nonconservative design methods for quadratic stabilization, \mathcal{H}_2 and \mathcal{H}_∞ control, all in terms of data-based linear matrix inequalities. In contrast to previous work, the dimensions of our decision variables are independent of the time horizon of the experiment. Our approach, thus, enables control design from large datasets.

Index Terms—Data-driven control, LMIs, robust control, uncertain systems.

I. INTRODUCTION

IN THIS article, we study the problem of designing control laws for an unknown dynamical system using noisy data. This general problem exists for a long time, but has seen a renewed surge of interest over the past few years. The problem can be approached via different angles, for example, using combined system identification and model-based control, or by computing control laws from data without the intermediate modeling step. We will contribute to the second category of methods, aiming at control design directly from noisy data.

One of the main challenges in this area is to come up with robust control laws that guarantee stability and performance of the unknown system despite the inherent uncertainty caused by noisy data. Even though there are several recent contributions addressing this issue, there are multiple open questions.

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Henk J. van Waarde is with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence and the Engineering and Technology Institute Groningen, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: h.j.van.waarde@rug.nl).

M. Kanat Camlibel is with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: m.k.camlibel@rug.nl).

Mehran Mesbahi is with the William E. Boeing Department of Aeronautics and Astronautics, University of Washington, Seattle, WA 98195 USA (e-mail: mesbahi@aa.washington.edu).

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In fact, one of the unsolved problems is to come up with *nonconservative* control design strategies using only a finite number of data samples.

We will tackle this problem by providing necessary and sufficient conditions on noisy data, under which controllers can be obtained. As a consequence, our ensuing control technique is nonconservative and also shown to be tractable from a computational point of view. The technical ingredient that enables our design is a new generalization of the classical S-lemma [1], [2], which will be proven in this article. We will formulate our control problems using the general data informativity framework, as introduced in [3]. As such, the results developed in this article can be seen as a natural extension of those in [3] to noisy data.

A. Literature on Data-Driven Control

The literature on data-driven control is expanding rapidly. Our account of previous work is, therefore, not exhaustive, but we note that additional references can be found in the survey [4]. We mention contributions to data-driven optimal control [5]–[16], proportional–integral–derivative control [17], [18], predictive control [19]–[23], and nonlinear control [24]–[28]. Some of these techniques are iterative in nature: the controller is updated online when new data are presented. Examples of this include policy iteration methods [29] and iterative feedback tuning [30]. Other methods are one-shot in the sense that the controller is constructed offline from a batch of data. We mention, for instance, virtual reference feedback tuning [31] and methods based on Willems' fundamental lemma [32] (see also [33]). The latter line of work has been quite fruitful, with contributions ranging from output matching [34] and control by interconnection [35], to data-enabled predictive control [36] and a data-based closed-loop system parameterization [37]. This parameterization has been used for stabilizing and optimal control design using data-based linear matrix inequalities (LMIs) [37]. We also mention the extension [38] studying linear–quadratic regulator using noisy data and the article [39] for a closed-loop parameterization using noisy data. Additional recent research directions include data-driven control of networks [40], [41] and the interplay between data-guided control and model reduction [42].

B. Review of the S-Lemma

First proven by Yakubovich [1], the *S-lemma* is a classical result in control theory and optimization [2]. The result revolves around the question when the nonnegativity of one quadratic function implies that of another. The crux of the S-lemma is that

this seemingly difficult implication is equivalent to the feasibility of an LMI in a scalar variable, called a *multiplier*. The act of replacing the implication by an LMI is often referred to as the *S-procedure*. The S-procedure is more generally applicable in situations where one quadratic inequality is implied by *multiple* quadratic inequalities [43]. In this case, multiple scalar multipliers are used. It is well known, however, that the S-procedure is conservative in general, although there exist special cases in which losslessness (according to S-lemma) can be shown [2, Sec. 3]. A generalized S-lemma involving more general types of multipliers was considered in [44] and [45] and was shown to be nonconservative under extra assumptions.

The classical S-lemma and the contributions mentioned above all deal with vector variables. However, for reasons that will become clear in Section II, we need a type of S-lemma that is applicable to quadratic functions of *matrix* variables. Such a result has been reported for specific quadratic functions in [46, Th. 3.3]. In the special case that variables are *bounded*, an S-lemma for matrix variables can also be derived by combining the so-called *full block S-procedure* [47]–[50] with results from the literature on LMI relaxations [51]–[53]. These specific results are, however, less suited for the application of data-driven control that we have in mind. Therefore, in this article, we derive general matrix S-lemmas for both strict and nonstrict inequalities. Our matrix S-lemma for nonstrict inequalities is a direct generalization of the classical S-lemma [1]. It also recovers the result from [46] as a special case. As a corollary of our matrix S-lemma for a strict inequality, we recover the S-lemma derived from the general theory on LMI relaxations [51].

C. Our Contributions

The core of our approach is to formulate data-driven control as the problem of deciding whether one quadratic matrix inequality (QMI) is implied by another one. Our first contribution is to extend the classical S-lemma to quadratic matrix inequalities. Our second contribution is to apply these results to data-driven control. In particular, we come up with design procedures for quadratic stabilization, \mathcal{H}_2 control, and \mathcal{H}_∞ control.

Throughout this article, we will assume no statistics on the noise, but we will work with general bounded disturbances. We are thereby inspired by recent papers [37], [39] that formalize the assumption of bounded disturbances in terms of quadratic matrix inequalities. In fact, we will work with an assumption on the noise that is closely related to that of [39] and is more general than the assumption in [37]. In terms of control design, our approach completely differs from the above papers. In fact, instead of working with data-based parameterizations of closed-loop systems [37]–[39], we will work with a representation of all *open-loop* systems explaining the data, akin to the framework of [3]. We believe that our approach is attractive for the following reasons.

- 1) We provide robust guarantees on the stability and performance of the unknown data-generating system. The design involves data-guided LMIs that are tractable from a computational point of view and are easy to implement.
- 2) By virtue of our matrix S-lemma, the design method is *nonconservative*. This is in contrast with previous LMI

formulations in [37] and [39] that provide sufficient conditions for controller design.

- 3) Last but not least, the variables involved in our method are independent of the time horizon of the experiment. Our approach is, thus, applicable to large datasets. This is an advantage over closed-loop system parameterizations [37], [39], which become computationally intractable when applied to big data.

D. Outline of This Article

In Section II, we will formulate the problem. Section III contains our results on the matrix S-lemma. These results are then applied to data-driven stabilization in Section IV and to data-driven \mathcal{H}_2 control and \mathcal{H}_∞ control in Section V. In Section VI, we provide simulation examples. Finally, Section VII concludes this article.

II. PROBLEM OF DATA-DRIVEN STABILIZATION

Consider the linear time-invariant system

$$\mathbf{x}(t+1) = A_s \mathbf{x}(t) + B_s \mathbf{u}(t) + \mathbf{w}(t) \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ denotes the state, $\mathbf{u} \in \mathbb{R}^m$ is the input, and $\mathbf{w} \in \mathbb{R}^n$ is an unknown noise term. The matrices $A_s \in \mathbb{R}^{n \times n}$ and $B_s \in \mathbb{R}^{n \times m}$ denote the unknown state and input matrices, respectively. Our goal is to design stabilizing controllers for (1) on the basis of a finite number of measurements of the state and input of the system. To this end, suppose that we measure state and input data on a time interval¹ and collect these samples in the matrices

$$\begin{aligned} X &:= [x(0) \quad x(1) \quad \cdots \quad x(T)] \\ U_- &:= [u(0) \quad u(1) \quad \cdots \quad u(T-1)]. \end{aligned}$$

By defining the matrices

$$\begin{aligned} X_+ &:= [x(1) \quad x(2) \quad \cdots \quad x(T)] \\ X_- &:= [x(0) \quad x(1) \quad \cdots \quad x(T-1)] \\ W_- &:= [w(0) \quad w(1) \quad \cdots \quad w(T-1)] \end{aligned}$$

we clearly have

$$X_+ = A_s X_- + B_s U_- + W_- \quad (2)$$

We emphasize that the system matrices A_s and B_s as well as the noise term W_- are *unknown*, while X and U_- are measured. Before we introduce the problem, we will explain our assumption on the noise W_- .

A. Assumption on the Noise

We will formalize our assumption on the noise in terms of a QMI.

Assumption 1: The noise samples $w(0), w(1), \dots, w(T-1)$, collected in the matrix W_- , satisfy the bound

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geq 0 \quad (3)$$

¹All our results are still true for data collected on multiple intervals, see [3, Ex. 2] for more details on how to arrange the data matrices in this case.

for known matrices $\Phi_{11} = \Phi_{11}^\top$, Φ_{12} and $\Phi_{22} = \Phi_{22}^\top < 0$.

