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A Unifying Framework for Strong Structural Controllability

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Abstract—This article deals with strong structural controllability of linear systems. In contrast to the existing work, the structured systems studied in this article have a so-called zero/nonzero/arbitrary structure, which means that some of the entries are equal to zero, some of the entries are arbitrary but nonzero, and the remaining entries are arbitrary (zero or nonzero). We formalize this in terms of pattern matrices, whose entries are either fixed zero, arbitrary nonzero, or arbitrary. We establish necessary and sufficient algebraic conditions for strong structural controllability in terms of full rank tests of certain pattern matrices. We also give a necessary and sufficient graph-theoretic condition for the full rank property of a given pattern matrix. This graph-theoretic condition makes use of a new color change rule that is introduced in this article. Based on these two results, we then establish a necessary and sufficient graph-theoretic condition for strong structural controllability. Moreover, we relate our results to those that exist in the literature and explain how our results generalize previous work.

Index Terms—Network controllability, pattern matrices, strong structural controllability, structured system.

I. INTRODUCTION

Controllability is a fundamental concept in systems and control. For linear time-invariant systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

controllability can be verified using the Kalman rank test or the Hautus test [1]. Often, the exact values of the entries in the matrices A and B are not known, but the underlying interconnection structure between the input and state variables is known exactly.

In order to formalize this, Mayeda and Yamada have introduced a framework, in which, instead of a fixed pair of real matrices, only the zero/nonzero structure of A and B is given [2]. This means that each entry of these matrices is known to be either a *fixed zero* or an *arbitrary nonzero* real number. Given this zero/nonzero structure, they then study controllability of the family of systems, for which the state and input matrices have this zero/nonzero structure. In this setup, this family of systems is called *strongly structurally controllable* if all members of the family are controllable in the classical sense [2].

Most of the existing literature up to now has considered strong structural controllability under the above rather restrictive assumption that for each of the entries of the system matrices, there are only

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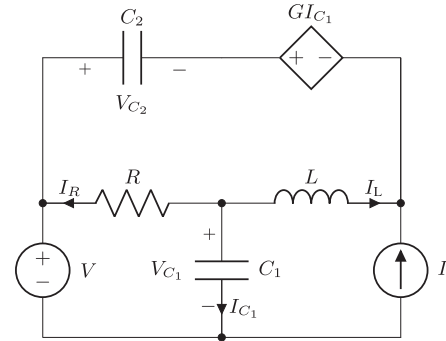


Fig. 1. Example of electrical circuit.

two possibilities: it is either a fixed zero, or an arbitrary nonzero value [2]–[9]. There are, however, many scenarios, in which, in addition to these two possibilities, there is a third possibility, namely, that a given entry is not a fixed zero or nonzero, but can take *any real value*. In such a scenario, it is not possible to represent the system using a zero/nonzero structure, but a third possibility needs to be taken into account. To illustrate this, consider the following example.

Example 1: The electrical circuit in Fig. 1 consists of a resistor, two capacitors, an inductor, an independent voltage source, an independent current source, and a current-controlled voltage source. Assume that the parameters R, C_1, C_2 , and L are positive but not known exactly. We denote the current through R, L , and C_1 by I_R, I_L , and I_{C_1} , respectively, and the voltage across C_1 and C_2 by V_{C_1} and V_{C_2} , respectively. The current-controlled voltage source is represented by GI_{C_1} with gain G assumed to be positive. Define the state vector as $x = [V_{C_1} V_{C_2} I_L]^T$ and the input as $u = [V I]^T$. By Kirchhoff's current and voltage laws, the circuit is represented by a linear time-invariant system (1) with

$$A = \begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ \frac{R-G}{RL} & \frac{1}{L} & -\frac{G}{L} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{RC_1} & 0 \\ 0 & -\frac{1}{C_2} \\ \frac{G-R}{RL} & 0 \end{bmatrix}. \quad (2)$$

Recall that the parameters $R, C_1, C_2, L > 0$ are not known exactly. This means that the matrices in (2) are not known exactly, but we do know that they have the following structure. First, some entries are *fixed zeros*. Second, some of the entries are always *nonzero*, for instance, the entry with value $-\frac{1}{RC_1}$. The third type of entries, those with value $\frac{R-G}{RL}$ and $\frac{G-R}{RL}$, can be either zero (if $R = G$) or nonzero. Since the system matrices in this example do not have a zero/nonzero structure, the existing tests for strong structural controllability [2]–[8] are not applicable.

A similar problem as in Example 1 appears in the context of linear networked systems. Strong structural controllability of such systems has been well studied [7], [8], [10]–[12]. In the setup of these references, the weights on the edges of the network graph are unknown, while the network graph itself is known. Under the assumption that the edge weights are arbitrary but nonzero, linear networked systems can, thus, be regarded as systems with a given zero/nonzero structure. This

zero/nonzero structure is determined by the network graph, in the sense that nonzero entries in the system matrices correspond to edges in the network graph. However, often even exact knowledge of the network graph is not available, in the sense that it is unknown whether certain edges in the graph exist or not. This issue of missing knowledge appears, for example, in social networks [13], the world wide web [14], biological networks [15], [16], and ecological systems [17]. Another cause for uncertainty about the network graph might be malicious attacks and unintentional failures. This issue is encountered in transportation networks [18], sensor networks [19] and gas networks [20].

To conclude, both in the context of modeling physical systems and in representing networked systems, capturing the system simply by a zero/nonzero structure is not always possible, and a more general concept of system structure is required. The papers [10], [11], [21]–[24] study classes¹ of zero/nonzero/arbitrary patterns in the context of strong structural controllability. However, necessary and sufficient conditions for strong structural controllability of general zero/nonzero/arbitrary patterns have not yet been established.

The goal of this article is to provide such general necessary and sufficient conditions. In particular, our main contributions are the following.

- 1) We extend the notion of zero/nonzero structure to a more general *zero/nonzero/arbitrary structure* and formalize this structure in terms of suitable pattern matrices.
- 2) We establish necessary and sufficient conditions for strong structural controllability for families of systems with a given zero/nonzero/arbitrary structure. These conditions are of an algebraic nature and can be verified by a rank test on two pattern matrices.
- 3) We provide a graph-theoretic condition for a given pattern matrix to have full row rank. This condition can be verified using a new *color change rule*, that will be defined in this article.
- 4) We establish a graph-theoretic test for strong structural controllability for the new families of structured systems.
- 5) Finally, we relate our results to those existing in the literature by showing how existing results can be recovered from those we present in this article. We find that seemingly incomparable results of [8] and [10] follow from our main results, which reveals an overarching theory. For these reasons, our article can be seen as a unifying approach to strong structural controllability of linear time-invariant systems without parameter dependencies.

