Blending of mathematics and physics

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Chapter 6

General discussion

In order to understand advanced physics topics, undergraduate students need to be proficient in both the mathematical and physical concepts involved and they should be able to relate these different concepts in their reasoning. However, we know from literature that this is often difficult for students (e.g. Bollen et al., 2016; Gupta et al., 2007; Modir et al., 2019; Ryan et al., 2018; Wilcox et al., 2013; Wilcox & Pollock, 2015). Therefore, in this dissertation, we added to this growing body of knowledge by further investigating student reasoning when blending mathematics and physics and by formulating possible solutions to foster this blending process in instruction.

We approached the interplay of mathematics and physics in student reasoning from a blending perspective, which is associated with the conceptual blending framework (Fauconnier & Turner, 2003b). We chose heat transfer in a one-dimensional physical system as the setting of our research because this is a topic situated in the undergraduate curriculum where advanced mathematical concepts have to be blended with physical concepts. We addressed two aims: developing a deeper understanding of reasoning processes when students combine mathematics and physics (theory-oriented aim), and building on this understanding to develop an instructional approach that scaffolds the blending of mathematics and physics in student reasoning (practice-oriented aim). This resulted in three studies, on which we reported in four chapters. We first built a theoretical basis and eventually developed and evaluated a tutorial. The development of this instructional material not only built on that theoretical basis, but also contributed to it.

In this final chapter, we highlight the main findings of the presented studies. Next, we discuss the role of the conceptual blending framework in this dissertation. We formulate implications for instruction and finish with some suggestions for future research.

6.1 Main findings

In Figure 6.1, we summarize the main findings per chapter. In this section, we discuss each chapter briefly.

6.1.1 Study 1: Difficulties with boundary conditions for the diffusion equation

As a first step, we explored student understanding and reasoning in the context of heat transfer and particle diffusion and their respective description in terms of the heat/diffusion equation. The aim was to explore several instances where blending plays a role in student reasoning in this context. We conducted task-based, think aloud interviews with six students from the Bachelor programmes of Physics and

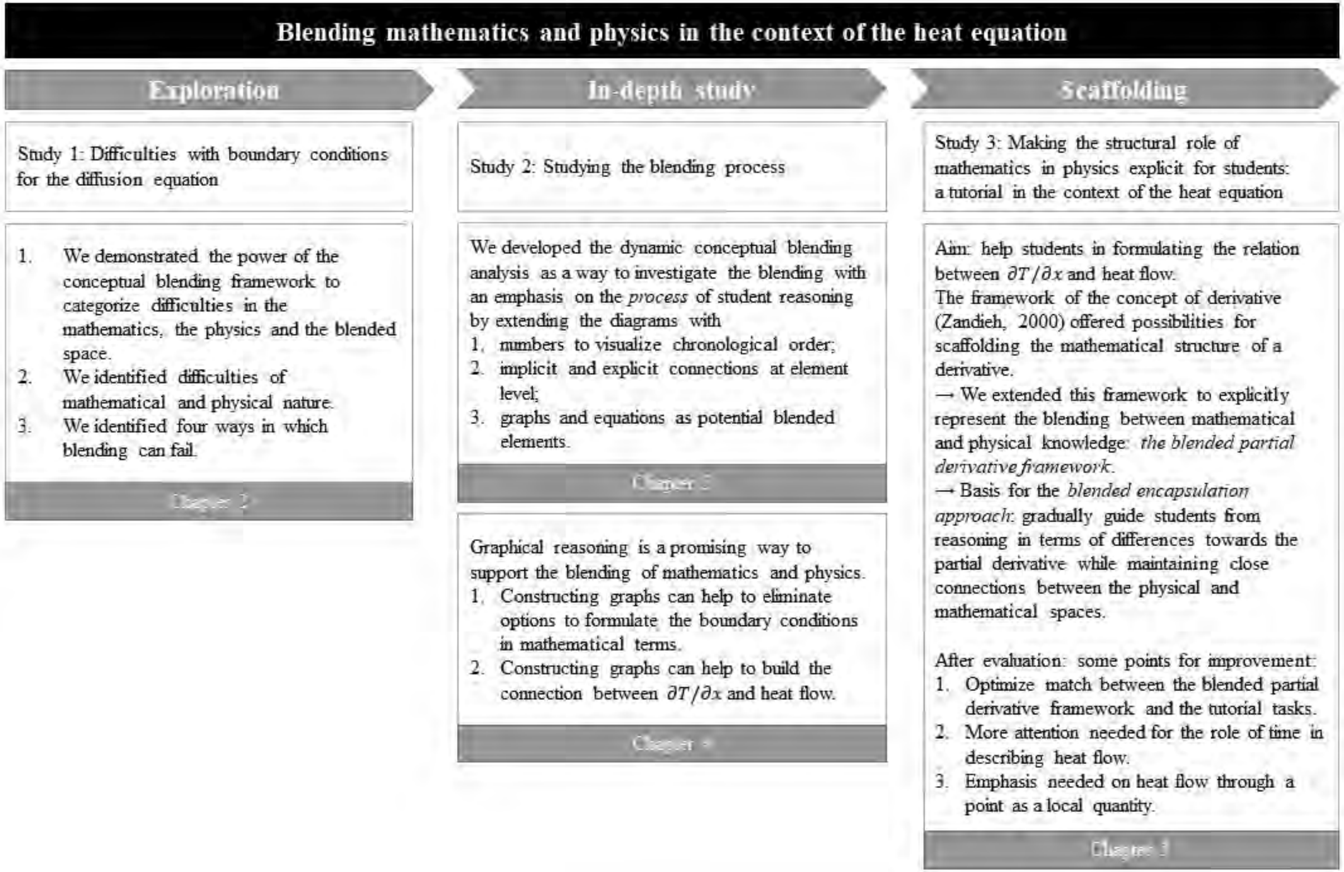

Figure 6.1: Overview of the research findings per chapter.