Note that the negative definiteness of Φ_{22} ensures that the set of noise matrices W_- satisfying (3) is bounded. In the special case $\Phi_{12} = 0$ and $\Phi_{22} = -I$, (3) reduces to

$$W_- W_-^\top = \sum_{t=0}^{T-1} w(t)w(t)^\top \leq \Phi_{11}. \quad (4)$$

Inequality (4) has the interpretation that the energy of w is bounded on the finite-time interval $[0, T-1]$. If w is a random variable, its *sample covariance matrix* is given by

$$\frac{1}{T} W_- \left(I - \frac{1}{T} J \right) W_-^\top$$

where J is the matrix of ones. Thus, (3) can also capture known bounds on the sample covariance by the choices $\Phi_{12} = 0$ and $\Phi_{22} = -\frac{1}{T}(I - \frac{1}{T}J)$. We emphasize, however, that we do not make any assumptions on the statistics of w and work with the general bound (3) instead. Note that [37, Asm. 5] is a special case of Assumption 1 for the choices $\Phi_{11} = \gamma X_+ X_+^\top$ with $\gamma > 0$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$. We remark that norm bounds on the individual noise samples $w(t)$ also give rise to bounds of the form (3), although this may lead to some conservatism. Indeed, note that $\|w(t)\|_2^2 \leq \epsilon$ implies that $w(t)w(t)^\top \leq \epsilon I$ for all t . As such, the bound (4) is satisfied for $\Phi_{11} = T\epsilon I$.

Remark 1: Note that the noise model in Assumption 1 is the “transposed” of the model in [39], in the sense that we penalize, e.g., the term $W_- \Phi_{22} W_-^\top$ instead of a term $W_-^\top Q_w W_-$. In some cases, these two different noise models are actually equivalent. For example, if $\Phi_{11} > 0$ and $\Phi_{12} = 0$, then (3) can be written via a Schur complement argument as

$$\begin{bmatrix} \Phi_{11} & W_- \\ W_-^\top & -\Phi_{22}^{-1} \end{bmatrix} \geq 0.$$

In turn, this is equivalent to $-\Phi_{22}^{-1} - W_-^\top \Phi_{11}^{-1} W_- \geq 0$, which is of the same form as [39].

Remark 2: In some cases, we may know *a priori* that the noise w does not directly affect the entire state space, but is contained in a subspace. This prior knowledge can be captured by the noise model in Assumption 1. Indeed, W_- is of the form $W_- = E\hat{W}_-$ for some $\hat{W}_- \in \mathbb{R}^{r \times T}$ satisfying

$$\begin{bmatrix} I \\ \hat{W}_-^\top \end{bmatrix}^\top \begin{bmatrix} \hat{\Phi}_{11} & \hat{\Phi}_{12} \\ \hat{\Phi}_{12}^\top & \hat{\Phi}_{22} \end{bmatrix} \begin{bmatrix} I \\ \hat{W}_-^\top \end{bmatrix} \geq 0$$

if and only if

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} E\hat{\Phi}_{11}E^\top & E\hat{\Phi}_{12} \\ \hat{\Phi}_{12}^\top E^\top & \hat{\Phi}_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geq 0.$$

Thus, the conclusion is that we can incorporate the knowledge that $W_- \in \text{im } E$ by appropriate choices of the Φ -matrices in (3). Showing the above claim is straightforward: note that the “only if” statement follows by pre- and postmultiplication with E and E^\top , respectively. The “if” part follows by noting that $x \in \ker E^\top$ implies $x^\top W_- \hat{\Phi}_{22} W_-^\top x \geq 0$; thus, $W_-^\top x = 0$. Hence, $\ker E^\top \subseteq \ker W_-^\top$; equivalently, $\text{im } W_- \subseteq \text{im } E$.

B. Problem Formulation

We will follow the general framework for data-driven analysis and control in [3]. To this end, we define the set of all systems (A, B) explaining the data (U_-, X) , i.e., all (A, B) satisfying

$$X_+ = AX_- + BU_- + W_- \quad (5)$$

for some W_- satisfying (3). We denote this set by Σ :

$$\Sigma := \{(A, B) \mid (5) \text{ holds for some } W_- \text{ satisfying (3)}\}.$$

We can only guarantee that a state feedback $u = Kx$ stabilizes the true system (A_s, B_s) if it stabilizes *all* systems in Σ . This motivates the following definition of *informative* data. Loosely speaking, data are called informative if they enable the design of a controller that stabilizes all systems in Σ (and thus, the unknown (A_s, B_s)).

Definition 3: Assume that (U_-, X) satisfies (2) for some W_- satisfying Assumption 1. The data (U_-, X) are called *informative for quadratic stabilization* if there exists a feedback gain K and a matrix $P = P^\top > 0$ such that

$$P - (A + BK)P(A + BK)^\top > 0 \quad (6)$$

for all $(A, B) \in \Sigma$.

Note that, in particular, we are interested in *quadratic stabilization*, and we ask for a *common* Lyapunov matrix P for all $(A, B) \in \Sigma$. We will not treat (A, B) -dependent Lyapunov matrices in this article, but consider this case for future work instead.

Definition 3 leads to two natural problems. First, we are interested in the question under which conditions the data are informative. We formalize this in the following problem.

Problem 1 (Informativity): Assume that (U_-, X) satisfies (2) for some W_- satisfying Assumption 1. Find necessary and sufficient conditions under which the data (U_-, X) are informative for quadratic stabilization.

The second problem is a design issue: we are interested in procedures to come up with a feedback that stabilizes all systems in Σ .

Problem 2 (Control design): Assume that (U_-, X) satisfies (2) for some W_- satisfying Assumption 1. If the data (U_-, X) are informative for quadratic stabilization, find a stabilizing feedback gain K such that (6) is satisfied for all $(A, B) \in \Sigma$.

In addition to data-driven stabilization, we are also interested in including performance specifications. Natural extensions to Problems 1 and 2 will be discussed in Section V.

C. Our Approach

In what follows, we will outline our strategy for solving Problems 1 and 2. Let $(A, B) \in \Sigma$ and rewrite (5) as

$$W_- = X_+ - AX_- - BU_- \quad (7)$$

Recall that by Assumption 1, we have

$$\begin{bmatrix} I \\ W_-^\top \end{bmatrix}^\top \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I \\ W_-^\top \end{bmatrix} \geq 0.$$

By substitution of (7), this yields

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} \geq 0. \quad (8)$$

This shows that A and B satisfy a QMI of the form (8).² In fact, the set Σ of all systems explaining the data can be equivalently characterized in terms of (8), as asserted in the following lemma.

Lemma 4: We have that $\Sigma = \{(A, B) \mid (8) \text{ is satisfied}\}$.

Proof: Suppose that $(A, B) \in \Sigma$. Then, (7) is satisfied for some W_- satisfying (3). This means that (8) holds. As such

$$\Sigma \subseteq \{(A, B) \mid (8) \text{ is satisfied}\}.$$

To prove the reverse inclusion, let (A, B) be such that (8) is satisfied. Define $W_- := X_+ - AX_- - BU_-$. By (8), W_- satisfies the assumption (3). Since (5) holds for (A, B) by construction, we conclude that $(A, B) \in \Sigma$. ■

By Lemma 4, the set Σ of systems explaining the data is characterized by a QMI in (A, B) . Next, we turn our attention to the design condition (6). Suppose that we fix³ a Lyapunov matrix $P = P^\top > 0$ and a feedback gain K . Note that inequality (6) is equivalent to

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0 \quad (9)$$

which is yet another QMI in A and B . Therefore, Problem 1 essentially boils down to understanding under which conditions the QMI (9) holds for all (A, B) satisfying the QMI (8). Data-driven stabilization, thus, naturally leads to the following fundamental question:

When does one QMI imply another QMI?

The familiar reader will immediately recognize the similarity between the above question and the statement of the so-called *S-lemma* [2]. In fact, the S-lemma provides conditions under which the nonnegativity of one quadratic function implies that of another one. This motivates the following section, in which we generalize the S-lemma to matrix variables.

III. MATRIX-VALUED S-LEMMA

In this section, we present a new S-lemma with matrix variables. Before we do so, we provide a brief recap on the classical S-lemma.

A. Recap of the Classical S-Lemma

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *quadratic* if it can be written in the form

$$f(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^\top \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \quad (10)$$

²We note that quadratic uncertainty descriptions have also arisen in the papers [54]–[56] studying data-driven control under the assumption that w is a normally distributed process noise.

³We make this hypothesis purely to explain the ideas behind our approach. In fact, in Section IV, we show how P and K can be computed from data.

for some $M_{11} \in \mathbb{R}$, $M_{12} \in \mathbb{R}^{1 \times n}$, and $M_{22} = M_{22}^\top \in \mathbb{R}^{n \times n}$. A homogeneous quadratic function of the form $f(x) = x^\top M_{22} x$ is called a *quadratic form*. The following theorem describes the celebrated S-lemma, proven by Yakubovich [1] (see also [2, Th. 2.2]).