We conclude this section by giving a brief account of research lines that are related to strong structural controllability but that will not be pursued in this article. The concept of weak structural controllability was introduced by Lin [25] and has been studied extensively (see [25]–[31]). Another, more recent, line of work focuses on structural controllability of systems, for which there are *dependencies* among the arbitrary entries of the system matrices [32], [33]. An important special case of dependencies among parameters arises when the state matrix is constrained to be symmetric, which was considered in [11], [34], and [35]. The problem of *minimal input selection* for controllability has also been well studied (see e.g., [36]–[39]). Strong structural controllability was also studied for *time-varying systems* in [40], and conditions for controllability were established for both discrete-time and continuous-time systems. Finally, weak and strong structural *targeted* controllability have been investigated in [41] and [22], [42], respectively.

The outline of the rest of this article is as follows. In Section II, we present some preliminaries. Next, in Section III, we formulate the main problem treated in this article. Then, in Section IV, we state our main results. Section V contains a comparison of our results with

¹In [10], [11], and [21]–[23], a special structure where only the diagonal entries of the state matrix are arbitrary entries (typically arising from a network context) was studied. In [24], the authors call zero/nonzero/arbitrary structure a “selective structure.”

previous work. In Section VI, we state proofs of the main results. Finally, Section VII concludes this article.

II. PRELIMINARIES

Let \mathbb{R} and \mathbb{C} denote the fields of real and complex numbers, respectively. The spaces of n -dimensional real and complex vectors are denoted by \mathbb{R}^n and \mathbb{C}^n , respectively. Likewise, the space of $n \times m$ real matrices is denoted by $\mathbb{R}^{n \times m}$.

Moreover, I and 0 will denote the identity and zero matrix of appropriate dimensions, respectively.

In this article, we will use the so-called pattern matrices. By a pattern matrix, we mean a matrix with entries in the set of symbols $\{0, *, ?\}$. These symbols will be given a meaning in the following.

The set of all $p \times q$ pattern matrices will be denoted by $\{0, *, ?\}^{p \times q}$. For a given $p \times q$ pattern matrix \mathcal{M} , we define the *pattern class* of \mathcal{M} as

$$\mathcal{P}(\mathcal{M}) := \{M \in \mathbb{R}^{p \times q} \mid M_{ij} = 0 \text{ if } \mathcal{M}_{ij} = 0, \\ M_{ij} \neq 0 \text{ if } \mathcal{M}_{ij} = *\}.$$

This means that for a matrix $M \in \mathcal{P}(\mathcal{M})$, the entry M_{ij} is either: 1) *zero* if $\mathcal{M}_{ij} = 0$; 2) *nonzero* if $\mathcal{M}_{ij} = *$; or 3) *arbitrary* (zero or nonzero) if $\mathcal{M}_{ij} = ?$.

III. PROBLEM FORMULATION

Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3)$$

where the system matrix A is in $\mathcal{P}(\mathcal{A})$ and the input matrix B is in $\mathcal{P}(\mathcal{B})$, and where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input.

We will call the family of systems (3) a *structured system*. To simplify the notation, we denote this structured system by the ordered pair of pattern matrices $(\mathcal{A}, \mathcal{B})$.

Example 2: Consider the electrical circuit discussed in Example 1. Recall that this was modeled as the state-space system (2), in which the entries of the system matrix and input matrix were either fixed zeros, strictly nonzero, or undetermined. This can be represented as a structured system $(\mathcal{A}, \mathcal{B})$ with pattern matrices

$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}. \quad (4)$$

In this article, we will study structural controllability of structured systems. In particular, we will focus on strong structural controllability, which is defined as follows.

Definition 3: The system $(\mathcal{A}, \mathcal{B})$ is called *strongly structurally controllable* if the pair (A, B) is controllable for all $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$.

The problem that we will investigate in this article is stated as follows.

Problem 4: Given two pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$, provide necessary and sufficient conditions, under which $(\mathcal{A}, \mathcal{B})$ is strongly structurally controllable.

In the remainder of this article, we will simply call the structured system $(\mathcal{A}, \mathcal{B})$ *controllable* if it is strongly structurally controllable.

Remark 5: In addition to strong structural controllability, *weak structural controllability* has also been studied extensively. This concept was introduced by Lin [25]. Instead of requiring *all* systems in a family associated with a given structured system to be controllable, weak structural controllability only asks for the existence of at least one controllable member of that family (see [25]–[27]). In these references, conditions were established for weak structural controllability of structured systems, in which the pattern matrices only contain 0 or ? entries. The question then arises: Is it possible to generalize the results from [25]–[27] to structured systems in the context of our article, with more general pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and

$\mathcal{B} \in \{0, *, ?\}^{n \times m}$? Indeed, it turns out that the results in [25]–[27] can immediately be applied to assess weak structural controllability of our more general structured systems. To show this, for given pattern matrices $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$, we define two new pattern matrices $\mathcal{A}' \in \{0, ?\}^{n \times n}$ and $\mathcal{B}' \in \{0, ?\}^{n \times m}$ as follows: $\mathcal{A}'_{ij} = 0 \iff \mathcal{A}_{ij} = 0$ and $\mathcal{B}'_{ij} = 0 \iff \mathcal{B}_{ij} = 0$. The new structured system $(\mathcal{A}', \mathcal{B}')$ is now a structured system of the form studied in [25]–[27]. Using the fact that weak structural controllability is a generic property [26], it can then be shown that weak structural controllability of $(\mathcal{A}', \mathcal{B}')$ is equivalent to that of $(\mathcal{A}, \mathcal{B})$. In other words, weak structural controllability of general $(\mathcal{A}, \mathcal{B})$ can be verified using the conditions established in previous work [25]–[27].

IV. MAIN RESULTS

In this section, the main results of this article will be stated. First, we will establish an algebraic condition for controllability of a given structured system. This condition states that controllability of a structured system is equivalent to full rank conditions on two pattern matrices associated with the system. Second, a graph-theoretic condition for a given pattern matrix to have full row rank will be given in terms of a so-called *color change rule*. Finally, based on the above algebraic condition and the graph-theoretic condition, we will establish a graph-theoretic necessary and sufficient condition for controllability of a structured system.

Our first main result is a rank test for controllability of a structured system. In the following, we say that a pattern matrix \mathcal{M} has *full row rank* if every matrix $M \in \mathcal{P}(\mathcal{M})$ has full row rank.

Theorem 6: The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the following two conditions hold.