Mathematics at KU Leuven and six students from the Bachelor programme of Physics at the University of Groningen. We narrowed our focus in the analysis to boundary conditions because from a preliminary analysis of the data this appeared to be a topic where blending mathematical and physical knowledge was challenging.

We identified a set of difficulties and demonstrated the power of the conceptual blending framework to categorize these difficulties in the mathematics, the physics and the blended space. We identified difficulties in both input spaces, i.e. of mathematical and of physical nature. Furthermore, we also identified different ways in which the blending process can fail. This indicates that even with well-developed input spaces, students might encounter problems in solving a task because they are unable to blend these input spaces in a correct way. This confirms that teaching students mathematics first and expecting them to transfer this to their physics courses is not sufficient (Karam et al., 2019).

The identified difficulties can be used as a starting point to design teaching/learning activities at a later stage. In our developed tutorial, which is partly presented in Chapter 5, we focused on some of the difficulties that were identified in Chapter 2:

- we incorporated a task focusing on the role of boundary conditions in solving the partial differential equation;
- we incorporated boundary conditions of different forms in order to shift the focus away from "$T = 0$";
- we paid explicit attention to reasoning with functions of two variables in the tasks and used graphical representations to help students with this;
- we designed tasks that intend to elicit the difference between $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$;
- we provided scaffolding in the structure of the tutorial to guide the blending process.

We note that, to the best of our knowledge, there has not been done any research yet on student understanding of the physical meaning associated with boundary conditions for the heat/diffusion equation. With this study, we set a first step in exploring this topic. We only conducted interviews with a small population (twelve students), but our analysis was detailed and deep. However, the interview data showed a lot of variation, and the identified difficulties might just be the tip of the iceberg. More research on this topic is necessary to reach saturation. The identified ways of failed blending in this study gave only a first indication of what is happening during a blending process and which obstacles occur. We therefore set up the second study to look deeper into this blending process.

6.1.2 Study 2: Studying the blending process

We developed a second task-based interview, which solely focused on boundary conditions for the heat equation. In the interview tasks, we provided mathematically formulated boundary conditions and asked students to discuss their physical meaning and vice versa. We conducted this second round of interviews with four pairs of physics and mathematics majors from KU Leuven. This interview study formed the basis for two papers, presented in Chapters 3 and 4. First, we developed the research instrument to study the blending process in detail: dynamic conceptual blending diagrams. Second, we used this research instrument to investigate the role of graphs in blending mathematical and physical concepts.

Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics

Blending diagrams in PER and MER have mainly been used to visualize the final product of student reasoning instead of the process (Bing & Redish, 2007; Bollen et al., 2016; Gerson & Walter, 2008; Hu & Rebello, 2013). In Chapter 3, we proposed the dynamic blending diagram (DBD) as a way to represent and analyze student reasoning with a focus on the reasoning *process*. By presenting two case studies, we illustrated the construction of a DBD from interview transcripts.

In order to visualize the reasoning *process* in a diagram, there are some specific additions in these DBDs compared to other blending diagrams in PER and MER:

1. We add a number to each element in the mental spaces to visualize chronological order.
2. We draw connections at element level and distinguish between explicit and implicit connections.
3. We give a special role to graphs and equations in the diagrams because of their potential as blended elements. When a student uses a graph or an equation in combination with physical meaning, we consider it as an indication of blending.

Furthermore, we also imported the elements from the transcript into the blending diagram as literally as possible to minimize bias and interpretations.

Doing a DBD analysis enables a researcher to visualize and analyze the reasoning in a very detailed and fine-grained way. By identifying individual connections between elements, we can judge the degree of integration in the reasoning. Numbering the elements enables the reader to follow the line of reasoning in the diagram. However, it is important to note that the DBD analysis is very extensive and time-consuming. This means it cannot be used for quick analysis or directly as a teaching tool.

Whereas in earlier work in PER the blended space gives a comprehensive overview of the final product of the blending process, this is slightly different in our DBD approach, where the blending has to be interpreted in a broader way than just the elements in the blended space. Indeed, the blended space contains elements that combine mathematics and physics ideas and graphs and equations that carry physical meaning, but also the existing connections between elements in the physics and mathematics input spaces are signs that blending is taking place.

Role of graphs in understanding the connection between $\frac{\partial T}{\partial x}$ and heat flow from a dynamic conceptual blending perspective

Graphs have a special position in DBDs. A graph can be a purely mathematical element, but once the student assigns physical meaning to it, it becomes a blended element. In Chapter 4, we investigated the role of graph construction in the blending process, using the dynamic blending analysis. Specifically, we investigated how graph construction helped students in blending the physical and mathematical meaning of the partial derivatives $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$.