Theorem 5 (S-lemma): Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be quadratic functions. Suppose that there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) > 0$. Then, $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ such that $g(x) \geq 0$ if and only if there exists a scalar $\alpha \geq 0$ such that

$$f(x) - \alpha g(x) \geq 0 \quad \forall x \in \mathbb{R}^n. \quad (11)$$

We note that the functions f and g are not assumed to be convex. As such, it appears to be difficult to check the condition $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ satisfying $g(x) \geq 0$. The importance of the S-lemma lies in the fact that the characterization (11) of this condition is equivalent to an LMI

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} - \alpha \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix} \geq 0$$

in the scalar variable $\alpha \geq 0$. Here, the matrices $N_{11} \in \mathbb{R}$, $N_{12} \in \mathbb{R}^{1 \times n}$, and $N_{22} \in \mathbb{R}^{n \times n}$ define the quadratic function g analogous to (10).

The scalar α is called a *multiplier*, and the assumption $g(\bar{x}) > 0$ for some $\bar{x} \in \mathbb{R}^n$ is often referred to as the *Slater condition*. This assumption is necessary in the sense that Theorem 5 is false without it. To show this by means of an example, one can take, e.g., $f(x) = x^\top A x$ and $g(x) = -x^\top B x$ with A and B as in the example of [57, p. 4476]. A version of the S-lemma, where g satisfies a strict inequality, has been presented in [2, Th. 7.8]. We will reformulate the result in the following theorem.

Theorem 6 (Strict S-lemma): Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be quadratic forms. Suppose that there exists an $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) > 0$. Then, $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ such that $g(x) > 0$ if and only if there exists a scalar $\alpha \geq 0$ such that

$$f(x) - \alpha g(x) \geq 0 \quad \forall x \in \mathbb{R}^n.$$

Note that Theorem 6 is stated with two multipliers in [2, Th. 7.8]. However, the inclusion of the Slater condition allows us to state Theorem 6 with a single multiplier α .

B. S-Lemma With Matrix Variables

Next, we aim at generalizing Theorems 5 and 6 to quadratic functions of the form

$$\begin{bmatrix} I \\ X \end{bmatrix}^\top \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \text{ and } \begin{bmatrix} I \\ X \end{bmatrix}^\top \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} \quad (12)$$

where $X \in \mathbb{R}^{n \times k}$ is a *matrix variable*, and the partitioned matrices $M, N \in \mathbb{R}^{(k+n) \times (k+n)}$ are real and symmetric. As our first step, the following theorem provides an S-lemma for homogeneous quadratic functions of the form $X^\top M X$ and $X^\top N X$. Naturally, instead of the nonnegativity of functions in the classical S-lemma, we now consider the positive (semi)definiteness of quadratic functions of matrix variables.

Theorem 7 (Homogeneous matrix S-lemma): Let $M, N \in \mathbb{R}^{n \times n}$ be symmetric matrices and assume that $\bar{X}^\top N \bar{X} > 0$ for some $\bar{X} \in \mathbb{R}^{n \times k}$. The following statements are equivalent.

- i) $X^\top MX \geq 0$ for all $X \in \mathbb{R}^{n \times k}$ such that $X^\top NX \geq 0$.
- ii) $X^\top MX \geq 0$ for all $X \in \mathbb{R}^{n \times k}$ such that $X^\top NX > 0$.
- iii) There exists a scalar $\alpha \geq 0$ such that $M - \alpha N \geq 0$.

Remark 8: The assumption on the existence of \bar{X} such that $\bar{X}^\top N \bar{X} > 0$ is a natural generalization of the Slater condition in Theorems 5 and 6. The assumption is again necessary in the sense that Theorem 7 is false without it. Nonetheless, it can be shown that the assumption can be weakened if one is interested only in the equivalence of (i) and (iii). In fact, one can show using similar arguments as in the proof of Theorem 7 that (i) \iff (iii) under the assumption that $\exists \bar{x} \in \mathbb{R}^n$ such that $\bar{x}^\top N \bar{x} > 0$, i.e., under the ‘‘standard’’ Slater condition.

Proof of Theorem 7: It is clear that (i) \implies (ii) and (iii) \implies (i). As such, it suffices to prove the implication (ii) \implies (iii). To this end, suppose that (ii) holds. Let $x \in \mathbb{R}^n$ be such that $x^\top Nx > 0$. We want to prove that $x^\top Mx \geq 0$ so that we can apply Theorem 6. Choose a vector $v \in \mathbb{R}^k$ such that $\|v\| = 1$. Next, we define the matrix $X \in \mathbb{R}^{n \times k}$ as $X := \epsilon \bar{X} + xv^\top$ for $\epsilon \neq 0$. Clearly, $X^\top NX$ is equal to

$$\epsilon^2 \bar{X}^\top N \bar{X} + \epsilon (\bar{X}^\top N xv^\top + vx^\top N \bar{X}) + (x^\top Nx)vv^\top.$$

We claim that $X^\top NX$ is positive definite for ϵ sufficiently small. To prove this claim, first, suppose that $y \in \mathbb{R}^k$ is nonzero and $v^\top y = 0$. Then, we obtain

$$y^\top X^\top NX y = \epsilon^2 y^\top \bar{X}^\top N \bar{X} y > 0.$$

Second, suppose that $y \in \mathbb{R}^k$ is nonzero and $v^\top y =: \beta \neq 0$. Then, $y^\top X^\top NX y$ is equal to

$$y^\top (\epsilon^2 \bar{X}^\top N \bar{X} + \epsilon (\bar{X}^\top N xv^\top + vx^\top N \bar{X})) y + (x^\top Nx)\beta^2$$

which is positive for ϵ sufficiently small since $\beta \neq 0$ and $x^\top Nx > 0$. We conclude that $X^\top NX > 0$ for ϵ sufficiently small. Now, by (ii), we conclude that $X^\top MX \geq 0$. Multiplication of the latter inequality from left by v^\top and right by v yields the inequality

$$\epsilon^2 v^\top \bar{X}^\top M \bar{X} v + \epsilon (v^\top \bar{X}^\top Mx + x^\top M \bar{X} v) + x^\top Mx \geq 0. \quad (13)$$

This implies that $x^\top Mx \geq 0$. Indeed, if $x^\top Mx < 0$, then there exists a sufficiently small $\epsilon \neq 0$ such that

$$\epsilon^2 v^\top \bar{X}^\top M \bar{X} v + \epsilon (v^\top \bar{X}^\top Mx + x^\top M \bar{X} v) + x^\top Mx < 0$$

which contradicts (13). To conclude, we have shown that $x^\top Mx \geq 0$ for all $x \in \mathbb{R}^n$ such that $x^\top Nx > 0$. By Theorem 6, condition (iii) is satisfied. This proves the theorem. \blacksquare

Next, we build on Theorem 7 by introducing a general (inhomogeneous) S-lemma with matrix variables. The following theorem is one of the main results of this section.

Theorem 9 (Matrix S-lemma): Let $M, N \in \mathbb{R}^{(k+n) \times (k+n)}$ be symmetric matrices, and assume that there exists some matrix $\bar{Z} \in \mathbb{R}^{n \times k}$ such that

$$\begin{bmatrix} I \\ \bar{Z} \end{bmatrix}^\top N \begin{bmatrix} I \\ \bar{Z} \end{bmatrix} > 0. \quad (14)$$

Then, the following statements are equivalent:

- I) $\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0 \forall Z \in \mathbb{R}^{n \times k}$ with $\begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0$.
- II) $\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0 \forall Z \in \mathbb{R}^{n \times k}$ with $\begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} > 0$.
- III) There exists a scalar $\alpha \geq 0$ such that $M - \alpha N \geq 0$.

Note that for $k = 1$, assumption (14) reduces to the standard Slater condition. In this case, Theorem 9 recovers Theorems 5 and 6 in the following sense: the equivalence of (I) and (III) is the statement of Theorem 5. The equivalence of (II) and (III) generalizes Theorem 6 for quadratic forms to general quadratic functions.

