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Bifurcations of basins of attraction from the view point of prime ends

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Abstract

In dynamical systems examples are common in which two or more attractors coexist, and in such cases the basin boundary is nonempty and the basins often have fractal basin boundaries. The purpose of this paper is to describe the structure and properties of unbounded basins and their boundaries for two-dimensional diffeomorphisms. Frequently, if not always, there is a periodic saddle on the boundary that is accessible from the basin. Carathéodory and many others developed an approach in which an open set (in our case a basin) is compactified using so-called prime end theory. Under the prime end compactification of the basin, boundary points of the basin (prime ends) can be characterized as either type 1, 2, 3, or 4. In all well-known examples, most points are of type 1. Many two-dimensional basins have a *basin cell*, that is, a trapping region whose boundary consists of pieces of the stable and unstable manifolds of a well chosen periodic orbit. Then the basin consists of a central body (the basin cell) and a finite number of channels attached to it, and the basin boundary is fractal. We present a result that says {a basin has a basin cell} if and only if {every prime end that is defined by a chain of unbounded regions (in the basin) is a prime end of type 3 and furthermore all other prime ends are of type 1}. We also prove as a parameter is varied, the basin cell for a basin B is created (or destroyed) if and only if either there is a saddle node bifurcation or the basin B has a prime end that is defined by a chain of unbounded regions and is a prime end of either type 2 or type 4.

1. Introduction

One of the goals of dynamical systems is to determine the global structure for ever more complicated dynamical systems. These "global" structures include the boundaries of basins. For two-dimensional maps including the Hénon map and the time-2π map of the forced damped pendulum differential equation and forced Duffing differential equation, basins and their boundaries have been studied quite extensively; see, for example, [21] and references therein. Frequently, accessible periodic points play a crucial role in describing phenomena of basin boundaries. For an open set U, a point $p \in \partial U$ is *accessible* from U if there exists a half-open path $\gamma : [0, 1) \to U$ such that $\lim_{t\to 1} \gamma(t) = p$. We refer to $\partial \bar{B}$ as the basin boundary. When the basin boundary $\partial \bar{B}$ is fractal (i.e., it contains a homoclinic point),

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only a relatively small subset of $\partial\bar{B}$ consists of accessible points, and generally no accessible points that are accessible from B will be accessible from another basin. We say that $p \in M$ is *B-accessible* if $p \in \partial\bar{B}$ and p is accessible from the interior of $\bar{B}$. If one point of a periodic orbit is B-accessible, then so are the other points, so we refer to these orbits as B-accessible periodic orbits. Accessible points on basin boundaries for area contracting dynamical systems have been studied previously by Alligood and Yorke [1] and they presented the result that (under certain conditions) every B-accessible point is on the stable manifold of some m-periodic saddle, for some $m \in \mathbb{N}$.

In dynamical systems examples are common in which three or more basins of attraction exist. It is possible to have three or more basins, such that for every basin boundary point x, each open neighborhood of x intersects each of the basins. When this last situation occurs, the boundaries have a complicated structure. This phenomenon does occur naturally in simple dynamical systems. To describe this phenomenon, Kennedy and Yorke [12] introduced the notion of "Wada property" for dynamical systems. They say, the basins of attraction have the *Wada property* if there are at least three basins and each point that is on the boundary of any basin is on the boundary of every basin. They state the following result. Assume that p is a periodic saddle point such that the unstable manifold of p intersects every basin and the stable manifold of p is dense in each of the basin boundaries. Then the basins have the Wada property. In Refs. [16,17], we introduced the notions of Wada basin (a point x is a *Wada point* if every open neighborhood of x has a nonempty intersection with at least three basins, and basin B is a *Wada basin* if every $x \in \partial\bar{B}$ is a Wada point) and presented a theorem guaranteeing the occurrence of Wada basins. Our results show that sometimes some basins of a dynamical system are Wada basins while other basins are not. In order to prove that theorem, we introduced the basic notion of basin cell. (A *basin cell* is a trapping region whose boundary consists of pieces of the stable and unstable manifolds of a B-accessible periodic orbit and it determines the structure of the corresponding basin.) Basin cells allow us to discuss the global structure of basin boundaries for many choices of parameters in well known two-dimensional maps including the Hénon map and the time-2π map of the forced damped pendulum differential equation. The goal of this paper is to investigate the structure of basin boundaries and to provide (1) necessary and sufficient conditions for a basin B in terms of prime ends of B under which the basin B has a basin cell and (2) necessary and sufficient conditions for a basin B in terms of prime ends of B under which a basin cell for B is created (or destroyed) while a parameter is varied over an interval.

Let M denote either $\mathbb{R}^2$ or the cylinder $\mathbb{R} \times S^1$. Let $F : M \to M$ be a C^1-diffeomorphism. In Fig. 1(a), parts of three basins of attraction are shown. Each basin is unbounded with infinite area. The basins shown are open, connected, simply connected sets. These basins are highly convoluted. We can characterize such a convoluted set B by examining the type of prime ends of B that are defined by chains of unbounded regions; see Section 3.2 for the definition of prime end. Each of the three basins of Fig. 1(a) is a d_B-unbounded region. If B is any of these three basins, then there is a prime end of B (defined by a chain of d_B-unbounded regions) that is a prime end of type 3; see Section 2 for the definition of the metric d_B. Our first result in this paper implies that each basin B shown in Fig. 1(a) has the property: each prime end of B that is defined by a chain of d_B-unbounded regions, is a prime end of type 3 and all other prime ends of B are of type 1. We want to point out that there is an open set of C^1-diffeomorphisms $g : M \to M$ which have unbounded basins with this property. Our goal is to determine when basins B have the property that each prime end of B that is defined by a chain of d_B-unbounded regions, is a prime end of type 3 and all other prime ends of B are of type 1.

Basin of a trapping region. A basin (for F) is usually defined to be the set of points x for which $\omega(x)$ is contained in a specified compact attractor. Of course, the attractor is contained in a compact trapping region. In this paper we take a slightly more general approach of emphasizing the role of trapping region. By a *compact region* we mean a simply connected, connected compact set with nonempty interior. The trapping regions we are interested in have piecewise smooth boundaries. We say that a compact region Q is a *trapping region* (for F) if $F(Q) \subset Q$ and $F(Q) \neq Q$. (Note that we do not require that $F(Q)$ is in the interior of Q.) If Q is a trapping region, then we define the *basin of Q* to be the set of points which eventually map into the interior of Q. In this paper, a set B is a *basin* if it is the basin of some trapping region. This modified definition avoids the problem of determining the attractors of a system. For our choices of a trapping region Q for the black basin in Fig. 1(a), there is an attracting period-2 orbit in Q and also a saddle fixed point. If a basin of attraction contains an attracting fixed point and no other attracting periodic orbits, then the trapping region may include periodic saddles or even invariant Cantor sets, so the orbits of some points in the basin will not converge to the fixed point attractor. Let B denote a basin. A point $x \in M$ is a *boundary point* of B if $x \in \bar{B} \setminus \mathrm{Int}(\bar{B})$. The *boundary* of B is the set $\partial\bar{B} = \bar{B} \setminus \mathrm{Int}(\bar{B})$. (Notice that this concept of boundary is slightly