The choice to focus on the partial derivatives $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$ originated partly from the finding in Chapter 2 that students have difficulties distinguishing between the (physical and mathematical) meanings of these partial derivatives. In the second interview round, we saw this finding confirmed, but we also observed that some students tried to find meaning using graphical reasoning. The analysis with DBDs is very suited to investigate the role of these graphs in the reasoning process because it is so detailed and visual.

First, we observed that constructing graphs sometimes helped students to eliminate options to formulate the boundary conditions in mathematical terms. By sketching graphs of the temperature distribution in the system at different times, students saw that for some situations the temperature at the boundaries was changing over time, which helped them seeing that $\frac{\partial T}{\partial t}$ could not be equal to zero and made it an inadequate boundary condition.

Second, we observed that graphs have the potential to help students to build the connection between $\frac{\partial T}{\partial x}$ and heat flow. For all student pairs, the graphs helped in formulating a relation between $\frac{\partial T}{\partial x}$ and heat flow. For two pairs, however, this was quite a superficial relation. These two pairs of students used the $T(x)$-graphs at fixed times and referred to the slope in the x-direction and, as such, showed to understand the graphical meaning of the partial derivative. However, they did not interpret the partial derivative as being related to a temperature difference between nearby positions.

Instead, they immediately jumped to heat flow without visible reasoning. One group showed deeper understanding. They started from $\frac{\partial T}{\partial x}$ as the slope of the $T(x)$-graph at a specific time t, but they explicitly interpreted it in terms of temperature differences, which eventually led them towards heat flow. The graphical interpretation of the partial derivative as slope appeared helpful for all pairs to build the connection between $\frac{\partial T}{\partial x}$ and heat flow, but without showing a good insight in the concept of partial derivative (Zandieh, 2000) and the way this partial derivative relates to its physical meaning in terms of temperature gradient, students will not connect the mathematical and the physical meaning.

In conclusion, even though graph construction helped students to progress in their reasoning, in most cases it was not sufficient to arrive at a complete understanding of the mathematical and physical aspects of the task. This finding taught us a lot about how students understand the relation between $\frac{\partial T}{\partial x}$ and heat flow and it inspired the development of the tutorial in study 3 (Chapter 5).

6.1.3 Study 3: Making the structural role of mathematics in physics explicit for students: a tutorial in the context of the heat equation

In the last study, we developed a tutorial that aimed to foster the blending of mathematics and physics in student reasoning about heat transfer and boundary conditions for one-dimensional physical systems. We formulated three design principles that formed the basis of our design, all aiming to promote the blending of mathematics and physics: giving explicit attention to both the mathematical and the physical aspects, stimulating graphical reasoning, and guiding students to encapsulate the partial derivative in close connection to its physical meaning (blended encapsulation).

In Chapter 5, we reported on the development and analysis of the part of the tutorial where we introduce students to the relation between $\frac{\partial T}{\partial x}$ and heat flow. We established in Chapters 2 and 4 that this is a conceptually difficult relation. We also established that graphical reasoning can help students, but that more scaffolding is necessary. The framework for the concept of derivative (Zandieh, 2000) with its layers of process-object pairs offered possibilities for scaffolding the mathematical structure of a derivative. We extended this framework to explicitly include the blending between mathematical and physical knowledge that has to take place when reasoning about the relation between $\frac{\partial T}{\partial x}$ and heat flow. We call this extended framework the *blended partial derivative framework* (see Figures 5.2 and 5.3). This framework formed the basis for the blended encapsulation approach, in which we gradually guide students from reasoning in terms of differences towards reasoning in terms of the partial derivative while maintaining close connections between the physical and mathematical spaces.

We conducted teaching-learning interviews with three groups of three mathematics or physics majors from KU Leuven. This data allowed us to make a first evaluation of the blended encapsulation approach. Generally, we conclude that this approach can potentially help students in recognizing that temperature differences lead to heat flow and how these temperature differences between positions can be described using the concept of a partial derivative of temperature with respect to position. We observed this intended line of reasoning in two out of the three interviewed groups. However, there are still some weaknesses in the current design that need to be optimized in order for the approach to reach its full potential.

Apart from this general evaluation of the blended encapsulation approach, we identified two topics within our tutorial that appeared to be conceptually difficult for students. Our current design did not suffice in guiding students through these topics. However, the data helped us to define the difficulties with these topics more precisely and to suggest some ways in which this can be tackled in instruction.

The first observation is that two out of the three groups struggled with the role of time in the description of heat flow, despite our intentions in the design. We observe that the word ‘flow’ induces reasoning in terms of time. However, the crucial point here is that, when discussing heat flow, it is correct to reason in terms of rate of change of *energy/heat* over time, but not in terms of rate of change of *temperature* over time.

The second conceptually difficult topic was the difference between reasoning about the physical system at macroscopic and microscopic level. We did not anticipate on this finding in our design, but we established that it plays an important role in student reasoning. Two out of the three groups of students did not reason in terms of the heat flow through the boundary (microscopic) but rather in terms of the temperature of and heat flow through the entire system (macroscopic). This approach hindered them in formulating mathematical boundary conditions for the case where the boundary was isolated.