Proof of Theorem 9: Clearly, (I) \implies (II) and (III) \implies (I). Thus, it suffices to prove that (II) \implies (III). Our strategy will be to show that (II) implies statement (ii) of Theorem 7. To this end, suppose that (II) holds, and let $X \in \mathbb{R}^{(k+n) \times k}$ be such that $X^\top NX > 0$. Partition X as

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1 \in \mathbb{R}^{k \times k}$ and $X_2 \in \mathbb{R}^{n \times k}$. Clearly, for all sufficiently small $\epsilon > 0$, we have

$$\begin{bmatrix} X_1 + \epsilon I \\ X_2 \end{bmatrix}^\top N \begin{bmatrix} X_1 + \epsilon I \\ X_2 \end{bmatrix} > 0.$$

Also note that $X_1 + \epsilon I$ is nonsingular for all sufficiently small $\epsilon > 0$. This implies that

$$\begin{bmatrix} I \\ X_2(X_1 + \epsilon I)^{-1} \end{bmatrix}^\top N \begin{bmatrix} I \\ X_2(X_1 + \epsilon I)^{-1} \end{bmatrix} > 0.$$

By (II), we have

$$\begin{bmatrix} I \\ X_2(X_1 + \epsilon I)^{-1} \end{bmatrix}^\top M \begin{bmatrix} I \\ X_2(X_1 + \epsilon I)^{-1} \end{bmatrix} \geq 0$$

equivalently

$$\begin{bmatrix} X_1 + \epsilon I \\ X_2 \end{bmatrix}^\top M \begin{bmatrix} X_1 + \epsilon I \\ X_2 \end{bmatrix} \geq 0$$

for all $\epsilon > 0$ sufficiently small. By taking the limit $\epsilon \downarrow 0$, we conclude that $X^\top MX \geq 0$. Therefore, statement (ii) [equivalently, statement (iii)] of Theorem 7 is satisfied. This means that (III) holds, which proves the theorem. \blacksquare

As a special case of Theorem 9, we recover [46, Th. 3.3].

Corollary 10: The QMI $M_{11} + M_{12}Z + Z^\top M_{12}^\top + Z^\top M_{22}Z \geq 0$ holds for all $Z \in \mathbb{R}^{n \times k}$ satisfying $I - Z^\top DZ \geq 0$ if and only if there exists a scalar $\alpha \geq 0$ such that

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} - \alpha \begin{bmatrix} I & 0 \\ 0 & -D \end{bmatrix} \geq 0.$$

Proof: Note that the generalized Slater condition (14) is satisfied (one can choose e.g., $\bar{Z} = 0$). Thus, the statement follows from Theorem 9. \blacksquare

Theorem 9 provides a natural generalization of the S-lemma to matrix variables. However, note that for the application that we have in mind, we need a slightly different version of the theorem. Indeed, note that in the data-driven context of Section II, a *strict* inequality (9) must hold for all (A, B) satisfying a nonstrict inequality (8). As such, we need to extend Theorem 9 to the case when the inequality involving M is *strict*. Before we do so, we introduce the shorthand notation

$$\mathcal{S}_N := \left\{ Z \in \mathbb{R}^{n \times k} \mid \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0 \right\}.$$

Theorem 11 (Strict matrix S-lemma): Let M and N be symmetric matrices in $\mathbb{R}^{(k+n) \times (k+n)}$. Assume that \mathcal{S}_N is bounded and that there exists some matrix $\bar{Z} \in \mathbb{R}^{n \times k}$ satisfying (14). Then, we have that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ for all } Z \in \mathcal{S}_N \quad (15)$$

if and only if there exists $\alpha \geq 0$ such that $M - \alpha N > 0$.

Proof: The “if” part is clear, so we focus on proving the “only if” part. Suppose that (15) holds. We claim that there exists an $\epsilon > 0$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top (M - \epsilon I) \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ for all } Z \in \mathcal{S}_N. \quad (16)$$

Suppose that this is not the case. Then, there exists a sequence $\{\epsilon_i\}$ with $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$ with the property that for each i , there exists $Z_i \in \mathcal{S}_N$ such that

$$\begin{bmatrix} I \\ Z_i \end{bmatrix}^\top (M - \epsilon_i I) \begin{bmatrix} I \\ Z_i \end{bmatrix} \not> 0.$$

Since \mathcal{S}_N is bounded, the sequence $\{Z_i\}$ is bounded. As such, by the Bolzano–Weierstrass theorem, it contains a converging subsequence with limit, say, Z^* . We conclude that

$$\begin{bmatrix} I \\ Z^* \end{bmatrix}^\top M \begin{bmatrix} I \\ Z^* \end{bmatrix} \not> 0.$$

Note that \mathcal{S}_N is closed, and thus, $Z^* \in \mathcal{S}_N$. Since (15) holds, we arrive at a contradiction. Therefore, we conclude that there exists an $\epsilon > 0$ such that (16) holds. In particular, this implies the existence of $\epsilon > 0$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top (M - \epsilon I) \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0 \text{ for all } Z \in \mathcal{S}_N.$$

Now, by Theorem 9, there exists an $\alpha \geq 0$ such that

$$(M - \epsilon I) - \alpha N \geq 0.$$

We conclude that $M - \alpha N > 0$, which proves the theorem. ■

At this point, it is worthwhile to point out a relation between Theorem 11 and the literature on LMI relaxations in robust control (see [51]–[53]). In fact, we can derive a type of matrix S-lemma from the general theory in [51]. As it turns out, this

matrix S-lemma is also a corollary of Theorem 11. To obtain the result, we substitute $A = 0$, $B = I$, $W(x) = -M$, and

$$\Delta = \left\{ \Delta \mid \begin{bmatrix} \Delta \\ I \end{bmatrix}^\top N \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0 \right\}$$

into [51, eq. (1.2)]. Then, we combine the full block S-procedure (cf. [49], [51, p. 367]) with the fact that the LMI relaxation of [51] is exact for a single full block [51, Th. 5.3]. This yields the following result.

Corollary 12: Let $M, N \in \mathbb{R}^{(k+n) \times (k+n)}$ be symmetric matrices, partitioned as in (12). Assume that N is nonsingular, $N_{11} \geq 0$, and $N_{22} < 0$. Then, we have that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ for all } Z \in \mathcal{S}_N$$

if and only if there exists $\alpha \geq 0$ such that $M - \alpha N > 0$.

Proof: We will show that the assumptions on N imply the assumptions of Theorem 11. First of all, $N_{22} < 0$ implies that \mathcal{S}_N is bounded. Second, the nonsingularity of N and N_{22} implies that the Schur complement $N_{11} - N_{12}N_{22}^{-1}N_{12}^\top$ is nonsingular, and since $N_{11} \geq 0$ and $N_{22} < 0$, we have $N_{11} - N_{12}N_{22}^{-1}N_{12}^\top > 0$. Thus, the matrix $\bar{Z} := -N_{22}^{-1}N_{12}^\top$ satisfies the Slater condition (14). The result now follows from Theorem 11. ■

It turns out that we can even state Theorem 11 without the boundedness assumption if some more structure on the matrices M and N is given. In fact, we have the following result.

Theorem 13: Let $M, N \in \mathbb{R}^{(k+n) \times (k+n)}$ be symmetric matrices, partitioned as in (12). Assume that $M_{22} \leq 0$, $N_{22} \leq 0$, and $\ker N_{22} \subseteq \ker N_{12}$. Suppose that there exists some matrix $\bar{Z} \in \mathbb{R}^{n \times k}$ satisfying (14). Then, we have that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \text{ for all } Z \in \mathcal{S}_N \quad (17)$$

if and only if there exist $\alpha \geq 0$ and $\beta > 0$ such that

$$M - \alpha N \geq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: The “if” part is clear, so we focus on proving the “only if” statement. Suppose that (17) holds. We will first prove that $\ker N_{22} \subseteq \ker M_{22}$ and $\ker N_{22} \subseteq \ker M_{12}$. Let $Z \in \mathcal{S}_N$ and $\hat{Z} \in \mathbb{R}^{n \times k}$ be such that $N_{22}\hat{Z} = 0$. By the hypothesis $\ker N_{22} \subseteq \ker N_{12}$ we have $Z + \gamma\hat{Z} \in \mathcal{S}_N$ for any $\gamma \in \mathbb{R}$. Thus, we obtain

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} + \gamma(M_{12}\hat{Z} + (M_{12}\hat{Z})^\top) + \gamma^2\hat{Z}^\top M_{22}\hat{Z} > 0. \quad (18)$$

This implies that $M_{22}\hat{Z} = 0$. Indeed, recall that $M_{22} \leq 0$. Thus, if $M_{22}\hat{Z} \neq 0$, then there exists a sufficiently large γ such that (18) is violated. Similarly, we conclude that $M_{12}\hat{Z} = 0$. Therefore, we have shown that

$$\ker N_{22} \subseteq \ker M_{22} \text{ and } \ker N_{22} \subseteq \ker M_{12}. \quad (19)$$

Subsequently, we claim that there exists a $\beta > 0$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \left(M - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \quad \text{for all } Z \in \mathcal{S}_N. \quad (20)$$

If this claim is not true, then there exists a sequence $\{\beta_i\}$ such that $\beta_i \rightarrow 0$ and for all i there exists $Z_i \in \mathcal{S}_N$ such that

$$\begin{bmatrix} I \\ Z_i \end{bmatrix}^\top \left(M - \begin{bmatrix} \beta_i I & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ Z_i \end{bmatrix} \not\geq 0. \quad (21)$$