- 1) The pattern matrix $[\mathcal{A} \ \mathcal{B}]$ has full row rank.
- 2) The pattern matrix $[\bar{\mathcal{A}} \ \mathcal{B}]$ has full row rank, where $\bar{\mathcal{A}}$ is the pattern matrix obtained from \mathcal{A} by modifying the diagonal entries of \mathcal{A} as follows:

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0 \\ ? & \text{otherwise.} \end{cases} \quad (5)$$

We note here that the two rank conditions in Theorem 6 are independent, in the sense that one does not imply the other in general. To show that the first rank condition does not imply the second, consider the pattern matrices \mathcal{A} , the corresponding $\bar{\mathcal{A}}$, and \mathcal{B} given by

$$\mathcal{A} = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}, \bar{\mathcal{A}} = \begin{bmatrix} ? & * \\ 0 & * \end{bmatrix}, \text{ and } \mathcal{B} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

It is evident that the pattern matrix $[\mathcal{A} \ \mathcal{B}]$ has full row rank. However, for the choice

$$\bar{\mathcal{A}} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{P}(\bar{\mathcal{A}}) \text{ and } \mathcal{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{P}(\mathcal{B})$$

the matrix $[\bar{\mathcal{A}} \ \mathcal{B}]$ does not have full row rank.

To show that the second condition does not imply the first one, consider the pattern matrix \mathcal{A} , the corresponding $\bar{\mathcal{A}}$, and \mathcal{B} given by

$$\mathcal{A} = \begin{bmatrix} ? & 0 \\ * & 0 \end{bmatrix}, \bar{\mathcal{A}} = \begin{bmatrix} ? & 0 \\ * & * \end{bmatrix}, \text{ and } \mathcal{B} = \begin{bmatrix} * \\ * \end{bmatrix}.$$

Obviously, the pattern matrix $[\bar{\mathcal{A}} \ \mathcal{B}]$ has full row rank. However, for the choice

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathcal{P}(\mathcal{A}) \text{ and } \mathcal{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{P}(\mathcal{B})$$

we see that $[\mathcal{A} \ \mathcal{B}]$ does not have full row rank.

Next, we discuss a noteworthy special case, in which the first rank condition in Theorem 6 is implied by the second one. Indeed, if none of

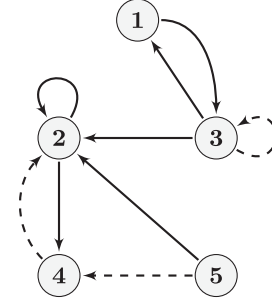


Fig. 2. Graph $G(\mathcal{M})$ associated with \mathcal{M} .

the diagonal entries of \mathcal{A} is zero, it follows from (5) that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{P}(\bar{\mathcal{A}})$. Hence, we obtain the following corollary to Theorem 6.

Corollary 7: Suppose that none of the diagonal entries of \mathcal{A} is zero. Let $\bar{\mathcal{A}}$ be as defined in (5). The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if $[\bar{\mathcal{A}} \ \mathcal{B}]$ has full row rank.

Note that both $[\mathcal{A} \ \mathcal{B}]$ and $[\bar{\mathcal{A}} \ \mathcal{B}]$ appearing in Theorem 6 are $n \times (n + m)$ pattern matrices. Next, we will develop a graph-theoretic test for checking whether a given pattern matrix has full rank. To do so, we first need to introduce some terminology.

Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leq q$. We associate a directed graph $G(\mathcal{M}) = (V, E)$ with \mathcal{M} as follows. Take as node set $V = \{1, 2, \dots, q\}$ and define the edge set $E \subseteq V \times V$ such that $(j, i) \in E$ if and only if $\mathcal{M}_{ij} = *$ or $\mathcal{M}_{ij} = ?$. If $(i, j) \in E$, then we call j an *out-neighbor* of i . Also, in order to distinguish between $*$ and $?$ entries in \mathcal{M} , we define two subsets E_* and $E_?$ of the edge set E as follows: $(j, i) \in E_*$ if and only if $\mathcal{M}_{ij} = *$ and $(j, i) \in E_?$ if and only if $\mathcal{M}_{ij} = ?$. Then, obviously, $E = E_* \cup E_?$ and $E_* \cap E_? = \emptyset$. To visualize this, we use solid and dashed arrows to represent edges in E_* and $E_?$, respectively.

Example 8: As an example, consider the pattern matrix \mathcal{M} given by

$$\mathcal{M} = \begin{bmatrix} 0 & 0 & * & 0 & 0 \\ 0 & * & * & ? & * \\ * & 0 & ? & 0 & 0 \\ 0 & * & 0 & 0 & ? \end{bmatrix}.$$

The associated directed graph $G(\mathcal{M})$ is then given in Fig. 2.

Next, we introduce the notion of *colorability* for $G(\mathcal{M})$ given as follows:

- 1) initially, color all nodes of $G(\mathcal{M})$ white;
- 2) if a node i has exactly one white out-neighbor j and $(i, j) \in E_*$, we change the color of j to black;
- 3) repeat step 2 until no more color changes are possible.

The graph $G(\mathcal{M})$ is called *colorable* if the nodes $1, 2, \dots, p$ are colored black following the procedure above. Note that the remaining nodes $p + 1, \dots, q$ can never be colored black, since they have no incoming edges.

We refer to step 2 in the above procedure as the *color change rule*. Similar color change rules have appeared in the literature before (see e.g., [8], [10], and [43]). Unlike some of these rules, node i in step 2 does not need to be black in order to change the color of a neighboring node.

Example 9: Consider the pattern matrix \mathcal{M} given by

$$\mathcal{M} = \begin{bmatrix} * & 0 & 0 & 0 & * & 0 \\ 0 & ? & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 & 0 \\ 0 & ? & * & * & 0 & 0 \end{bmatrix}.$$

The directed graph $G(\mathcal{M})$ associated with \mathcal{M} is depicted in Fig. 3(a). By repeated application of the color change rule, as shown in

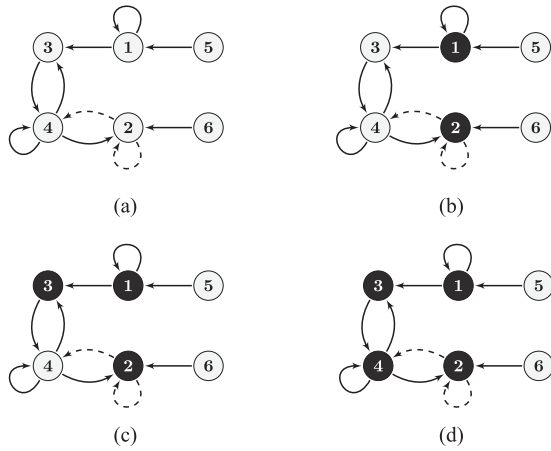


Fig. 3. Example of a colorable graph. (a) Graph $G(\mathcal{M})$. (b) Node 5 colors 1 and Node 6 colors 2. (c) Node 1 colors 3. (d) Node 3 colors 4.