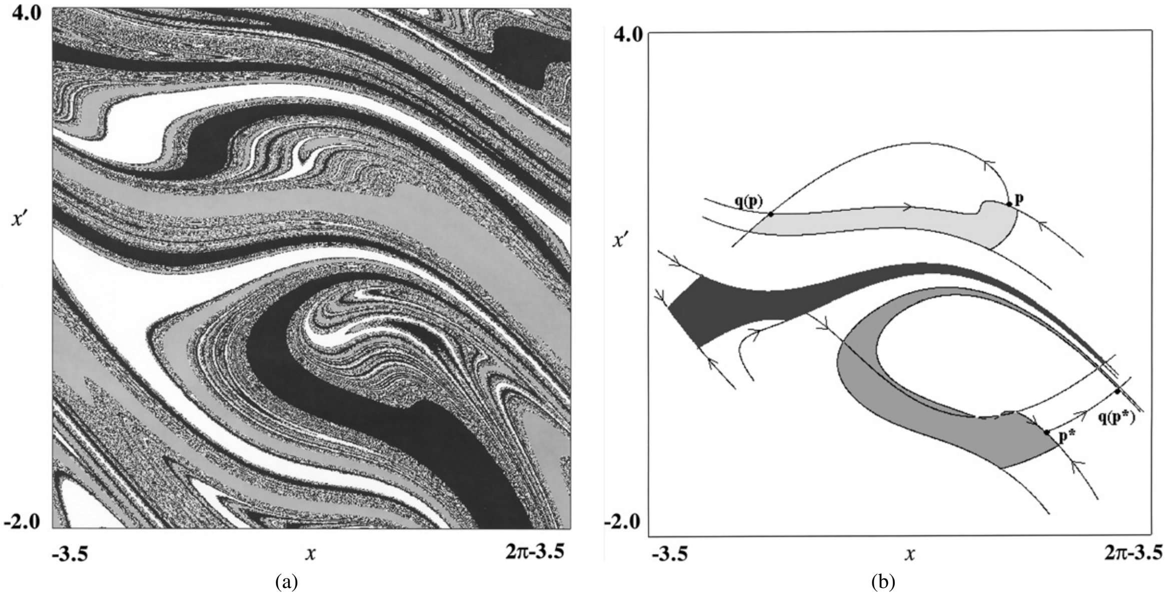

Fig. 1. Basins of attraction and basin cells. Fig. 1(a) displays three basins of the time-2π map of the forced damped pendulum differential equation $x''(t) + 0.2x'(t) + \sin x(t) = 1.66\cos(t)$. They are grey, white and black. All three basins have basin cells which are shown in Fig. 1(b). The uppermost basin cell is in the grey region, the middle basin cell is in the white basin; the lower basin cell is in the black basin. The middle basin cell is generated by a period-3 orbit and so it has six sides and is diffeomorphic to Fig. 2. The two other basins in Fig. 1(a) have a basin cell generated by a saddle-hyperbolic period-2 orbit and so have four sides. A basin cell determines both the structure of its basin and the global structure of the corresponding basin boundary. In the uppermost basin cell, one of the two period-2 points has been labeled by p, and its corresponding primary homoclinic point $q(p)$ is a corner point of the basin cell. Similarly, in the lower basin cell, one of the two period-2 points has been labeled by p^*, and its corresponding primary homoclinic point $q(p^*)$ is a corner point of the basin cell. The result in this paper implies that for each of the basins the limit set of any diverging path in that basin is the basin's entire boundary.

different from the notion of the topological boundary, ∂B.) We say that $\partial \bar{B}$ is a *fractal basin boundary* if it contains a transversal homoclinic point.

Prime ends. Study [24] and Carathéodory [6] introduced in 1913 the notions of 'end' and 'prime end' for a bounded, simply connected region R in the complex plane. According to their expositions, the prime ends are axiomatically defined building blocks (for a compactification) that provide a substitute for the points in the boundary ∂R so that $\bar{R}$ would be homeomorphic to a closed disk. To each 'prime end $\mathcal{P}$', two subsets of ∂R are assigned to $\mathcal{P}$: the 'impression' of $\mathcal{P}$ and the 'principal set' of $\mathcal{P}$. The '*principal set*' of $\mathcal{P}$ can be thought of as the intersection of all limit sets of half-open paths Γ in R converging to $\mathcal{P}$, and the '*impression*' can be thought of as the union of all limit sets of half-open paths Γ in R converging to $\mathcal{P}$. Hence, there are four kinds of these 'prime ends'. The 'prime end' $\mathcal{P}$ is called of *type 1* if its impression consists of a single point; $\mathcal{P}$ is of *type 2* if its principal set is a single point but its impression is a nondegenerate continuum; $\mathcal{P}$ is of *type 3* if its principal set and its impression coincide and its principal set is a nondegenerate continuum; and $\mathcal{P}$ is of *type 4* if its principal set is a nondegenerate continuum and its complement in its impression contains a nondegenerate continuum (see Section 3.2 for details on prime ends).

There exist many articles concerning which types do occur simultaneously. For example, Carathéodory [6] showed that all prime ends of R are of type 1 if and only if ∂R is a Jordan curve. Denjoy [8] constructed an artificial example so that all prime ends are of type 3. Collingwood [7] proved that the union of type 1 and type 3 is residual in the space of prime ends. Therefore, one cannot construct examples for which all prime ends are either of type 2 or of type 4. Piranian [23] constructed an example utilizing a square and line segments; the collection of prime ends of the resulting open region in the figure he created includes all four types of prime ends. The majority of papers in dynamical systems theory utilizing prime end theory as a tool, exploit especially the prime end rotation number. In contrast, we are interested in what types of (ideal) prime ends are involved for basins having a basin cell or for which a basin cell is created or destroyed.

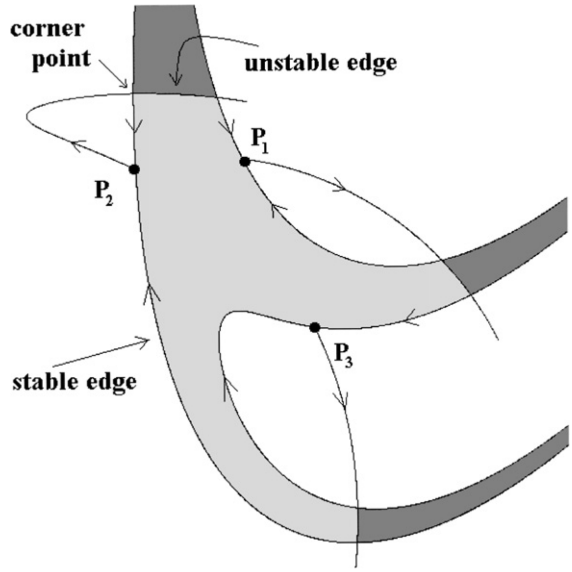

Fig. 2. Basin cell. A basin cell (as shown in light grey) for a two-dimensional diffeomorphism F is a trapping region whose boundary consists of pieces of the stable and unstable manifolds of some period-m orbit P. In this figure, $m = 3$. It follows that this basin cell has $2m$ sides, namely m *stable edges* and m *unstable edges* (pieces of the stable and unstable manifolds of the points of P). We also say that the orbit P *generates* the basin cell. Each of the three dark grey regions is in the basin and is the initial part of a channel (see text) of the basin. The basin has a fractal basin boundary since three of the six corner points of the basin cell are homoclinic points.