Define $\mathcal{V} := \{Z \in \mathbb{R}^{n \times k} \mid N_{22}Z = 0\}$. Write Z_i as $Z_i = Z_i^1 + Z_i^2$, where $Z_i^1 \in \mathcal{V}^\perp$ and $Z_i^2 \in \mathcal{V}$. By the hypothesis $\ker N_{22} \subseteq \ker N_{12}$, we see that $Z_i^1 \in \mathcal{S}_N$. Next, we claim that the sequence $\{Z_i^1\}$ is bounded. We will prove this claim by contradiction. Thus, suppose that $\{Z_i^1\}$ is unbounded. Clearly, the sequence

$$\left\{ \frac{Z_i^1}{\|Z_i^1\|} \right\}$$

is bounded. By the Bolzano–Weierstrass theorem, it thus has a convergent subsequence with limit, say Z_* . Note that

$$\frac{1}{\|Z_i^1\|^2} (N_{11} + N_{12}Z_i^1 + (N_{12}Z_i^1)^\top + (Z_i^1)^\top N_{22}Z_i^1) \geq 0.$$

By taking the limit along the subsequence as $i \rightarrow \infty$, we get $Z_*^\top N_{22}Z_* \geq 0$. Using the fact that $N_{22} \leq 0$, we conclude that $Z_* \in \mathcal{V}$. Since $Z_i^1 \in \mathcal{V}^\perp$ for all i , also $\frac{Z_i^1}{\|Z_i^1\|} \in \mathcal{V}^\perp$, and thus, $Z_* \in \mathcal{V}^\perp$. Therefore, we conclude that both $Z_* \in \mathcal{V}$ and $Z_* \in \mathcal{V}^\perp$, i.e., $Z_* = 0$. This is a contradiction since $\frac{Z_i^1}{\|Z_i^1\|}$ has norm 1 for all i . We conclude that the sequence $\{Z_i^1\}$ is bounded. It, thus, contains a convergent subsequence with limit, say Z_∞ . Note that \mathcal{S}_N is closed, and thus, $Z_\infty \in \mathcal{S}_N$. By (19) and (21), we conclude that

$$\begin{bmatrix} I \\ Z_i^1 \end{bmatrix}^\top \left(M - \begin{bmatrix} \beta_i I & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ Z_i^1 \end{bmatrix} \not\geq 0$$

for all i . We take the limit as $i \rightarrow \infty$, which yields

$$\begin{bmatrix} I \\ Z_\infty \end{bmatrix}^\top M \begin{bmatrix} I \\ Z_\infty \end{bmatrix} \not\geq 0.$$

As $Z_\infty \in \mathcal{S}_N$ this contradicts (17). As such, we conclude that there exists $\beta > 0$ such that (20) holds. In particular, there exists $\beta > 0$ such that

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top \left(M - \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0 \quad \text{for all } Z \in \mathcal{S}_N.$$

The theorem now follows by application of Theorem 9. \blacksquare

IV. DATA-DRIVEN STABILIZATION REVISITED

In this section, we apply the theory from Section III to data-driven stabilization, i.e., to Problems 1 and 2 defined in Section II. To this end, for given $P = P^\top > 0$ and K , we define the partitioned matrices

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} := \begin{bmatrix} P & 0 & 0 \\ 0 & -P & -PK^\top \\ 0 & -KP & -KPK^\top \end{bmatrix} \quad (22)$$

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{12}^\top & N_{22} \end{bmatrix} := \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ -0 & -X_- \\ 0 & -U_- \end{bmatrix}^\top. \quad (23)$$

Recall from Section II that data-driven stabilization entails deciding whether (9) holds for all (A, B) satisfying (8). In terms of the matrices M and N as defined above, we thus have to decide whether

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} > 0 \quad \forall Z \in \mathbb{R}^{(n+m) \times n} \quad \text{with} \quad \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \geq 0. \quad (24)$$

Here, Z is given by

$$Z := \begin{bmatrix} A^\top \\ B^\top \end{bmatrix}.$$

The idea is now to apply Theorem 13. To this end, we have to verify its assumptions. In particular, we will check that $M_{22} \leq 0$, $N_{22} \leq 0$ and $\ker N_{22} \subseteq \ker N_{12}$. Note that

$$M_{22} = - \begin{bmatrix} I \\ K \end{bmatrix} P \begin{bmatrix} I \\ K \end{bmatrix}^\top \leq 0, \quad N_{22} = \begin{bmatrix} X_- \\ U_- \end{bmatrix} \Phi_{22} \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \leq 0$$

because $P > 0$ and $\Phi_{22} < 0$. Since Φ_{22} is nonsingular, we also see that

$$\ker N_{22} = \ker \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top$$

$$\ker N_{12} = \ker \left((\Phi_{12} + X_+ \Phi_{22}) \begin{bmatrix} X_- \\ U_- \end{bmatrix}^\top \right)$$

and thus $\ker N_{22} \subseteq \ker N_{12}$. We conclude that the assumptions of Theorem 13 are satisfied. We assume that the generalized Slater condition (14) holds for N in (23). Then, Theorem 13 asserts that (24) holds if and only if there exist scalars $\alpha \geq 0$

$$\begin{bmatrix} P - \beta I & 0 & 0 & 0 \\ 0 & -P & -L^\top & 0 \\ 0 & -L & 0 & L \\ 0 & 0 & L^\top & P \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \end{bmatrix}^\top \geq 0. \quad (\text{FS})$$

and $\beta > 0$ such that

$$M - \alpha N \geq \begin{bmatrix} \beta I & 0 \\ 0 & 0 \end{bmatrix}. \quad (25)$$

From a design point of view, the matrices P and K that appear in M are not given. However, the idea is now to *compute* matrices P , K and scalars α and β such that (25) holds. In fact, by the above discussion, the data (U_-, X) are informative for quadratic stabilization *if and only if* there exists an $n \times n$ matrix $P = P^\top > 0$, a $K \in \mathbb{R}^{m \times n}$, and two scalars $\alpha \geq 0$ and $\beta > 0$ such that (25) holds. We note that (25) (in particular, M) is not linear in P and K . Nonetheless, by a rather standard change of variables and a Schur complement argument, we can transform (25) into an LMI. We summarize our progress in the following theorem, which is the main result of this section.

Theorem 14: Assume that the generalized Slater condition (14) holds for N in (23) and some $Z \in \mathbb{R}^{(n+m) \times n}$. Then, the data (U_-, X) are informative for quadratic stabilization if and only if there exists an $n \times n$ matrix $P = P^\top > 0$, an $L \in \mathbb{R}^{m \times n}$, and scalars $\alpha \geq 0$ and $\beta > 0$ satisfying (FS) shown at the bottom of the previous page.

Moreover, if P and L satisfy (FS), then $K := LP^{-1}$ is a stabilizing feedback gain for all $(A, B) \in \Sigma$.

Proof: To prove the “if” statement, suppose that there exist P , L , α , and β satisfying (FS). Define $K := LP^{-1}$. By computing the Schur complement of (FS) with respect to its fourth diagonal block, we obtain (25). As such, (24) holds. We conclude that the data (U_-, X) are informative for quadratic stabilization, and $K = LP^{-1}$ is indeed a stabilizing controller for all $(A, B) \in \Sigma$.

Conversely, to prove the “only if” statement, suppose that the data (U_-, X) are informative for quadratic stabilization. This means that there exist $P = P^\top > 0$ and K such that (24) holds. By Theorem 13, there exist $\alpha \geq 0$ and $\beta > 0$ satisfying (25). Finally, by defining $L := KP$ and using a Schur complement argument, we conclude that (FS) is feasible. ■

Theorem 14 provides a powerful necessary *and* sufficient condition under which quadratically stabilizing controllers can be obtained from noisy data. The assumption (14) puts a mild condition on the data matrices appearing in (23). It is satisfied whenever N has at least n positive eigenvalues, a condition that is simple to verify from given data. So far, this condition was satisfied in all of our numerical experiments; see Section VI for more details.⁴ Theorem 14 leads to an effective design procedure for obtaining stabilizing controllers directly from data. Indeed, the approach entails solving the LMI (FS) for P , L , α , and β and computing a controller as $K = LP^{-1}$. We now discuss some of the features of our control design procedure.

- 1) First of all, we note that the procedure is *nonconservative* since Theorem 14 provides a necessary and sufficient condition for obtaining quadratically stabilizing controllers from data.
- 2) We believe that our approach based on the set Σ of *open-loop* systems provides a valuable alternative to the data-based closed-loop system parameterizations of [37,

Th. 2] and [39, Th. 4]. Indeed, in the case of noisy data, it was recognized that certain linear constraints [39, eq. (3)] defining these closed-loop systems were difficult to incorporate in the control design.⁵ Our design procedure does not suffer from the above problem. In fact, the constraint [39, eq. (3)] is *automatically incorporated* in our control design approach.