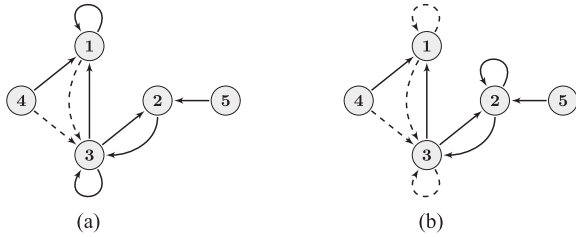


Fig. 4. Graphs associated with the circuit in Example 1. (a) Graph $G([\mathcal{A} \ \mathcal{B}])$. (b) Graph $G([\bar{\mathcal{A}} \ \mathcal{B}])$.

Fig. 3(b)–(d), we obtain the derived set $\mathcal{D} = \{1, 2, 3, 4\}$. Hence, $G(\mathcal{M})$ is colorable.

The following theorem now provides a necessary and sufficient graph-theoretic condition for a given pattern matrix to have full row rank.

Theorem 10: Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leq q$. Then, \mathcal{M} has full row rank if and only if $G(\mathcal{M})$ is colorable.

It is clear from the definition of the color change rule that colorability of a given graph can be checked in polynomial time.

Finally, based on the rank test in Theorem 6 and the result in Theorem 10, the following necessary and sufficient graph-theoretic condition for controllability of a given structured system is obtained.

Theorem 11: Let $\mathcal{A} \in \{0, *, ?\}^{n \times n}$ and $\mathcal{B} \in \{0, *, ?\}^{n \times m}$ be pattern matrices. Also, let $\bar{\mathcal{A}}$ be obtained from \mathcal{A} by modifying the diagonal entries of \mathcal{A} as follows:

$$\bar{\mathcal{A}}_{ii} := \begin{cases} * & \text{if } \mathcal{A}_{ii} = 0 \\ ? & \text{otherwise.} \end{cases} \quad (6)$$

Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $G([\mathcal{A} \ \mathcal{B}])$ and $G([\bar{\mathcal{A}} \ \mathcal{B}])$ are colorable.

As an example, we study controllability of the electrical circuit discussed in Example 1.

Example 12: According to Example 2, the electrical circuit depicted in Fig. 1 can be modeled as a structured system of the form (3). For this example, we have

$$\mathcal{A} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & * \\ ? & * & * \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} * & 0 \\ 0 & * \\ ? & 0 \end{bmatrix}, \quad \text{and} \quad \bar{\mathcal{A}} = \begin{bmatrix} ? & 0 & * \\ 0 & * & * \\ ? & * & ? \end{bmatrix}.$$

TABLE I
GRAPH-THEORETIC CONDITIONS ARE ABBREVIATED BY “GTC” AND ALGEBRAIC CONDITIONS BY “AC”

Ref.	\mathcal{A}	\mathcal{B}	Conditions	
			GTC	AC
[2]	$\{0, *\}^{n \times n}$	$\{0, *\}^{n \times 1}$	✓	–
[3]			–	✓
[4]		$\{0, *\}^{n \times m}$	✓	–
[5]			✓	✓
[6]			✓	✓
[7]			–	✓
[8]		particular $\{0, *\}^{n \times m}$	–	✓
[10]			particular $\{0, *, ?\}^{n \times n}$	✓

The graphs $G([\mathcal{A} \ \mathcal{B}])$ and $G([\bar{\mathcal{A}} \ \mathcal{B}])$ are depicted in Fig. 4(a) and (b), respectively. Both graphs are colorable. Indeed, node 5 colors 2, node 2 colors 3, and finally node 3 colors 1 in both graphs. Therefore, the system $(\mathcal{A}, \mathcal{B})$ is controllable by Theorem 11.

By applying Theorem 11 to the special case discussed in Corollary 7, we obtain the following.

Corollary 13: Suppose that none of the diagonal entries of \mathcal{A} is zero. Let $\bar{\mathcal{A}}$ be defined as in (6). Then, the system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if $G([\bar{\mathcal{A}} \ \mathcal{B}])$ is colorable.

To conclude this section, the results we have obtained for controllability lead to an interesting observation in the context of structural stabilizability. We say that a structured system $(\mathcal{A}, \mathcal{B})$ is *stabilizable* if the pair $(\mathcal{A}, \mathcal{B})$ is stabilizable for all $A \in \mathcal{P}(\mathcal{A})$ and $B \in \mathcal{P}(\mathcal{B})$.

Theorem 14: The system $(\mathcal{A}, \mathcal{B})$ is stabilizable if and only if it is controllable.

V. DISCUSSION OF EXISTING RESULTS

In this section, we compare our results with those existing in the literature. We focus on the most relevant related work [2]–[8], [10]. The structured systems studied in these references are all special cases of those we study in this article. In Table I, we summarize the different pattern matrices \mathcal{A} and \mathcal{B} studied in these references. We also include the type of conditions that were developed, i.e., either graph-theoretic, algebraic or both. Note that the papers [7], [8], and [10] study controllability in a network context, where the pattern matrix \mathcal{B} has a particular structure in the sense that each column has exactly one $*$ -entry, and each row has at most one $*$ -entry. Additionally, the paper [10] considers a particular class of systems where the diagonal entries of \mathcal{A} are all $?$ and none of the off-diagonal entries is $?$. In the following two subsections, we elaborate on the existing graph-theoretic conditions and algebraic conditions, respectively. In both sections, we also compare these results to the present work.

A. Graph-Theoretic Conditions

The graph-theoretic conditions provided in [2, Th. 1] for the single-input case ($m = 1$) and extended to the multi-input case in [4, Satz 3] are based on the graph $G = (V, E)$ associated with a pattern matrix $[\mathcal{A} \ \mathcal{B}]$, where $\mathcal{A} \in \{0, *\}^{n \times n}$ and $\mathcal{B} \in \{0, *\}^{n \times m}$. Note that $V = \{1, 2, \dots, n + m\}$ in this case. The graph-theoretic characterization in [4, Satz 3] (or in [2, Th. 1] if $m = 1$) consists of three conditions. The first one requires checking the so-called accessibility of each node in $\{1, 2, \dots, n\}$ from the nodes in $\{n + 1, n + 2, \dots, n + m\}$. The remaining two conditions require checking certain relations for all subsets of $\{1, 2, \dots, n\}$. As such, the computational complexity of checking these conditions is at least exponential in n . Note that, in contrast, the computational complexity of checking the colorability conditions of our Theorem 11 is polynomial in n .

Paper [2] provides another set of graph-theoretic conditions, stated, more specifically, in [2, Th. 2] (only for the case $m = 1$). As argued in [2, p. 135], this theorem performs better than [2, Th. 1] for sparse graphs. Essentially, the conditions given in [2, Th. 2] require checking the existence of a unique serial buds cactus as well as nonexistence

of certain cycles within the graph G . How these conditions can be checked in an algorithmic manner is not clear, whereas the colorability conditions given in Theorem 11 can be checked by a simple algorithm.