Basin cell and results. When we introduced the notion of "basin cell" [16,17] (see also Fig. 2 and its caption), we showed that when such cells exist, fractal basin boundaries can be characterized robustly. Robust structures (that is, those structures that persist under small (smooth) perturbations in the system) are particularly valuable in studying nonlinear dynamics where many structures (like chaotic attractors) can often be destroyed by arbitrarily small (smooth) perturbations. In our approach to the theory of basins, we examine a trapping region whose boundary consists of alternating pieces of the stable and unstable manifolds of certain saddle periodic points. These periodic orbits are said to *generate* this trapping region. If a single periodic orbit generates a trapping region, then it is called a *basin cell*; see Section 3.1 for some properties. A typical trapping region which is a basin cell is shown in Fig. 2. Only a few (if any) of the infinitely many periodic orbits in the boundary of the basin may generate a basin cell. Of course, these periodic orbits are boundary points of the basin cell. We say that a basin B *has a basin cell* if B is the basin of a trapping region that is a basin cell. Basin cells reveal a great deal about the structure of the corresponding basin. For example, the six-sided basin cell of Fig. 2 (the light grey region) is generated by a periodic orbit of period 3. The corresponding basin can be viewed as the central body (basin cell) plus three channels (dark grey) that connect to it. These channels are infinitely long and wind in a very complicated pattern without crossing each other. The channels may vary greatly in thickness but must occasionally get quite thin as they wander back and forth (see the three regions in Fig. 1(a)). Our first result says that under mild hypotheses, if B is a basin, then B has the property that each prime end of B that is defined by a chain of d_B-unbounded regions, is a prime end of type 3 and all other prime ends of B are of type 1 if and only if B has a basin cell (see Section 2 for the theorem). Our last result says that under mild hypotheses, if B_μ is a basin depending on a parameter μ that has a basin cell C_μ for $a < \mu < b$ and B_μ has no basin cell for $\mu \in \{a, b\}$, then at $\mu = a$, the basin B_μ has the property that the basin cell C_μ is created if and only if either there is a saddle node bifurcation or there is a prime end of B_μ that is defined by a chain of d_B-unbounded regions, that is either of type 2 or of type 4. A similar result holds for $\mu = b$, at which a basin cell for B_μ is destroyed. The sequence of bifurcations leading to this phenomenon, will be investigated in the near future.

Remarks. Figure 1(a) displays three basins, B_1 (white region), B_2 (grey region), and B_3 (black region). B_1 has a basin cell (the middle basin cell shown in Fig. 1(b)) that is diffeomorphic to the light grey object in Fig. 2. Each of the two other basins B_2 and B_3 has a basin cell generated by a period-2 orbit. These are the uppermost basin cell and lower basin cell in Fig. 1(b). On the other hand, there are many basins that have no basin cells. For example, if the basin's boundary is not fractal, then it follows that there are no homoclinic points on the boundary and there is no basin cell. Previously, we exploited basin cells as a tool for proving a basin B is a Wada basin. (Basin B is a Wada basin if every basin boundary point of B is also the boundary point of at least two other basins.) In [17] we showed by utilizing properties of basin cells that each of the three basins B_1, B_2 and B_3 in Fig. 1 is a Wada basin and that the boundaries of all three basins coincide. Hence, the three basins have the property that each neighborhood of each point on the boundary of any of the basins intersects all three basins. Therefore, applying the result of [19], the limit set of every unbounded path in any of the three basins B_1, B_2 or B_3 equals $\partial \bar{B}_1 = \partial \bar{B}_2 = \partial \bar{B}_3$.

Overview. The organization of the paper is as follows. The main results for basins of attraction from the view point of prime ends are stated in Section 2. Section 3 contains preliminaries, Section 3.1 contains preliminaries on basin cells and reviews some properties of basin cells, and Section 3.2 contains preliminaries on prime ends and reviews some properties. The proofs of the results are given in Section 4. The pictures in this paper were made using *Dynamics* [18].

2. Main results

Let all of the tangent spaces of M be equipped with an Euclidean inner product $\langle\cdot,\cdot\rangle$. Let $F : M \to M$ be a C^1-diffeomorphism. We assume that all tangencies are *generic*, that is, the manifolds intersect at isolated points but are locally noncrossing, and we assume that no two independent tangencies occur simultaneously. Let $B \subset M$ be an open, simply connected, connected set such that $B = \mathrm{Int}(\bar{B})$. Let $\Gamma : [0,1) \to B$ be a half-open path. We say that $q \in M$ is a *limit point of* Γ, if for every open neighborhood U of q and every $0 < \varepsilon < 1$, there exists t such that $1 - \varepsilon \leqslant t < 1$ and $\Gamma(t) \in U$. We call the collection of all limit points of Γ, the *limit set of* Γ. A point $x \in M$ is B*-accessible* $\Leftrightarrow$ (1) $x \in \partial\bar{B}$ and (2) there exists a path $\gamma : [0,1) \to \mathrm{Int}\,\bar{B}$ (the interior of $\bar{B}$) such that $\lim_{t\to 1}\gamma(t) = x$, that is, the limit set of γ is a single point.

2.1. A path metric d_B and d_B-unbounded regions

For our purposes, the notion of distance between two points in B is needed. A key element for compactifying open sets in the plane is the concept of path metric. For any differentiable path $\varphi : [0,1] \to B$, define the length $\ell(\varphi)$ of φ by $\ell(\varphi) = \int_0^1 |\varphi'(s)|\,\mathrm{d}s = \int_0^1 \sqrt{\langle\varphi'(s),\varphi'(s)\rangle}\,\mathrm{d}s$.

For every pair of points $p, q \in B$, define the path metric $d_B(p,q)$ between p and q to be the infimum of $\ell(\varphi)$ taken over all C^1-paths φ in B having $\varphi(0) = p$ and $\varphi(1) = q$. Note that two points p, q in B might be close in the usual sense but every path lying entirely in B might be quite long, so $d_B(p,q)$ would be large. By the path Γ in B is d_B-diverging we mean that $\lim_{t\to 1} d_B(\Gamma(0),\Gamma(t)) = \infty$, that is, for every $K > 0$ there exists $0 < t_K < 1$ such that for all $t \in (t_K, 1)$, $d_B(\Gamma(0),\Gamma(t)) > K$. Note that this definition does not imply that $d_B(\Gamma(0),\Gamma(t_1)) < d_B(\Gamma(0),\Gamma(t_2))$ for all $0 < t_1 < t_2 < 1$. By the path Γ in B is d_B*-unbounded* we mean that for every $K > 0$ there exists $0 < t_K < 1$ such that $d_B(\Gamma(0),\Gamma(t_K)) > K$, so $\sup_{t\in[0,1)} d_B(\Gamma(0),\Gamma(t)) = \infty$. Note that a d_B-diverging path is d_B-unbounded but a d_B-unbounded path need not be a d_B-diverging path. Define B^{acc} = the completion of B with respect to the metric d_B. The metric for the completion is also denoted by d_B. The completion of an open set need not be identical with its closure, so B^{acc} need not be identical to $\bar{B}$ (the closure of B in M). In our investigations, the B-accessible points are $B^{\mathrm{acc}} \setminus B$, so the set B^{acc} is the union of B and all B-accessible points.

A region R in B is called d_B*-unbounded* if it contains a d_B-unbounded path (that is, there exists a d_B-unbounded path $\Gamma : [0,1) \to B$ such that $\mathrm{Im}(\Gamma) = \{\Gamma(t)\colon 0 \leqslant t < 1\} \subset R$), and it is d_B*-bounded* if it is not d_B-unbounded. Note that a region R in B is d_B-unbounded if there is a d_B-diverging path in R, but some (d_B-unbounded) regions have d_B-unbounded paths but no d_B-diverging paths.