We hypothesize that the idea of blended encapsulation might also be useful for other physical concepts that are modelled by a derivative or a (partial) derivative. Moreover, it might be interesting to develop a similar approach for phenomena described by other mathematical concepts that have such a layered structure, e.g. an integral. This opens the possibility of incorporating the approach in many situations where physical concepts are described mathematically.

6.2 The different roles of conceptual blending in this dissertation

The conceptual blending framework is central in this dissertation. It is an upcoming framework in physics and mathematics education research (PER and MER) (Close & Scherr, 2015; Dreyfus et al., 2015; Edwards, 2009; Gregorcic & Haglund, 2021; Hutchins, 2005; Podolefsky & Finkelstein, 2007; Wittmann, 2010; Yoon et al., 2011; Zandieh et al., 2014) because it offers a way to think about reasoning processes in which students have to combine elements from different mental spaces, which can be disciplines, but also different topics within a discipline. In PER especially, the conceptual blending framework is gaining momentum to interpret the interplay of mathematics and physics in student reasoning (Bain et al., 2018; Bing & Redish, 2007; Bollen et al., 2016; Hu & Rebello, 2013; Huynh & Sayre, 2019; Schermerhorn, 2018; Taylor & Loverude, 2018). We opted for this framework because it matches well with the way we view the interplay of mathematics and physics at the undergraduate level: mathematics and physics are considered equally important, none solely serving the other one, and connections have to be made in both directions. Concepts and structures of mathematics have to be combined with physical knowledge and intuition, which then ideally results in an enhanced understanding of both.

The role of the conceptual blending framework as an analytical framework evolved over the course of this dissertation. In Chapter 2, conceptual blending provided a way to structure the different difficulties we identified. We formulated definitions of the three mental spaces: the physics, mathematics and blended space, and used these to organize the difficulties in the categories 'difficulties of physical nature', 'difficulties of mathematical nature', and 'difficulties with the blending of both'. These categories originated from the conceptual blending framework, but the role of the framework was rather limited.

In Chapter 3 we extended the use of the blending framework with the aim to visualize and analyze the reasoning process. In PER and MER literature, the focus was primarily on visualizing the product of the reasoning, while we focus on the process. We developed the methodology of constructing dynamic blending diagrams and optimized the definitions of the mental spaces. In Chapter 4, we applied this analysis method to study the role of graphs in the blending process.

One of the most challenging steps in the research documented in Chapters 2 and 3 was defining the different mental spaces. Most studies in physics and mathematics education research discussed in this dissertation did not explicitly define the mental spaces that were used (e.g. Bain et al., 2018; Bing & Redish, 2007; Bollen et al., 2016; Gerson & Walter, 2008; Hu & Rebello, 2013; Huynh & Sayre, 2019). However, as we are interested in the way students combine elements from different mental spaces, it is important to define these spaces explicitly. We were inspired by the way Sirnoorkar et al. (2016) formulated definitions for mathematical and physical propositions in a typical derivation in physics, and used this as the basis for the definitions of our input spaces. After several cycles of data analysis, both in Chapters 2 and 3, we arrived at the categorization shown in Table 3.1.

As mentioned before, graphs and equations play a special role in the categorization: when used in an abstract, mathematical way, they are categorized as part of the mathematics space, but when they are used to discuss a specific physical situation, they serve as mathematical representations of physical meaning and, hence, we consider them as blended elements. The place of graphs and equations in a blending diagram is something that we have seen developing throughout the different implementations of the blending framework in PER. Bing and Redish (2007) already placed equations in the blended space when they were used in combination with physical meaning. Hu and Rebello (2013) accounted for the double role an equation can have. They formulated a symbolic input space, which represents the technical role of mathematics in physics and contains symbols and equations. These formulas can also have a place in the blended space, but only if there is physical meaning attached to the symbols and structures in the equation. In the work of Gerson and Walter (2008) we observed the same mechanism, but with graphs. When a graph was used in its abstract (mathematical) form, it was part of an input space, but when the graph had contextual meaning (the problem was about water levels) attached to it, it became part of the blended space. In our work, we formalized this trend in the characterization of the different mental spaces.

In Chapters 2, 3 and 4, conceptual blending was our primary analytical framework. It served well to analyse the reasoning process when students combine mathematical and physical knowledge. However, once we wanted to provide more scaffolding to that blending process in instructional materials, like we did in Chapter 5, we needed another framework to complement the conceptual blending perspective. In our specific case, we wanted to guide students in relating $\frac{\partial T}{\partial x}$ and heat flow. The conceptual blending perspective is still crucial: we want to foster connections between a mathematical and a physical concept. However, to provide the necessary scaffolding, this needed to be complemented by a framework that offers a way to break the mathematical concept of a (partial) derivative down into several steps or layers that help guiding the blending process. We started from a MER framework that describes the different aspects of the concept of derivative (Zandieh, 2000). We then extended this framework, having the blending perspective in mind, to incorporate the relation between the mathematical concept of a partial derivative $\frac{\partial T}{\partial x}$ and the physical concept of heat flow. The resulting *blended partial derivative framework* (see Figures 5.2 and 5.3) formed the basis of the instructional approach we proposed: blended encapsulation. To our knowledge, conceptual blending has not been used yet in PER or MER to develop instructional materials. With this work, we set a first and promising step in the development of a curriculum aimed at fostering meaningful blending of mathematics and physics.