- 3) The variables P , L , α , and β are *independent* of the time horizon T of the experiment. In fact, note that $P \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{m \times n}$, and $\alpha, \beta \in \mathbb{R}$. Also, the LMI (FS) is of dimension $(3n + m) \times (3n + m)$ and thus independent of T . As such, our approach fundamentally differs from the design methods in [37] and [39], where certain decision variables have dimension $T \times n$; cf. [37, Th. 6] and [39, Cor. 6]. We believe that our T -independent design method will play a crucial role in control design from larger datasets. We note that the collection of big datasets is often unavoidable, for example, because the signal-to-noise ratio is small, or because the data-generating system is large scale.

Remark 15: We note that under the extra assumption

$$\text{rank} \begin{bmatrix} X_- \\ U_- \end{bmatrix} = n + m \quad (26)$$

it is possible to prove a variant Theorem 14, in which the nonstrict inequality is replaced by a strict inequality, and the term $-\beta I$ is removed. This can be done by invoking Theorem 11, which is possible since (26) implies that the set Σ is bounded. The reason is that the coefficient matrix N_{22} defining the quadratic term in (8) is negative definite if (26) holds.

Here, we chose to state and prove Theorem 14 in the slightly more general setting without assuming (26). In the discussion preceding Theorem 14, we have verified the assumptions of Theorem 13 for M and N in (22) and (23). In particular, this implies that the subspace inclusions (19) hold, and thus, $\ker[X_-^\top \ U_-^\top] \subseteq \ker[I \ K^\top]$, equivalently

$$\text{im} \begin{bmatrix} I \\ K \end{bmatrix} \subseteq \text{im} \begin{bmatrix} X_- \\ U_- \end{bmatrix}. \quad (27)$$

Therefore, any controller K that stabilizes the systems in Σ is necessarily of the form (27). This generalizes [3, Lem. 15] to the case of noisy data.

V. INCLUSION OF PERFORMANCE SPECIFICATIONS

In this section, we extend our data-driven stabilization result by including different performance specifications. In particular, we will treat the \mathcal{H}_2 and \mathcal{H}_∞ control problems, thereby illustrating the general applicability of the theory in Section III.

A. \mathcal{H}_2 Control

As before, consider the unknown system (1). We associate to (1) a performance output

$$z(t) = Cx(t) + Du(t) \quad (28)$$

⁴In addition, we remark that even if the generalized Slater condition does not hold, the “if” statement of Theorem 14 remains true.

⁵In fact, it was mentioned in [58] that involving the condition [39, eq. (3)] in design procedures is still an open problem.

inequalities of (\mathcal{H}_ϵ) are satisfied by definition of Z . In addition, we know that (32) and (33) hold for all $(A, B) \in \Sigma$. By (32), the second LMI of (\mathcal{H}_ϵ) is satisfied. To prove that the first LMI of (\mathcal{H}_ϵ) also holds, we want to apply Theorem 13. Note that we have already verified the assumptions of this theorem for the matrix N in (23); see the discussion preceding Theorem 14. In addition, we note that

$$M_{22} = - \begin{bmatrix} Y \\ L \end{bmatrix} (Y - C_{Y,L}^\top C_{Y,L})^{-1} \begin{bmatrix} Y \\ L \end{bmatrix}^\top \leq 0$$

since $Y - C_{Y,L}^\top C_{Y,L} > 0$. Hence, Theorem 13 is applicable. We conclude that there exist $\alpha \geq 0$ and $\beta > 0$ such that (35) holds. Using a Schur complement argument, we see that Y , L , α , and β satisfy the first LMI of (\mathcal{H}_ϵ) . Thus, (\mathcal{H}_ϵ) is feasible, which proves the theorem. \blacksquare

Remark 18: If we know *a priori* that the noise w is contained in a subspace, say $\text{im } E$, then this information can easily be exploited in the \mathcal{H}_2 controller design. In fact, we only need to replace the LMI involving Z by

$$\begin{bmatrix} Z & E^\top \\ E & Y \end{bmatrix} \geq 0.$$

We recall that prior knowledge of $w \in \text{im } E$, if available, can also be captured by our noise model (see Remark 2). A natural choice is thus to use E both in the noise model (3) and in the LMI (\mathcal{H}_ϵ) . However, we remark that this is not necessary: the noise in the experiment may come from a different subspace than the disturbances that are attenuated by the \mathcal{H}_2 controller.

B. \mathcal{H}_∞ Control

In this section, we will turn our attention to the \mathcal{H}_∞ control problem. As before, consider system (1) with performance output (28). For any $(A, B) \in \Sigma$, the feedback $u = Kx$ yields the system (29) with transfer matrix from w to z given by $G(z)$. We will denote the \mathcal{H}_∞ norm of $G(z)$ by $\|G(z)\|_{\mathcal{H}_\infty}$. Let $\gamma > 0$. By [59, Th. 4.6.6(iii)], the matrix $A + BK$ is stable and $\|G(z)\|_{\mathcal{H}_\infty} < \gamma$ if and only if there exists a matrix $P = P^\top > 0$ such that

$$P - A_K^\top \left(P^{-1} - \frac{1}{\gamma^2} I \right)^{-1} A_K - C_K^\top C_K > 0 \quad (36)$$

$$P^{-1} - \frac{1}{\gamma^2} I > 0 \quad (37)$$

where we have defined $A_K := A + BK$ and $C_K := C + DK$. We now have the following definition of informativity for \mathcal{H}_∞ control.

Definition 19: Assume that (U_-, X) satisfies (2) for some W_- satisfying Assumption 1. The data (U_-, X) are *informative for \mathcal{H}_∞ control* with performance γ if there exist matrices $P = P^\top > 0$ and K such that (36) and (37) hold for all $(A, B) \in \Sigma$.

By pre- and postmultiplication of (36) by P^{-1} , we obtain

$$Y - A_{Y,L}^\top \left(Y - \frac{1}{\gamma^2} I \right)^{-1} A_{Y,L} - C_{Y,L}^\top C_{Y,L} > 0$$

$$Y - \frac{1}{\gamma^2} I > 0$$

where the matrices $Y := P^{-1}$, $L := KY$, $A_{Y,L} := AY + BL$, and $C_{Y,L} := CY + DL$ are defined as in the \mathcal{H}_2 problem. Note that the first of these inequalities can again be written in the—by now familiar—form

$$\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} Y - C_{Y,L}^\top C_{Y,L} & 0 \\ 0 & - \begin{bmatrix} Y \\ L \end{bmatrix} Z \begin{bmatrix} Y \\ L \end{bmatrix}^\top \end{bmatrix} \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix} > 0$$

where $Z := (Y - \frac{1}{\gamma^2} I)^{-1}$. We thus have the following theorem.

Theorem 20: Assume that the generalized Slater condition (14) holds for N in (23) and some $\bar{Z} \in \mathbb{R}^{(n+m) \times n}$. Then, the data (U_-, X) are informative for \mathcal{H}_∞ control with performance γ if and only if there exist matrices $Y = Y^\top > 0$ and L , and scalars $\alpha \geq 0$ and $\beta > 0$ satisfying (\mathcal{H}_∞) .

Moreover, if Y and L satisfy (\mathcal{H}_∞) , then $K := LY^{-1}$ is such that $A + BK$ is stable and $\|G(z)\|_{\mathcal{H}_\infty} < \gamma$ for all $(A, B) \in \Sigma$.

The proof of Theorem 20 is based on Theorem 13. It follows similar steps as the proof of Theorem 17 and is, therefore, not reported here.

VI. EXAMPLES

In this section, we illustrate our theoretical results by examples and numerical simulations.