2.2. Statement of the main results

When exploring dynamical systems numerically, frequently one encounters the situation that the map involved has properties which can be reduced to the following. Let $F : M \to M$ be an orientation preserving C^1-diffeomorphism for which

(T) there exists a connected, simply connected, trapping region T whose basin $B_T \subset M$ has the properties B_T is d_B-unbounded, $\mathrm{Int}(\bar{B}_T)$ is connected and simply connected, and $\bar{B}_T \neq M$, and
(S) there exists exactly one B_T-accessible periodic orbit in $\partial \bar{B}_T$, denoted by P, and P is saddle-hyperbolic of (smallest) period m.

We refer to such an F as an *ST-diffeomorphism*. We write $P = \{p_i\}_{i=1}^{m}$, and $B = \mathrm{Int}(\bar{B}_T)$. Note that $\partial B = \partial \bar{B}_T$ by definition of B. Our first result of this paper relates basin cells to the fact that prime ends of the corresponding basin B defined by a chain of d_B-unbounded regions are of type 3. See Section 3.2 for details on prime ends.

Theorem BC-PE3 *(Basin cells and prime ends of type 3). Let $F : M \to M$ be an ST-diffeomorphism. Then*:

B has a basin cell $\Leftrightarrow$ B has m (ideal) prime ends of type 3, *and all other prime ends of B are prime ends of type* 1.

Note 1. Theorem BC-PE3 allows that there are homoclinic tangencies.

Theorem HT-PE2 *(Homoclinic tangencies and prime ends of type 2). Let $F : M \to M$ be an ST-diffeomorphism. Then*:

The inner unstable branch of a point of P (the branch lying wholly in B) has a tangency with the stable branch of a (possible different) point of P $\Leftrightarrow$ every prime end of B is an accessible prime end and B has exactly m prime ends of type 2 *each having a point of P as its principal point.*

Note 2a. The impression of the type 2 prime end in Theorem HT-PE2 equals the closure of a stable branch of some point $p \in P$.

Note 2b. The sequence of nested, d_B-unbounded regions in B defining the prime end in Theorem HT-PE2 has the property that each of the regions contains d_B-unbounded but no d_B-diverging paths.

The outer unstable branch $W^u_{\mathrm{out}}(p_j)$ of $p_j \in P$ has a *tangency inside $\bar{B}$ at q_{ij}* with the stable branch of $p_i \Leftrightarrow$ there exists an arc $A \subset W^u_{\mathrm{out}}(p_j)$ of positive length such that $q_{ij} \in A$, q_{ij} is not an end point of A and $A \setminus \{q_{ij}\} \subset B$.

Theorem HT-PE4 *(Homoclinic tangencies and prime ends of type 4). Let $F : M \to M$ be an ST-diffeomorphism. Assume that the inner unstable branch of every point of P lies wholly in B. Then*:

The outer unstable branch of $p \in P$ crosses a stable branch of p and has a tangency inside $\bar{B}$ with another stable branch of some (possibly different) point p' of P and has no crossings with that branch $\Leftrightarrow$
(a) *there exist segments in the stable manifolds of P and outer unstable branches of P that together form a cell having* 2*m edges, and*
(b) *B has m (ideal) prime ends of type* 4, *and all other prime ends of B are prime ends of type* 1.

Note 3. In Theorem HT-PE4, the impression of a prime end of type 4 equals the closure of the stable manifold of a point of P and its principal set is the closure of a stable branch of that point which is a proper subset of the closure of the stable manifold.

The following theorem concerns a saddle-hyperbolic periodic orbit that generates a basin cell for a parameter in some maximal interval in the parameter space. While the parameter is varied through any of the end points of that parameter interval, then either there is a saddle-node bifurcation or the types of the prime ends of the basin represented by sequences of d_B-unbounded regions are changing. In other words, the basin cells are created or destroyed either by a saddle-node bifurcation or there is a change in the type of prime ends of the basin represented by sequences of d_B-unbounded regions.

Theorem CDBC *(Creation and destruction of basin cells). Let $F_\mu : M \to M$ be a one-parameter family of orientation preserving C^1-diffeomorphisms such that for $a \leqslant \mu \leqslant b$ $(a, b \in \mathbb{R})$ the map F_μ satisfies*:

- (T_μ) *There exists a connected, simply connected, trapping region T_μ whose basin $B_\mu \subset M$ has the properties B_μ is d_B-unbounded,* $\mathrm{Int}(\bar{B}_\mu)$ *is connected and simply connected, and $\bar{B}_\mu \neq M$.*
- (S_μ) *There exists a B_μ-accessible periodic orbit in $\partial \bar{B}_\mu$, denoted by P_μ. The orbit P_μ is saddle-hyperbolic of* (*smallest*) *period m.*
- (M) *For $a < \mu < b$ the B_μ-accessible periodic orbit P_μ generates a basin cell, and for $\mu \in \{a, b\}$, P_μ does not generate a basin cell.*

Then at $\mu = a$, either

(1) *The map F_μ has a forward saddle node bifurcation at P_μ, or*
(2) *The basin B_μ has either a prime end of type* 2 *or a prime end of type* 4.

Note 4a. A similar result holds for $\mu = b$, only in (1) the forward saddle node bifurcation is replaced by backward saddle node bifurcation.

Note 4b. For $a < \mu < b$, the basin B_μ has m (ideal) prime ends of type 3.

Notations. For a set $D \subset M$, we write $\bar{D}$ for its closure in M. We write $P_\mu = \{p_i\}_{i=1}^m$, and $B_\mu = \mathrm{Int}\,\overline{(B(T_\mu))}$. Note that $\partial B_\mu = \partial \overline{B(T_\mu)}$ by definition. B^{acc} is the completion of B in the path metric d_B, and it equals the union of B and the B-accessible points.

3. Preliminaries

Let $F : M \to M$ be a C^1-diffeomorphism. Let p be a saddle-hyperbolic fixed point of F, that is, the eigenvalues λ and μ of the Jacobian matrix $DF(p)$ satisfy $|\lambda| < 1 < |\mu|$. The *stable manifold* $W^S(p)$ of p is the set $W^S(p) = \{x \in M\colon F^n(x) \to p \text{ as } n \to \infty\}$, and the *unstable manifold* $W^u(p)$ of p is the set $W^u(p) = \{x \in M\colon F^n(x) \to p \text{ as } n \to -\infty\}$. A point $q \in M$ is a *homoclinic point* with respect to p if and only if (a) $q \neq p$, and (b) $\lim_{n\to\infty} F^n(q) = p$ and $\lim_{n\to\infty} F^{-n}(q) = p$. For $x, y \in W^S(p)$, we denote the closed segment in $W^S(p)$ with end points x and y by $S_p[x, y]$. For $x, y \in W^u(p)$, we denote the closed segment in $W^u(p)$ with end points x and y by $U_p[x, y]$. A homoclinic point q of F with respect to the fixed point p is called a *primary homoclinic point* if and only if $S_p[p, q] \cap U_p[p, q] = \{p, q\}$. Primary homoclinic points always exist whenever there exist homoclinic points, see Palis and Takens [22].