6.3 Implications for instruction

In the different studies presented in this dissertation, we discussed their relevance for the research literature. In this section, we want to bring together the practical implications for instruction. Although the research presented in this dissertation was done in a very particular context and on a very specific topic, we think it is important to reflect on the implications of our findings for instruction. Making the step from discipline-specific research findings to actual impact on teaching practice is not always trivial, but it starts by thinking about possible implications and recommendations. Overall, we recommend to...

1. not only teach mathematics and physics in a separate and sequential way, but also complement it with a blended approach to teaching and learning mathematics and physics. As shown in this dissertation, a blended approach also has the potential to strengthen the understanding of the individual disciplines.
2. make the structural role of mathematics in physics explicit in teaching. It should be made explicit for students **why** a certain mathematical description fits a certain physical concept. Instruction should stimulate students to think about the meaning of mathematical concepts in physical contexts by providing time and practice for this.
3. use a combination of different representations, like graphs, equations and verbal descriptions. It challenges students to approach the same content in different ways and form connections. Specifically, we established that stimulating graphical reasoning can support the blending process.
4. also offer exercises that ask for conceptual and semi-quantitative reasoning. This challenges the students to think about the relevant mathematical and physical concepts without hiding behind the technical calculations that are often a big part of problem solving.
5. design learning environments in such a way that they stimulate active engagement among the students.

In what follows, we ground these recommendations in the context of the research presented in this dissertation. However, these general recommendations have a wider scope than the specific context of boundary conditions for the heat equation.

6.3.1 Teaching mathematics and physics in a blended way

Research shows that students have difficulties with the mathematics used in physics at all levels of education. It has been established that teaching more mathematics as a prerequisite for physics, and expecting students to transfer that mathematical knowledge to a physics context, is not solving the whole problem (Karam et al., 2019). Often in physics courses, the blending with mathematics is not taught in an explicit way. We might even say that the blending of mathematics and physics is taken for granted; as if it will happen naturally. Based on the findings presented in this dissertation, we see potential in an approach to teaching and learning mathematics and physics based on conceptual blending.

First, we claim that when a student's mathematics and/or physics space(s) is/are not entirely developed, instruction based on blending can have a positive effect on the development of these input spaces. We regularly observed that students connect mathematical and physical knowledge, and thus blend, and in that process return to their physics or mathematics space and extend it, resolve an issue, or change their initial thought. We underpin this claim with an example from Chapter 2 and discuss how we included it in our instructional design in Chapter 5. We observed that some students had difficulties reasoning with functions of two variables (Chapter 2). One could argue that this implies a problem in their prior mathematical knowledge (in the mathematics space) and that this topic needs more attention in their Calculus course. However, this alone would probably not solve the problem (e.g. Eichenlaub & Redish, 2019; Greca & de Ataíde, 2019; Redish & Kuo, 2015). In Chapter 5, we presented a possible way to account for this specific mathematical difficulty in a conceptual blending format. At different points in the tutorial (see Appendix C), we developed tasks that explicitly focus on the fact that $T(x,t)$ is a function of two variables, which therefore has two different partial derivatives. In these tasks, the aim is to form connections between mathematical and physical knowledge. We asked students for instance to:

- construct two-dimensional graphs of both the temperature distribution $T(x)$ at different times and the evolution of the temperature $T(t)$ at different positions based on their physical understanding of heat transfer in a one-dimensional system;
- check if these two perspectives ($T(x)$ and $T(t)$) are in agreement with a provided three-dimensional plot;
- reason about the physical and mathematical meaning of both $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial t}$.

Important is that all these tasks focused on the blending of mathematical and physical knowledge, but that we also had an indirect mathematical goal in mind: strengthening students' reasoning with functions of two variables. This supports the idea that in teaching, we should not limit ourselves to developing the separate spaces, but that offering students opportunities to make blending explicit can also result in an extended understanding of mathematical and physical knowledge separately.

Second, we claim that even if both input spaces, the mathematics and physics space, are well developed, this does not necessarily result in a fruitful blend. This implies that instruction should explicitly focus on the blending process too. We illustrate this with two examples from this dissertation.

A common problem we reported in the different chapters was that some students described heat flow by using $\frac{\partial T}{\partial t}$. Often, this is not merely a mathematical or a physical problem. Many of our students had an understanding of heat flow as a physical process and they also had an understanding of $\frac{\partial T}{\partial t}$ as a derivative, which expresses the slope of the $T(t)$-graph. The problem seems to be situated in the blending process. Therefore, it does not help to only invest more in the separate instruction of the mathematical and physical concepts involved. Instruction should rather focus on the blending process, where mathematical and physical knowledge is integrated. We tackled this problem with the blended encapsulation approach in our tutorial. Moreover, the tutorial also generally stimulated the integration of mathematical and physical knowledge (see Appendix C).