On top of the advantages of computational complexity, the conditions provided in Theorem 11 are more attractive because of their conceptual simplicity. Indeed, colorability is a simpler and more intuitive notion than those appearing in the results of [2] and [4].

Yet another graph-theoretical characterization is provided in [6, Th. 5]. In order to verify the conditions of [6, Th. 5], one needs to check whether a unique spanning cycle family with certain properties exists in $\binom{n+m}{n}$ directed graphs obtained from the pattern matrices \mathcal{A} and \mathcal{B} . Needless to say, checking the conditions of Theorem 11 is much easier than checking these conditions.

Also in the context of networked systems, graph-theoretic conditions for strong structural controllability have been obtained (see e.g., [7], [8], and [10]). To elaborate further on the relationship between the work on networked systems and this article, we first need to explain the framework of the papers [7], [8], and [10]. The starting point of these papers is a directed graph $H = (W, F)$, where $W = \{1, 2, \dots, n\}$ denotes the node set and F the edge set. The graphs considered in [7] and [8] are so-called loop graphs, which are graphs that are allowed to contain self-loops, whereas graphs in [10] do not allow self-loops. Apart from the graph H , these papers consider a subset of the node set W , the so-called leader set, say $W_L = \{w_1, w_2, \dots, w_m\}$. Based on the graph H and W_L , papers [7], [8], and [10] introduce systems of the form (3), where the pattern matrix \mathcal{B} is defined by

$$\mathcal{B}_{ij} = \begin{cases} *, & \text{if } i = w_j \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

for $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$. In [7] and [8], the pattern matrix \mathcal{A} is defined by

$$\mathcal{A}_{ij} = \begin{cases} *, & \text{if } (j, i) \in F \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

whereas in [10], the pattern matrix \mathcal{A} is defined by

$$\mathcal{A}_{ij} = \begin{cases} *, & \text{if } (j, i) \in F \\ ?, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

for $i, j \in \{1, 2, \dots, n\}$.

In [7], the authors first define two bipartite graphs obtained from the pattern matrices \mathcal{A} and \mathcal{B} . Then, they show in [7, Th. 5] that $(\mathcal{A}, \mathcal{B})$ is strongly structurally controllable if and only if there exist so-called constrained matchings with certain properties in these bipartite graphs. Later, in [8, Th. 5.4], an equivalence between the existence of constrained matchings and so-called zero forcing sets for loop graphs was established. To explain this in more detail, we need to introduce the notion of zero forcing that was originally studied in the context of minimal rank problems (see e.g., [43]).

Let $H = (W, F)$ be a directed loop graph and $S \subseteq W$. Color all nodes in S black and the others white.

If a node i (of any color) has exactly one white out-neighbor j , we change the color of j to black and write $i \rightarrow j$. If all the nodes in W can be colored black by repeated application of this color change rule, we say that S is a *loopy zero forcing set* for H . Given a loopy zero forcing set, we can list the color changes in the order in which they were performed to color all nodes black. This list is called a chronological list of color changes.

In order to quote [8, Th. 5.5], we need two more definitions. Define $W_{\text{loop}} \subseteq W$ to be the subset of all nodes with self-loops and let H^* be the graph obtained from H by placing a self-loop at every node.

Theorem 15 (see [8, Th. 5.5]): Let H be a directed loop graph and W_L be a leader set. Consider the pattern matrices defined in (7) and

(8). Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the following conditions hold.

- 1) W_L is a loopy zero forcing set for H .
- 2) W_L is a loopy zero forcing set for H^* , for which there is a chronological list of color changes that does not contain a color change of the form $i \rightarrow i$ with $i \in W_{\text{loop}}$.

A result similar to this theorem was obtained in [10] for controllability of pattern matrices defined by (7) and (9) that are obtained from a graph H without self-loops. However, in order to deal with this class of pattern matrices, paper [10] introduces a slightly different notion of zero forcing to be defined below.

Let $H = (W, F)$ be a directed graph without self-loops and $S \subseteq W$. Color all nodes in S black and the others white. If a *black* node i has exactly one white out-neighbor j , we change the color of j to black. If all the nodes in W can be colored black by repeated application of this color change rule, we say that S is a *ordinary zero forcing set* for H .

We now state the graph-theoretic characterization of controllability established in [10].

Theorem 16 (see [10, Th. IV.4]): Let H be a directed graph without self-loops and W_L be a leader set. Consider the pattern matrices given by (7) and (9). Then, the structured system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if W_L is an ordinary zero forcing set for H .

Even though Theorems 15 and 16 present conditions that are similar in nature, it is not possible to compare these results immediately as they deal with two different and nonoverlapping system classes. Indeed, the pattern matrices considered in [8] [given by (8)] *do not contain any ? entries*, whereas those studied in [10] [given by (9)] contain *only ? entries on their diagonals*.

Next, we will show that the conditions of Theorem 11 are equivalent to those of Theorems 15 and 16 if specialized to the corresponding pattern matrices. This will shed light on the relationship between these results based on the different zero forcing notions.

We start with Theorem 15. According to our color change rule, the nodes belonging to W_L will be colored black in both $G([\mathcal{A} \ \mathcal{B}])$ and $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$ because \mathcal{B} is a pattern matrix with structure defined by (7). Since \mathcal{A} does not contain ? entries, $G([\mathcal{A} \ \mathcal{B}])$ is colorable if and only if W_L is a loopy zero forcing set for $G(\mathcal{A})$. By noting that $H = G(\mathcal{A})$, we see that the first condition in Theorem 11 is equivalent to that of Theorem 15. Now, let the pattern matrix \mathcal{A}^* be such that $H^* = G(\mathcal{A}^*)$. Since $W_{\text{loop}} = \{i \mid \bar{\mathcal{A}}_{ii} = ?\}$, we see that $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$ is colorable if and only if the second condition of Theorem 15 holds. Thus, the second condition of Theorem 11 is equivalent to that of Theorem 15.

Now, we turn attention to Theorem 16. It follows from (6) and (9) that $\bar{\mathcal{A}} = \mathcal{A}$, i.e., graphs $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$ and $G([\mathcal{A} \ \mathcal{B}])$ are the same. As in the discussion above, the nodes belonging to W_L will be colored black in $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$ because \mathcal{B} is a pattern matrix with the structure defined by (7). According to our color change rule, a white node can never color any other white node in $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$, since $(i, i) \in E_{\bar{\mathcal{A}}}$ for every node i of $G(\bar{\mathcal{A}})$. This means that $G([\bar{\mathcal{A}} \ \bar{\mathcal{B}}])$ is colorable if and only if W_L is an ordinary zero forcing set for $G(\bar{\mathcal{A}})$. By noting that $H = G(\mathcal{A}) = G(\bar{\mathcal{A}})$, we see that the conditions in Theorem 11 are equivalent to the single condition of Theorem 16.