3.1. Preliminaries on basin cells

For clarity, we repeat the brief definition in the introduction, adding important details. For any basin, there are many choices of a trapping region, but there is at most one periodic orbit that generates a basin cell for that basin. If a periodic orbit generates a basin cell, there are a countable number of ways of choosing its basin cell. A *cell* is a connected, simply connected, compact region such as a disk. A cell C is called a *manifold cell* if the boundary of C is piecewise smooth and there exists a saddle-hyperbolic periodic orbit P such that (a) the boundary of C consists alternately of pieces of the stable manifold $W^S(P)$ and unstable manifold $W^u(P)$ of the periodic orbit P, and (b) every point $x \in \partial C$ that is on both the stable and unstable manifolds of P is a point of transverse intersection of $W^S(P)$ and $W^u(P)$. See Fig. 2 for $m = 3$. In this case, we also say that the cell C is a "manifold cell for P", or, the cell C is *generated by the orbit* P, or also, the orbit P *generates* the manifold cell C.

For a manifold cell C, each of the sides of C that is in the stable manifold of P is called a *stable edge* of C and each of the sides of C that is in the unstable manifold of P is called an *unstable edge* of the cell C. The common point of a stable and an unstable edge of a cell C is called a *corner point* of the cell C. Note that each of the corner points of a cell C generated by a periodic orbit P is either (1) a periodic point, (2) a primary homoclinic point, or (3) a homoclinic point being a heteroclinic point for F^m $(m \geqslant 2)$. Note that for a fixed point p, if p is not a corner point of a cell C then both corner points are homoclinic points, one on each stable manifold branch. In this paper, a *basin* *cell* is a manifold cell C which is a trapping region, so $F(C) \subset C$ and $F(C) \neq C$. Hence, the basin consists of one component.

In the proofs of the theorems, we use the following results on basin cells. The first result gives a criterion that guarantees that the manifold cell generated by a saddle-hyperbolic periodic orbit is a basin cell. The second result concerns certain fractal basin boundaries and says that a basin of attraction B has a basin cell if and only if every d_B-diverging path in basin B has the entire boundary $\partial \bar{B}$ as its limit set. This property reflects a complete entangled basin boundary.

Basic Basin Cell Proposition. *Let $P = \{p_k\}_{1 \leqslant k \leqslant m}$, be a saddle-hyperbolic periodic orbit of F that generates a manifold cell C. Assume that C satisfies the following conditions*:

(a) *P is contained in ∂C*;
(b) *for every integer k $(1 \leqslant k \leqslant m)$, p_k is not a corner point of C*;
(c) *F maps each of the unstable edges of C into C*; *and*
(d) *C has $2m$ edges* (*that is, m stable and m unstable edges*).

Then C is a basin cell.

Proof. All that is needed is to show that C is a trapping region. For a detailed proof, see [17]. □

Theorem BC-LSDP *(Basin cells and limit sets of diverging paths). Let $F : M \to M$ be a C^1-diffeomorphism for which there exists a connected, simply connected, trapping region T.*

Let $B \subset M$ be the basin of the trapping region T such that:

(a) *$\bar{B} \neq T$ and $\bar{B} \neq M$*;
(b) *there exists exactly one B-accessible periodic orbit P*;
(c) *the orbit P is saddle-hyperbolic*;
(d) *the orbit P has a homoclinic point, and the stable manifold of P has no tangency with the unstable manifold of P.*

Then:

B has a basin cell $\Leftrightarrow$ (1) there are diverging paths in B, and (2) for every diverging path Γ in B, the limit set of Γ equals the boundary of B.

Proof. For a proof, see [19]. □

We now present a property in which the inner unstable branches of B-accessible periodic points play a crucial role. It says that if P is the sole B-accessible periodic orbit, then the collection of all B-accessible points coincides with the stable manifold of P. We note that the cell in the Inner Unstable Branch Proposition below may not be a manifold cell and therefore a basin cell, since the intersection need not be transversal. Notice that for every $p \in P$, there is an unstable branch of p in $\bar{B}$, since P is B-accessible. The next result concerns the case when the branch is in B.

Recall that an orientation preserving C^1-diffeomorphism $F : M \to M$ is an S_0T_0-diffeomorphism if

(T_0) there exists a connected, simply connected, trapping region T_0 whose basin $B_T \subset M$ has the properties B_T is d_B-unbounded, $\mathrm{Int}(\bar{B}_T)$ is connected and simply connected, and $\bar{B}_T \neq M$, and
(S_0) there exists exactly one B_T-accessible periodic orbit in $\partial \bar{B}_T$, denoted by P, and P is saddle-hyperbolic of (smallest) period m.

Inner Unstable Branch Proposition. *Let $F : M \to M$ be an S_0T_0-diffeomorphism such that* (1) *P generates a cell with $2m$ edges and m corner points being primary homoclinic points of P, and* (2) *there is exactly one B-accessible periodic orbit which is P. If the inner unstable branches of P are contained in B, then the collection of B-accessible points coincides with the stable manifold of P.*

Proof. For a proof, see [20]. □

3.2. Preliminaries on prime ends

Formalizing and developing ideas of Study [24], Carathéodory [6] introduced in 1913 the notions of 'end' and 'prime end' for a bounded, simply connected region R in the complex plane. According to his exposition, the prime ends are axiomatically defined building blocks (for a compactification) that provide a substitute for the points in the boundary ∂R. Koebe's 1915 paper [13] used the more descriptive term of 'boundary element' for 'prime end'. Freudenthal [9,10] investigated the compactification of topological spaces by adjoining a set (of topological dimension 0) of 'ideal points': the adjoined objects are certain equivalence classes of open sets and Freudenthal called them 'ends'. He [11] showed that the Carathéodory theory of prime ends nicely fits in his "theory of ends". In 1982, Mather [14] wrote an excellent exposition on some topological consequences of Carathéodory's theory on prime ends. During the last two decades, a variety of papers used the prime end rotation number of Carathéodory's theory as a tool for proving certain results in dynamical systems [3,5,4,1,25,2].

Let $F: M \to M$ be an orientation preserving C^1-diffeomorphism. For our purpose, let $B \subset M$ be an open, connected, simply connected region such that $F(B) = B$, $B = \mathrm{Int}(\bar{B})$, B is d_B-unbounded, and $\bar{B} \neq M$, where the bar denotes the closure in M. Write $\partial B = \bar{B} \setminus B$. Let B^{acc} denote the union of B and the B-accessible points. The set B^{acc} is the completion of B in the path metric d_B. Our terminology (for Carathéodory's theory) is essentially similar to Mather's [14].