We observed a similar pattern in students' understanding of the concept of temperature gradient in Chapter 5. Our interpretation there was that the students had difficulties transferring their mathematical knowledge about the concept of gradient to the physics context of one-dimensional heat transfer, while the system is described using a function with two variables $T(x,t)$. The different nature of the physical variables position and time plays a crucial role here. The problem here is situated in the formation of the blend, where this mathematical concept needs to be incorporated in the physical situation it models. In mathematics, the gradient of a function $f(x,y)$ of two variables is given by the 2D-vector $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. In the context of the one-dimensional heat equation, however, the independent variables x and t have different roles and the temperature gradient is not given by $(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial t})$, but by $\frac{\partial T}{\partial x}$. Merely teaching students about gradient in a mathematics context will not solve this problem. Instruction should focus on scaffolding the process of blending the mathematical and physical knowledge involved.

6.3.2 Teaching the structural role of mathematics in physics

As mentioned in Chapter 1, the role of mathematics is twofold (Uhden et al., 2012). There is the technical role, the algorithmic use of mathematics, where mathematics is seen as an external instrument, a tool without any physical content. Second, there is the structural role, where mathematical structure penetrates into the construction of the physical concept itself. The structural role of mathematics in physics provides the biggest challenge in learning and understanding physics, and also in teaching physics. Often, this structural role stays implicit in the teaching and learning process (Greca & de Ataíde, 2019). We recommend to make the structural role of mathematics in physics explicit in teaching. It should be made explicit for students **why** a certain mathematical description fits a certain physical concept.

In Chapter 5, we focused on blended encapsulation as a teaching approach and its implementation in a tutorial. Blended encapsulation is an approach to learning mathematics and physics in a blended way. Furthermore, it brings something else to the table: it is an approach that guides students through the process of concept construction in which the structural role of mathematics in physics is made explicit.

We established at several instances in this dissertation that students had difficulties formulating the relation between $\frac{\partial T}{\partial x}$ and heat flow. In Chapter 4, we observed a student pair that used the concept of partial derivative in different representations (symbolical and graphical) and connected this to thinking in terms of temperature *differences*. These students in some way 'peeled' the layers of the concept of partial derivative until they arrived at a level at which they were comfortable to make the blend with physics.

Zandieh's framework for the concept of derivative offered a very well-suited overview of all the aspects of the understanding of the concept of derivative in different representations. The framework makes the underlying structure of a derivative explicit. Therefore, it was the perfect starting point to make the structural role of mathematics in physics explicit.

6.3.3 Graphical reasoning as a way to support blending

Our third recommendation is to use a combination of different representations, like graphs, equations and verbal descriptions. It challenges students to approach the same content in different ways and form connections. Research shows that the use of multiple representations might enhance student learning and supports the development of a deeper conceptual understanding (e.g. Ainsworth, 1999; Klein, Viiri, Mozaffari, Dengel, & Kuhn, 2018; Podolefsky & Finkelstein, 2006; Von Korff & Rebello, 2012).

In this dissertation we established specifically that stimulating students to also reason in a graphical way instead of solely focusing on symbolic representations, can foster the blending of mathematics and physics and help to deepen their understanding. This contributes to the more general consensus in research that implementing multiple representations in physics instruction is valuable.

As mentioned before, in the work presented in Chapters 2 and 3 we already recognized the importance of graphs as elements that can form a bridge between mathematical and physical knowledge and thus can foster blending. As Rodriguez et al. (2019, p. 1) stated, "successful interpretation of a graph involves a combination of mathematical expertise and discipline-based content to reason about the relationship between the variables and to describe the phenomena represented".

In our own data, we saw that students actively constructed graphs in their attempts to connect mathematical and physical understanding, which was in agreement with the central position of graphs in our DBDs. Therefore, in Chapter 4 we investigated the role of graph construction in the blending process in more detail. We established that constructing graphs sometimes helped students to eliminate options to formulate the boundary conditions in mathematical terms. Second, we observed that graphs have the potential to help build the connection between $\frac{\partial T}{\partial x}$ and heat flow. The concept of slope showed to be very useful to connect the mathematical partial derivative $\frac{\partial T}{\partial x}$ to its physical meaning as expressing the temperature gradient, which in turn is connected to the concept of heat flow. This graphical approach showed to be helpful, but often not sufficient to formulate the relation entirely.

Graphical reasoning was also central to the approach of blended encapsulation. We extended Zandieh's framework based on our conceptual blending perspective as we developed it over the course of the dissertation. Therefore, graphs are considered to help students in blending mathematical and physical knowledge. Again, we see this confirmed in our results as all groups used the graphical elements in their reasoning (see graphical column in Figures 5.12, 5.13, and 5.14).

Apart from the success stories, we also saw proof that graphical reasoning alone does not always guarantee a successful blending of mathematics and physics. In the reasoning of both groups 1 and 3 in Chapter 5, we observed that graphical reasoning eventually led to the correct answer: the students noticed that the tangent lines in the x-direction at the boundaries are horizontal, which led them to the correct boundary conditions of the form $\frac{\partial T}{\partial x} = 0$. However, they were both unable to interpret their answer in a physical way, neither in terms of temperature, nor in terms of heat flow. We can conclude that we have shown that graphical reasoning has positive effects on the blending process, but of course does not always guarantee success. Therefore, we suggest incorporating tasks on graphical reasoning and graphing in instructional materials, and looking for other, complementary ways to support student reasoning.