B. Algebraic Conditions

In this subsection, we will compare our rank tests for strong structural controllability with those provided in [5], [7], and [10]. More precisely, we will show that the rank tests in Theorem 6 reduce to those in [5], [7], and [10] for the corresponding special cases of pattern matrices.

An algebraic condition for controllability of $(\mathcal{A}, \mathcal{B})$ was provided in [5, Th. 2] for $\mathcal{A} \in \{0, *\}^{n \times n}$ and $\mathcal{B} \in \{0, *\}^{n \times m}$. Later, these conditions were reformulated in [7, Th. 3]. These conditions rely on a matrix property that will be defined next for pattern matrices that may also contain ? entries.

Definition 17: Consider a pattern matrix $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ with $p \leq q$. The matrix \mathcal{M} is said to be of Form III if there exist two

permutation matrices P_1 and P_2 such that

$$P_1 \mathcal{M} P_2 = \begin{bmatrix} \otimes & \cdots & \otimes & * & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \ddots & \ddots & \vdots \\ \otimes & \cdots & \otimes & \cdots & \otimes & * & 0 \\ \otimes & \cdots & \otimes & \cdots & \otimes & \otimes & * \end{bmatrix} \quad (10)$$

where the symbol \otimes indicates an entry that can be either 0, $*$ or $?$.

The aforementioned algebraic conditions are stated next.

Theorem 18 (see [7, Th. 3]): Let $\mathcal{A} \in \{0, *\}^{n \times n}$ and $\mathcal{B} \in \{0, *\}^{n \times m}$ be two pattern matrices. Also, let \mathcal{A}^* be the pattern matrix obtained from \mathcal{A} by replacing all diagonal entries by $*$. The system $(\mathcal{A}, \mathcal{B})$ is controllable if and only if the following two conditions hold.

- 1) The matrix $[\mathcal{A} \ \mathcal{B}]$ is of Form III.
- 2) The matrix $[\mathcal{A}^* \ \mathcal{B}]$ is of Form III with the additional property that $*$ entries appearing in (10) do not originate from diagonal elements in \mathcal{A} that are $*$ entries.

It can be shown that our algebraic conditions in Theorem 6 are equivalent to those in Theorem 18 for the special case of pattern matrices that only contain 0 and $*$ entries. Recall that it follows from Theorem 6 that $(\mathcal{A}, \mathcal{B})$ is controllable if and only if both $[\mathcal{A} \ \mathcal{B}]$ and $[\bar{\mathcal{A}} \ \mathcal{B}]$ have full row rank, where $\bar{\mathcal{A}}$ is given in (6). To relate our algebraic conditions with the ones in Theorem 18, we need the following lemma.

Lemma 19: Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ with $p \leq q$. Then, \mathcal{M} has full row rank if and only if \mathcal{M} is of Form III.

From Lemma 19, it immediately follows that $[\mathcal{A} \ \mathcal{B}]$ has full row rank if and only if $[\bar{\mathcal{A}} \ \mathcal{B}]$ is of Form III. Hence, the first condition of Theorem 6 is equivalent to that of Theorem 18. We will now also show that $[\bar{\mathcal{A}} \ \mathcal{B}]$ has full row rank if and only if the second condition of Theorem 18 holds. From Lemma 19, we have that $[\bar{\mathcal{A}} \ \mathcal{B}]$ has full row rank if and only if $[\bar{\mathcal{A}} \ \mathcal{B}]$ is of Form III. By definition of $\bar{\mathcal{A}}$ and \mathcal{A}^* , it follows that $\bar{\mathcal{A}}_{ij} = \mathcal{A}_{ij}^*$ for all $i \neq j$. If $\mathcal{A}_{ii} = 0$ then both $\bar{\mathcal{A}}_{ii} = *$ and $\mathcal{A}_{ii}^* = *$. On the other hand, if $\mathcal{A}_{ii} = *$, then $\bar{\mathcal{A}}_{ii} = ?$ and $\mathcal{A}_{ii}^* = *$. To sum up, $\bar{\mathcal{A}}_{ij} \neq \mathcal{A}_{ij}^*$ if and only if $i = j$ and $\mathcal{A}_{ii} = *$. In other words, all entries of $\bar{\mathcal{A}}$ and \mathcal{A}^* are the same, except for those that correspond to the diagonal elements of \mathcal{A} that are $*$ entries. Hence, there exist two permutation matrices P_1 and P_2 such that all entries of the matrices $P_1[\bar{\mathcal{A}} \ \mathcal{B}]P_2$ and $P_1[\mathcal{A}^* \ \mathcal{B}]P_2$ are the same, except those that originate from diagonal elements of \mathcal{A} that are $*$ entries. This implies that $[\bar{\mathcal{A}} \ \mathcal{B}]$ is of Form III if and only if $[\mathcal{A}^* \ \mathcal{B}]$ is of Form III with the additional property that the $*$ entries in (10) do not originate from diagonal elements in \mathcal{A} that are $*$ entries. In other words, the second conditions of Theorems 6 and 18 are equivalent. Since also the first conditions in these theorems are equivalent, we conclude that the algebraic conditions in Theorem 6 are equivalent to those in Theorem 18 for the special case, in which $\mathcal{A} \in \{0, *\}^{n \times n}$ and $\mathcal{B} \in \{0, *\}^{n \times m}$.

A different algebraic condition was introduced in [10] for systems defined on simple directed graphs. The pattern matrices of such systems can be represented by \mathcal{A} and \mathcal{B} given by (9) and (7), respectively. The algebraic condition referred to above is then stated as follows.

Theorem 20 (see [10, Lem. IV.1]): Consider the pattern matrices \mathcal{A} and \mathcal{B} given by (9) and (7), respectively. Then, $(\mathcal{A}, \mathcal{B})$ is controllable if and only if $[\mathcal{A} \ \mathcal{B}]$ has full row rank.

In order to see that this theorem follows from Corollary 7, note that $\mathcal{A} = \bar{\mathcal{A}}$, since all diagonal entries of \mathcal{A} are $?$'s.