We first discuss prime points and prime ends. For an open set $V \subset B$, the boundary of V in (the topology of B) is denoted by $\partial_B V$. We say V is *simple* if $V \subset B$, V is open and connected, and $\partial_B V$ is a curve of nonzero finite length with no self-intersections. Note that $\partial_B V$ can have end points that are in $B^{\mathrm{acc}} \setminus B$. Write $\overline{\partial_B V}$ for the curve plus its end points (if any). For simple sets U, V, we say V *divides* U if $V \subset U$ and a $\overline{\partial_B U} \cap \overline{\partial_B V} = \emptyset$. A *chain* is a sequence $\{V_n\}_{n=1}^{\infty}$ of simple sets such that V_n divides V_m for each $n > m$. From now on, we write $\{V_n\}$ for $\{V_n\}_{n=1}^{\infty}$. A chain $\{V_n\}$ *divides* an open set U if for some n, V_n divides U. If $\{U_n\}$ and $\{V_n\}$ are two chains in B, then: the chain $\{V_n\}$ *divides* the chain $\{U_n\} \Leftrightarrow \{V_n\}$ divides each element U_n of the chain $\{U_n\}$. (Note that $\{V_n\}$ *divides* the chain $\{U_n\} \Leftrightarrow$ for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $V_m \subset U_n$.) Two chains $\{U_n\}$ and $\{V_n\}$ are *equivalent* if and only if $\{U_n\}$ divides $\{V_n\}$ and $\{V_n\}$ divides $\{U_n\}$ A chain $\{V_n\}$ in B is a *prime chain* if and only if for every chain $\{U_n\}$ in B for which $\{U_n\}$ divides $\{V_n\}$ it must be the case that $\{U_n\}$ and $\{V_n\}$ are equivalent. If $\{V_n\}$ is a prime chain, then $\bigcap_{n=1}^{\infty} V_n$ contains at most one point. A *prime point* of B is an equivalence class of prime chains of B. In order to define 'prime ends', we first consider prime points having an additional property. We say a simple set is *trivial* if $\partial_B V$ is a closed curve. For any chain $\{V_n\}$, it follows that if V_n is trivial for some n, then V_m is trivial for $m > n$. If a chain contains a trivial element, then we call its prime point a *trivial prime point* of B. If a chain contains no trivial elements, then its prime point is called a *prime end* of B.

We now discuss the topology on the collection of prime points. Let $\hat{B}$ denote the set of prime points of B. There can exist prime ends of B with representative chain $\{V_n\}$ such that V_n is d_B-unbounded (in the path metric d_B) for all $n \in \mathbb{N}$. Such a point is called an *ideal point*. (Ideal boundary points are known to compactify a topological space, see Freudenthal [9,10].) The topology on $\hat{B}$ is described as follows. For every open set $U \subset B$ and $\eta \in \hat{B}$, the prime point η *divides* $U \Leftrightarrow$ if $\{V_n\}$ is a chain representing η, then, for some m, $V_m \subset U$. Let $[U]_{\mathrm{div}}$ denote the set of prime points dividing U, so $[U]_{\mathrm{div}} = \{\eta \in \hat{B}\colon\ \eta \text{ divides } U\}$. Clearly, if U and W are open subsets of B, then $[U \cap W]_{\mathrm{div}} = [U]_{\mathrm{div}} \cap [W]_{\mathrm{div}}$. Thus, $\{[U]_{\mathrm{div}}\colon\ U \text{ is an open subset of } B\}$ is the basis of a topology on $\hat{B}$. From now on, we consider $\hat{B}$ as a topological space, provided with this topology. Convergence of a sequence of prime ends in this topology is given by: $\lim_{n\to\infty} \eta_n = \eta$ given a representative chain $\{V_m\}$ of η, for each m there is an N such that η_n divides V_m for all $n \geqslant N$. With this topology, $\hat{B}$ is compact and is homeomorphic with a closed disk, and the collection of prime ends in $\hat{B}$ is homeomorphic with a circle. We call $\hat{B}$ together with this topology, the (Carathéodory–Freudenthal) *prime end compactification* of B. (Unlike Freudenthal, Study and Carathéodory's theory required that B be bounded.) From now on, we write $\hat{B}$ for the prime end compactification of B and we write $\partial\hat{B}$ for the collection of prime ends of B.

Carathéodory associated to any $\eta \in \partial\hat{B}$ two subsets of ∂B: the 'impression of η', and the 'principal set of η'. For $\eta \in \partial\hat{B}$ and any chain $\{V_n\}$ in B defining η, we write $\mathrm{Imp}(\eta) = \bigcap_{n=1}^{\infty} \bar{V}_n$. The set $\mathrm{Imp}(\eta)$ is called the *impression* of η and it is independent of the chain chosen. A point $x \in M$ is said to be a *principal point* of $\eta \in \partial\hat{B}$ if and only if there is a chain $\{V_n\}$ in B which represents η, such that $\lim_{n\to\infty} \partial_B V_n = x$. For $\eta \in \partial\hat{B}$, the set of all principal points of η

is called the *principal set* of η, denoted by $\mathrm{Prin}(\eta)$. It is obvious that $\mathrm{Prin}(\eta) \subset \mathrm{Imp}(\eta)$. For every prime end $\eta \in \partial\hat{B}$, both $\mathrm{Imp}(\eta)$ and $\mathrm{Prin}(\eta)$ are closed. Not every point in the impression of a prime end need be a principal point. Note that $p \in \mathrm{Imp}(\eta)$ is a principal point of $\eta \Leftrightarrow$ there is a chain $\{V_n\}_{n=1}^{\infty}$ in B which represents η, such that for every open neighborhood U of p there exists $n \in \mathbb{N}$ such that U contains the cross-cut $\partial_B V_n$.

By *a half-open path* in a topological space S we mean a continuous mapping of the half-open interval $[0, 1)$ into S. Let $\Gamma : [0, 1) \to M$ be a half-open path. We say that Γ is a *path in* B if $\Gamma(t) \in B$ for all $0 \leqslant t < 1$. For $a \in S$, we will say a is a *limit point* of $\Gamma(t)$ (as $t \to 1$), if for each neighborhood U of a and each $\varepsilon > 0$, there exists t such that $1 - \varepsilon \leqslant t < 1$ and $\Gamma(t) \in U$. We call the set of limit points of $\Gamma(t)$ (as $t \to 1$), the *S-limit set of Γ*, and denote it by S-limset(Γ). The path Γ is called a *half-open arc* if it has no self-intersections, and it is called an *end-cut* if Γ is injective and $\Gamma(t)$ converges to a point in ∂B as $t \to 1$. Let $\gamma : [0, 1] \to M$ be continuous. The path γ is called a *cross-cut* of B if $\gamma(0, 1) \subset B$; $\gamma(0), \gamma(1) \in \partial B$; $\gamma(0) \neq \gamma(1)$; and γ is injective. The following properties concerning limit sets and accessible prime ends can be found in [6] or [14].

Limit Sets of Paths. Let $\eta \in \partial\hat{B}$ be any prime end of B. Then:

(a) for every half-open path Γ in B such that $\hat{B}$-limset$(\Gamma) = \eta$, the limit set of Γ satisfies $\mathrm{Prin}(\eta) \subset M$-limset$(\Gamma) \subset \mathrm{Imp}(\eta)$;
(b) there exists a half-open arc Γ in B such that $\hat{B}$-limset$(\Gamma) = \eta$ and M-limset$(\Gamma) = \mathrm{Imp}(\eta)$;
(c) there exists a half-open arc Γ in B such that $\hat{B}$-limset$(\Gamma) = \eta$ and M-limset$(\Gamma) = \mathrm{Prin}(\eta)$.

Degenerate Principal Set. For every prime end $\eta \in \partial\bar{B}$, there exists an end-cut γ such that $\gamma(t) \to \eta$ in $\hat{B}$ as $t \to 1 \Leftrightarrow$ the principal set $\mathrm{Prin}(\eta)$ consists of a single point. A prime end whose principal set consists of a single point, is said to be *accessible*. Note that the B-accessible points correspond to accessible prime ends.

Accessible Prime Ends. The accessible prime ends of B are dense in $\partial\hat{B}$.

The impression of a prime end is either a single B-accessible point or is a continuum contained in ∂B. The impression of a prime end contains at least one principal point, and if it does contain exactly one principal point then this point is B-accessible. A prime end whose impression contains more than one principal point contains a continuum of such points but may have no other points. The set of points which are not principal points, if it exists, contains a nondegenerate continuum.