6.3.4 Stimulating conceptual and semi-quantitative reasoning

Bollen (2017, p. 225) stated in his dissertation: "Many university physics courses focus strongly on mathematical derivations and calculations, and take student understanding in basic concepts for granted". On the other hand, PER literature and this dissertation show that students do not always understand the underlying physical and mathematical concepts. This indicates that explicit attention at the conceptual level is necessary. A more conceptual approach has the ability to confront students with their own level of understanding.

Therefore, in all of our interviews and in the developed tutorial, we formulated questions and tasks that foster conceptual or semi-quantitative reasoning (i.e. reasoning in terms of mathematical and physical concepts without the need to perform extensive calculations). We observed students that were comfortable with solving the partial differential equation with the algorithmic technique of separation of variables, but realized while solving our questions that they had gaps in their understanding. Therefore, we recommend in general to also offer exercises that make use of conceptual and semi-quantitative reasoning. This challenges the students to think about the relevant mathematical and physical concepts without hiding behind the technical calculations that are often a big part of problem solving. This is in line with other research studies that report on interview studies or tutorial development that target a more conceptual approach (e.g. Baily, Dubson, & Pollock, 2013; Bollen, van Kampen, & De Cock, 2018; Mazur, 1997; McDermott, Shaffer, & Rosenquist, 1996; Porter & Heckler, 2020; Wosilait, Heron, Shaffer, & McDermott, 1998).

6.3.5 Stimulating active engagement

The effect of conventional instruction is often small (Heron, 2015). In recent decades, there is a movement towards activating teaching methods at the university level (e.g. Biggs & Tang, 2011; Mazur, 1997; McDermott & Shaffer, 2002). This is related to a constructivist perspective on learning, which goes back at least to Piaget. In this view, learners construct knowledge through their own activity. They interpret concepts and principles in terms of the 'schemata' that they have already developed. Teaching is not considered a way of transmitting knowledge, but of engaging students in active learning, building their knowledge in terms of what they already understand. Biggs and Tang (2011) state that active engagement results in higher level learning activities, for example in terms of Bloom's taxonomy, and to higher achievement, which is essentially the goal of instruction. Most demonstrably effective instructional strategies are 'active', but not all active-learning instructional methods are effective (Freeman et al., 2014).

In the tutorial, we designed the worksheet to stimulate active engagement. We opted for a tutorial format, i.e. worksheet-based group-learning activities created to help students make sense of physics (McDermott et al., 1994; Scherr, 2009), with a focus on conceptual reasoning. The tutorial format is a way to induce active learning while respecting existing instructional constraints (Heron, 2019). By letting students work through the worksheets, we are teaching by questioning and not by telling. We established that the tasks evoked discussion among the students. This way, we give students the opportunity to construct answers themselves, reflect on their developing understanding, and develop reasoning ability.

6.4 Future research

The research presented in this dissertation added to the understanding of student reasoning when blending mathematics and physics. We have defined our research questions in a fairly specific context: undergraduate student reasoning about heat transfer in a one-dimensional system. This context proved to be a fruitful environment for our investigation, and there is certainly much more to find out. In this section, we formulate some ideas for further research.

In general, the first step in moving towards a research-based curriculum is to study student difficulties and student understanding of a certain topic. In a next step, it is then possible to use these findings to purposefully design educational interventions. Therefore, a first big theme for future research is the further investigation of student difficulties and student understanding. As mentioned in section 6.1.1, we set a first step in exploring student difficulties in the context of boundary conditions for the heat/diffusion equation. The interview data showed a lot of variation, and the identified difficulties might just be the tip of the iceberg. More research on this topic is necessary to reach saturation.

Related to this, it would be of interest to also investigate student understanding of and difficulties with other aspects related to the heat equation (or, by extension, other physical partial differential equations). In this dissertation we have focused on boundary conditions and the physical meaning of $\frac{\partial T}{\partial x}$. In future research, it would be of interest to investigate students' physical sense-making of the different terms in the heat equation ($\frac{\partial T}{\partial t}$ and $\frac{\partial^2 T}{\partial x^2}$) and the way the heat equation can be used to reason about what happens in a physical system. We have touched upon aspects of this in our data collection by asking students to explain the different terms in the equation and to reason (semi-) qualitatively about the heat transfer or diffusion process (see questions in Appendix A). This data from the first interview round could help shed light on this in the future.

We limited our research to one-dimensional physical systems. Another way to extend this line of research is to also include two- or three-dimensional physical systems and investigate if our findings still hold. In 1D, the variables x and t each have a different physical role. We observed that this caused difficulties for some of our students. In 2D and 3D, there would be multiple variables in a similar physical role (x, y, and z). This might have an impact.

A second big theme for future research is the role of graphs and graphical reasoning in the blending of mathematics and physics. Starting from the research presented in this dissertation, we can formulate several recommendations for future research.