A. Stabilization Using Bounds on the Noise Samples

Consider an unstable system of the form (1) with A_s and B_s given by

$$A_s = \begin{bmatrix} 0.850 & -0.038 & -0.380 \\ 0.735 & 0.815 & 1.594 \\ -0.664 & 0.697 & -0.064 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 1.431 & 0.705 \\ 1.620 & -1.129 \\ 0.913 & 0.369 \end{bmatrix}.$$

$$\begin{bmatrix} Y - \beta I & 0 & 0 & 0 & C_{Y,L}^\top \\ 0 & 0 & 0 & Y & 0 \\ 0 & 0 & 0 & L & 0 \\ 0 & Y & L^\top & Y - \frac{1}{\gamma^2} I & 0 \\ C_{Y,L} & 0 & 0 & 0 & I \end{bmatrix} - \alpha \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^\top & \Phi_{22} \end{bmatrix} \begin{bmatrix} I & X_+ \\ 0 & -X_- \\ 0 & -U_- \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top \geq 0, \quad Y - \frac{1}{\gamma^2} I > 0. \quad (\mathcal{H}_\infty)$$

In this example, we assume that the noise samples $w(t)$ are bounded in norm as $\|w(t)\|_2^2 \leq \epsilon$ for all t . As explained in Section II, we can capture this prior knowledge using the noise model (3) with $\Phi_{11} = T\epsilon I$, $\Phi_{12} = 0$, and $\Phi_{22} = I$. We pick a time horizon of $T = 20$ and draw the entries of the inputs and initial state randomly from a Gaussian distribution with zero mean and unit variance. The noise samples are drawn uniformly at random from the ball $\{w \in \mathbb{R}^3 \mid \|w\|_2^2 \leq \epsilon\}$. We aim at constructing stabilizing controllers from the input/state data for various values of ϵ . In particular, we investigate six different noise levels: $\epsilon \in \{0.5, 1, 1.5, 2, 2.2, 2.4\}$. For each noise level, we generate 100 datasets using the method described above. We check the generalized Slater condition (14) by verifying that N in (23) has three positive eigenvalues; this turns out to be true for all 600 datasets. For each noise level, we record the percentage of datasets from which a stabilizing controller was found for (A_s, B_s) using the formulation (FS). We display the results in the following table:

$\epsilon = 0.5$	$\epsilon = 1$	$\epsilon = 1.5$	$\epsilon = 2$	$\epsilon = 2.2$	$\epsilon = 2.4$
100%	96%	90%	82%	75%	73%

For $\epsilon = 0.5$, we find a stabilizing controller in all 100 cases. When the noise level increases, the percentage of datasets for which the LMI (FS) is feasible decreases. The interpretation is that by increasing the noise we enlarge the set of explaining systems Σ . It thus becomes harder to simultaneously stabilize the systems in Σ . Nonetheless, even for the larger noise level of $\epsilon = 2.4$, we find a stabilizing controller in 73 out of the 100 datasets.

B. \mathcal{H}_2 Control of a Fighter Aircraft

We consider a state-space model of a fighter aircraft [59, Ex. 10.1.2]. In particular, we discretize the model of [59] using a sampling time of 0.01, which results in the (unstable) system of the form (1) with A_s and B_s given by

$$\begin{bmatrix} 1.000 & -0.374 & -0.190 & -0.321 & 0.056 & -0.026 \\ 0.000 & 0.982 & 0.010 & -0.000 & -0.003 & 0.001 \\ 0.000 & 0.115 & 0.975 & -0.000 & -0.269 & 0.191 \\ 0.000 & 0.001 & 0.010 & 1.000 & -0.001 & 0.001 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.741 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.741 \end{bmatrix}$$

$$\begin{bmatrix} 0.007 & 0.000 & -0.043 & 0.000 & 0.259 & 0.000 \\ -0.003 & 0.000 & 0.030 & 0.000 & 0.000 & 0.259 \end{bmatrix}^\top$$

respectively. We consider the performance output as in (28) with

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $D = 0$. First, we look for the smallest γ such that (30) is feasible for (A_s, B_s) . This minimum value of γ is 1.000 and can be regarded as a benchmark: no data-driven method can perform better than the model-based solution using full knowledge of (A_s, B_s) .

Of course, our goal is not to use the knowledge of (A_s, B_s) but to seek a data-driven solution instead. Therefore, we collect $T =$

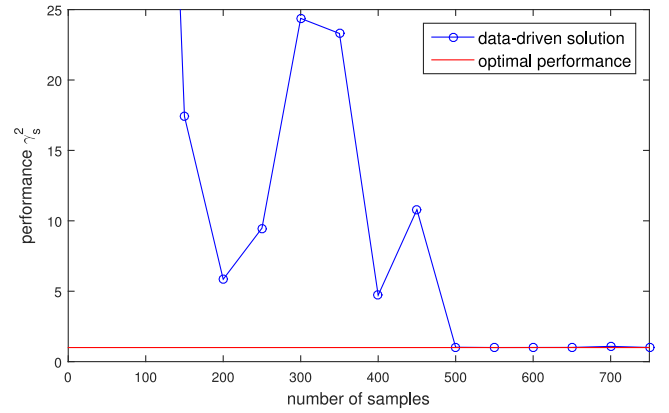


Fig. 1. Achieved \mathcal{H}_2 performance of the true system in feedback with a data-based controller (blue) and the optimal (model-based) performance of the true system (red).

750 input and state samples of (1). The entries of the inputs and initial state were drawn randomly from a Gaussian distribution with zero mean and unit variance. Also, the noise samples were drawn randomly from a Gaussian distribution, with zero mean and variance σ^2 with $\sigma = 0.005$. In this example, we assume knowledge of a bound on the energy of the noise as

$$W_- W_-^\top \leq 1.35 T \sigma^2 I. \quad (38)$$

We verified that this bound is satisfied for the generated noise sequence. In addition, we verified that the matrix N in (23) has six positive eigenvalues; thus, the generalized Slater condition (14) holds.

Next, we want to compute an \mathcal{H}_2 controller for the unknown system using the generated data. We do so by minimizing γ subject to (\mathcal{H}_ϵ) . This is a semidefinite program that we solve in MATLAB, using Yalmip [60] with Mosek as an LMI solver. The obtained controller K stabilizes the original system (A_s, B_s) . In addition, the system, in feedback with K , has an \mathcal{H}_2 norm of γ_s , where $\gamma_s^2 = 1.007$. We note that this is almost identical to the smallest possible \mathcal{H}_2 norm of 1.000.

Subsequently, we repeat the above experiment using only a part of our dataset. In particular, we compute an \mathcal{H}_2 controller via the semidefinite program as before, using only the first i samples of X_+ , X_- , and U_- for $i = 50, 100, \dots, 750$. We display the results in Fig. 1.

In each of the cases, a stabilizing controller was found from data. However, the performance of these controllers when applied to the true system varies and is quite poor for $i < 500$. Starting from $i = 500$ and onward, the performance is close to the optimal performance of the true system.

Next, we investigate what happens when we increase the variance σ^2 of the noise. First, we take $\sigma = 0.05$. We again generate 750 data samples and assume the same bound on the noise. The \mathcal{H}_2 controller achieves a performance of $\gamma_s^2 = 1.146$ when interconnected to the true system. Increasing the variance of the noise has the effect that the set Σ of explaining systems becomes larger. As such, it is more difficult to control all systems in Σ resulting in a slightly larger γ_s . This behavior becomes

even more apparent when increasing the variance of the noise to $\sigma = 0.5$. In this case, we obtain a controller that yields a performance of $\gamma_s^2 = 3.579$. Increasing σ even more to $\sigma = 1$ results in infeasibility of the LMI's (\mathcal{H}_ϵ) for any γ ; the set of explaining systems has become too large for a quadratically stabilizing controller to exist.

We remark that the size of the set Σ does not only depend on the variance of the noise, but also on the available bound on the noise. Throughout this example, we have used the bound (38). However, if we reconsider the case of $\sigma = 0.5$ with the tighter bound $W_- W_-^\top \leq 1.22 T \sigma^2 I$ (which is also satisfied in this example), we obtain a controller with better performance $\gamma_s^2 = 2.706$. This illustrates the simple fact that data-driven controllers not only depend on the particular design strategy, but also on the *prior knowledge* on the noise.

We conclude the example with a remark on the dimension of the variables involved in the formulation (\mathcal{H}_ϵ). The symmetric matrices Y and Z both have 21 free variables. The matrix L contains 12 variables, and α and β are both scalar variables. Thus, the total number of variables is 56. The size of the largest LMI in (\mathcal{H}_ϵ) is 21×21 . We emphasize that our approach is based directly on the set Σ of open-loop systems and avoids the parameterization of closed-loop systems, as employed in [37] and [39]. Such parameterizations involve decision variables of dimension $T \times n$, which would result in at least 4500 variables in this example.

C. Comparison With Related Results

As we have mentioned before, one of the advantages of our approach compared to existing work is that our LMI condition provides necessary and sufficient conditions for data-driven quadratic stabilization. The purpose of this example is to demonstrate that our results can thus lead to stabilizing controllers even in situations where the results from [37] and [39] cannot.