Carathéodory classified the prime ends into four kinds. Let $\eta \in \partial\hat{B}$ be a prime end of B. η is a *prime end of type* 1 $\Leftrightarrow$ $\mathrm{Imp}(\eta)$ consists of a single (principal) point which is B-accessible; η is a *prime end of type* 2 $\Leftrightarrow$ $\mathrm{Imp}(\eta)$ is a nondegenerate continuum and contains only one principal point which is B-accessible; η is a *prime end of type* 3 $\Leftrightarrow$ $\mathrm{Imp}(\eta) = \mathrm{Prin}(\eta)$ is a nondegenerate continuum; η is a *prime end of type* 4 $\Leftrightarrow$ $\mathrm{Prin}(\eta)$ is a nondegenerate continuum and $\mathrm{Imp}(\eta) \setminus \mathrm{Prin}(\eta)$ contains a nondegenerate continuum.

Freudenthal [11] pointed out that ideal points correspond to Carathéodory's notion of prime end. We call an ideal point an *ideal prime end* if it is an inaccessible prime end. In other words, an ideal prime end is a prime end defined by a sequence of nested d_B-unbounded regions in B and its principal set is a nondegenerate continuum. Hence, an ideal prime end is either of type 3 or of type 4. We note that if the chain $\{V_n\}$ defines an ideal prime end of B, then $\{F(V_n)\}$ defines also an ideal prime end of B. Furthermore, if $\{V_n\}$ defines an ideal prime end of B, then (i) for any subset A of B that is bounded in the path metric d_B, there exists n such that $V_n \cap A = \emptyset$, and (ii) if $\gamma(t)$ converges to some point in the $\mathbb{R}^2$ topology and γ is in B, then γ intersects at most finitely many of V_n.

Let $\mathcal{F}$ be an ideal prime end, and let $\{V_n\}$ be a chain that represents the prime end $\mathcal{F}$. Let $\Gamma : [0, 1) \to B$ be a path in B. We say that the path Γ is *converging to* $\mathcal{F}$ if Γ intersects all cross-cuts $\partial_B V_n$ except finitely many. By the Limit Set of Paths property, $\mathrm{Prin}(\mathcal{F}) = \bigcap\{M\text{-limset}(\Gamma)\colon$ half-open path Γ is in V_1 and Γ converges to $\mathcal{F}\}$, and $\mathrm{Imp}(\mathcal{F}) = \bigcup\{M\text{-limset}(\Gamma)\colon$ half-open path Γ is in V_1 and Γ converges to $\mathcal{F}\}$. We call the path Γ a *principal path* if the limit set of Γ coincides with the principal set of $\mathcal{F}$. Hence, $\mathcal{F}$ is of type 3 if and only if every path $\Gamma : [0, 1) \to V_1$ converging to $\mathcal{F}$ is a principal path; $\mathcal{F}$ is of type 4 if and only if there exists a path $\Gamma : [0, 1) \to V_1$ converging to $\mathcal{F}$ which is not a principal path, or equivalently, $\mathcal{F}$ is of type 4 if and only there exist paths $\Gamma_{1,2} : [0, 1) \to V_1$ converging to $\mathcal{F}$ such that (a) Γ_1 is a principal path, (b) M-limset$(\Gamma_1) \subset M$-limset(Γ_2), and (c) M-limset$(\Gamma_1) \neq M$-limset(Γ_2).

Recall that the collection of prime ends $\partial\hat{B}$ of B is homeomorphic with a circle. Let $\partial\hat{F}$ denote the map on the space of prime ends $\partial\hat{B}$ induced by F as follows. Since F is an orientation-preserving diffeomorphism, $\partial\hat{B}$ inherits orientation from M and F induces an orientation-preserving homeomorphism $\partial\hat{F}:\partial\hat{B}\to\partial\hat{B}$ on the circle of prime ends defined by the following: given $\eta\in\partial\hat{B}$ with defining chain $\{V_n\}$, then $\partial\hat{F}(\eta)$ is the prime end with defining chain $\{F(V_n)\}$. Note that ideal prime ends get mapped to ideal prime ends, that is, if $\mathcal{F}$ is an ideal prime end, then $\partial\hat{F}(\mathcal{F})$ is also an ideal prime end, and in some cases ($m=1$) $\partial\hat{F}(\mathcal{F})=\mathcal{F}$.

4. Proofs

For proofs of Theorems HT-PE2 and HT-PE4, see [20]. The notation in [20] is somewhat different, but it is equivalent to the current notation. In this section, we prove Theorems BC-PE3 and DCBC.

Proof of Theorem BC-PE3 (*Basin cells and prime ends of type 3*). Assume the hypotheses and notations of the theorem.

($\Rightarrow$) Assume that B has a basin cell. Recall that the choice of basin cell for basin B is not unique. In particular, if C is a basin cell for B, then $F(C)$ is also a basin cell. This implies immediately that the orbit P has a homoclinic point. In the ($\Rightarrow$) part of the proof of Theorem BC-LSDP in [19], the assumption "the stable manifold of P has no tangency with the unstable manifold of P" has not been used. Following the proof we obtain that (1) there are diverging paths in B, and (2) for every diverging path $\Gamma:[0,1)\to B$, the limit set of Γ equals the boundary $\partial\bar{B}$. If an inner unstable branch of a point of P (a branch that lies wholly in $\bar{B}$) has a tangency with a stable manifold of P, then B has d_B-unbounded, but not d_B-diverging paths, and by applying Theorem BC-LSDP, B has no basin cell. Hence, the inner branches have no tangencies with the stable manifold of P. By the Inner Unstable Branch Proposition, it follows that each point on the stable manifold of P is B-accessible.

Let η be a prime end of B and let $\{V_n\}$ be a chain in B defining η. If V_1 is a d_B-bounded region, then V_n is a d_B-bounded region for all n, and $\{\overline{\partial_B V_n}\}$ converges to a B-accessible point. Hence, if η is represented by a sequence of d_B-bounded regions in B, then η is an accessible prime end and it is of type 1. Assume from now on that V_n is a d_B-unbounded region for all n. Then the end points of $\overline{\partial_B V_n}$ are in two different stable branches of P and if $m\geqslant 2$ these end points are in stable branches of two different points of P. Let Γ be a path in B converging to η. The path Γ is a d_B-unbounded path and by applying Theorem BC-LSDP, we have that Γ is a diverging path in B, and the limit set of Γ equals the boundary $\partial\bar{B}$. Hence, Γ is a principal path. Since Γ is arbitrarily given, every path converging to η is a principal path. Therefore, η is an ideal prime end of type 3. Similarly, for $1\leqslant i\leqslant m-1$, the chains $\{F^i(V_n)\}$ define an ideal prime end of type 3, and the chain $\{F^m(V_n)\}$ represents η. The conclusion is that basin B has m ideal prime ends of type 3 and all other prime ends of B are of type 1.