VI. PROOFS

A. Proof of Theorem 6

To prove the “only if” part, assume that $(\mathcal{A}, \mathcal{B})$ is controllable. By the Hautus test [1, Th. 3.13] and the definition of strong structural controllability, it follows that $[A - \lambda I \ B]$ has full row rank for all $(A, B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$ and all $\lambda \in \mathbb{C}$. By substitution of $\lambda = 0$, we conclude that condition 1 is satisfied. To prove that condition 2 also holds, suppose that $x^T[\bar{A} \ B] = 0$ for some pair $(\bar{A}, B) \in \mathcal{P}(\bar{\mathcal{A}}) \times$

$\mathcal{P}(\mathcal{B})$ and $x \in \mathbb{R}^n$. We want to prove that $x = 0$. Let $\alpha \in \mathbb{R}$ be a nonzero real number such that $\alpha \notin \{\bar{A}_{ii} \mid i \text{ is such that } \bar{A}_{ii} = *\}$. Then define a nonsingular diagonal matrix $X \in \mathbb{R}^{n \times n}$ as

$$X_{ii} = \begin{cases} 1, & \text{if } \bar{A}_{ii} = ? \\ \alpha/\bar{A}_{ii}, & \text{if } \bar{A}_{ii} = *. \end{cases}$$

It is clear that $\bar{A}X \in \mathcal{P}(\bar{\mathcal{A}})$ and $x^T[\bar{A}X \ B] = 0$. Furthermore, by the choice of α and X , we obtain $\hat{A} := \bar{A}X - \alpha I \in \mathcal{P}(\mathcal{A})$. By assumption, $[\hat{A} + \alpha I \ B]$ has full row rank (by substitution of $\lambda = -\alpha$). In other words, $[\bar{A}X \ B]$ has full row rank and therefore $x = 0$. We conclude that condition 2 is satisfied.

To prove the “if” part, assume that conditions 1 and 2 are satisfied. Suppose that $z^H[A - \lambda I \ B] = 0$ for some $(A, B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$ and $(\lambda, z) \in \mathbb{C} \times \mathbb{C}^n$, and z^H denotes the conjugate transpose of z . We want to prove that $z = 0$. Note that if $\lambda = 0$, it readily follows that $z = 0$ by condition 1. Therefore, it remains to be shown that $z = 0$ if $\lambda \neq 0$. To this end, write $z = \xi + j\eta$, where $\xi, \eta \in \mathbb{R}^n$ and j denotes the imaginary unit. Next, let $\alpha \in \mathbb{R}$ be a nonzero real number such that

$$\alpha \notin \left\{ -\frac{\xi_i}{\eta_i} \mid \eta_i \neq 0 \right\} \cup \left\{ -\frac{(\xi^T A)_i}{(\eta^T A)_i} \mid (\eta^T A)_i \neq 0 \right\}.$$

We define $x := \xi + \alpha\eta$. Now, we claim the following:

- (a) $x_i = 0$ if and only if $z_i = 0$;
- (b) $x_i = 0$ if and only if $(x^T A)_i = 0$.

Note that (a) follows directly from the definition of x and the choice of α . To prove the “only if” part of (b), suppose that $x_i = 0$. By (a), this implies that $z_i = 0$. Since $z^H A = \lambda z^H$, we see that $(z^H A)_i = 0$. Equivalently, $((\xi^T - j\eta^T)A)_i = 0$. Therefore, both $(\xi^T A)_i = 0$ and $(\eta^T A)_i = 0$. We conclude that $(x^T A)_i = ((\xi^T + \alpha\eta^T)A)_i = 0$.

To prove the “if” part of (b), suppose that $(x^T A)_i = 0$. This means that $((\xi^T + \alpha\eta^T)A)_i = 0$. Equivalently, $(\xi^T A)_i + \alpha(\eta^T A)_i = 0$. By the choice of α , this implies that $(\xi^T A)_i = (\eta^T A)_i = 0$. We conclude that $(z^H A)_i = 0$. Recall that $z^H A = \lambda z^H$, where λ was assumed to be nonzero. This implies that $z_i = 0$. Again, using (a), we conclude that $x_i = 0$. This proves (b).

Next, we define the diagonal matrix $X' \in \mathbb{R}^{n \times n}$ as

$$X'_{ii} = \begin{cases} 1, & \text{if } x_i = 0 \\ \frac{(x^T A)_i}{x_i}, & \text{otherwise.} \end{cases}$$

We know that X' is nonsingular by (b). By definition of X' , we have $x^T A = x^T X'$. Furthermore, as $z^H B = 0$, we obtain $\xi^T B = \eta^T B = 0$ and, therefore, $x^T B = 0$. Hence, $x^T[A - X' \ B] = 0$. Since X' is nonsingular, it follows that $A - X' \in \mathcal{P}(\bar{\mathcal{A}})$. By condition 2, this means that $x = 0$. Finally, we conclude that $z = 0$ using (a). ■

B. Proof of Theorem 10

To prove Theorem 10, we need the following auxiliary result.

Lemma 21: Let $\mathcal{M} \in \{0, *, ?\}^{p \times q}$ be a pattern matrix with $p \leq q$. Consider the directed graph $G(\mathcal{M})$. Suppose that each node is colored white or black. Let $D \in \mathbb{R}^{p \times p}$ be the diagonal matrix defined by

$$D_{kk} = \begin{cases} 1, & \text{if node } k \text{ is black} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose further that $j \in \{1, 2, \dots, p\}$ is a node for which there exists a node $i \in \{1, 2, \dots, p\}$, possibly identical to j , such that j is the only white out-neighbor of i and $(i, j) \in E_*$. Then, for all $M \in \mathcal{P}(\mathcal{M})$, we have that $[M \ D]$ has full row rank if and only if $[M \ D + e_j e_j^T]$ has full row rank, where e_j denotes the j th column of I .

Proof: The “only if” part is trivial. To prove the “if” part, suppose that $M \in \mathcal{P}(\mathcal{M})$ and $[M \ D + e_j e_j^T]$ has full row rank. Let $z \in \mathbb{R}^p$ be such that $z^T[M \ D] = 0$. Our aim is to show that $z_j = 0$. Indeed, if z_j is zero, then $z^T[M \ D + e_j e_j^T] = z^T[M \ D] = 0$, and hence, z must be zero. This would prove that $[M \ D]$ has full row rank. We

will distinguish two cases: $i = j$ and $i \neq j$. Suppose first that $i = j$. Let $\beta, \omega \subseteq \{1, 2, \dots, p\}$ be defined as the index sets $\beta = \{k \mid k \neq j \text{ and } k \text{ is black}\}$ and $\omega = \{\ell \mid \ell \neq j \text{ and } \ell \text{ is white}\}$. In the following, to simplify the notations, for a given vector $z \in \mathbb{R}^p$ and a given index set $\alpha \subseteq \{1, 2, \dots, p\}$, we define $z_\alpha := \{x \in \mathbb{R}^{|\alpha|} \mid x_i = z_{\alpha(i)}, i \in \{1, 2, \dots, |\alpha|\}\}$, where $|\alpha|$ is the cardinality of α . From $z^T M = 0$, we get

$$z_j M_{jj} + z_\beta^T M_{\beta j} + z_\omega^T M_{\omega j} = 0. \quad (11)$$

Since j is the only white out-neighbor of itself, we must have that M_{jj} is nonzero and that $M_{\omega j}$ is a zero vector. Moreover, it follows from $z^T D = 0$ that z_β must be a zero vector. Therefore, (11) implies that z_j must be zero.