Consider the system (1), where $A_s = 1$ and $B_s = 1$. Suppose that $T = 3$ and the noise matrix is given by

$$W_- = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We collect the data samples

$$X = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\ U_- = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

Throughout the example, we assume that we have access to the noise bound $W_- W_-^\top \leq 1$. Note that this bound is indeed satisfied, and that it can be captured using Assumption 1 by the choices $\Phi_{11} = 1$, $\Phi_{12} = 0$, and $\Phi_{22} = -I$. We also note that by Remark 1, this noise bound can be captured by [39, As. 3] with the choices $Q_w = -1$, $S_w = 0$, and $R_w = I$. Finally, we note that the noise bound can be captured by [37, Asm. 5] with the choice $\gamma = 1$. As such, we can compare the design methods reported in Theorem 14 of this article with the approaches in [39, Cor. 6, Rem. 7] and [37, Th. 6].⁶

⁶We note that [37] studies stabilization in the setting that w represents a bounded nonlinearity. The interpretation of w , however, is not important for

We start by applying Theorem 14 to the data in this example. Note that $X_- = [0 \ 0 \ 1]$ and $X_+ = [0 \ 1 \ 0]$. It can be easily verified that $(P, L, \alpha, \beta) = (0.9, -1.35, 1.1, 0.18)$ is a solution to (FS). In addition, the Slater condition is satisfied in this example. As such, we conclude by Theorem 14 that the controller $K = L/P = -1.5$ is stabilizing for all $(A, B) \in \Sigma$. In particular, we see that the true closed-loop system matrix -0.5 is stable.

We will now investigate the design method of [37, Th. 6]. This approach involves finding a matrix Q and a scalar $\alpha > 0$ such that $X_- Q$ is symmetric and

$$\begin{bmatrix} X_- Q - \alpha X_+ X_+^\top & X_+ Q \\ Q^\top X_+^\top & X_- Q \end{bmatrix} > 0, \quad \begin{bmatrix} I & Q \\ Q^\top & X_- Q \end{bmatrix} > 0 \quad (39)$$

$$\frac{\alpha^2}{4 + 2\alpha} > \gamma \quad (40)$$

where we recall that $\gamma = 1$ in this example. We will now show that (39) and (40) do not have a solution (Q, α) , and thus, the design procedure from [37, Th. 6] cannot find a controller that is guaranteed to stabilize the true system. To see this, note that (40) and $\alpha > 0$ imply $\alpha > 1 + \sqrt{5}$. We write $Q = [q_1 \ q_2 \ q_3]^\top$ and note that by the upper left block of the first matrix in (39), we have $q_3 > \alpha$ and, thus, $q_3 > 1$. Now, by taking the Schur complement of the second matrix in (39) with respect to the upper left block, we obtain $q_3 > q_1^2 + q_2^2 + q_3^2$. However, this inequality cannot be satisfied since $q_3 > 1$. As such, we conclude that (39) and (40) do not have a solution.

Next, we turn our attention to the design procedure of [39, Rem. 7]. For $S_w = 0$, this procedure boils down to finding a solution (\mathcal{Y}, M) to

$$\begin{bmatrix} -\mathcal{Y} & 0 & M^\top X_+^\top & M^\top \\ 0 & Q_w & 1 & 0 \\ X_+ M & 1 & -\mathcal{Y} & 0 \\ M & 0 & 0 & -R_w^{-1} \end{bmatrix} < 0 \quad (41)$$

$$X_- M = \mathcal{Y} \quad (42)$$

where we recall that $Q_w = -1$ and $R_w = I$ in this example. We will now show that (41) and (42) do not have a solution (\mathcal{Y}, M) . To see this, note that (42) implies $M = [m_1 \ m_2 \ \mathcal{Y}]^\top$ with $m_1, m_2 \in \mathbb{R}$. The negative definiteness of the submatrix of (41) consisting of the second and third rows and columns imply $\mathcal{Y} > 1$. Moreover, by inspection of the first and fourth row and column block of (41), we see that $-\mathcal{Y} + M^\top M < 0$ and, thus, $-\mathcal{Y} + m_1^2 + m_2^2 + \mathcal{Y}^2 < 0$. This inequality, however, cannot be satisfied as $\mathcal{Y} > 1$. As such, we see that (41) and (42) do not have a solution (\mathcal{Y}, M) .

We conclude that Theorem 14 can be successfully applied even in situations, in which the design procedures of [39] and [37] do not lead to controllers that are guaranteed to stabilize the true system.

this comparison. In fact, our results (as well as those in [39]) are applicable to general bounded disturbances, hence also to disturbances resulting from bounded nonlinearities.

VII. DISCUSSION AND CONCLUSION

REFERENCES

We have studied the problem of obtaining feedback controllers from noisy data. The essence of our approach has been to formulate data-driven control as the problem of determining when one QMI implies another one. To get a grip on this fundamental question, we have generalized the classical S-lemma [1] to matrix variables. The implication involving quadratic matrix inequalities is thereby *equivalent* to an LMI in a scalar variable. We have established several versions of the matrix S-lemma, for both strict and nonstrict inequalities. These matrix S-lemmas are interesting in their own right and generalize existing S-lemmas [2] as well as a theorem involving quadratic matrix inequalities [46].

We have followed up by applying our matrix S-lemma to data-driven control. In particular, we have given necessary and sufficient conditions under which stabilizing, \mathcal{H}_2 , and \mathcal{H}_∞ controllers can be obtained from noisy data. Our control design revolves around data-guided LMIs, which can be solved efficiently using modern LMI solvers. In addition to being nonconservative, an attractive feature of our design procedure is that decision variables are *independent* of the time horizon of the experiment.

So far, we have only applied the matrix S-lemma involving a strict inequality (see Theorems 11 and 13) to data-driven control. However, we are convinced that also the matrix S-lemma with *nonstrict* inequalities (see Theorem 9) will find applications, for example, in the verification of dissipativity properties from data [58].

The noise model that we have employed is flexible and can describe, e.g., constant disturbances, energy bounded noise, and norm bounds on noise samples. If one is only interested in the latter, however, we expect that more specific control techniques are possible. In fact, analogous to (8), we can write the inequality $w(t)^\top w(t) \leq \epsilon$ as

$$\underbrace{\begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}^\top \begin{bmatrix} I & x(t+1) \\ 0 & -x(t) \\ 0 & -u(t) \end{bmatrix} \begin{bmatrix} \epsilon I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & x(t+1) \\ 0 & -x(t) \\ 0 & -u(t) \end{bmatrix}^\top \begin{bmatrix} I \\ A^\top \\ B^\top \end{bmatrix}}_{:=N_t} \geq 0.$$

In the spirit of the S-procedure, one could thus design a stabilizing controller by computing⁷ matrices $P = P^\top > 0$ and K , and *multiple* nonnegative scalars $\alpha_0, \alpha_1, \dots, \alpha_{T-1}$ such that

$$M - \sum_{t=0}^{T-1} \alpha_t N_t > 0$$

with M given by (22). We will consider norm-bounded noise samples in more detail in future work.

Yet, another idea for future work is to extend the current results for state-feedback design to data-driven dynamic output feedback design. Specifically, it would be interesting to see whether the matrix S-lemmas can be applied to obtain dynamic output feedback controllers from a finite set of noisy *input-output* samples.

⁷This procedure is likely to be conservative, however, since the classical S-lemma is in general conservative for more than two quadratic functions [2].

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Henk J. van Waarde received the master's (*summa cum laude*) and Ph.D. (*cum laude*) degrees in applied mathematics from the University of Groningen, Groningen, The Netherlands, in 2017 and 2020, respectively.

He was a Visiting Researcher with the University of Washington, Seattle, WA USA, in 2019–2020. His research interests include data-driven control, networks of dynamical systems, system identification and identifiability, robust and optimal control, and the geometric theory of linear systems.



M. Kanat Camlibel (Member, IEEE) received the Ph.D. degree in mathematical theory of systems and control from Tilburg University, Tilburg, The Netherlands, in 2001.

He is currently an Associate Professor with the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Groningen, The Netherlands, where he was an Assistant Professor between 2007 and 2013. From 2001 to 2007, he held Postdoctoral/Assistant Professor positions with the University of Groningen, Tilburg University, and Eindhoven Technical University, Eindhoven, The Netherlands. His research interests include differential variational inequalities, complementarity problems, optimization, piecewise affine dynamical systems, switched linear systems, constrained linear systems, multiagent systems, model reduction, geometric theory of linear systems, and data-driven control.



Mehran Mesbahi (Fellow, IEEE) received the Ph.D. degree in electrical engineering from the University of Southern California, Los Angeles, CA, USA, in 1996.

He was a member of the Guidance, Navigation, and Analysis Group, Jet Propulsion Laboratory, Pasadena, CA, from 1996 to 2000, and an Assistant Professor of Aerospace Engineering and Mechanics with the University of Minnesota, Minneapolis, MN, USA, from 2000 to 2002. He is currently a Professor of Aeronautics and Astronautics, an Adjunct Professor of Electrical and Computer Engineering and Mathematics, and the Executive Director of the Joint Center for Aerospace Technology Innovation, University of Washington, Seattle, WA, USA. His research interests include aerospace systems, control and learning, and networks.

Dr. Mesbahi was the recipient of the NSF CAREER Award, the NASA Space Act Award, the UW Distinguished Teaching Award, the Graduate Instructor of the Year Award, and the UW College of Engineering Innovator Award.