($\Leftarrow$) Assume that B has m ideal prime ends of type 3, and all other prime ends of B are prime ends of type 1. For every path $\Gamma:[0,1)\to B$ that is converging to an ideal prime of B, Γ is a diverging path and the limit set of Γ is the boundary $\partial\bar{B}$. This implies that the closure of each of the stable branches of points of P equals $\partial\bar{B}$. For every $p\in P$, it follows that the outer unstable branch of p, denoted by $W^u_{\text{out}}(p)$, has a transversal intersection with a stable manifold branch of p. If $W^u_{\text{out}}(p)$ does not intersect transversely another stable branch of a point of P, then either (1) there is a tangency between $W^u_{\text{out}}(p)$ and a stable branch such that P generates a cell, or $W^u_{\text{out}}(p)$ does not intersect another stable branch. In case (1), applying Theorem HT-PE4 gives that B has m prime ends of type 4. In case (2), let η be a prime end of B defined by a chain of d_B-unbounded regions $\{V_n\}$ and the end points of the corresponding cross-cuts $\overline{\partial_B V_n}$ are in two different stable branches. The closure of these two stable branches do not coincide, and it follows in a straightforward manner that there are diverging paths in B having different limit sets. This implies that B has a prime end of type 4. Since each prime end of B is either of type 1 or of type 3, this cannot occur. Hence, it follows that $W^u_{\text{out}}(p)$ has a transversal intersection with a stable manifold branch of p as well as a transversal intersection with another stable branch of a point of P.

If an inner unstable branch of P (the unstable branch that is contained in $\bar{B}$) has a tangency with a stable branch of some (possibly different) point of P, then by Theorem HT-PE2, B has a prime end of type 2. Since all prime ends of B are either of type 3 or of type 1, such an inner tangency does not occur. By the Inner Unstable Branch Proposition, it follows that each point on the stable manifold of P is B-accessible.

We now want to show that B has a basin cell. By a minor variation of the Cross-cut Existence Lemma and the Unbounded Component Lemma in [19], we get the following. For $p\in P$, there exists an arc γ in $W^u_{\text{out}}(p)\cap B^{\text{acc}}$ such that γ is a cross-cut of B that connects two different stable manifold branches of the orbit P, and $B^{\mathrm{acc}} \setminus \gamma$ has a d_B-unbounded component D_0 that contains no point of P. For such an arc γ in $W^u_{\mathrm{out}}(p) \cap B^{\mathrm{acc}}$, the points of intersection of $W^u_{\mathrm{out}}(p)$ and the stable manifold of P at the end points of γ may be nontransversal. Then the cross-cut γ cannot be an unstable edge of a basin cell even in case when P generates a basin cell. However, the following result claims that B has a basin cell generated by P.

Basin Cell Existence Lemma. *Let γ be an arc in $W^u_{\mathrm{out}}(p) \cap B^{\mathrm{acc}}$ such that γ is a cross-cut of B that connects two different stable manifold branches of the orbit P, and $B^{\mathrm{acc}} \setminus \gamma$ has a d_B-unbounded component D_0 that contains no point of P. Let $A \subset W^u_{\mathrm{out}}(p)$ be an arc such that*

(a) *A includes the cross-cut γ,*
(b) *$W^u_{\mathrm{out}}(p)$ intersects two stable branches of points of P transversally at the end points of A, and*
(c) *there is no arc in $W^u_{\mathrm{out}}(p)$ of smaller length satisfying* (a) *and* (b).

Let D^* be the collection of points in B on one side of A such that $D^* \cap D_0$ is d_B-unbounded. Then $B^{\mathrm{acc}} \setminus \bigcup_{n=0}^{m-1} F^n(D^*)$ is a basin cell.

Remark. If $A = \gamma$, then $D^* = D_0$. If $A \neq \gamma$, then D^* consists of a d_B-unbounded component and at least one, but at most finitely many, d_B-bounded components.

Proof. Assume the hypotheses and notations of the lemma. Write $A_0 = A$ and for $0 \leqslant k \leqslant m-1$, define $A_k = F^k(A_0)$. Note that A_k includes a cross-cut of B^{acc}, where $0 \leqslant k \leqslant m-1$. Hence, for every k $(0 \leqslant k \leqslant m-1)$, there is an arc S_k in $W^S(F^k(p))$ such that the union of the arcs $\bigcup_{k=0}^{m-1}(S_k \cup A_k)$ in the stable and unstable manifolds of P constitute a closed curve. Since B is simply connected and since $W^u_{\mathrm{out}}(p)$ intersects two stable branches of points of P transversally at the end points of A, the bounded region enclosed by this closed curve is a manifold cell, denoted by C.

Let U be an unstable edge of C. Thus $U = A_k$, for some $0 \leqslant k \leqslant m-1$. Since $U \subset B^{\mathrm{acc}}$, $F(U) \subset B^{\mathrm{acc}}$. By construction, either $F(U)$ is an unstable edge of C or $F(U)$ does not intersect any of the unstable edges of C. Let a and b be the end points of U. Obviously, $F(a) \in C$, $F(b) \in C$. If $F(a)$ is a corner point of C, then $F(b)$ is a corner point of C and $F(U)$ is an unstable edge of C. If $F(a)$ is not a corner point of C then $F(b)$ is not a corner point of C and, since B is simply connected, $F(U) \subset C$. This implies that $F(U)$ is not an unstable edge of C. Since U is an arbitrary unstable edge of C, we now have that every unstable edge of C is mapped into C under F (and also $F(\partial C) \subset C$). Since C is a manifold cell, by the Basic Basin Cell Proposition, C is a basin cell. □

This completes the proof of the theorem. □

Proof of Theorem CDBC (*Creation and destruction of basin cells*). Assume the hypotheses and notations of the theorem.

For $a < \mu < b$, the periodic orbit P_μ generates a basin cell for basin B_μ. In particular, if C_μ denotes the basin cell, then (a) the boundary of C_μ consists alternately of finitely many pieces of the stable manifold $W^S(P_\mu)$ and unstable manifold $W^u(P_\mu)$ of the periodic orbit P_μ, and (b) every point $x \in \partial C_\mu$ that is on both the stable and unstable manifolds of P_μ is a point of transverse intersection of $W^S(P_\mu)$ and $W^u(P_\mu)$. Recall that the choice of a basin cell is not unique. In particular, if C_μ is a basin cell, then $F_\mu(C_\mu)$ is also a basin cell. For $\mu = a$, either there is a saddle-node bifurcation at P_a, or there is no saddle-node bifurcation at P_a. If there is a saddle-node bifurcation at P_a then it is a forward saddle-node bifurcation from which periodic saddle-hyperbolic orbit P_μ and a periodic orbit attractor are emerging for $\mu > a$. An example of this phenomenon that a saddle-hyperbolic periodic orbit created at a saddle-node bifurcation generates a basin cell has been presented in [15]. This proves case (1) of the theorem.

From now on, we assume there is no saddle-node bifurcation at P_a. For $a < \mu < b$ and $p_\mu \in P_\mu$, the outer unstable manifold branch $W^u_{\mathrm{out}}(p_\mu)$ intersects a stable manifold branch of p_μ transversally as well as at least one other stable branch of a (possibly different) point $p'_\mu \in P_\mu$ transversally. Since P_μ generates a basin cell for $a < \mu < b$, and the manifolds of points of P_μ depend C^1 on μ, P_a generates a cell. The result of Theorem BC-PE3 says that basin B has a basin cell $\Leftrightarrow$ every prime end of B defined by a chain of d_B-unbounded regions is of type 3 and all other prime ends of B are of type 1. This implies for basin B_a the following. Basin B_a has no basin cell $\Leftrightarrow$ basin B_a has at least one prime end defined by a chain of d_B-unbounded regions that is not of type 3. Since every prime end defined by a chain of d_B-unbounded regions can only be of type 2, 3 or 4, it follows that basin B_a has either a prime end of type 2 or of type 4. This proves case (2) of the theorem. □

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