In this dissertation, we focused mostly on the use of two-dimensional graphs ($T(x)$ for several values of t, or $T(t)$ for several values of x) because it is quite easy for students to make sketches of this. We established positive effects of including this type of graphical reasoning on the blending process. In the tutorial in study 3, we also included three-dimensional computer plots which could be rotated by the students during their problem solving process. This was a first exploration of the implementation of three-dimensional graphs and has not been formally analysed yet. We observed that the students actively used the three-dimensional plot and they rotated it depending on the task and the perspective they found best suited for that task. In future research, we recommend to further investigate how such three-dimensional plots can be used in instruction to help students reason about a physical quantity depending on two variables.

Moreover, also in the case of one-dimensional physical systems described by a function of two variables (e.g. $T(x,t)$) there is still a lot of room for further investigation on graphical reasoning. In a next step, it would be interesting to bring variation into these graphical representations (e.g. students' sketches, pre-made graphs, 2D graphs, 3D graphs, animations, etc.) and investigate which format or combination of formats would be best suited to stimulate the blending of mathematics and physics in student reasoning.

Relating back to our suggestion to also investigate student understanding of 2D or 3D systems that are described by the heat equation, we should acknowledge that it becomes harder for students to sketch and use graphs related to functions $T(x,y,t)$ and $T(x,y,z,t)$ by hand. Therefore, it is a challenge for research to find ways in which we can still benefit from the positive effects of fostering graphical reasoning in these more difficult contexts. An option would be to make use of computer software. We set a first step in this direction with our 3D plots that we used in the tutorials. It would be interesting to investigate the benefits of using computer simulations to foster conceptual understanding, or even to let students create computer simulations themselves.

A third theme for future research is related to the blended encapsulation approach that guided our tutorial design. This approach parses the mathematical concept of a (partial) derivative into different layers (difference, ratio and limit) and identifies the connections to physical meaning at each layer. We hypothesize first of all that the idea of blended encapsulation might also be useful for other physical concepts that are modelled by a derivative or a (partial) derivative. Second, the approach might also be suited for physical phenomena described by other mathematical concepts with a similar layered structure, e.g. an integral. This opens the possibility of incorporating the approach in many situations where physical concepts are described mathematically.

Apart from the specific recommendations for future research presented in this section, we generally recommend further research on the interplay of mathematics and physics in student reasoning, as this is one of the most important research domains within PER. In this dissertation, we showed that the conceptual blending framework offers new perspectives on the widely recognized 'transfer' problem, in which students have difficulties transferring their mathematical knowledge to a physics context.

Methodologically, we developed a tool, *dynamic conceptual blending analysis*, based on the conceptual blending framework to analyse student reasoning and make connections between mathematical and physical knowledge in student reasoning explicit. On a theoretical level, the conceptual blending framework offers new ways to think about this transfer problem. When framing it as a transfer problem, it implies an inherent direction: the knowledge has to be transferred from the mathematical domain to the physical domain. However, when looking at the problem from a blending perspective, both directions are legitimate. This also accounts for what we observe in our data: students go back and forth between their mathematics and physics space, and blending can happen at any time in that process. In the future, it would be worthwhile to further explore the potential of the conceptual blending framework to study the interplay of mathematics and physics in student reasoning.

Referring back to section 1.4, where we introduced the original framework of conceptual blending, there are some aspects of the framework that have been little to not yet incorporated in PER and MER.

In the original framework, Fauconnier and Turner distinguish between three mechanisms to develop emergent structure in the blend. The first one is composition, which says that during blending elements from the input spaces are composed to provide relations that do not exist in the separate input spaces. The second one is completion, in which the person adds new elements based on background models that are brought into the blend unconsciously. Third, there is elaboration, which refers to treating the blend as a simulation and running it imaginatively, which creates new insights. These three mechanisms have not been identified nor used, to our knowledge, in any study in PER nor MER. Because Fauconnier and Turner present these three mechanisms as the processes that thrive the blending process, it implies that identifying and stimulating these processes is important in fostering blending. In the context of the interplay between mathematics and physics in student reasoning, it remains a challenge to, in the first place, identify these processes. In a second phase, these could be crucial in future educational development that intends to stimulate the blending between mathematics and physics.

The construct of generic space has been proven difficult to incorporate in adaptations in PER and MER. The generic space was originally defined as describing the underlying structure of the input spaces, identifying commonalities in content and structure (Fauconnier & Turner, 2003b). In recent conferences like PERC 2020 (PER) and RUME 2020 (MER), the role of the generic space in analysing student reasoning has been a topic for discussion. Until now, only Schermerhorn (2018) and Gregorcic and Haglund (2021) implemented the generic space in their work. The generic space is related to the mechanism of composition, where elements from both input spaces with a similar function are mapped onto each other in the blending process. Based on the PER and MER literature, we see that the first steps in exploring the role of the generic space have been taken, but that the mechanisms underlying the blending process is virgin territory in the context of PER and MER.

To conclude, in this dissertation we added to the understanding of the process of student reasoning when blending mathematics and physics. We explored the value of the conceptual blending perspective to investigate the interplay of mathematical and physical knowledge. With this approach, we definitely set some steps forward in this dissertation, but we also showed that there is much more to discover, learn and improve. It is crucial to extend the knowledge about the way students reason in order to work towards a research-based curriculum.