Next, suppose that $i \neq j$. Let $\beta, \omega \subseteq \{1, 2, \dots, p\}$ be defined as the index sets $\beta = \{k \mid k \neq i, k \neq j, \text{ and } k \text{ is black}\}$ and $\omega = \{\ell \mid \ell \neq i, \ell \neq j, \text{ and } \ell \text{ is white}\}$. From $z^T M = 0$, we now get

$$z_i M_{ii} + z_j M_{ji} + z_\beta^T M_{\beta i} + z_\omega^T M_{\omega i} = 0. \quad (12)$$

Since j is the only white out-neighbor of i , we must have that M_{ji} is nonzero and that $M_{\omega i}$ is a zero vector. Moreover, it follows from $z^T D = 0$ that z_β must be a zero vector. Therefore, (12) implies that

$$z_i M_{ii} + z_j M_{ji} = 0. \quad (13)$$

Now, we distinguish two cases: i is black and i is white. If i is black, then we have that z_i is zero because $z^T D = 0$. Therefore, (13) implies that $z_j = 0$ as desired. Finally, if i is white, then we have that $M_{ii} = 0$ for otherwise i would have two white out-neighbors. Again, (13) implies that z_j is zero. This completes the proof. ■

Now, we can give the proof of Theorem 10.

Proof of Theorem 10: To prove the “if” part, suppose that $G(\mathcal{M})$ is colorable. Let $M \in \mathcal{P}(\mathcal{M})$. By repeated application of Lemma 21, it follows that M has full row rank if and only if $[M \ J]$ has full row rank, which is obviously true. Therefore, we conclude that M has full row rank.

To prove the “only if” part, suppose that \mathcal{M} has full row rank, but $G(\mathcal{M})$ is not colorable. Let C be the set of nodes that are colored black by repeated application of the color change rule until no more color changes are possible. Then, C is a *strict* subset of $\{1, 2, \dots, p\}$. Thus, possibly after reordering the nodes, we can partition \mathcal{M} as

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix}$$

where the rows of the matrix \mathcal{M}_1 correspond to the nodes in C and that of the matrix \mathcal{M}_2 correspond to the remaining white nodes. Note that $C = \emptyset$ means that $\mathcal{M}_2 = \mathcal{M}$ and \mathcal{M}_1 is absent. Since no more color changes are possible, there is no column of \mathcal{M}_2 that has exactly one $*$ entry, while all other entries are 0. Therefore, for any column of \mathcal{M}_2 , we have one of the following three cases.

- All entries are 0.
- There exists exactly one $?$ entry while all other entries are 0.
- At least two entries belong to the set $\{*, ?\}$.

Consequently, there exists a matrix $M_2 \in \mathcal{P}(\mathcal{M}_2)$ such that its column sums are zero, that is $\mathbb{1}^T M_2 = 0$, where $\mathbb{1}$ denotes the vector of ones of appropriate size. Take any $M_1 \in \mathcal{P}(\mathcal{M}_1)$. Then

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \in \mathcal{P} \left(\begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \right) = \mathcal{P}(\mathcal{M})$$

satisfies $\begin{bmatrix} 0^T & \mathbb{1}^T \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = 0$. Hence, M does not have full row rank and we have reached a contradiction. ■

C. Proof of Theorem 11

By Theorems 6 and 10, we have that $[A \ B]$ is controllable if and only if $G([A \ B])$ and $G([\bar{A} \ \bar{B}])$ are colorable. ■

D. Proof of Theorem 14

The “if” part is evident. Therefore, it is enough to prove the “only if” part. Suppose that the system (A, B) is stabilizable. Let $(A, B) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B})$. Then, (A, B) is stabilizable. Note that $-A \in \mathcal{P}(\mathcal{A})$. Therefore, both (A, B) and $(-A, B)$ are stabilizable. It follows from the Hautus test for stabilizability (see e.g., [1, Th. 3.32]) that (A, B) is controllable. Consequently, the system (A, B) is controllable. ■

E. Proof of Lemma 19

Since the “if” part is evident, it remains to prove the “only if” part. Suppose that \mathcal{M} has full row rank. From Theorem 10, it follows that $G(\mathcal{M})$ is colorable. In particular, there exist $i \in \{1, 2, \dots, q\}$ and $j \in \{1, 2, \dots, p\}$ such that $M_{ji} = *$ and $M_{ki} = 0$ for all $k \neq j$. Therefore, we can find permutation matrices P_1 and P_2 such that

$$P_1 \mathcal{M} P_2 = \left[\begin{array}{c|c} \mathcal{M}' & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hline r \otimes \cdots \otimes * & * \end{array} \right]$$

where the symbol \otimes indicates an entry that can be either 0, $*$ or $?$. Note that M has full row rank for all $M \in \mathcal{P}(\mathcal{M})$ if and only if M' has full row rank for all $M \in \mathcal{P}(\mathcal{M}')$. Therefore, repeated application of the argument above results in permutation matrices P_1 and P_2 such that (10) holds. ■

VII. CONCLUSION

In most of the existing literature on strong structural controllability of structured systems, a zero/nonzero structure of the system matrices is assumed to be given. However, in many physical systems or linear networked systems, apart from fixed zero entries and nonzero entries, we need to allow a third kind of entries, namely those that can take arbitrary (zero or nonzero) values. To deal with this, we have extended the notion of zero/nonzero structure to what we have called zero/nonzero/arbitrary structure. We have formalized this more general class of structured systems using pattern matrices containing fixed zero, arbitrary nonzero, and arbitrary entries. In this setup, we have established necessary and sufficient algebraic conditions for strong structural controllability of these systems in terms of full rank tests on two associated pattern matrices. Moreover, a necessary and sufficient graph-theoretic condition for a given pattern matrix to have full row rank has been given in terms of a new color change rule. We have then established a graph-theoretic test for strong structural controllability of the new class of structured systems. Finally, we have shown how our results generalize previous work. We have also shown that some existing results [8], [10] that are seemingly incomparable to ours can be put in our framework, thus unveiling an overarching theory.

In addition to strong structural controllability, weak structural controllability and strong structural stabilizability of structured systems with zero/nonzero/arbitrary structures have been briefly analyzed. We have shown that weak structural controllability of our structured systems can be checked using tests that already exist in the literature. We have also shown that a structured system with a zero/nonzero/arbitrary structure is strongly structurally stabilizable if and only if it is strongly structurally controllable.

It would be interesting to adopt our new framework of structured systems to other problem areas in systems and control, such as network identification [44] or fault detection and isolation [45]. This is left as a possibility for future research.

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